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Research Paper

Multiplicative character sums and Kloosterman sheaves

Nicholas M. Katz [a,2], Pham Huu Tiep [b,*,1,2]

[a] Department of Mathematics, Princeton University, Princeton, NJ 08544, United States of America

[b] Department of Mathematics, Rutgers University, Piscataway, NJ 08854, United States of America

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ABSTRACT

We are given a prime p, a power q of p, and a prime to p integer a with $q > a \geq 2$. For a nontrivial multiplicative character χ, we consider the one parameter family of character sums

$$t \mapsto -\sum_{x} \chi(x^q + x^a - t),$$

which are the traces of a local system on the $\mathbb{G}_m/\mathbb{F}_p(\chi)$ of nonzero t's. We show that this local system is the pullback of a Kloosterman sheaf $\mathcal{K}_{q,a,\rho}$ (any ρ with $\rho^{q-a} = \chi$), and determine the geometric monodromy group G_{geom} of this $\mathcal{K}$. We also determine G_{geom} for the universal family $\mathcal{F}_{\chi,e}$ of sums $-\sum_x \chi(f_e(x))$, as f_e runs over degree e polynomials with all distinct roots. These local systems $\mathcal{F}_{\chi,e}$ were the main focus of [14, Chapter 4], and our new results for $\mathcal{F}_{\chi,e}$ are the complete determination of G_{geom} in the cases where G_{geom} is finite.

* Corresponding author.
E-mail addresses: nmk@math.princeton.edu (N.M. Katz), tiep@math.rutgers.edu (P.H. Tiep).

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Contents

1. Introduction

We work in characteristic $p > 0$. Fix $q > 1$ a power of p, and a a prime to p integer with $q > a \geq 2$. Denote by $f(x)$ the polynomial

$$f(x) := x^q + x^a.$$

For any $t \neq 0$ in $\overline{\mathbb{F}_p}$, the polynomial in the variable x

$$f(x) - t$$

has all distinct zeroes in $\overline{\mathbb{F}_p}$ (because the only zero of the derivative f' is $x = 0$, and $f(0) = 0$).

Fix an integer $d \geq 2$ which is prime to p. We will consider the one-parameter (parameter t) of "superelliptic" curves $\mathcal{C}_t$ with affine equation

$$C_t : y^d = x^q + x^a - t.$$

For $t \neq 0$, this is the complement of a single point at ∞ in a projective, smooth, geometrically connected curve of genus g with $2g = (d-1)(q-1)$. We denote by

$$\pi : \mathcal{C} \to \mathbb{G}_m$$

this family (in either its affine form, as given above, or in its projective form: it will not matter for consideration of H^1_c). Fix a prime $\ell \neq p$, so that we can speak of $\overline{\mathbb{Q}_\ell}$-adic cohomology. Then we get a local system

$$\mathcal{G} := R^1\pi_!(\overline{\mathbb{Q}_\ell})$$

on $\mathbb{G}_m/\mathbb{F}_p$.

Over the field $\mathbb{F}_p(\mu_d)$ (concretely this is the field $\mathbb{F}_{p^r}$ for r the multiplicative order of p mod d), the group μ_d acts fiberwise on this family of curves, by $\zeta : (x, y) \mapsto (x, \zeta y)$. This action breaks the local system $\mathcal{G}$, when pulled back to $\mathbb{G}_m/\mathbb{F}_p(\mu_d)$, into eigenspaces, one for each $\overline{\mathbb{Q}_\ell}^\times$-valued character χ of μ_d. The trivial eigenspace $\mathcal{G}_{\mathbb{1}} = 0$, and for each character $\chi \neq \mathbb{1}$ of μ_d, the χ-eigenspace $\mathcal{G}_\chi$ has rank $q - 1$. By means of the surjection

$$\mathbb{F}_p(\mu_d)^\times = \mathbb{F}_{p^r}^\times \twoheadrightarrow \mu_d, x \mapsto x^{(p^r-1)/d},$$

we view characters of μ_d as characters of $\mathbb{F}_p(\mu_d)^\times$ of order dividing d.

Given a character χ of $\mathbb{F}_p(\mu_d)^\times$, for any finite extension $k/\mathbb{F}_p(\mu_d)$ we can form the character

$$\chi_k := \chi \circ \mathrm{Norm}_{k/\mathbb{F}_p(\mu_d)}$$

of the multiplicative group $k^\times$.

For $k/\mathbb{F}_p(\mu_d)$ a finite extension, $\chi \neq \mathbb{1}$ a character of order dividing d, and $t \in k^\times$, we have the Lefschetz trace formula for the local system $\mathcal{G}_\chi$:

$$\mathrm{Trace}(\mathrm{Frob}_{t,k}|\mathcal{G}_\chi) = -\sum_{x \in k} \chi_k(f(x) - t). \tag{1.0.1}$$

These traces are the "multiplicative character sums" of the title. It will turn out that the local systems $\mathcal{G}_\chi$ which give rise to them are closely related to certain Kloosterman sheaves, and it is this relation which will allow us to compute the geometric monodromy groups of the $\mathcal{G}_\chi$.

More precisely, for a given character χ of $\mathbb{F}_p(\mu_d)^\times$ of order d, we choose a multiplicative character ρ of $k := \mathbb{F}_p(\mu_{d(q-a)})$ with

$$\rho^{q-a} = \chi_k,$$

and construct a Kloosterman sheaf $\mathcal{K}_{q,a,\rho}$ on $\mathbb{G}_m/k$, see (3.2.1), whose Kummer $[q - a]^\star$ pullback is $\mathcal{G}_\chi$. We refer to [22, §1.2] for a general discussion of Kloosterman and hypergeometric sheaves.

For fixed q, a, χ, we are concerned with the family $\mathcal{G}_\chi$, whose trace function is

$$t \mapsto -\sum_x \chi(x^q + x^a - t).$$

[Strictly speaking, for each finite extension $L/\mathbb{F}_p(\mu_d)$ and each $t \in L^\times$, we consider the sum $-\sum_{x \in L} \chi_L(x^q + x^a - t)$.]

Suppose instead we fix invertible scalars $\alpha, \beta \in \overline{\mathbb{F}_p}$ and look at the family

$$t \mapsto -\sum_x \chi(\alpha x^q + \beta x^a - t).$$

This family is no more general. Indeed, if we choose a scalar λ with $\lambda^{a-q} = \alpha/\beta$, then after the change of variable $x \mapsto \lambda x$, we obtain

$$\chi(\alpha\lambda^q x^q + \beta\lambda^a x^a - t) = \chi(\alpha\lambda^q)\chi(x^q + x^a - t/(\alpha\lambda^q)).$$

Over a big enough extension of $\mathbb{F}_p(\alpha, \beta, \lambda)$ we have $\chi(\alpha\lambda^q) = 1$, so we are dealing with a harmless pullback on the t line of our original family.

Our main results include the determination of the geometric monodromy groups $G_{\rm geom}$ of the Kloosterman sheaves $\mathcal{K}_{q,a,\rho}$ (as defined in (3.2.1)), and of their Kummer pullbacks $\mathcal{G}_\chi$, see Theorems 4.2, 6.1, 6.2 for the finite cases, and Theorems 8.1, 8.2, 8.3 for the infinite cases. We also compute the groups $G_{\rm geom}$ for the universal families $\mathcal{F}_{\chi,e}$ of character sums $-\sum_x \chi(f_e(x))$, as f_e runs over degree e polynomials with all distinct roots, see also (5.0.1). These local systems $\mathcal{F}_{\chi,e}$ were the main focus of [14, Chapter 4], and our new results for $\mathcal{F}_{\chi,e}$ are the complete determination of $G_{\rm geom}$ in the cases where $G_{\rm geom}$ is finite, see Theorems 7.1 and 7.2.

2. Analysis of the local system $\mathcal{F} := f_\star\overline{\mathbb{Q}_\ell}/\overline{\mathbb{Q}_\ell}$

Our first task is to calculate the local monodromy at 0 of $f_\star\overline{\mathbb{Q}_\ell}$. Over $\overline{\mathbb{F}_p[[t]]}$, the polynomial $f(x) - t = x^q + x^a - t$ has $q - a$ roots which, at time $t = 0$, are the $q - a$ nonzero roots of $x^q + x^a$, as one sees immediately from either Hensel or Newton. The other a roots may be seen as follows. Over $\overline{\mathbb{F}_p}[[x]]$, $x^q + x^a$ is an $a^{\rm th}$ power of a uniformizer $u = x +$ higher terms, and our equation becomes $u^a = t$. Thus the $I(0)$-representation of $f_\star\overline{\mathbb{Q}_\ell}$ is the direct sum

$$(q - a \text{ copies of } \mathbb{1}) \oplus (\oplus_{\rho:\rho^a=\mathbb{1}}\mathcal{L}_\rho).$$

And so the $I(0)$-representation of $\mathcal{F}$ is the direct sum

$$(q - a \text{ copies of } \mathbb{1}) \oplus (\oplus_{\rho:\rho^a=\mathbb{1}\neq\rho}\mathcal{L}_\rho).$$

In particular, $\mathcal{F}$ is tame at 0.

By Frobenius reciprocity, the only $\mathcal{L}_\rho$ which can occur in the $I(\infty)$-representation of $f_\star\overline{\mathbb{Q}_\ell}$ are those for which $f^\star\mathcal{L}_\rho$ occurs in the $I(\infty)$-representation of $\overline{\mathbb{Q}_\ell}$ (necessarily with multiplicity 1 if it occurs at all, simply because $\overline{\mathbb{Q}_\ell}$ has rank one). Because f has degree q, these are the ρ with $\rho^q = \mathbb{1}$, which is to say only $\rho = \mathbb{1}$. Thus the $I(\infty)$-representation of $\mathcal{F} := f_\star\overline{\mathbb{Q}_\ell}/\overline{\mathbb{Q}_\ell}$ is totally wild.

We next calculate the Swan conductor $\mathsf{Swan}_\infty(\mathcal{F})$.

Lemma 2.1. $\mathsf{Swan}_\infty(\mathcal{F}) = q - a$.

Proof. We use the fact that $\mathsf{Swan}_\infty(\mathcal{F}) = \mathsf{Swan}_\infty(f_\star\overline{\mathbb{Q}_\ell})$. On the one hand, by the Leray spectral sequence for f, we have an equality of Euler-Poincaré characteristics

$$1 = \mathsf{EP}(\mathbb{A}^1, \overline{\mathbb{Q}_\ell}) = \mathsf{EP}(\mathbb{A}^1, f_\star\overline{\mathbb{Q}_\ell}) = \mathsf{EP}(\mathbb{G}_m, f_\star\overline{\mathbb{Q}_\ell}) + \#\{x : f(x) = 0\}.$$

Because $f_\star\overline{\mathbb{Q}_\ell}$ is lisse on $\mathbb{G}_m$ and tame at 0, we have

$$\mathsf{EP}(\mathbb{G}_m, f_\star\overline{\mathbb{Q}_\ell}) = -\mathsf{Swan}_\infty(f_\star\overline{\mathbb{Q}_\ell})$$

and

$$\#\{x : f(x) = 0\} = 1 + q - a.$$

Thus

$$1 = -\mathsf{Swan}_\infty(f_\star\overline{\mathbb{Q}_\ell}) + 1 + q - a,$$

and we are done. $\square$

Lemma 2.2. *The lisse sheaf $\mathcal{F}|\mathbb{G}_m$ has a descent through the $[q-a]$ power map, to the $[t \mapsto (-1)^a t]^\star$ pullback of Sawin's Kloosterman sheaf $\mathcal{K}l(\mathsf{Char}(q-a), \mathsf{Char}_{\mathrm{ntriv}}(a))$.*

Proof. The trace function of $\mathcal{F}$ at time $t \in k^\times$ is

$$-1 + \#\{x \in k : f(x) = t\} = -1 + \#\{x \in k : f(x^q) = t^q\} = -1 + \#\{x \in k : f(x) = t^q\},$$

where the first equality holds because f has coefficients in $\mathbb{F}_p$, and the second equality holds because $x \mapsto x^q$ is bijective on k. We now massage $f(x) = t^q$ by taking

$$x^q + x^a = t^q$$

and making the change of variable $x \mapsto tx$ to write the equation as

$$t^q x^q + t^a x^a = t^q, \text{ i.e., } x^q + t^{a-q} x^a = 1,$$

which is the $[q-a]^\star$ pullback of the equation

$$x^q + t^{-1} x^a = 1.$$

This shows that $\mathcal{F}$ is $[q-a]^\star\mathcal{K}$ for some lisse $\mathcal{K}$ on $\mathbb{G}_m$.

Now consider Sawin's Kloosterman sheaf [21, 9.2], which at time $t \in k^\times$ is -1 plus the number of k-solutions of

$$x^{q-a}(1-x)^a = t.$$

We rewrite this first as

$$x^q((1-x)/x)^a t, \text{ i.e., } x^q(1/x-1)^a = t,$$

then by $x \mapsto 1/x$ as

$$(1/x^q)(x-1)^a = t,$$

then by $x \mapsto x+1$ as

$$(1/(x^q+1))x^a = t, \text{ i.e., } (1/t)x^a = x^q+1, \text{ i.e., } x^q - (1/t)x^a + 1 = 0,$$

and finally, by $x \mapsto -x$, as

$$-x^q - (1/t)(-x)^a + 1 = 0, \text{ i.e., } x^q + (1/t)((-1)^a x^a = 1,$$

which is the $[t \mapsto (-1)^a t]$ pullback of the trace function of our descent.

We will show below that $\mathcal{F}$ is geometrically irreducible, and so a fortiori arithmetically irreducible. As it has the same trace function as $[q-a]^\star\mathcal{K}$, it follows by Chebotarev that $[q-a]^\star\mathcal{K}$ is also arithmetically irreducible, and that $\mathcal{F}$ and $[q-a]^\star\mathcal{K}$ are arithmetically isomorphic. □

Lemma 2.3. *Let $A > B \geq 2$ be two integers with* $\gcd(A,B) = 1$. *In any characteristic p, for $f(x) = x^A + x^B$,*

$$\mathcal{F}_{A,B} := f_\star\overline{\mathbb{Q}_\ell}/\overline{\mathbb{Q}_\ell}$$

is geometrically irreducible on any dense open set of $\mathbb{A}^1$ on which it is lisse.

Proof. In general, to prove that $f_\star\overline{\mathbb{Q}_\ell}/\overline{\mathbb{Q}_\ell}$ is geometrically reducible for a given polynomial f, it is equivalent to prove that the two-variable polynomial $f(x) - f(y)$ is the product of powers of at least two distinct geometrically irreducible polynomials. So it suffices to show that the polynomial

$$(f(x) - f(y))/(x-y)$$

is geometrically irreducible. We will show this for $f(x) = x^A + x^B$, i.e., that the polynomial

$$(x^A - y^A)/(x-y) + (x^B - y^B)/(x-y)$$

is geometrically irreducible, or equivalently, that the affine curve defined by its vanishing is geometrically irreducible. This affine curve is the complement of finitely many points in the projective curve defined by the vanishing of the three-variable polynomial

$$(x^A - y^A)/(x-y) + z^{A-B}(x^B - y^B)/(x-y).$$

So it is equivalent to show that this last polynomial is geometrically irreducible.

Suppose first that $p \nmid A$. Choose a root of unity ζ_A of order A. Then $x - \zeta_A y$ is a factor of $(x^A - y^A)/(x-y)$ with multiplicity one, and does not occur in $(x^B - y^B)/(x-y)$. Thus our polynomial in Z is Eisenstein for the prime $x - \zeta_A y$ in the UFD $\overline{\mathbb{F}_p}[x, y]$.

Suppose next that $p|A$. Because $\gcd(A, B) = 1$, we must have $p \nmid B$. In this case, our polynomial in z is the palindrome of an Eisenstein polynomial for the prime $x - \zeta_B y$ in the UFD $\overline{\mathbb{F}_p}[x, y]$. □

Remark 2.4. What happens to Lemma 2.3 in the case $B = 1$? Here the three-variable polynomial in question is

$$(x^A - y^A)/(x-y) + z^{A-1}.$$

If $p \nmid A$, this is again Eisenstein.

On the other hand, it A is a power q of p, this polynomial is

$$(x-y)^{q-1} + z^{q-1},$$

which is always reducible (except in the trivial case $q = 2$).

However, if $p|A$ but A is not a power of p, then this polynomial is irreducible. For if $A = nq$ with $n \geq 2, p \nmid n$, then our polynomial is

$$(x^{nq} - y^{nq})/(x-y) + z^{nq-1}.$$

But

$$(x^{nq} - y^{nq})/(x-y) = (x^n - y^n)^q/(x-y),$$

which contains $x - \zeta_n y$ with multiplicity q. As $\gcd(nq-1, q) = 1$, the extension defined by this polynomial in z is fully ramified over the prime $x - \zeta_n y$. So in this case also the polynomial is irreducible.

