# FINITE MONODROMY FOR KLOOSTERMAN SHEAVES AND FOR UNIVERSAL LAURENT FAMILIES

NICHOLAS M. KATZ, ANTONIO ROJAS-LEÓN, AND PHAM HUU TIEP

*Dedicated to Gabriel Navarro on the occasion of his sixtieth birthday*

Abstract. We study the universal families of multi-parameter systems of one-variable Laurent polynomials, with the aim of determining which of these have finite geometric monodromy groups, and what these finite monodromy groups are. We also classify all primitive Kloosterman sheaves in rank at least 11 which have finite geometric monodromy groups, and determine their monodromy groups.

## Contents



## 1. Introduction

Fix a prime $p$, and an auxiliary prime $\ell \neq p$. Fix a finite extension $k/\mathbb{F}_p$, a (possibly trivial) $\overline{\mathbb{Q}_\ell}$-valued multiplicative character of $k^\times$, and a nontrivial $\overline{\mathbb{Q}_\ell}$-valued additive character $\psi$ of $\mathbb{F}_p$, extended to an nontrivial additive character $\psi_k$ of $k$ by composition with the trace $\mathrm{Trace}_{k/\mathbb{F}_p}$. [Of course these characters take values in a cyclotomic field inside $\overline{\mathbb{Q}_\ell}$.] Given a Laurent polynomial $f(1/x)+g(x)$ with $f,g\in k[x]$, of prime to $p$ degrees $a,b$ respectively, it essentially goes back to

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Weil (see [HD], [Weil], and [De1, Ex. 3.5]), that, under any complex embedding, the sum

$$\sum_{x\in k^\times} \psi_k\big(f(1/x)+g(x)\big)\chi(x)$$

has absolute value $\le (a+b)\sqrt{\#k}$. More precisely, the cohomology groups

$$H^i_c\big(\mathbb{G}