Returning now to $f(x) = x^q + x^a$, we now have the following information about its $\mathcal{F}$.

Theorem 2.5. *$\mathcal{F}$ is lisse of rank $q-1$ on $\mathbb{G}_m$ and geometrically irreducible. Its $I(\infty)$-representation has all slopes $(q-a)/(q-1)$, and its $I(0)$-representation is*

$$(q - a \text{ copies of } \mathbb{1}) \oplus (\oplus_{\nu:\nu^a=\mathbb{1},\nu\neq\mathbb{1}}\mathcal{L}_\nu).$$

Lemma 2.6. *The Fourier Transform $\mathrm{FT}_\psi(\mathcal{F})$ is lisse of rank $a-1$ on $\mathbb{G}_m$, its $I(0)$-representation has all slopes $(q-a)/(a-1)$ and its $I(\infty)$-representation is $\oplus_{\nu:\nu^a=\mathbb{1},\nu\neq\mathbb{1}}\mathcal{L}_\nu$.*

Proof. This is a straightforward application of Laumon's results on the local monodromy of FT_ψ. □

3. $\mathcal{G}_\chi$ via middle additive convolution

We continue with $f(x) = x^q + x^a$ and its local system $\mathcal{F}$ as defined in §2.

For any nontrivial multiplicative character χ, the middle additive convolution $\mathcal{G}_\chi := \mathcal{F} \star_{mid,+} \mathcal{L}_\chi$ (see [14, (6.1.2)]) is geometrically irreducible, lisse of rank $q-1$ on $\mathbb{G}_m$, and its trace function at time $t \in k^\times$, for $k/\mathbb{F}_p(\chi)$ a finite extension, is given by

$$\mathrm{Trace}(\mathrm{Frob}_{t,k}|\mathcal{G}_\chi) = -\sum_{x \in k} \chi_k(t - x^q - x^a),$$

[Notice that after the (at worst a quadratic) extension of the ground field $\mathbb{F}_p(\chi)$ to $\mathbb{F}_p(\chi, \sqrt{\chi(-1)})$, we always have $\chi_k(-1) = 1$, in which case the trace function is also given by

$$\mathrm{Trace}(\mathrm{Frob}_{t,k}|\mathcal{G}_\chi) = -\sum_{x \in k} \chi_k(x^q + x^a - t),$$

cf. (1.0.1).] By its relation to Fourier transform, cf. [12, 2.10.5, 3.3.5] and Laumon's theory of local Fourier Transform [23], one knows that $\mathcal{G}_\chi$ has all $I(\infty)$-slopes $(q-a)/(q-1)$. To describe its $I(0)$-representation, denoted $\mathcal{G}_\chi(0)$, we use the fact that the quotient space $\mathcal{G}_\chi(0)/\mathcal{G}_\chi(0)^{I(0)}$ is given by

$$\mathcal{G}_\chi(0)/\mathcal{G}_\chi(0)^{I(0)} \cong \mathcal{L}_\chi \otimes (\mathcal{F}(0)/\mathcal{F}(0)^{I(0)}) = \mathcal{L}_\chi \otimes (\oplus_{\nu:\nu^a = \mathbb{1} \neq \nu} \mathcal{L}_\nu).$$

Suppose first that $\chi^a \neq \mathbb{1}$. Then the $I(0)$-representation of $\mathcal{G}_\chi$ is

$$(q - a \text{ copies of } \mathbb{1}) \oplus \mathcal{L}_\chi \otimes (\oplus_{\nu:\nu^a = \mathbb{1} \neq \nu} \mathcal{L}_\nu).$$

But if $\chi^a = \mathbb{1}$, with χ nontrivial, then χ occurs among the $\mathcal{L}_\nu$, so in this case the $I(0)$-representation of $\mathcal{G}_\chi$ is

$$(q - a - 1 \text{ copies of } \mathbb{1}) \oplus Unip(2) \oplus (\oplus_{\nu:\nu^a = \mathbb{1} \neq \nu, \nu \neq \chi} \mathcal{L}_\nu).$$

Lemma 3.1. *The lisse sheaf $\mathcal{G}_\chi|\mathbb{G}_m$ has a descent through the $[q-a]$ power map, to a (multiplicative translate of a) Kloosterman sheaf, denoted $\mathcal{K}$.*

Proof. The statement is geometric, but we can show the existence of such a descent by working on $\mathbb{G}_m/\mathbb{F}_p(\chi,$ all $(q-a)^{\mathrm{th}}$ roots of $\chi)$. For $f(x) = x^a + x^q$, the trace function of $\mathcal{G}_\chi$ is

$$t \mapsto -\sum_x \chi(t - f(x)).$$

There is a unique character χ_0 with $\chi_0^q = \chi$. In terms of χ_0, the trace function is

$$t \mapsto -\sum_x \chi_0(t^q - f(x)^q) = -\sum_x \chi_0(t^q - f(x^q)) = -\sum_x \chi_0(t^q - f(x)),$$

the first equality because $f(x) \in \mathbb{F}_p[x]$, the second because $x \mapsto x^q$ is bijective on each finite extension of $\mathbb{F}_p$.

So the trace function is

$$t \mapsto -\sum_x \chi_0(t^q - x^a - x^q) = (\text{by } x \mapsto tx) \;\; -\sum_x \chi_0(t^q - t^a x^a - t^q x^q)$$
$$= -\chi_0(t^q) \sum_x \chi_0(1 - t^{a-q} x^a - x^q).$$

Choose a $(q-a)^{\text{th}}$ root ρ of χ. Recalling that $\chi_0^q = \chi$, we see that this is the $[q-a]^\star$ pullback of the local system $\mathcal{K}$ whose trace function is

$$t \mapsto -\rho(t) \sum_x \chi_0(1 - x^a/t - x^q).$$

This $\mathcal{K}$ is visibly lisse on $\mathbb{G}_m$ of rank $q-1$. Because $[q-a]^\star\mathcal{K} = \mathcal{G}_\chi$, $\mathcal{K}$ is tame at 0 and has all ∞-slopes $1/(q-1)$, so by [9, 8.7.1], $\mathcal{K}$ is a multiplicative translate of a Kloosterman sheaf. □

We now give a different argument, which explicitly exhibits a Kloosterman sheaf whose existence was shown in Lemma 3.1.

Theorem 3.2. *Choose a multiplicative character ρ with $\rho^{q-a} = \chi$. Then on $\mathbb{G}_m$ we have a geometric isomorphism of $\mathcal{G}_\chi$ with (a multiplicative translate of) the Kummer pullback $[q-a]^\star\mathcal{K}$ of the Kloosterman sheaf*

$$\mathcal{K} = \mathcal{K}_{q,a,\rho} := \mathcal{K}l(\mathsf{Char}(q-a) \cup \rho\mathsf{Char}_{\text{ntriv}}(a)) \tag{3.2.1}$$

Proof. Denote by $j : \mathbb{G}_m \subset \mathbb{A}^1$ the inclusion. The first key point is that, by [11, 5.5.1], $\mathcal{G}_\chi$ is the additive ! convolution of $\mathcal{F}$ with $\mathcal{L}_\chi$. Because $\mathcal{F}$ is totally wild at ∞, [12, 2.9.4] shows that $\mathcal{G}_\chi$ is isomorphic to the middle additive convolution of $\mathcal{F}$ with $\mathcal{L}_\chi$. Then by [12, 2.10.5] we have

$$\mathrm{FT}(j_\star\mathcal{G}_\chi) \cong j_\star(\mathcal{L}_{\overline{\chi}} \otimes j^\star(\mathrm{FT}(\mathcal{F}))).$$

The second key point is that, by Lemma 2.2, we have an isomorphism

$$\mathcal{F} \cong j_\star[q-a]^\star\mathcal{K}l(\mathsf{Char}(q-a) \cup \mathsf{Char}_{\text{ntriv}}(a))$$

up to a multiplicative translate. We then apply [10, Theorem 9.3.2], which tells us how to compute the Fourier transform of the Kummer pullback of a hypergeometric sheaf. In the case at hand, it gives isomorphisms (up to a multiplicative translate)

$$\begin{aligned}\mathrm{FT}(\mathcal{F}) &\cong \mathrm{FT}\big(j_\star[q-a]^\star\mathcal{K}l(\mathsf{Char}(q-a)\cup\mathsf{Char}_{\mathrm{ntriv}}(a))\big)\\ &\cong j_\star[q-a]^\star\big(\mathbf{Cancel}(\mathcal{H}yp(\mathsf{Char}(q-a);\mathsf{Char}(q-a)\cup\mathsf{Char}_{\mathrm{ntriv}}(a)))\big)\\ &\cong j_\star[q-a]^\star(\mathcal{H}yp(\varnothing;\mathsf{Char}_{\mathrm{ntriv}}(a)).\end{aligned}$$

Then recalling that $\rho^{q-a}=\chi$, we get

$$\mathrm{FT}(j_\star\mathcal{G}_\chi)\cong j_\star(\mathcal{L}_{\overline{\chi}}\otimes\mathrm{FT}(\mathcal{F}))\cong j_\star[q-a]^\star(\mathcal{H}yp(\varnothing;\overline{\rho}\mathsf{Char}_{\mathrm{ntriv}}(a)).$$

By Fourier inversion, and a second application of [10, Theorem 9.3.2], we get, up to a multiplicative translate,

$$j_\star\mathcal{G}_\chi\cong j_\star[q-a]^\star\mathcal{K}l(\mathsf{Char}(q-a)\cup\rho\mathsf{Char}_{\mathrm{ntriv}}(a)).\quad\square$$

Theorem 3.3. *For any a with $q>a\geq 2$ which is prime to p, and for any multiplicative character ρ, the Kloosterman sheaf*

$$\mathcal{K}=\mathcal{K}_{q,a,\rho}=\mathcal{K}l(\mathsf{Char}(q-a)\cup\rho\mathsf{Char}_{\mathrm{ntriv}}(a))$$

is primitive, with the single exceptional case $a=q-1$ and $\rho=\mathbb{1}$. In the exceptional case, $\mathcal{K}=\mathcal{K}l(\mathsf{Char}(q-1))=[q-1]_\star\mathcal{L}_\psi$ is Kummer induced. If $(a,\rho)\neq(q-1,\mathbb{1})$, and $(q,\rho^{q-a})\neq(5,\mathbb{1})$, then $\mathcal{K}$ satisfies the condition $(\mathbf{S}+)$ *as defined in [22, Definition 1.1.4].*

Proof. (a) We will assume $(a,\rho)\neq(q-1,\mathbb{1})$. First we prove the primitivity. By Pink's criterion [8, Lemma 11], a Kloosterman sheaf is induced if and only if it is Kummer induced. So we must show that for any nontrivial character χ, the Kloosterman sheaf $\mathcal{L}_\chi\otimes\mathcal{K}$ is not isomorphic to $\mathcal{K}$.

Suppose first that $q-a\geq 2$, and that such a χ exists. After $[q-a]^\star$ pullback, $\mathcal{L}_{\chi^{q-a}}\otimes[q-a]^\star\mathcal{K}$ has χ^{q-a} occurring at least $q-a$ times in the $I(0)$-representation. But in $\mathcal{G}_\chi$, the only character occurring more than once is $\mathbb{1}$. Thus this χ has order dividing $q-a$. Then $\mathsf{Char}(q-a)$ is stable by multiplication by χ, and hence $\rho\mathsf{Char}_{\mathrm{ntriv}}(a)$ must be stable by multiplication by χ, and hence $\mathsf{Char}_{\mathrm{ntriv}}(a)$ must be stable by multiplication by χ. Choose Λ of order a. Then $\chi\Lambda$ must have order dividing a. If so, then χ has order dividing a. But $\gcd(a,q-a)=1$, and hence χ is trivial.

Suppose next that $q-a=1$, i.e., that $a=q-1$. So $\mathcal{K}$ is $\mathcal{K}l(\mathbb{1},\rho\mathsf{Char}_{\mathrm{ntriv}}(q-1))$. We have $q-2>1$ except in the case $q=3$, $a=2$, where our Kloosterman is $\mathcal{K}l(\mathbb{1},\rho\chi_2)$, which is induced precisely when $\rho=\mathbb{1}$. When $q-2>1$, we argue as follows. Looking at the $[q-1]^\star$ pullback, whose $I(0)$-characters are

$$\mathbb{1}\text{ (once)},\quad\rho^{q-1}\ (q-2\text{ times}),$$

we see that any χ must have $\chi^{q-1}\rho^{q-1}$ occurring (at least) $q-2$ times, so $\chi^{q-1}\rho^{q-1}$ must be ρ^{q-1}. Therefore $\chi^{q-1}=\mathbb{1}$. So if χ is nontrivial, then $\chi\in\mathsf{Char}_{\mathrm{ntriv}}(q-1)$. But χ occurs once in $\mathcal{L}_\chi \otimes \mathcal{K}$, so if χ is nontrivial then $\chi \in \rho\mathsf{Char}_{\rm ntriv}(q-1)$. Therefore $\rho^{q-1} = \mathbb{1}$. If $\rho \neq \mathbb{1}$, then $\mathbb{1} \in \rho\mathsf{Char}_{\rm ntriv}(q-1)$, and hence $\mathbb{1}$ occurs twice in the character list of $\mathcal{K}$. So in this $\rho \neq \mathbb{1}$ case, $\mathbb{1}$ occurs twice, but no other character occurs twice, whereas in $\mathcal{L}_\chi \otimes \mathcal{K}$, χ occurs twice. Thus $\mathcal{K}$ is primitive so long as $\rho \neq \mathbb{1}$.

(b) By [21, Theorem 1.7], $\mathcal{K}$ satisfies ($\mathbf{S}+$) unless $q = 5$ (note that $p = 3$ when $q = 9$). It remains to consider the case $q = p = 5$, in which case we still know by [21, Lemma 2.3] that $\mathcal{K}$ is tensor indecomposable. By way of contradiction, assume that $\mathcal{K}$ is 2-tensor induced, i.e. $G_{\rm geom}$ permutes the 2 tensor factors U_1, U_2 of a tensor decomposition $V = U_1 \otimes U_2$ of the underlying representation. Let g_0 denote a generator of the image of $I(0)$ in $G_{\rm geom}$, with spectrum $\mu_{q-a} \cup \alpha(\mu_a \smallsetminus \{1\})$ for some root of unity α (note that g_0 has finite order). Also let H denote the subgroup consisting of all elements in $G_{\rm geom}$ that fix both U_1 and U_2, so that $[G_{\rm geom} : H] = 2$ and H is a closed normal subgroup of $G_{\rm geom}$. Hence if $g_0 \in H$, then $G_{\rm geom} = H$ by [21, Theorem 4.1], a contradiction. So g_0 flips U_1 and U_2. Arguing as in the proof of [22, Proposition 5.2.1], we see that there are $u, v \in \mathbb{C}^\times$ such that the spectrum of g_0 is $\{u, v, w, -w\}$ (with counting multiplicities), where $w^2 = uv$. We will now show that $\rho^{q-a} = \mathbb{1}$.

Suppose $a = 2$. Since μ_3 cannot contain both w and $-w$, we may assume that $w = -\alpha$, and then $\alpha = -w \in \mu_3$. Hence $\alpha^3 = 1$ and $\rho^3 = \mathbb{1}$.

Suppose $a = 3$. If $\{w, -w\} \cap \mu_2 = \varnothing$, then $\{w, -w\} = \alpha(\mu_3 \smallsetminus \{1\})$, which implies $\zeta_3^2 = -\zeta_3$, a contradiction. So we may assume $w \in \mu_2$, whence $\{w, -w\} = \mu_2$, and $\{u, v\} = \alpha(\mu_3 \smallsetminus \{1\})$. In this case, $1 = w^2 = uv = \alpha^2$, and hence $\rho^2 = \mathbb{1}$.

Suppose $a = 4$. First assume that both w and $-w$ belong to $\alpha(\mu_4 \smallsetminus \{1\})$. Then $\{w, -w\} = \{\alpha\zeta_4, \alpha\zeta_4^3\}$, which implies that $uv = w^2 = -\alpha^2$. Also, $\{u, v\} = \{1, -\alpha\}$, so $uv = -\alpha$. Thus $\alpha = 1$, which means $\rho = \mathbb{1}$, the exceptional case $(a, \rho) = (q-1, \mathbb{1})$. So we may assume that $w = 1$, whence $uv = w^2 = 1$, and $\{-1, u, v\} = \{-\alpha, \alpha\zeta_4, \alpha\zeta_4^3\}$, which implies that $uv = \alpha^3$ and thus $\alpha^3 = 1$. We also have $1 = (-1)^4 = (\alpha\zeta_4^j)^4 = \alpha^4$ (for some j). It follows that $\alpha = 1$, again the exceptional case.

Note that when $q = 5$, $a = 2, 3$, and $\rho^{q-a} = \mathbb{1}$, $\mathcal{K}$ is 2-tensor induced. Tensoring with a Kummer sheaf $\mathcal{L}_{\bar{\rho}}$, which does not affect being 2-tensor induced, we reduce to the case $\rho = \mathbb{1}$. In this case our $\mathcal{K}$ is

$$\mathcal{K}_0 := \mathcal{K}l(\mathsf{Char}(2), \mathsf{Char}_{\rm ntriv}(3)) = \mathcal{K}l(\chi_3, \overline{\chi_3}, \mathbb{1}, \chi_2),$$

which by [16, Theorem 6.3] is 2-tensor induced. Alternatively, by [21, Theorem 9.3], $G_{{\rm geom},\mathcal{K}_0}$ is S_5 acting on its deleted permutation module $\mathbb{C}^4$, and this action is 2-tensor induced: the normal subgroup A_5 acts on $\mathbb{C}^4$ as $\mathrm{SL}_2(5)$ acting on the tensor product of the two inequivalent 2-dimensional irreducible representations, and these two factors are permuted by S_5. $\square$

Lemma 3.4. *For any a with $q > a \geq 2$ which is prime to p, and for any multiplicative character ρ, the Kloosterman sheaf*

$$\mathcal{K} = \mathcal{K}_{q,a,\rho} = \mathcal{K}l(\mathsf{Char}(q-a) \cup \rho\mathsf{Char}_{\rm ntriv}(a))$$

has geometric determinant $\mathcal{L}_\Lambda$, with $\Lambda = \chi_2\rho^{a-1}$ if q is odd, and with $\Lambda = \rho^{a-1}$ if q is even.

Proof. For any Kloosterman sheaf of rank ≥ 2, its geometric determinant is $\mathcal{L}_\Lambda$, for Λ the product of its characters. Thus

$$\Lambda = \rho^{a-1}\Big(\prod_{\sigma\in\mathsf{Char}_{\rm ntriv}(a)} \sigma\Big)\Big(\prod_{\tau\in\mathsf{Char}(q-a)} \tau\Big).$$

When q is odd, precisely one of $a, q-a$ is odd (simply because their sum, being q, is odd). So one of $\prod_{\sigma\in\mathsf{Char}_{\rm ntriv}(a)} \sigma$ or $\prod_{\tau\in\mathsf{Char}(q-a)} \tau$ is χ_2, and the other is $\mathbb{1}$. When q is even, then a (and hence $q-a$) is odd (because $p \nmid a$), so both $\prod_{\sigma\in\mathsf{Char}_{\rm ntriv}(a)} \sigma$ and $\prod_{\tau\in\mathsf{Char}(q-a)} \tau$ are $\mathbb{1}$. $\square$

Lemma 3.5. *For q a power of an odd prime p, $2 \leq a < q$ a prime to p integer, and $\chi = \chi_2$ the quadratic character, the sheaf $\mathcal{G}_\chi|(\mathbb{G}_m/\mathbb{F}_p)$, whose trace function is*

$$t \mapsto -\sum_x \chi(x^q + x^a - t),$$

is, geometrically, symplectically self-dual.

Proof. Indeed, the sheaf $\mathcal{G}_\chi|\mathbb{G}_m$ is the $R^1\mathrm{pr}_!\overline{\mathbb{Q}_\ell}$ for the family, over $\mathbb{G}_m/\mathbb{F}_p$, of genus $(q-1)/2$ curves of affine equation

$$y^2 = x^q + x^a - t,$$

(with a single point at ∞). So the result, for lack of a better reference, is a special case of [20, 9.1.14 or 9.1.16]. $\square$

Theorem 3.6. *Let χ be nontrivial, and $\rho^{q-a} = \chi$. Suppose that $q \neq 3$ and that the Kloosterman sheaf*

$$\mathcal{K} = \mathcal{K}_{q,a,\rho} = \mathcal{K}l(\mathsf{Char}(q-a) \cup \rho\mathsf{Char}_{\rm ntriv}(a))$$

has infinite $G_{\rm geom}$. Then $\mathcal{K}$ is Lie self-dual only when χ has order 2 and in this case, the duality is symplectic by Lemma 3.5.

Proof. We exclude $q = 3$, because then $\mathcal{K}$ has rank 2. As $\mathcal{K}$ is geometrically irreducible, $G^\circ_{\rm geom}$ is a semisimple (by [4, 1.3.8]) subgroup of GL_2. So if $G_{\rm geom}$ is infinite, then $G^\circ_{\rm geom}$ must be $\mathrm{SL}_2 = \mathrm{Sp}_2$, hence $\mathcal{K}$ is indeed Lie self-dual.

By Theorem 3.3, $\mathcal{K}$ satisfies (**S**+). So if in addition G_{geom} is infinite, then by [22, Lemma 1.1.6] $\mathcal{K}$ is Lie irreducible.

Suppose now that $\mathcal{K}$ is Lie self-dual. Then $\mathcal{K}$ and its dual are Lie-isomorphic. As $\mathcal{K}$ is tame at 0 and has all ∞-slopes < 1, then by [22, Proposition 2.4.6] there exists a multiplicative character Λ and an isomorphism $\mathcal{K}^\vee \cong \mathcal{L}_\Lambda \otimes \mathcal{K}$. Taking a square root σ of Λ, we get that $\mathcal{L}_\sigma \otimes \mathcal{K}$ is self-dual. Taking the $[q-a]^\star$ pullback, and putting $\tau := \sigma^{q-a}$, we get that $\mathcal{L}_\tau \otimes \mathcal{G}_\chi$ is self-dual. Therefore the list of characters occurring in the $I(0)$-representation of $\mathcal{L}_\tau \otimes \mathcal{G}_\chi$ is stable by inversion. This list of characters is

$$(\tau \text{ repeated } q-a \text{ times}, \ \tau\chi\mathsf{Char}_{\text{ntriv}}(a)).$$

Suppose first that $q-a \geq 2$. Then τ is the only character occurring more than once. The dual list is

$$1/\tau \text{ repeated } q-a \text{ times}, \ (1/(\tau\chi))\mathsf{Char}_{\text{ntriv}}(a)).$$

In this list, $1/\tau$ is the only character occurring more than once. So we must have $\tau = 1/\tau$, i.e., either $\tau = \mathbb{1}$ or p is odd and $\tau = \chi_2$, the quadratic character. Thus if $\mathcal{L}_\sigma \otimes \mathcal{K}$ is self-dual, either $\sigma^{q-a} = \mathbb{1}$ or p is odd and $\sigma^{q-a} = \chi_2$. But tensoring with $\mathcal{L}_{\chi_2}$ does not alter self-duality. Hence $\mathcal{G}_\chi$ is self-dual. Its list of $I(0)$-characters is

$$(\mathbb{1} \text{ repeated } q-a \text{ times}, \ \chi\mathsf{Char}_{\text{ntriv}}(a)).$$

The list for its dual is

$$(\mathbb{1} \text{ repeated } q-a \text{ times}, \ (1/\chi)\mathsf{Char}_{\text{ntriv}}(a)).$$

Comparing the two lists, we get an equality of lists

$$\chi\mathsf{Char}_{\text{ntriv}}(a) = (1/\chi)\mathsf{Char}_{\text{ntriv}}(a), \text{ i.e., } \chi^2\mathsf{Char}_{\text{ntriv}}(a) = \mathsf{Char}_{\text{ntriv}}(a).$$

If $\chi^2 = \mathbb{1}$, we are in the case with χ of order 2. If $\chi^2 \neq \mathbb{1}$, pick a character Λ of "exact" order a (possible because $a \geq 2$ and $p \nmid a$). Then $\chi^2\Lambda$ is some element $\eta \in \mathsf{Char}_{\text{ntriv}}(a)$, and $\eta \neq \Lambda$. Thus $\chi^2 (= \eta/\Lambda) \in \mathsf{Char}_{\text{ntriv}}(a)$. But in that case, $1/\chi^2 \in \mathsf{Char}_{\text{ntriv}}(a)$, and hence $\mathbb{1} \in \chi^2\mathsf{Char}_{\text{ntriv}}(a)$, contradicting $\chi^2\mathsf{Char}_{\text{ntriv}}(a) = \mathsf{Char}_{\text{ntriv}}(a)$.

It remains to treat the case when $q - a = 1$. Thus $\mathcal{K} = \mathcal{G}_\chi$ is

$$\mathcal{K}l(\mathbb{1}, \ \chi\mathsf{Char}_{\text{ntriv}}(q-1)).$$

Suppose first that $\chi \in \mathsf{Char}_{\text{ntriv}}(q-1)$. As $\mathcal{K}$ is Lie self-dual, $\mathcal{L}_\sigma \otimes \mathcal{K}$ is self-dual. As its character list has $\mathbb{1}$ as the only character occurring more than once, we see, just as above, that $\sigma^2 = \mathbb{1}$, and then that $\mathcal{K}$ is self-dual. Comparing character lists for $\mathcal{K}$

and its dual, we get the equality of lists $\chi\mathsf{Char}_{\mathrm{ntriv}}(q-1) = (1/\chi)\mathsf{Char}_{\mathrm{ntriv}}(q-1)$, i.e., $\chi^2\mathsf{Char}_{\mathrm{ntriv}}(q-1) = \mathsf{Char}_{\mathrm{ntriv}}(q-1)$, and, just as above, this is possible only when $\chi^2 = \mathbb{1}$.

Suppose now that $\chi^{q-1} \neq \mathbb{1}$. Then after the $[q-1]^\star$ pullback, the $I(0)$ action is the diagonal action

$$(\mathbb{1}, \chi^{q-1} \text{ repeated } q-2 \ \text{ times}).$$

This element is a scalar multiple of a pseudoreflection of determinant $(1/\chi)^{q-1}$. Thus G°_{geom} is normalized by a pseudoreflection of determinant $(1/\chi)^{q-1}$. If χ^{q-1} has order ≥ 3, then $G^\circ_{\mathrm{geom}} = \mathrm{SL}$, and hence $\mathcal{K}$ is not self-dual. If χ^{q-1} is the quadratic character, then p is odd and G°_{geom} is either SO or SL. We must rule out the SO case.

To do this, we argue as follows. If $\mathcal{K}$ is Lie self-dual, then $\mathcal{L}_\tau \otimes \mathcal{K}$ is self-dual. Its list of $I(0)$-characters is

$$(\tau,\ \tau\chi\mathsf{Char}_{\mathrm{ntriv}}(q-1)).$$

The dual list is

$$(1/\tau,\ 1/(\tau\chi)\mathsf{Char}_{\mathrm{ntriv}}(q-1)).$$

Comparing the lists, either $1/\tau = \tau$, or $1/\tau \in \tau\chi\mathsf{Char}_{\mathrm{ntriv}}(q-1)$. Suppose first that $1/\tau = \tau$. Then either $\tau = \mathbb{1}$ or $p \neq 2$ and $\tau = \chi_2$. As tensoring with $\mathcal{L}_{\chi_2}$ does not alter self-duality, we find that $\mathcal{K}$ is self-dual. Comparing the lists of characters, just as above we get the equality of lists

$$\chi\mathsf{Char}_{\mathrm{ntriv}}(a) = (1/\chi)\mathsf{Char}_{\mathrm{ntriv}}(a), \text{ i.e., } \chi^2\mathsf{Char}_{\mathrm{ntriv}}(a) = \mathsf{Char}_{\mathrm{ntriv}}(a).$$

So either $\chi^2 = \mathbb{1}$, or just as above we get a contradiction.

Suppose next that $1/\tau \in \tau\chi\mathsf{Char}_{\mathrm{ntriv}}(q-1)$. Then $\mathbb{1} \in \tau^2\chi\mathsf{Char}_{\mathrm{ntriv}}(q-1)$. Thus $\tau^2\chi \in \mathsf{Char}_{\mathrm{ntriv}}(q-1)$. But $\chi^{q-1} = \chi_2$, hence $\tau^{2(q-1)}\chi^{2(q-1)} = \chi_2$, hence $(\tau\chi)^{q-1}$ is a square root of χ_2, say

$$(\tau\chi)^{q-1} = \Lambda_4,$$

with Λ_4 a character of order 4. Then with the notation

$$\mathsf{Char}(q-1, \rho) := \{\sigma : \sigma^{q-1} = \rho\},$$

we have

$$\chi\mathsf{Char}_{\mathrm{ntriv}}(q-1) = \mathsf{Char}(q-1, \chi_2) \setminus \{\chi\},$$

and

$$\tau\chi\mathsf{Char}_{\rm ntriv}(q-1) = \mathsf{Char}(q-1, \Lambda_4) \smallsetminus \{\tau\chi\}.$$

But the set of characters

$$(\tau, \tau\chi\mathsf{Char}_{\rm ntriv}(q-1)) = (\tau, \mathsf{Char}(q-1, \Lambda_4) \smallsetminus \{\tau\chi\})$$

is to be stable by inversion. The dual set of characters is

$$(1/\tau, \mathsf{Char}(q-1, 1/\Lambda_4) \smallsetminus \{1/\tau\chi\}).$$

But the two sets $\mathsf{Char}(q-1, \Lambda_4)$ and $\mathsf{Char}(q-1, 1/\Lambda_4)$ are disjoint. As soon as $q > 3$, this is impossible. Indeed, each of the $q-2$ elements of the first list which lie $\mathsf{Char}(q-1, \Lambda_4)$ must be equal to the unique element of the second list which does not lie in $\mathsf{Char}(q-1, 1/\Lambda_4)$. If this happens, then $q-2 = 1$, so $q = 3$, an excluded case. □

In the course of proving Theorem 3.6, we showed that if $\mathcal{K} := \mathcal{K}_{q,a,\rho}$ is Lie self-dual, then $\rho^{q-a} = \chi_2$, and hence $[q-a]^\star\mathcal{K}$ is $\mathcal{G}_{\chi_2}$, whose $G_{\rm geom}$ is Sp_{q-1}. Then $G_{{\rm geom},\mathcal{K}}$ normalizes Sp_{q-1} in the ambient GL_{q-1}, so lies in $\mathrm{CSp}_{q-1} := \mathbb{G}_m * \mathrm{Sp}_{q-1}$, the group of symplectic similitudes. Forming the square of the "$\mathbb{G}_m$" factor is called the *multiplicator homomorphism*, which sends every element to its *conformal multiplier*

$$\mathrm{mult} : \mathbb{G}_m\mathrm{Sp}_{q-1} \to \mathbb{G}_m, \quad \lambda g \mapsto \lambda^2$$

for $g \in \mathrm{Sp}$ and λ a nonzero scalar. Its kernel is Sp_{q-1}.

Corollary 3.7. *In the situation of Theorem 3.6, suppose q is odd, and $\rho^{q-a} = \chi_2$. Denote by N the order of ρ^2. Then for $\mathcal{K} := \mathcal{K}_{q,a,\rho}$, the tensor product $\mathcal{L}_{\overline{\rho}} \otimes \mathcal{K}$ is symplectically self-dual, and*

$$G_{{\rm geom},\mathcal{K}} = \left\{ g \in \mathrm{CSp}_{q-1} | \mathrm{mult}(g)^N = 1 \right\}.$$

Proof. Indeed, $\overline{\rho}\mathsf{Char}(q-a)$ is the set $\mathsf{Char}(q-a, \chi_2) := \{\Lambda | \Lambda^{q-a} = \chi_2\}$, a set visibly stable under complex conjugation. Thus

$$\mathcal{L}_{\overline{\rho}} \otimes \mathcal{K} = \mathcal{K}l(\mathsf{Char}(q-a, \chi_2), \mathsf{Char}_{\rm ntriv}(a)) := \mathcal{K}^{sdual}$$

is self-dual, so its $G_{\rm geom}$ is Sp_{q-1}. Thus $\mathcal{K} = \mathcal{L}_\rho \otimes \mathcal{K}^{sdual}$ has $G_{{\rm geom},\mathcal{K}} = \mathcal{L}_\rho \otimes \mathrm{Sp}_{q-1}$, and its multiplicator group is $\mathcal{L}_{\rho^2}$. □

4. Monodromy groups of $\mathcal{K}_{q,a,\rho}$ when $q \geq 11$: finite cases

We begin by recalling the following standard fact, which we will apply to relate the $G_{\rm geom}$ groups of Kloosterman sheaves and of their Kummer pullbacks, e.g. the $\mathcal{G}_\chi$.

Lemma 4.1. *For a lisse sheaf $\mathcal{K}$ on $\mathbb{G}_m/\overline{\mathbb{F}_p}$, and an integer $d \geq 1$ with $p \nmid d$, $\mathcal{K}$ has finite $G_{\rm geom}$ if and only if its Kummer pullback $[d]^\star\mathcal{K}$ has finite $G_{\rm geom}$. More precisely, $G_{{\rm geom},[d]^\star\mathcal{K}} \lhd G_{{\rm geom},\mathcal{K}}$ with quotient cyclic of order dividing d.*

Theorem 4.2. *For any $a \in \mathbb{Z}_{\geq 2}$ with $q = p^f \geq 11$ and $a < q$ coprime to p, and for any multiplicative character ρ, consider the Kloosterman sheaf*

$$\mathcal{K} = \mathcal{K}_{q,a,\rho} = \mathcal{K}l(\mathsf{Char}(q-a) \cup \rho\mathsf{Char}_{\rm ntriv}(a)).$$

Assume in addition that $(a,\rho) \neq (q-1,\mathbb{1})$. Then the geometric monodromy group $G_\rho = G_{\rm geom}$ of $\mathcal{K}$ is finite if and only if $\rho^{q-a} = \mathbb{1}$, in which case we have $\mathcal{K} = \mathcal{K}_{q,a,\mathbb{1}} \otimes \mathcal{L}_\rho$, and $\mathcal{K}_{q,a,\mathbb{1}}$ has its $G_{\rm geom} := G_{\mathbb{1}}$ given by $G_{\mathbb{1}} = \mathsf{A}_q$ if $p = 2$ and $G_{\mathbb{1}} = {\rm Sym}_q$ if $p > 2$.

Proof. (a) By Theorem 3.3 $\mathcal{K}$ satisfies (**S**+). Assume in addition that $\rho^{q-a} = \mathbb{1}$. Then $\mathcal{K} = \mathcal{K}_{q,a,\mathbb{1}} \otimes \mathcal{L}_\rho$, and $\mathcal{K}_{q,a,\mathbb{1}} = \mathcal{K}l(\mathsf{Char}(q-a) \cup \mathsf{Char}_{\rm ntriv}(a))$ is a Sawin sheaf as defined in [21, (9.3.1)]. Hence the conclusion holds by part (ii) of the proof of [21, Theorem 9.3].

(b) We will now assume that

$$\rho^{q-a} \neq \mathbb{1}, \tag{4.2.1}$$

and that $G = G_\rho$ is finite. Furthermore, in parts (b)–(d) of this proof, we will assume

$$q \geq 13.$$

This implies that $\mathcal{K}$ has rank $D = q-1 \geq 12$, and hence we can apply [19, Theorem 3.5] to $(\mathcal{K}, G)$. Now, none of $D = p^f - 1$ and $2D+1 = 2p^f - 1$ is divisible by p, ruling out cases (i), (iii), (iv) and (vi) of [19, Theorem 3.5]. Next, in the case of (vii) or (viii) of [19, Theorem 3.5], $D = 12$, so $q = p = 13$ and

$$X := \mathsf{Char}(13-a) \cup \rho\mathsf{Char}_{\rm ntriv}(a) = Y := \gamma\big(\mathsf{Char}(16) \smallsetminus \{\chi_8^{0,1,4,7}\}\big)$$

for some multiplicative character γ. Clearly, the ratio of any two characters in Y has order dividing 16. If $3 \leq a \leq 11$, then X contains a pair of two characters with ratio of order $13-a$ and a pair of two characters with ratio of order a, implying $(13-a)|16$ and $a|16$, which is impossible. If $a = 2$, then X contains a pair of two characters with ratio of order 11, again a contradiction. If $a = 12$, then X contains a pair of two characters with ratio of order 12, a contradiction.

(c) Assume we are in the case of [19, Theorem 3.5(v)]. Then

$$p^f - 1 = D = (r^{a+b} - 1)/(r+1)$$

for some power $r = p^g \geq p$ and odd integers $a, b \geq 1$. Comparing the p-part of $D+1$, we get $p^f = r = p^g$, i.e. $r = q$, and hence $a = b = 1$. Now we have

$$X := \mathsf{Char}(q-a) \cup \rho\mathsf{Char}_{\mathrm{ntriv}}(a) = Y := \gamma\big(\mathsf{Char}(q+1) \smallsetminus \{\alpha, \beta\}\big)$$

for some multiplicative characters $\alpha \neq \beta$, γ, with $\alpha, \beta \in \mathsf{Char}(q+1)$. Clearly, the ratio of any two characters in Y has order dividing $q+1$. If $3 \leq a \leq q-2$, then X contains a pair of two characters with ratio of order $q-a$ and a pair of two characters with ratio of order a, implying $(q-a)|(q+1)$ and $a|(q+1)$. In such a case, $q-a \leq (q+1)/2$, so $q+1 > a \geq (q-1)/2 > (q+1)/3$, and hence $a = (q+1)/2$. We now have $(q+1)/3 < (q-1)/2 = q-a < (q+1)/2$, contrary to $(q-a)|(q+1)$. If $a = 2$, then X contains a pair of two characters with ratio of order $q-2$ which is not a divisor of $q+1$. If $a = q-1$, then X contains a pair of two characters with ratio of order $q-1$, which is again a contradiction.

(d) We have shown that $\mathcal{K}$ satisfies [19, Theorem 3.5(ii)]. In fact, as shown in part (b) of the proof of [19, Theorem 3.5], our arguments also apply to the more general situation where $D > 8$ and $G^{(\infty)} = \mathsf{A}_{D+1}$ acting via it deleted permutation module. In particular, there is some prime to p integer $2 \leq k \leq (q-1)/2$ such that

$$X := \mathsf{Char}(q-a) \sqcup \rho\mathsf{Char}_{\mathrm{ntriv}}(a) = Y := \gamma\big(\mathsf{Char}_{\mathrm{ntriv}}(q-k) \sqcup \mathsf{Char}_{\mathrm{ntriv}}(k) \sqcup \{\mathbb{1}\}\big)$$

for some multiplicative character γ. List the $q-k$ characters $\gamma\zeta^i$ of Y, where i is in $\mathbb{Z}/(q-k)\mathbb{Z}$ and ζ has order $q-k$, as $q-k$ consecutive points on the circle, and color the point corresponding to $\gamma\zeta^i$ red if $\gamma\zeta^i \in \mathsf{Char}(q-a)$ and blue otherwise. Also fix a character ν of order a.

(d1) First suppose that two consecutive points $\gamma\zeta^i$ and $\gamma\zeta^{i+1}$ are both red. It follows that its ratio ζ belongs to $\mathsf{Char}(q-a)$, and so $(q-k)|(q-a)$. Since $q-k \geq (q+1)/2 > (q-a)/2$, we must have that $q-k = q-a$, i.e. $k = a$. Now,

$$\mathbb{1} = (\gamma\zeta^i)^{q-a} = \gamma^{q-a}, \tag{4.2.2}$$

which in turn implies that $\gamma\mathsf{Char}(q-a) = \mathsf{Char}(q-a)$ and thus

$$\gamma\mathsf{Char}_{\mathrm{ntriv}}(a) = \rho\mathsf{Char}_{\mathrm{ntriv}}(a). \tag{4.2.3}$$

As $a \geq 2$ and $\mathsf{Char}(a) = \langle\nu\rangle$, we have $\gamma\nu = \rho\nu^j$ for some $j \in \mathbb{Z}$, i.e. $\gamma = \rho\nu^{j-1}$. Multiplying (4.2.3) by $\bar{\rho}$, we get

$$\mathsf{Char}(a) \smallsetminus \{\mathbb{1}\} = \mathsf{Char}_{\mathrm{ntriv}}(a) = \nu^{j-1}\mathsf{Char}_{\mathrm{ntriv}}(a) = \mathsf{Char}(a) \smallsetminus \{\nu^{j-1}\},$$

whence $\nu^{j-1} = \mathbb{1}$ and $\gamma = \rho$. But in this case $\rho^{q-a} = \mathbb{1}$ by (4.2.2), and this contradicts (4.2.1).

(d2) Now suppose that two consecutive points $\gamma\zeta^i$ and $\gamma\zeta^{i+1}$ are both blue. It follows that its ratio ζ belongs to $\mathsf{Char}(a)$, and so $(q-k)|a$. Since $q-k \geq (q+1)/2 > a/2$, we must have that

$$a = q - k,$$

and

$$X = \mathsf{Char}(k) \sqcup \rho\mathsf{Char}_{\mathrm{ntriv}}(q-k) = Y = \gamma\mathsf{Char}(q-k) \sqcup \gamma\mathsf{Char}_{\mathrm{ntriv}}(k).$$

We can write the blue-labeled character $\gamma\zeta^i$ as $\rho\zeta^j$ for some $j \in \mathbb{Z}/(q-k)\mathbb{Z}$. As $\zeta \in \mathsf{Char}(q-k)$, multiplying by ζ repeatedly we see that $\gamma\mathsf{Char}(q-k) = \rho\mathsf{Char}(q-k)$, and thus

$$\rho\mathsf{Char}_{\mathrm{ntriv}}(q-k) \text{ covers all but one character } \gamma\zeta^l \text{ of } \gamma\mathsf{Char}(q-k). \tag{4.2.4}$$

In this case,

$$\gamma\zeta^l \text{ belongs to } \mathsf{Char}(k), \tag{4.2.5}$$

and $\gamma\mathsf{Char}_{\mathrm{ntriv}}(k) \subset \mathsf{Char}(k)$. Fixing a character $\lambda \in \mathsf{Char}_{\mathrm{ntriv}}(k)$ (recalling $k \geq 2$), we then see that $\mathbb{1} = (\gamma\lambda)^k = \gamma^k$, i.e. $\gamma \in \mathsf{Char}(k)$. It then follows from (4.2.5) that $\zeta^{lk} = \mathbb{1}$, and so lk is divisible by the order $q-k$ of ζ. Since $\gcd(k, q-k) = \gcd(k,q) = 1$, we have $\zeta^l = \mathbb{1}$. Hence $\gamma \in \mathsf{Char}(k)$ again by (4.2.5), and

$$\rho\mathsf{Char}_{\mathrm{ntriv}}(q-k) = \gamma\mathsf{Char}_{\mathrm{ntriv}}(q-k)$$

by (4.2.4). As $\gamma = \rho\zeta^{j-i}$ and $\zeta \in \mathsf{Char}(q-k)$, multiplying by $\bar{\rho}$ we get

$$\mathsf{Char}(q-k) \smallsetminus \{\mathbb{1}\} = \mathsf{Char}_{\mathrm{ntriv}}(q-k) = \zeta^{j-i}\mathsf{Char}_{\mathrm{ntriv}}(q-k) = \mathsf{Char}(q-k) \smallsetminus \{\zeta^{j-i}\},$$

i.e. $\zeta^{j-i} = \mathbb{1}$, and hence $\rho = \gamma \in \mathsf{Char}(k)$. But in this case $\mathbb{1} = \rho^k = \gamma^k = \gamma^{q-a}$, and this again contradicts (4.2.1).

(d3) We have shown that the $q-k$ points on the circle must have alternating colors; in particular, $2|(q-k)$. Without loss of generality, we may assume $\gamma\zeta^i \in \mathsf{Char}(q-a)$ when $i \in 2\mathbb{Z}/(q-k)\mathbb{Z}$, and $\gamma\zeta^i \in \rho\mathsf{Char}_{\mathrm{ntriv}}(a)$ when $i \in (2\mathbb{Z}+1)/(q-k)\mathbb{Z}$. Since $q-k \geq (q+1)/2 > 3$, we have $\gamma, \gamma\zeta^2 \in \mathsf{Char}(q-a)$ and $\gamma\zeta, \gamma\zeta^3 \in \mathsf{Char}(a)$. Considering the ratios between these characters and recalling that ζ has order $q-k$, we see that $(q-k)/2$ divides both a and $q-a$. Thus $(q-k)/2$ divides $a + (q-a) - (q-k) = k$, and hence $\gcd(k, q-k) \geq (q-k)/2 > 1$, a contradiction.

(e) It remains to consider the case $q = 11$, i.e. $D = 10$. Condition ($\mathbf{S}+$) implies by [21, Lemma 1.1] that G is an almost quasisimple group; in particular, $L = G^{(\infty)}$ is a quasisimple group acting irreducibly on the underlying representation $V = \mathbb{C}^{10}$. As mentioned in (d), we may assume that L is not A_{11} acting via its deleted permutation module. By [21, Proposition 4.8], we have

$$p = 11 \text{ divides the order of } G/\mathbf{Z}(G) \hookrightarrow \mathrm{Aut}(L).$$

These constraints on L imply by [7] that

$$L = \mathrm{PSL}_2(11),\quad \mathrm{SU}_5(2),\quad M_{11},\quad 2\cdot M_{12},\quad 2\cdot M_{22}.$$

Let $\langle g_0\rangle$ denote the image of $I(0)$ in G; in particular, g_0 is an element with simple spectrum on V, whose shape is determined by $\mathcal{K}$. We will show that

$$\bar{\mathsf{o}}(g_0) \geq 18; \tag{4.2.6}$$

more precisely, either $4 \leq a \leq 7$ and $\bar{\mathsf{o}}(g_0) \geq 28$, or $a = 3, 8$ and $\bar{\mathsf{o}}(g_0) \geq 24$, or $a = 2, 9$ and $9|\bar{\mathsf{o}}(g_0) \geq 18$, or $a = 10$ and $10|\bar{\mathsf{o}}(g_0) \geq 20$. Indeed, the ratio of any two characters in $\mathcal{K}$ has order dividing $N = \bar{\mathsf{o}}(g_0)$. Now, if $3 \leq a \leq 9$, then $\mathcal{K}$ contains two characters with ratio of order $q - a$ and two characters with ratio of order a, implying $N \geq a(11-a) \geq 18$ as $\gcd(a, q-a) = 1$, and the refined statement of (4.2.6) also follows in this case. Suppose $a = 2$, so that $\mathcal{K} = \mathcal{K}l(\mathsf{Char}(9) \sqcup \{\rho\chi_2\})$. As G is finite, $\rho\chi_2 \notin \mathsf{Char}(9)$. As N is divisible by 9 and by the order of $\rho\chi_2$, we get $N \geq 18$. Suppose $a = 10$, so that $\mathcal{K} = \mathcal{K}l(\rho\mathsf{Char}_{\mathrm{ntriv}}(10) \sqcup \{\mathbb{1}\})$. Then $10|N$. If $N = 10$ then $\mathcal{K} = \mathcal{K}l(\mathsf{Char}(10)) \otimes \mathcal{L}_\gamma$ and hence it would be imprimitive. So $N \geq 20$ in this case.

If $L = \mathrm{PSL}_2(11)$, then $\bar{\mathsf{o}}(g_0) \leq 12$ by [1], violating (4.2.6).

In the case $L = M_{11}$, $2\cdot M_{12}$, or $2\cdot M_{22}$, we have $\bar{\mathsf{o}}(g_0) \leq 14$ by [1], contrary to (4.2.6).

It remains to consider the case $L = \mathrm{SU}_5(2)$. Using [5] and (4.2.6), we see that either $\bar{\mathsf{o}}(g_0) = 18$ and $a = 2$, or $\bar{\mathsf{o}}(g_0) = 24$ and $a = 3, 8$. In the latter case, since $a = 3, 8$, the spectrum of g_0 on V must contain a full μ_8-coset or $(\mu_8 \smallsetminus \{1\})$-coset, which is not the case as one can check using [5]. In the former case, if $a = 2$ then the spectrum of g_0 on V must contain a full μ_9-coset, which is not the case as one can check using [5]. If $a = 9$, then the spectrum of g_0 on V must be $\alpha(\mu_9 \smallsetminus \{1\}) \sqcup \{\pm\beta\}$. Using [5] we can see that, after taking out a full $(\mu_9 \smallsetminus \{1\})$-coset of $\alpha = -1$ from the spectrum of g_0 on V, we are left with $\{\zeta_3, \zeta_3^2\}$ which are not $\pm\beta$ for any $\beta \in \mathbb{C}^\times$, a final contradiction. □

Corollary 4.3. *For χ nontrivial, $q \geq 11$, any $2 \leq a < q$ with $p \nmid a$, $\mathcal{G}_\chi$ has infinite G_{geom}.*

Proof. Immediate from Theorem 4.2 and Lemma 4.1. □

5. Universal families of arbitrary degree $e \geq 3$

Fix a nontrivial multiplicative character χ with $\chi^e \neq \mathbb{1}$. On the space $\mathbb{G}_m \times \mathbb{A}^e$, with coordinates $(a_e, a_{e-1}, \ldots, a_0)$ we have the "universal" polynomial of degree e, namely

$$f_{\mathrm{univ}} := a_e x^e + \sum_{i<e} a_i x^i,$$

whose discriminant $\Delta(f_{\mathrm{univ}})$ is a **nonzero** polynomial function on $\mathbb{G}_m \times \mathbb{A}^e$. It is nonzero because it is nonzero at an f of degree e which is irreducible over $\mathbb{F}_p$, for example. Over the open set $U := (\mathbb{G}_m \times \mathbb{A}^e)[1/\Delta]$, we have the local system $\mathcal{F}_{\chi,e}$ whose trace function is

$$f \mapsto -\sum_x \chi(f(x)). \tag{5.0.1}$$

Because $\chi^e \neq \mathbb{1}$, the local system $\mathcal{F}_{\chi,e}$ is lisse of rank $e-1$ and pure of weight one. By [11, 2.2], $\mathcal{F}_{\chi,e}$ is geometrically irreducible.

These local systems, where f is an n-variable polynomial, were the main focus of Chapter 5 of [14]. To deal with the case $n \geq 2$, it was assumed there that the leading form f_e of f was a Deligne polynomial, meaning that e was prime to p and that the projective locus $f_e = 0$ in $\mathbb{P}^{n-1}$ was smooth of dimension $n-2$. In that same chapter, the hypothesis $p \nmid e$ in the $n = 1$ case was carried over, but was sometimes not used, cf. the Introduction to [13] where this distinction between the $n = 1$ and $n \geq 2$ cases is discussed. The upshot is that in [14, Theorems 1.20.3 and 5.2.2], for the $M_{2,2}$ results and the Frobenius-Schur results in the $n = 1$ case, the only primes p that need to be excluded are those dividing the order of χ, but **not** those dividing e.

By [14, 1.20.3], we then have the following information about the moments of $\mathcal{F}_{\chi,e}$, so long as e is not too small (so that $\mathcal{P}(1,e)$, the space of polynomials of degree $\leq e$, which is $e+1$-separating, is sufficiently separating in the sense of [14, 1.1]).

Theorem 5.1. *We have the following moment evaluations.*

(i) *Suppose $e \geq 7$. If $\chi^e \neq \mathbb{1}$ and χ has order ≥ 3, then $M_{4,4}(\mathcal{F}_{\chi,e}) = 4!$ (indeed $M_{k,k} = k!$ for each $k \leq (e+1)/2$).*

(i-bis) *Suppose $e \geq 5$. If $\chi^e \neq \mathbb{1}$ and χ has order ≥ 3, then $M_{3,3}(\mathcal{F}_{\chi,e}) = 3!$.*

(ii) *Suppose $e \geq 9$. If χ has order 2 and $\chi^e \neq \mathbb{1}$ (i.e., e is odd), then $M_{4,4}(\mathcal{F}_{\chi,e}) = 8!! :=$ 1.3.5.7 (indeed $M_{k,k} = (2k)!!$ for each $k \leq (e-1)/2$).*

Also, as a part of [14, Theorem 5.2.2] we have

Theorem 5.2. *The local system $\mathcal{F}_{\chi,e}$ has finite $G_{\rm geom}$ in the following cases:*

(4,3) *χ has order 4, $e = 3$, $p \nmid 4$.*
(6,3) *χ has order 6, $e = 3$, $p \nmid 6$.*
(6,4) *χ has order 6, $e = 4$, $p \nmid 6$.*
(6,5) *χ has order 6, $e = 5$, $p \nmid 6$.*
(10,3) *χ has order 10, $e = 3$, $p \nmid 10$.*

6. Monodromy groups of $\mathcal{K} = \mathcal{K}_{q,a,\rho}$ when $2 < q < 11$: finite cases

Theorem 6.1. *For any $a \in \mathbb{Z}_{\geq 2}$ with $3 \leq q = p^f < 11$ and $a < q$ coprime to p, and for any multiplicative character ρ, consider the Kloosterman sheaf*

$$\mathcal{K} = \mathcal{K}_{q,a,\rho} = \mathcal{K}l(\mathsf{Char}(q-a) \cup \rho\mathsf{Char}_{\mathrm{ntriv}}(a)).$$

Assume in addition that $(a, \rho) \neq (q-1, \mathbb{1})$. *Then the geometric monodromy group* $G_\rho = G_{\mathrm{geom}}$ *of* $\mathcal{K}$ *is finite if* $\rho^{q-a} = \mathbb{1}$. *There are no additional cases in which* $G_\rho = G_{\mathrm{geom}}$ *of* $\mathcal{K}$ *is finite for* $q \in \{4, 7, 8, 9\}$. *The additional cases for* $q \in \{3, 5\}$ *are listed below.*

(i) $q = 5$, *and* $(a, |\rho|) = (2, 18)$, $(3, 12)$, *or* $(4, 6)$.
(ii) $q = 3$, *and* $(a, |\rho|) = (2, 4)$ *or* $(2, 10)$.

Proof. (a) We first explain why the additional cases each have finite G_{geom}. For this, we go back to the genesis of the Kloosterman sheaves in question, as having $[q-a]$ Kummer pullbacks which are the one-parameter families of character sums

$$t \mapsto -\sum_x \rho^{q-a}(x^q + x^a - t).$$

The five cases in question are then

$$t \mapsto \sum_x \chi_6(x^5 + x^2 - t), \text{char } p = 5$$

$$t \mapsto \sum_x \chi_6(x^5 + x^3 - t), \text{char } p = 5$$

$$t \mapsto \sum_x \chi_6(x^5 + x^4 - t), \text{char } p = 5$$

$$t \mapsto \sum_x \chi_4(x^3 + x^2 - t), \text{char } p = 3$$

$$t \mapsto \sum_x \chi_{10}(x^3 + x^2 - t), \text{char } p = 3.$$

Each of these has finite G_{geom}, as each is a subfamily of the universal family $\mathcal{F}_{\chi,e}$ of Theorem 5.2. As each of these G_{geom}'s is contained in the G_{geom} of "its" Kloosterman sheaf as a subgroup, the Kloosterman sheaf also has finite G_{geom}.

We will now explain why these are the only cases. We will also give independent proofs of the finiteness in the case $q = 5$, with $(a, |\rho|) = (2, 18)$ and in the cases $q = 3$, with $(a, |\rho|) = (2, 4)$ or $(2, 10)$. It would be of interest to have an independent proof of the finiteness in the case $q = 5$ with $(a, |\rho|) = (3, 12)$, or $(4, 6)$.

(b) The key observation is that, for the Kloosterman sheaf $\mathcal{K} = \mathcal{K}_{q,a,\rho}$ and for γ_0 a topological generator of the image of $I(0)$, the element $\gamma_0^{a(q-a)}$ acts either as the identity or as a quadratic element with eigenvalues

$$\big(\underbrace{1, \dots, 1}_{q-a \text{ times}}, \underbrace{\zeta_\rho^{a(q-a)}, \dots, \zeta_\rho^{a(q-a)}}_{a-1 \text{ times}}\big),$$

where for simplicity of notation we write

$$\zeta_\rho := \rho(\gamma_0).$$

If G_ρ is finite, then by Theorem 3.3 it is a finite primitive subgroup of $\mathrm{GL}_{q-1}(\mathbb{C})$. By Blichfeld's theorem [3], $\zeta_\rho^{a(q-a)}$ must have order ≤ 5. Moreover, by theorems of Wales [24] and Zalesski [25], $\zeta_\rho^{a(q-a)}$ has order ≤ 3 unless $q-a = a-1$. Finally, if either $q-a=1$ or $a-1=1$, but $q-a \neq a-1$, then by a theorem of Wales [24], if $\rho(\gamma_0)^{a(q-a)}$ has order 3, then $q-1 \leq 4$.

Notice also that ρ^a must be nontrivial, otherwise the trivial character $\mathbb{1}$ occurs twice in the list of characters of $\mathcal{K}$, which forces its G_ρ to be infinite.

Given these preliminaries, we begin the case by case analysis.

(c) Consider the case $q = 9$, and suppose G_ρ is finite. Assume in addition $a \neq 5$, so that $q-a \neq a-1$. Then $\zeta_\rho^{a(q-a)}$ has order ≤ 3. But in characteristic 3, all tame characters have order prime to 3, hence $\zeta_\rho^{a(q-a)}$ has order ≤ 2. When $a = 2$, then ζ_ρ^{14} has order one of $\{1, 2\}$. If its order is 1, then ζ_ρ has order dividing 14, so ζ_ρ has order one of $2, 7, 14$. But neither 2 nor 7 is not allowed, so the order is 14, and this fails the V-test (already over $\mathbb{F}_{3^6}$). [See [15, 13.2] and [18, Chapter 9] for a discussion of the V-test.] If the order is 2, then ζ_ρ has order dividing 28, but $2, 7$ are not allowed, and 14 has been eliminated. So the possible orders are 4 and 28. Each of these fails the V-test (in both cases over $\mathbb{F}_{3^6}$).

Suppose next that $a = 4$ (remember $a = 3$ is not allowed). Here ζ_ρ^{20} has order ≤ 2, so ζ_ρ has order dividing 40, but not dividing 4 or 5. So the possible orders of ζ_ρ are among $8, 10, 20, 40$. Each of these fails the V-test (in all cases over $\mathbb{F}_{3^4}$).

Suppose next that $a = 5$. So here ζ_ρ^{20} has order ≤ 5. If the order is among $1, 2, 4$, then ζ_ρ has order dividing 80, but not dividing 4 or 5. So its possible orders are $8, 16, 10, 20, 40, 80$. Each of these fails the V-test (in all cases over $\mathbb{F}_{3^4}$).

Suppose next that $a = 7$. Then just as in the $a = 2$ case, ζ_ρ has order dividing 28, but not dividing 2 or 7. So the possible orders are $4, 14, 28$. Each of these fails the V-test (in all cases over $\mathbb{F}_{3^6}$).

Suppose next that $a = 8$. Then ζ_ρ has order dividing 16 but not dividing 8, so the only possible order is 16. This fails the V-test (over $\mathbb{F}_{3^4}$). This finishes the $q = 9$ case.

(d) Consider next the $q = 8$ case. Here $\zeta_\rho^{a(q-a)}$ has odd order ≤ 3. Suppose first $a = 3$ (a is required to be odd). Then ζ_ρ has order dividing 45, but not dividing 3 or 5. So the possible orders are $9, 15, 45$. Each of these fails the V-test (in all cases over $\mathbb{F}_{2^8}$).

Suppose next that $a = 5$. Again here the possible orders are $9, 15, 45$. Each of these fails the V-test (in all cases over $\mathbb{F}_{2^8}$).

Suppose finally that $a = 7$. Then ζ_ρ has order dividing 21, but not dividing 7. So the possible orders are $3, 21$. Each of these fails the V-test (in both cases over $\mathbb{F}_{2^6}$). This finishes the $q = 8$ case.

(e) Consider next the $q = 7$ case. Let us begin with the hardest case, $a = 4$, where ζ_ρ^{12} has order ≤ 5. If the order divides 4, then ζ_ρ has order dividing 48, but not dividing

3 or 4. So the possible orders of ζ_ρ are $6, 12, 24, 48, 8, 16$. Each of these fails the V-test (in all cases over $\mathbb{F}_{7^8}$, though in the case of order 24 one needed to check more than one character of order 24: when ρ passed over $\mathbb{F}_{7^8}$, ρ^5 failed). If the order divides 5, then ζ_ρ has order dividing 60, but not dividing 3 or 4. So the possible orders of ζ_ρ are $5, 6, 10, 12, 15, 20, 30, 60$. Each of these fails the V-test over $\mathbb{F}_{7^4}$. If the order divides 3, then ζ_ρ has order dividing 36, but not dividing 3 or 4. So the possible orders of ζ_ρ are $6, 9, 12, 18, 36$. Each of these fails the V-test over $\mathbb{F}_{7^6}$.

Suppose next that $a = 2$. Then ζ_ρ^{10} has order ≤ 3, so ζ_ρ has order dividing either 20 or 30, but not dividing 2 or 5. So the possible orders of ζ_ρ are $4, 10, 20, 3, 6, 15, 30$. Each of these fails the V-test (in all cases over $\mathbb{F}_{7^4}$).

Suppose next that $a = 3$. Then ζ_ρ^{12} has order ≤ 3, so ζ_ρ has order dividing 24 or 36, but not dividing 3 or 4. So the possible orders of ζ_ρ are $6, 12, 24, 8, 9, 18, 36$. These each fail the V-test (the first four over $\mathbb{F}_{7^2}$, the last three over $\mathbb{F}_{7^6}$).

Suppose next $a = 5$. Then just as in the $a = 2$ case, the possible orders of ζ_ρ are $4, 10, 20, 3, 6, 15, 30$. Each of these fails the V-test (in all cases over $\mathbb{F}_{7^4}$).

Suppose finally that $a = 6$. Then ζ_ρ^6 has order ≤ 3, but here we have a scalar times a complex reflection. But no complex reflection in a primitive finite group can have order 3 unless the rank, here $q - 1 = 6$, is ≤ 4. So in fact ζ_ρ has order dividing 12, but not dividing 6. So the possible orders are $4, 12$. Each of these fails the V-test (in both cases over $\mathbb{F}_{7^2}$).

(f) Next we consider the possibility $q = 5$. Let us begin with the hardest case, $a = 3$, where ζ_ρ^6 has order ≤ 5. But tame characters in characteristic 5 have order prime to 5, so in fact ζ_ρ^6 has order ≤ 4. If the order of ζ_ρ^6 is 3, then ζ_ρ has order dividing 18, but not dividing 2 or 3. So the possible orders are $6, 9, 18$. Each of these fails the V-test (in all cases over $\mathbb{F}_{5^6}$). If this order divides 4, then ζ_ρ has order dividing 24, but not dividing 2 or 3. So the possible orders are $4, 6, 8, 12, 24$. Here all but order 12 fail the V-test. As mentioned in (a), the sheaf $\mathcal{K}$ in the case $(a, |\rho|) = (3, 12)$ indeed has finite G_{geom}.

Suppose now that $a = 2$. Here ζ_ρ^6 has order ≤ 3. If the order divides 2, then ρ has order dividing 12, but not dividing 2 or 3. So the possible orders are $4, 6, 12$. Each of these fails the V-test (in all cases over $\mathbb{F}_{5^2}$). If the order divides 3, then ζ_ρ has order dividing 18, but not dividing 2 or 3. So the possible orders are $6, 9, 18$. The orders $6, 9$ each fail the V-test (both over $\mathbb{F}_{5^6}$). We now give another explanation that the order 18 case indeed has a finite G_ρ. Here, $\mathcal{K} = \mathcal{K}l(\mathsf{Char}(3), \rho\chi_2) = \mathcal{K}l(\mathsf{Char}(3), \chi_9)$ which is $\mathcal{L}_{\chi_9} \otimes \mathcal{K}'$, where $\mathcal{K}' := \mathcal{K}l(\mathbb{1}, \mathsf{Char}(3, \overline{\chi_3}))$. Now the proof of [19, Proposition 8.5(ii)] shows that $\mathcal{K}'$ has finite monodromy, hence so does $\mathcal{K}$.

Next assume that $a = 4$. Then ζ_ρ^4 has order ≤ 3. So ζ_ρ has order dividing either 8 or 12, but not dividing 4. So the order of ζ_ρ is either 8 or one of $3, 6, 12$. Each of $3, 8, 12$ fails the V-test over $\mathbb{F}_{5^4}$. To finish the analysis of the remaining order 6 we next show that the sheaves for the two cases with $(a, |\rho|) = (3, 12)$, $(4, 6)$ are Kummer $\mathcal{L}_\sigma$ multiples of each other, so one has finite G_{geom} if and only if the other one does (see e.g. [18, Lemma 5.2]). To see this, let us begin with the $(a, |\rho|) = (4, 6)$ case. Fix a character χ_{12} of order 12. Then our sheaf is

$$\mathcal{K}l(\mathbb{1}, \chi_{12}^2 \mathsf{Char}_{\mathrm{ntriv}}(4)) = \mathcal{K}l(\mathbb{1}, \chi_{12}^{2+3,2+6,2+9}) = \mathcal{K}l(\chi_{12}^{0,5,8,11}).$$

If we tensor this last $\mathcal{K}$ with $(\mathcal{L}_{\overline{\chi_{12}}})^{\otimes 5}$, we get

$$\mathcal{K}l(\chi_{12}^{7,0,3,6}) = \mathcal{K}l(\mathsf{Char}(2), \overline{\chi_{12}}\mathsf{Char}_{\mathrm{ntriv}}(3)),$$

which is precisely the sheaf with $(a, |\rho|) = (3, 12)$.

(g) Suppose $q = 4$. The only allowed a is $a = 3$, and ζ_ρ^3 has odd order ≤ 3, so order dividing 3, and thus ζ_ρ has order dividing 9 but not dividing 3, so order 3. This fails the V-test (already over $\mathbb{F}_{2^2}$).

Assume now that $q = 3$. Here the only allowed a is $a = 2$, and the Kloosterman sheaf $\mathcal{K}_\rho$ is $\mathcal{K}l(\mathbb{1}, \rho\chi_2)$. Here ζ_ρ^2 have order ≤ 5 but prime to 3, so one of $1, 2, 4, 5$. So ζ_ρ has order dividing either 8 or 10, but not dividing 2. So the possible orders are $4, 8, 5, 10$. Here the orders $5, 8$ each fail the V-test (over $\mathbb{F}_{3^4}$).

For the orders 4 and 10, we give another explanation of finite monodromy.

Suppose $|\rho| = 10$. Then $\mathcal{K} = \mathcal{K}l(\mathbb{1}, \rho\chi_2)$ is the Kloosterman $\mathcal{K}l(\mathbb{1}, \chi_5)$ considered in the proof of [19, Proposition 8.8], which is shown there to have $G_{\mathrm{geom}} = C_5 \times \mathrm{SL}_2(5)$, the Shephard–Todd group $ST16$.

Suppose $|\rho| = 4$. Then $\mathcal{K} = \mathcal{K}l(\mathbb{1}, \rho\chi_2)$ is the Kloosterman $\mathcal{K}l(\mathbb{1}, \chi_4)$ considered in the proof of [19, Proposition 8.8], which is shown there to have $G_{\mathrm{geom}} = C_4 * 2\mathsf{S}_4$, the Shephard–Todd group $ST8$. □

Theorem 6.2. *In the five finite cases listed in Theorem 6.1, the geometric monodromy group $G = G_{\mathrm{geom}}$ of $\mathcal{K} = \mathcal{K}_{q,a,\rho}$ is given as follows.*

- (i) *Suppose $q = 3$ and $a = 2$. If $|\rho| = 4$ then $G = C_4 * 2\mathsf{S}_4$, the Shephard–Todd group $ST8$. If $|\rho| = 10$ then $G = C_5 * \mathrm{SL}_2(5)$, the Shephard–Todd group $ST16$.*
- (ii) *Suppose $q = 5$. If $(a, |\rho|) = (2, 18)$ then $G = C_9 \times \mathrm{Sp}_4(3)$. If $(a, |\rho|) = (3, 12)$ then $G = C_{12} * \mathrm{Sp}_4(3)$. If $(a, |\rho|) = (4, 6)$ then $G = \mathrm{Sp}_4(3)$.*

Proof. (i) is already proved in part (g) of the proof of Theorem 6.1. We will therefore assume $q = 5$.

(a) Suppose $(a, |\rho|) = (2, 18)$. Then $\mathcal{K} = \mathcal{K}l(\mathsf{Char}(3), \rho\chi_2) = \mathcal{K}l(\mathsf{Char}(3), \chi_9)$ is $\mathcal{L}_{\chi_9} \otimes \mathcal{K}'$, where $\mathcal{K}' := \mathcal{K}l(\mathbb{1}, \mathsf{Char}(3, \overline{\chi_3}))$. As shown in the proof of [19, Proposition 8.5(ii)], if H denotes the geometric monodromy group of $\mathcal{K}'$ and C is the subgroup of order 3 of $Z := \mathbf{Z}(\mathrm{GL}_4(\mathbb{C}))$, then $C * H = C_3 \times \mathrm{Sp}_4(3)$. It follows that $[H, H] = \mathrm{Sp}_4(3)$. On the other hand, by [18, Lemma 5.2] we have $ZG = ZH$ and $[G, G] = [H, H]$. It follows that $G = D * \mathrm{Sp}_4(3)$, for some finite cyclic subgroup D, of order say d, of Z. Note that a generator g_0 of the image of $I(0)$ in G has spectrum $\{\zeta_9^{0,1,3,6}\}$ on the underlying representation V for a primitive 9^{th} root of unity $\zeta_9 \in \mathbb{C}^\times$. In particular g_0 has determinant ζ_9 on V, which forces d to be divisible by 9 (as $\mathrm{Sp}_4(3)$ is perfect). It was also shown in the proof of [19, Proposition 8.5(ii)] that the field of traces of $\mathcal{K}'$ is contained in $\mathbb{Q}(\zeta_9^3)$, whence the field of traces of $\mathcal{K}$ is contained in $\mathbb{Q}(\zeta_9)$. As $D \cong C_d$, it follows that $\mu_d \subseteq \mathbb{Q}(\zeta_9)$, and so $d|18$. Since $\mathrm{Sp}_4(3)$ contains $-\mathrm{Id}_V$ and $9|d$, we conclude that $G = C_9 \times \mathrm{Sp}_4(3)$.

(b) We now consider $\mathcal{K}_3 := \mathcal{K}_{5,3,\rho}$ with $|\rho| = 12$ and $\mathcal{K}_4 := \mathcal{K}_{5,4,\rho'}$ with $|\rho'| = 6$. As explained in part (f) of the proof of Theorem 6.1, we have $\mathcal{K}_4 = \mathcal{K}l(\rho^{0,5,8,11})$ and, with choosing $\rho' = \rho^2$, $\mathcal{K}_3 = \mathcal{K}_4 \otimes \mathcal{L}_{\rho^7} = \mathcal{K}l(\rho^{0,3,6,7})$. Let G_i denote the geometric monodromy group of $\mathcal{K}_i$ for $i = 3, 4$, and let $Z := \mathbf{Z}(\mathrm{GL}_4(\mathbb{C}))$. Recall from Theorem 3.3 that G_3 satisfies $(\mathbf{S}+)$. For $i = 3, 4$, let g_i be a generator of the image of $I(0)$ in G_i. Then we may assume that g_3 has spectrum $\{\zeta_{12}^{0,3,6,7}\}$ on the underlying representation V for a primitive 12^{th} root of unity $\zeta_{12} \in \mathbb{C}^\times$. In particular, $h := g_3^4$ is a complex reflection of order 3. It follows from [21, Lemma 1.1] that G_3 is either almost quasisimple, or an extraspecial normalizer.

Suppose we are in the latter case. Then G_3 contains a normal irreducible subgroup $E = 2_\epsilon^{1+4}$ with $\epsilon = \pm$ or $E = C_4 * 2_+^{1+4}$, whence

$$E \lhd G_1 \le \mathbf{N}_{\mathrm{GL}(V)}(E) \le ZE \cdot \mathrm{Sp}_4(2),$$

since $\mathrm{Aut}(E) \cong (E/\mathbf{Z}(E)) \cdot \mathrm{O}_4^\epsilon(2)$ in the case $E = 2_\epsilon^{1+4}$ and $\mathrm{Aut}(E) \cong (E/\mathbf{Z}(E)) \cdot \mathrm{Sp}_4(2)$ otherwise. Now h induces an automorphism $\bar{h}$ of order 3 of E. Furthermore the Sylow 3-subgroups of $\mathrm{Sp}_4(2) \cong \mathsf{S}_6$ are elementary abelian of order 3^2. Pick such a subgroup $\bar{P}$ which contains $\bar{h}$. Then, working in ZG_3/ZE, we may assume that $\bar{P}$ is induced by

$$\mathbf{N}_{\mathrm{GL}(V)}(E_1 * E_2) \cong \mathbf{N}_{\mathrm{GL}_2(\mathbb{C})}(E_1) * \mathbf{N}_{\mathrm{GL}_2(\mathbb{C})}(E_2),$$

where $E_1 \cong E_2 \cong 2_-^{1+2}$ and $\mathbf{N}_{\mathrm{GL}_2(\mathbb{C})}(E_i) \cong \mathbf{Z}(\mathrm{GL}_2(\mathbb{C}))E_i \cdot \mathsf{S}_3$. In particular, h fixes a tensor decomposition $V = \mathbb{C}^2 \otimes \mathbb{C}^2$ of the underlying representation V of G_3, a contradiction (cf. part (i) of the proof of [19, Proposition 8.5]).

We have shown that G_3 is almost quasisimple: $S \lhd G_3/\mathbf{Z}(G_3) \le \mathrm{Aut}(S)$ for some finite non-abelian simple group S. As $\bar{\mathsf{o}}(g_3) = 12$, $\mathrm{Aut}(S)$ contains an element of order 12, and $h = g_3^4$ is a complex reflection of order 3. Furthermore, the quasisimple cover $G_3^{(\infty)}$ of S acts irreducibly on $V = \mathbb{C}^4$ according to $(\mathbf{S}+)$. Checking the possibilities for $G_3^{(\infty)}$ listed in [7], we see that $S \cong \mathsf{A}_7$ or $\mathrm{PSp}_4(3)$. Note that, a priori, the field of traces $\mathbb{K}$ of G_3 is contained in $\mathbb{Q}(\zeta_{12})$ (see [18, Proposition 6.1(ii)]) which does not contain $\sqrt{-7}$. This rules out the case $S = \mathsf{A}_7$ since in this case the field of traces of $G_3^{(\infty)} \cong 2\mathsf{A}_7$ would contain $\sqrt{-7}$. Hence $S = \mathrm{PSp}_4(3)$, $G_3^{(\infty)} \cong \mathrm{Sp}_4(3)$, and $G_3 = \mathbf{Z}(G_3)\mathrm{Sp}_4(3)$ since the outer automorphism of $\mathrm{Sp}_4(3)$ does not fix the isomorphism class of 4-dimensional irreducible representations of $\mathrm{Sp}_4(3)$. We can now find a finite cyclic subgroup C_e of Z, of order e such that $G_3 = C_e * \mathrm{Sp}_4(3)$. By [21, Theorem 4.1], G_i is the normal closure of $\langle g_i \rangle$ for $i = 3, 4$. In particular, $\det(G_3)$ is C_3 which implies that $3|e$ and $e|12$. Observe that g_3 has trace

$$1 + \zeta_{12}^3 + \zeta_{12}^6 + \zeta_{12}^7 = 1 + \zeta_{12}^3(1 + \zeta_{12}^4) - 1 = \zeta_{12}^3(-\zeta_{12}^8) = -\zeta_{12}^{11},$$

so in fact $\mathbb{K} = \mathbb{Q}(\zeta_{12})$. Now if $e \in \{3, 6\}$ then we would have $\mathbb{K} = \mathbb{Q}(\zeta_3)$, a contradiction. Hence we conclude that $G_3 = C_{12} * \mathrm{Sp}_4(3)$.

By [18, Lemma 5.2], we have $ZG_3 = ZG_4$ and $[G_3, G_3] = [G_4, G_4]$. It follows that $G_4 = C_d * \mathrm{Sp}_4(3)$ for some finite cyclic subgroup C_d of Z, of order d. Since g_4 has determinant 1, $\det(G_4) = 1$, which implies that $d|4$. Applying [18, Proposition 6.1(iii)] (with $N = 12$ and $L = \mathbb{Q}(\zeta_3)$), we see that the field of traces of G_4 is contained in $\mathbb{Q}(\zeta_3)$. This shows that $d \neq 4$, i.e. $d|2$. As $\mathrm{Sp}_4(3)$ has center of order 2, we conclude that $G_4 = \mathrm{Sp}_4(3)$. $\square$

7. Universal families of arbitrary degree: monodromy groups

In this section, we determine the geometric monodromy groups of the universal families $\mathcal{F}_{\chi,e}$ (see (5.0.1)) studied in [14, Chapter 5]. First, in view of the truth [6, Theorem 1.4] of Larsen's Eight Moment Conjecture, we have the following theorem.

Theorem 7.1. *We have the following results.*

(i) *If $e \geq 7$ and χ has order ≥ 3 with $\chi^e \neq \mathbb{1}$, then $G^\circ_{\mathrm{geom},\mathcal{F}_{\chi,e}} = \mathrm{SL}_{e-1}$.*
(i-bis) *If $e = 6$ and χ has order ≥ 3 with $\chi^e \neq \mathbb{1}$, then $G^\circ_{\mathrm{geom},\mathcal{F}_{\chi,e}} = \mathrm{SL}_{e-1}$.*
(i-ter) *If $e \in \{3, 4, 5\}$ and χ has order ≥ 3 with $\chi^e \neq \mathbb{1}$, then $G^\circ_{\mathrm{geom},\mathcal{F}_{\chi,e}} = \mathrm{SL}_{e-1}$ except in the following cases, in each of which G_{geom} is finite:*

$$e = 3, \chi \text{ of order } \in \{4, 6, 10\}, \text{ and } e \in \{4, 5\}, \chi \text{ of order } 6.$$

These are precisely the systems listed in Theorem 5.2.
(ii) *If $e \geq 9$ is odd and χ is the quadratic character, then $G_{\mathrm{geom},\mathcal{F}_{\chi,e}} = \mathrm{Sp}_{e-1}$.*
(ii-bis) *If $e \in \{7, 5, 3\}$ and χ is the quadratic character, then $G_{\mathrm{geom},\mathcal{F}_{\chi,e}} = \mathrm{Sp}_{e-1}$.*

Proof. Statements (i) and (ii) are immediate corollaries of Theorem [6, Theorem 1.4], applied via Theorem 5.1. For statement (i-bis), the rank is 5. In this rank 5 case, having $M_{3,3} = 3!$ is enough, by [6, Theorem 1.6], in which all problematic cases are in rank ≥ 6.

For statement (i-ter), in all cases we have $e + 1 \geq 4$, and hence $M_{2,2} = 2!$ by [14, 1.20.3]. So by Larsen's Alternative, either $G^\circ_{\mathrm{geom},\mathcal{F}_{\chi,e}} = \mathrm{SL}_{e-1}$ or G_{geom} is finite. When $p \nmid e$, then by [14, Theorem 5.2.2 1)], the only finite cases are those listed in Theorem 5.2. To prove that there are no more finite cases when $p|e$ with $e \in \{3, 4, 5\}$, it suffices to look at the one parameter subfamilies of $\mathcal{F}(\chi, e)$ given by

$$t \mapsto -\sum_x \chi(x^e + x^{e-1} - t).$$

Each of these one parameter families is a Kloosterman sheaf $\mathcal{K}$, since $q - a = 1$ in the notations of Theorem 3.2. By Theorem 6.1, $G_{\mathrm{geom},\mathcal{K}}$ is not finite except in the listed cases, and those are known to be finite, cf. Theorem 5.2.

For statement (ii-bis), in all cases we have $e + 1 \geq 4$, and with χ_2 we are self-dual, hence by [14, 1.20.3] we have $M_{2,2} = 4!! = 3$ if $e - 1 \geq 4$ and $M_{2,2} = 2! = 2$ if $e - 1 = 2$. So $G_{\rm geom}$ is either finite or is the desired Sp_{e-1}. When $p \nmid e$, we have $G_{\rm geom} = \mathrm{Sp}_{e-1}$ by [11, Theorem 5.4]. So it suffices to treat the cases when $p = e$ is one of $\{7, 5, 3\}$. For this, it suffices to treat the one-parameter system, parameter $t \neq 0$, in characteristic p, whose trace function is

$$t \mapsto -\sum_x \chi_2(x^p + x^2 - t).$$

That this system has $G_{\rm geom} = \mathrm{Sp}(p - 1)$ is proven in [17, Theorem 7.5]. □

Next we determine the finite geometric monodromy groups that occur in Theorem 7.1(i-ter).

By [11, Determinant Lemma 5.2.bis], we know that, in odd characteristic, the geometric determinant of $\mathcal{F}_{\chi,e}$ has order dividing the order of $\chi\chi_2$, and in characteristic 2, the geometric determinant of $\mathcal{F}_{\chi,e}$ has order dividing 2 times the (necessarily odd) order of χ. In this cited reference, where our "e" is called "d", there is no hypothesis that $p \nmid d$. Moreover, if $p \nmid d$ and p is odd, then the use of "weakly superMorse" polynomials, via [11, Key Lemma 5.7 (4) and (5), Lemma 5.9 (1), and Lemma 5.15] shows that the order of the determinant of $\mathcal{F}_{\chi,e}$ is equal to the order of $\chi\chi_2$.

Theorem 7.2. *Let $\mathcal{F} = \mathcal{F}_{\chi,e}$ be one of the universal systems with finite monodromy listed in Theorem 5.2. Let $V = \mathbb{C}^{e-1}$ denote the underlying representation, and let $\mathcal{G} = \mathrm{GL}(V)$. Then the following statements hold for its geometric monodromy group $G = G_{\rm geom}$.*

(i) *If $e = 5$ and $|\chi| = 6$, then $G = C_3 \times \mathrm{Sp}_4(3)$, the Shephard-Todd group $ST32$.*
(ii) *Suppose $e = 4$ and $|\chi| = 6$. Then $G = (E_3 \rtimes Q_8) \cdot 3$, where $E_3 = 3^{1+2}_+$, the extraspecial 3-group of order 27 and exponent 3, and $Q_8 = 2^{1+2}_-$ is the quaternion group of order 8.*
(iii) *Suppose $e = 3$ and $|\chi| \in \{4, 6\}$. Then $G \rhd Q_8 = 2^{1+2}_-$, $|\mathbf{Z}(G)|$ divides $|\chi|$, and $\mathbf{Z}(\mathcal{G})G = \mathbf{Z}(\mathcal{G})H$, where $H = \mathrm{SL}_2(3)$, or $|\chi| = 4$ and $H = \mathrm{GL}_2(3)$. If in addition $p = 3$ and $|\chi| = 4$, then $G = C_4 * 2\mathsf{S}_4$, the Shephard–Todd group $ST8$.*
(iv) *Suppose $e = 3$ and $|\chi| = 10$. Then either*

  (a) *$G = C_5 \times \mathrm{SL}_2(5)$, the Shephard–Todd group $ST16$, or*
  (b) *$p > 5$, $G \rhd Q_8 = 2^{1+2}_-$, and $\mathbf{Z}(\mathcal{G})G = \mathbf{Z}(\mathcal{G})H$, where either $H = \mathrm{GL}_2(3)$ or $H = \mathrm{SL}_2(3)$; in particular, $|G| \leq 240$.*

Proof. Let φ denote the G-character afforded by V.

(a) First we consider the case $e = 5$. Recall from Theorem 5.1(i-bis) that $M_{3,3}(G, V) = 6 = M_{3,3}(\mathcal{G}, V)$. Hence we can apply [2, Theorem 8(B)] to arrive at one of the two possibilities (A1) and (A2) described therein. It is clear that $\mathbb{Q}(\varphi) \subseteq \mathbb{Q}(\zeta_3)$, whence

$\mathbf{Z}(G) \le C_6$, and this rules out (A2) which has $\mathbf{Z}(G) \ge C_4$. Thus we are in case (A2), where $G = \mathbf{Z}(G) * \mathrm{Sp}_4(3)$; in particular, $|\mathbf{Z}(G)| = 2$ or 6. If $p > 5$, then $\det(\mathcal{F})$ has order 3, forcing $|\mathbf{Z}(G)| = 6$. Assume in addition that $p = 5$. Then, as explained in the proof of Theorem 6.1, the specialization of $\mathcal{F}$ with trace function $t \mapsto -\sum_x \chi(x^5+x^2-t)$ is the [3] Kummer pullback of the Kloosterman sheaf $\mathcal{K}_{5,2,\rho}$ where $\rho^3 = \chi$, and by Theorem 6.2(ii) this Kloosterman sheaf has geometric monodromy group $C_9 \times \mathrm{Sp}_4(3)$, with center of order 18. It follows that $|\mathbf{Z}(G)|$ is exactly 6. Thus we always have $G = C_3 \times \mathrm{Sp}_4(3)$, as stated in (i).

(b) In the remaining cases of $e = 3, 4$, we have $M_{2,2}(G,V) = 2! = M_{2,2}(\mathcal{G},V)$ by [14, 1.20.3], and we can apply [2, Theorem 10] to G.

First assume that $e = 4$. Using [2, Theorem 10(B)] we arrive at one of the possibilities (B1)–(B3) described therein. Also, $\mathbb{Q}(\varphi) \subseteq \mathbb{Q}(\zeta_3)$, and this rules out (B1) for which has $\mathbb{Q}(\varphi) \ni \sqrt{5}$, and (B2) for which $\mathbb{Q}(\varphi) \ni \sqrt{-7}$. Thus we are in case (B3): $G \rhd E_3 = 3^{1+2}_+$, $\mathbf{Z}(\mathcal{G})G = \mathbf{Z}(\mathcal{G})H$, where $H = E_3 \rtimes \mathrm{SL}_2(3)$ or $H = E_3 \times Q_8$. Recall that $p > 5$ here, so $\det(\mathcal{F})$ has order 3. Since the rank D is 3, this shows that $\mathbf{Z}(G) \hookrightarrow C_9$. But $\mathbb{Q}(\varphi) \subseteq \mathbb{Q}(\zeta_3)$ rules out the case $\mathbf{Z}(G) = C_9$, and $\mathbf{Z}(G) \ge \mathbf{Z}(E_3) = C_3$. So we have $\mathbf{Z}(G) = \mathbf{Z}(E_3)$. The equality $\mathbf{Z}(\mathcal{G})G = \mathbf{Z}(\mathcal{G})H$ implies that there is a surjection $\pi : G \twoheadrightarrow H/\mathbf{Z}(H)$, where $\mathrm{Ker}(\pi) = \mathbf{Z}(G)$; in particular, the Sylow 2-subgroups of $G/\mathbf{Z}(G)$, and hence of G, are isomorphic to Q_8. Suppose $H = E_3 \rtimes Q_8$. Then $|G| = 3|H/\mathbf{Z}(H)| = |H| = |E_3| \cdot |Q_8|$. Since $E_3 \lhd G$, this implies by the Schur–Zassenhaus theorem that $G = E_3 \rtimes Q_8$. But note that the determinant of any element $x \in E_3$ is 1, and of any element $y \in Q_8$ has order a power of 2. So $\det(G)$ cannot be of order 3, a contradiction. Thus $H = E_3 \rtimes \mathrm{SL}_2(3)$, and we still have $|G| = |H|$. Let $G_1 := G \cap \mathrm{SL}(V)$, a normal subgroup of G of index 3. The same arguments as before show that $G_1 = E_3 \rtimes Q_8$, and hence $G = (E_3 \rtimes Q_8) \cdot 3$ as stated in (ii).

Assume now that $e = 3$. Using [2, Theorem 10(A)] we arrive at one of the possibilities (A1)–(A3) described therein. The cases (A2) and (A3) are included in (iii) and (iv). In the case of (A1), we have $G = \mathbf{Z}(G) * \mathrm{SL}_2(5)$, whence $\mathbb{Q}(\varphi) \ni \sqrt{5}$, which is possible only when $|\chi| = 10$. Assume that $e = p = 3$ and $|\chi| = 10$. Then, as explained in the proof of Theorem 6.1, the specialization of $\mathcal{F}$ with trace function $t \mapsto -\sum_x \chi(x^3+x^2-t)$ is the Kloosterman sheaf $\mathcal{K}_{3,2,\chi}$, and by Theorem 6.2(i) this Kloosterman sheaf has geometric monodromy group $C_5 \times \mathrm{SL}_2(5)$, with center of order 10. On the other hand, $\mathcal{F}$ has $\mathbb{Q}(\varphi) \subseteq \mathbb{Q}(\zeta_5)$, which implies that $\mathbf{Z}(G) \le C_{10}$. It follows that $|\mathbf{Z}(G)|$ is exactly 10, and thus $G = C_5 \times \mathrm{SL}_2(5)$, as stated in (iv). If $(e, |\chi|) = (3, 10)$ but $p > 5$, then $\det(\mathcal{F})$ has order 5, whereas $\mathrm{SL}_2(5)$ has determinant 1 and $\mathbf{Z}(G) \hookrightarrow C_{10}$ (since $\mathbb{Q}(\varphi) \subseteq \mathbb{Q}(\zeta_5)$), so we conclude that $\mathbf{Z}(G) = C_{10}$, and hence $G = C_5 \times \mathrm{SL}_2(5)$ if we are in (A1).

Next, assume that $e = p = 3$, $|\chi| = 6$, but $H = \mathrm{GL}_2(3)$. Using the realization $H = 2\mathsf{S}_4$ and the equality $\mathbf{Z}(\mathcal{G})G = \mathbf{Z}(\mathcal{G})H$, we can write $g = \lambda h$ for some $\lambda \in \mathbb{C}^\times$ and some element $h \in H$ whose coset in $H/\mathbf{Z}(H)$ contains a 4-cycle of S_4; in particular, h has order 8 and trace $\sqrt{-2}$. Since both g and h have finite orders, λ is a root of unity and so $|\varphi(g)| = |\mathrm{Trace}(h)| = \sqrt{2}$. On the other hand, $\varphi(g)$ is an algebraic integer in $\mathbb{Q}(\zeta_3)$, so

$\varphi(g) = a + b\zeta_3$ for some $a, b \in \mathbb{Z}$, and thus $8 = 4|\varphi(g)|^2 = 4(a^2 - ab + b^2) = (2a - b)^2 + 3b^2$, i.e. 8 is a square modulo 3, a contradiction.

Finally, assume that $e = p = 3$ and $|\chi| = 4$. Then, as explained in the proof of Theorem 6.1, the specialization of $\mathcal{F}$ with trace function $t \mapsto -\sum_x \chi(x^3 + x^2 - t)$ is the Kloosterman sheaf $\mathcal{K}_{3,2,\chi}$, and by Theorem 6.2(i) this Kloosterman sheaf has geometric monodromy group $H := C_4 * 2\mathsf{S}_4$, with center of order 4. On the other hand, $\mathcal{F}$ has $\mathbb{Q}(\varphi) \subseteq \mathbb{Q}(\zeta_4)$, which implies that $\mathbf{Z}(G) \leq C_4$. It follows that $|\mathbf{Z}(G)|$ is exactly 4, and $|G/\mathbf{Z}(G)|$ is of order divisible by $|\mathsf{S}_4| = 24 = |\mathrm{PGL}_2(3)|$. Hence we conclude that $G = H = C_4 * 2\mathsf{S}_4$, as stated in (iii). □

We next rule out the possibility (iv)(b) in Theorem 7.2.

Proposition 7.3. *Suppose $e = 3$ and $|\chi| = 10$. In every characteristic $p \nmid 10$, G_{geom} for $\mathcal{F}_{\chi,3}$ is $C_5 \times \mathrm{SL}_2(5)$.*

Proof. First we observe that $\mathcal{F}_{\chi_{10},3}$ exists as a local system on the space $\mathcal{M}_3$ of cubics with all distinct roots. Explicitly, this is the open set of $(\mathbb{G}_m \times \mathbb{A}^3)/\mathbb{Z}[\zeta_{10}, 1/10]$, coordinates (a_3, a_2, a_1, a_0), on which the discriminant Δ of the universal cubic $a_3x^3 + a_2x^2 + a_1x + a_0$ is invertible. By [14, Theorem 5.2.2 1)], and [10, Theorem 8.18.2], the group $G_{\mathrm{geom},\mathbb{C}}$ for the pullback of $\mathcal{F}_{\chi,3}$ to the complex fiber $\mathcal{M}_{3,\mathbb{C}}$, for a chosen embedding of $\mathbb{Z}[\zeta_{10}, 1/10]$ into $\mathbb{C}$ (e.g., by sending ζ_{10} to $e^{2\pi i/10}$) is finite, and we have the following extra information:

(i) For every finite place $\mathcal{P}$ of $\mathbb{Z}[\zeta_{10}, 1/10]$, $G_{\mathrm{geom},\mathcal{P}}$, the G_{geom} for the pullback of $\mathcal{F}_{\chi,3}$ to the fiber of $\mathcal{M}_3$ over $\overline{\mathbb{F}_{\mathcal{P}}}$ is a subgroup of $G_{\mathrm{geom},\mathbb{C}}$.
(ii) For all but finitely many such finite places, $G_{\mathrm{geom},\mathcal{P}}$ is equal to $G_{\mathrm{geom},\mathbb{C}}$.

By Theorem 7.2(iv), the possible $G_{\mathrm{geom},\mathbb{C}}$ are on a short list of finite groups, of which $C_5 \times \mathrm{SL}_2(5)$ has the largest order, and this group is attained by $G_{\mathrm{geom},\mathcal{P}}$ for any $\mathcal{P}$ of characteristic $p = 3$. Therefore $G_{\mathrm{geom},\mathbb{C}}$ is $C_5 \times \mathrm{SL}_2(5)$. Now consider the one-parameter family of cubics (parameter t)

$$\mathcal{N}_3, \text{ given by } x^3 + x^2 - t$$

with discriminant $\Delta = -t(27t - 4)$, and the pullback of $\mathcal{F}_{\chi,3}$ to $\mathcal{N}_3$. For this one parameter pullback family, each $G_{\mathrm{geom},\mathcal{N}_3,\mathcal{P}}$ is a subgroup of $G_{\mathrm{geom},\mathcal{P}}$ (simply because it is a pullback), and for the same reason $G_{\mathrm{geom},\mathcal{N}_3,\mathbb{C}}$ is a subgroup of $G_{\mathrm{geom},\mathbb{C}} = C_5 \times \mathrm{SL}_2(5)$. But at places $\mathcal{P}$ of characteristic 3, we already know that $G_{\mathrm{geom},\mathcal{N}_3,\mathcal{P}} = C_5 \times \mathrm{SL}_2(5)$. Therefore $G_{\mathrm{geom},\mathcal{N}_3,\mathbb{C}} = C_5 \times \mathrm{SL}_2(5)$, and thus for almost every place $\mathcal{P}$, $G_{\mathrm{geom},\mathcal{N}_3,\mathcal{P}}$ is $C_5 \times \mathrm{SL}_2(5)$.

It remains to show that for **all** places $\mathcal{P}$ of characteristic $p > 5$, $G_{\mathrm{geom},\mathcal{N}_3,\mathcal{P}}$ has the same value. The key point here is that in these characteristics, the pullback of $\mathcal{F}_{\chi,3}$ to

$\mathcal{N}_3$ is lisse on $\mathbb{A}^1 \smallsetminus \{0, 4/27\}$. As the group $C_5 \times \mathrm{SL}_2(5)$ has order 600 which is prime to every $p > 5$, it follows that $\mathcal{F}_{\chi,3}$, pulled back to $\mathcal{N}_3[1/30]$, is lisse on $\mathbb{P}^1 \smallsetminus \{0, 4/27, \infty\}$ and tamely ramified along each of those three points. So the asserted constancy is a special case of the Tame Specialization Theorem [10, Theorem 8.17.14]. □

The same argument completes the $(e = 3, |\chi| = 4)$ case of Theorem 7.2.

Proposition 7.4. *Suppose $e = 3$ and $|\chi| = 4$. In every characteristic $p \neq 2$, G_{geom} for $\mathcal{F}_{\chi,3}$ is the Shephard–Todd group $ST8$.*

Proof. By Theorem 7.2(iii), in any characteristic $p \neq 2$, $G := G_{\mathrm{geom}}$ has $G/\mathbf{Z}(G) \cong H/\mathbf{Z}(H)$ is either $\mathrm{PGL}_2(3)$ or $\mathrm{PSL}_2(3)$, with $\mathbf{Z}(G) \leq C_4$. Thus G has order dividing $96 = |ST8|$. At any $\mathcal{P}$ of characteristic $p = 3$, we have $G_{\mathrm{geom},\mathcal{P}} = ST8$. Recall that $G_{\mathrm{geom},\mathcal{P}} \leq G_{\mathrm{geom},\mathbb{C}}$. But $G_{\mathrm{geom},\mathbb{C}}$ has order dividing 96, hence $G_{\mathrm{geom},\mathbb{C}} = ST8$. Then we have $G_{\mathrm{geom},\mathcal{P}} = ST8$ for almost all places $\mathcal{P}$. Now pass to the pullback of $\mathcal{F}_{\chi,3}$ to $\mathcal{N}_3[1/6]$, and repeat the use of the Tame Specialization Theorem [10, Theorem 8.17.14] to find that already on this subfamily, we have $G_{\mathrm{geom},\mathcal{N}_3,\mathcal{P}} = ST8$ at all places of characteristic $p \geq 5$. □

Finally we complete the case $\mathcal{F}_{\chi_6,3}$ in every characteristic $p \geq 5$.

Proposition 7.5. *Suppose $e = 3$ and $|\chi| = 6$. In every characteristic $p \neq 2, 3$, G_{geom} for $\mathcal{F}_{\chi,3}$ is $C_3 \times \mathrm{SL}_2(3)$.*

Proof. Fix a characteristic $p \neq 2, 3$, a finite extension $\mathbb{F}_q/\mathbb{F}_p$ with $q \equiv 1 \pmod 6$, and a character χ of $\mathbb{F}_q^\times$ of order 6. Then all Frobenius traces of $\mathcal{F}_{\chi,3}|\mathbb{F}_q$ lie in $\mathbb{Z}[\zeta_6] = \mathbb{Z}[\zeta_3]$. The geometric determinant of $\mathcal{F}_{\chi,3}|\mathbb{F}_q$ has order dividing 6, cf. [22, Proposition 2.3.2].

We next exhibit a nonzero constant $D \in \mathbb{Z}[\zeta_6]$ such that over $\mathbb{F}_{q^2}$, the Tate-twisted

$$\mathcal{G}_{\chi,3} := (\mathcal{F}_{\chi,3}|\mathbb{F}_{q^2}) \otimes (1/D)^{\deg/\mathbb{F}_{q^2}}$$

has arithmetic determinant of order dividing 6. With $\mathcal{M}_3$ the space of cubics with all distinct roots, we choose a single $f_0 \in \mathcal{M}(\mathbb{F}_q)$ (e.g., we might choose $f_0 := x^3 - x$) and take $D := \det(\mathrm{Frob}_{f,\mathbb{F}_q}|\mathcal{F}_{\chi,3})$. Because $\mathcal{F}_{\chi,3}$ has rank 2, we have, for this f_0,

$$\det(\mathrm{Frob}_{f_0,\mathbb{F}_{q^2}}|\mathcal{G}_{\chi,3}) = 1.$$

Because the geometric determinant of $\mathcal{F}_{\chi,3}|\mathbb{F}_q$, which is also the geometric determinant of $\mathcal{G}_{\chi,3}$, has order dividing 6, it follows that for any finite extension $L/\mathbb{F}_{q^2}$, and any $f \in \mathcal{M}(L)$, we have $\det(\mathrm{Frob}_{f,L}|\mathcal{G}_{\chi,3}) \in \mu_6$. This is precisely the statement that $\mathcal{G}_{\chi,3}$ has arithmetic determinant of order dividing 6.

Because $G = G_{\mathrm{geom}}$ for $\mathcal{G}_{\chi,3}$ is finite, and $\mathcal{G}_{\chi,3}$ has determinant which is arithmetically of finite order, it follows, see [10, 8.14.3.1], that G_{arith} for $\mathcal{G}_{\chi,3}$ is finite. By Theorem 7.2(iii), $G/\mathbf{Z}(G) \cong \mathrm{SL}_2(3)/\mathbf{Z}(\mathrm{SL}_2(3)) = \mathsf{A}_4$, and $\mathbf{Z}(G) = C_2$ or C_6. Once G_{arith}

is finite, then $G_{\mathrm{arith}}/G_{\mathrm{geom}}$ is a finite group which is geometrically trivial, so after some finite extension $L/\mathbb{F}_{q^2}$, the two groups coincide. Then over this L, there is an $f \in \mathcal{M}_3(L)$ whose $\mathrm{Frob}_{f,L}$ is the identity element in $G_{\mathrm{arith}} = G_{\mathrm{geom}}$, and moreover every Frobenius $\mathrm{Frob}_{f_1,L}$ lies in G_{geom}. Thus we have

$$\mathrm{Trace}(\mathrm{Frob}_{f,L}|\mathcal{G}_{\chi,3}) = 2.$$

For any $\alpha \in L^\times$, and any $f_1 \in \mathcal{M}_3(L)$, we have

$$\mathrm{Trace}(\mathrm{Frob}_{\alpha f_1,L}|\mathcal{G}_{\chi,3}) = \chi(\alpha)\mathrm{Trace}(\mathrm{Frob}_{f_1,L}|\mathcal{G}_{\chi,3}),$$

simply because for each $x \in L$, $\chi(\alpha f_1(x)) = \chi(\alpha)\chi(f_1(x))$. Choose $\alpha \in L^\times$ to be a generator of $L^\times$. Then at the point $\alpha f \in \mathcal{M}_3(L)$, we have

$$\mathrm{Trace}(\mathrm{Frob}_{\alpha f,L}|\mathcal{G}_{\chi,3}) = \chi(\alpha)\mathrm{Trace}(\mathrm{Frob}_{f,L}|\mathcal{G}_{\chi,3}) = 2\chi(\alpha).$$

Since $\chi(\alpha)$ is a primitive sixth root of unity, $\mathrm{Frob}_{\alpha f,L}$ is a central element of order 6, and thus $\mathbf{Z}(G) = C_6$. In this case, $G = \langle C_3 \times Q_8, g\rangle$, where $g = \beta h$ for some $\beta \in \mathbb{C}^\times$ and $\langle Q_8, h\rangle = H = \mathrm{SL}_2(3)$ with $|h| = 3$. Since $g^3 = \beta^3 h^3$ centralizes Q_8 and g, $g^3 \in C_6 = \mathbf{Z}(G)$, and so $\beta^{18} = 1$. Note that both g and h have traces in $\mathbb{Q}(\zeta_3)$, and the trace of h is nonzero. Hence $\beta \in \mathbb{Q}(\zeta_3)$, and we conclude that in fact $\beta^6 = 1$. Now we can multiply g by a suitable element in $\mathbf{Z}(G)$ to get that $g = h$, and thus $G = C_3 \times H = C_3 \times \mathrm{SL}_2(3)$. □

To summarize, we have now proved the following refined version of Theorem 7.2:

Theorem 7.6. *Let $\mathcal{F} = \mathcal{F}_{\chi,e}$ be one of the universal systems with finite monodromy listed in Theorem 5.2. Then the following statements hold for its geometric monodromy group $G = G_{\mathrm{geom}}$ in any characteristic $p \nmid |\chi|$.*

(i) *If $e = 5$ and $|\chi| = 6$, then $G = C_3 \times \mathrm{Sp}_4(3)$, the Shephard-Todd group $ST32$.*
(ii) *Suppose $e = 4$ and $|\chi| = 6$. Then $G = (E_3 \rtimes Q_8) \cdot 3$, where $E_3 = 3^{1+2}_+$, the extraspecial 3-group of order 27 and exponent 3, and $Q_8 = 2^{1+2}_-$ is the quaternion group of order 8.*
(iii) *Suppose $e = 3$ and $|\chi| = 4$. Then $G = C_4 * 2\mathsf{S}_4$, the Shephard–Todd group $ST8$.*
(iv) *Suppose $e = 3$ and $|\chi| = 6$. Then $G = C_3 \times \mathrm{SL}_2(3)$.*
(v) *Suppose $e = 3$ and $|\chi| = 10$. Then $G = C_5 \times \mathrm{SL}_2(5)$, the Shephard–Todd group $ST16$.*

8. Monodromy groups of $\mathcal{G}_\chi$: infinite cases

Theorem 8.1. *For any $a \in \mathbb{Z}_{\geq 2}$ with $q = p^f \geq 11$ and $a < q$ coprime to p, and for any multiplicative character ρ, consider the Kloosterman sheaf*

$$\mathcal{K} = \mathcal{K}_{q,a,\rho} = \mathcal{K}l(\mathsf{Char}(q-a) \cup \rho\mathsf{Char}_{\text{ntriv}}(a)).$$

Assume in addition that $(a, \rho) \neq (q-1, \mathbb{1})$. Then the geometric monodromy group $G = G_{\text{geom}}$ of $\mathcal{K}$ is infinite if and only if $\chi := \rho^{q-a} \neq \mathbb{1}$, in which case $G^\circ = \mathrm{SL}_{q-1}$ if $\chi \neq \chi_2$ and $G^\circ = \mathrm{Sp}_{q-1}$ if $\chi = \chi_2$.

Proof. (a) In one direction, by Theorem 4.2 we know that G is finite if and only if $\chi = \mathbb{1}$. So we may assume $\chi \neq \mathbb{1}$; and the rest of the proof is to identify the infinite group G. By Theorem 3.3, $\mathcal{K}$ is primitive, and hence it satisfies (**S**+) by [21, Theorem 1.7]. In particular, G° acts irreducibly on the underlying representation $V = \mathbb{C}^{q-1}$. Now, by Theorem 3.6, the G°-module V is self-dual precisely when $\chi = \chi_2$, in which case it is of symplectic type. Indeed, by Lemma 3.5 it is symplectic over the finite index subgroup $G_{\text{geom},\mathcal{G}_\chi}$ of G.

(b) Next we apply [22, Theorem 6.2.14] to $\mathcal{K}$, which is of rank $D = q-1 = p^f - 1 \geq 10$, and arrive at one of the following two cases.

(b1) In the first case, either $G^\circ = \mathrm{SL}_D$, or the G°-module V is self-dual and $G^\circ\{\mathrm{SO}_D, \mathrm{Sp}_D\}$. As mentioned above, the latter possibility occurs precisely when $\chi = \chi_2$. Moreover, if $\chi = \chi_2$, then $G^\circ \leq \mathrm{Sp}_D$, and hence $G^\circ = \mathrm{Sp}_D$. Thus if $\chi \neq \chi_2$ we must have $G^\circ = \mathrm{SL}_D$.

(b2) In the second case, $D = 15$, so $q = 2^4$, and G° is the image SL_6/μ_2 of $\mathrm{SL}_6 = \mathrm{SL}(U)$ acting on $V = \wedge^2(U)$. Since the outer (graph) automorphism of G° does not fix the isomorphism class of $\wedge^2(U)$, G can only induce inner automorphisms of G°. Thus $G = \mathbf{Z}(G)G^\circ$, where $Z := \mathbf{Z}(G)$ acts on V via scalars.

Let Q, respectively J, denote the image of $P(\infty)$, respectively $I(\infty)$, in G. Since $(D, p) = (15, 2)$, we know that $J = Q \rtimes \langle g \rangle$, where $V|_Q$ is the direct sum of 15 one-dimensional pairwise non-isomorphic Q-modules permuted transitively by $\langle g \rangle \cong C_{15}$. The same is true if we replace J and Q by $\tilde{J} := ZJ$ and $\tilde{Q} = ZQ$. Using $G = ZG^\circ$, we can write $\tilde{J} = ZJ_1$ and $\tilde{Q} = ZQ_1$, where $J_1 = \tilde{J} \cap G^\circ$ and $Q_1 = \tilde{Q} \cap G^\circ$. Pulling back from G° to SL_6, without loss we may replace J_1 by its full inverse image J_2 in SL_6, and Q_1 by its full inverse image Q_2 in SL_6. Now the subgroups J_2 and Q_2 of $\mathrm{SL}(U)$ acts on V as on $\wedge^2(U)$, and moreover J_2 acts irreducibly on V. If the J_2-module U is reducible: $U|_{J_2} = A \oplus B$ with $\dim A \geq \dim B \geq 1$ and $\dim A \geq 3$, then

$$V|_{J_2} \cong \wedge^2(A) \oplus (A \otimes B) \oplus \wedge^2(B),$$

with the first two summands always being nonzero, contrary to the irreducibility of $V|_{J_2}$. So J_2 acts irreducibly on U.

Given any element $x \in J_1 \leq \tilde{J} = ZJ = ZJ_1$, we can write $x = z_1 y$ with $z_1 \in Z$ and $y \in J$. Similarly, we can write $g = zg_1$ with $z \in Z$ and $g_1 \in J_1$. Recalling that $J = Q \rtimes \langle g \rangle \cong Q \rtimes C_{15}$, we can find an integer $0 \leq n \leq 14$ such that $y = g^n h$ where $h \in Q$. It follows that

$$x = z_1 y = g^n z_1 h = (zg_1)^n z_1 h = (g_1)^n (z^n z_1 h)$$

with

$$z^n z_1 h \in J_1 \cap ZQ_1 = \tilde{J} \cap G^\circ \cap ZQ_1 \leq G^\circ \cap ZQ_1 = Q_1.$$

This shows that $J_1 = \langle Q_1, g_1 \rangle$, with

$$g_1^{15} = z^{-15} \in Z \cap J_1 \leq J_1 \cap ZQ_1 = Q_1$$

(as $g^{15} = 1$). Letting g_2 be an inverse image of g_1 in $\mathrm{SL}(U)$, we now have that $J_2 = \langle g_2, Q_2 \rangle$ with $g_2^{15} \in Q_2$.

Also recall that Q is a 2-group. So modulo scalars, Q_2 is a 2-group, and hence cannot act irreducibly on $U = \mathbb{C}^6$. But J_2/Q_2 is cyclic of order dividing 15 as we just established. It follows from Clifford's theorem that $U|_{Q_2} = M_1 \oplus M_2 \oplus M_3$, a direct sum of three pairwise non-isomorphic 2-dimensional submodules permuted transitively by g_2, in this case, $J_2 = \langle g_2, Q_2 \rangle$ fixes the 3-dimensional subspace $\wedge^2(M_1) \oplus \wedge^2(M_2) \oplus \wedge^2(M_3)$ of $\wedge^2(U)$, contrary to its irreducible action on V. □

Theorem 8.2. *For any $a \in \mathbb{Z}_{\geq 2}$ with $3 \leq q = p^f < 11$ and $a < q$ coprime to p, and for any multiplicative character ρ, consider the Kloosterman sheaf*

$$\mathcal{K} = \mathcal{K}_{q,a,\rho} = \mathcal{K}l(\mathsf{Char}(q-a) \cup \rho\mathsf{Char}_{\mathrm{ntriv}}(a)).$$

Assume in addition that $(a, \rho) \neq (q-1, \mathbb{1})$. Then the geometric monodromy group $G = G_{\mathrm{geom}}$ of $\mathcal{K}$ is infinite if and only if both $\chi := \rho^{q-a} \neq \mathbb{1}$ and (q, a, ρ) is not listed in Theorem 6.1(i), (ii)*. If G_{geom} is infinite, then $G^\circ = \mathrm{SL}_{q-1}$ if $\chi \neq \chi_2$ and $G^\circ = \mathrm{Sp}_{q-1}$ if $\chi = \chi_2$.*

Proof. The first statement is already proved in Theorem 6.1. For the second statement, we have $\chi = \rho^{q-a} \neq \mathbb{1}$, and so the infinite group G satisfies $(\mathbf{S}+)$ by Theorem 3.3. This will allow us to apply [22, Theorem 6.2.14].

Consider the case $q = 3$, then $\mathcal{K}$ has rank 2, so the only possibility is $G^\circ = \mathrm{SL}_2$.

For $q = 4$, the rank is 3, so by [22, Theorem 6.2.14] the only possibilities are SL_3 and SO_3. But with $q = 4$, we are in characteristic 2, where χ cannot be χ_2, so by Theorem 3.6, the only possibility is SL_3.

Consider the case $q = 7$. Here the only case other than $\mathrm{SL}_6, \mathrm{SO}_6, \mathrm{Sp}_6$ in rank $D = 6$ is case (iv), which requires $p = 2$. So the result follows from Theorem 3.6.

Consider the case $q = 8$. Then the only case other than $\mathrm{SL}_7, \mathrm{SO}_7$ is G_2. But the G_2 case is Lie self-dual, possible only when $\chi = \chi_2$, not possible in characteristic 2.

Consider the case $q = 9$. Here we must rule out cases (vi) and (vii) of [22, Theorem 6.2.14]. Both of these cases, the adjoint representation of SL_3 and the spin representation for SO_7, are Lie self-dual, but in both cases the duality is orthogonal, whereas by Theorem 3.6, the duality is symplectic. So these cases do not occur. □

We now combine the results of Theorems 8.1 and 8.2 with Lemma 3.4.

Theorem 8.3. *Given $q = p^f \geq 3$ and $a \in \mathbb{Z}_{\geq 2}$ with $a < q$ coprime to p, and a multiplicative character ρ, consider the Kloosterman sheaf*

$$\mathcal{K} = \mathcal{K}_{q,a,\rho} = \mathcal{K}l(\mathsf{Char}(q-a) \cup \rho\mathsf{Char}_{\mathrm{ntriv}}(a)).$$

Suppose that $\chi := \rho^{q-a} \neq \mathbb{1}$, and that $\mathcal{K}$ has infinite G_{geom}. Define integers N and M as follows:

- *If q is odd, then $N :=$ the order of $\chi_2\rho^{a-1}$, and $M :=$ the order of $\chi_2^{q-a}\chi^{a-1}$.*
- *If q is even, then $N :=$ the order of ρ^{a-1}, and $M :=$ the order of χ^{a-1}.*

Then we have the following results.

(i) *If $\chi := \rho^{q-a} \neq \chi_2$, then $G_{\mathrm{geom},\mathcal{K}}$ is given by*

$$G_{\mathrm{geom},\mathcal{K}} = \left\{g \in \mathrm{GL}_{q-1} \middle| \det(g)^N = 1\right\},$$

and $G_{\mathrm{geom},\mathcal{G}_\chi}$ is given by

$$G_{\mathrm{geom},\mathcal{G}_\chi} = \left\{g \in \mathrm{GL}_{q-1} \middle| \det(g)^M = 1\right\}.$$

(ii) *If $\chi := \rho^{q-a} = \chi_2$, then for N_1 the order of ρ^2 we have*

$$G_{\mathrm{geom},\mathcal{K}} = \{g \in \mathrm{CSp}_{q-1} | \mathrm{mult}(g)^{N_1} = 1\},$$

and

$$G_{\mathrm{geom},\mathcal{G}_\chi} = \mathrm{Sp}_{q-1}.$$

Proof. In the cases where $G^\circ = \mathrm{SL}$, the only groups between SL and GL are those specified by their determinants. In the cases when $G^\circ = \mathrm{Sp}$, then $\chi = \chi_2$, and statement (ii) is proven in Corollary 3.7. □

Data availability

No data was used for the research described in the article.

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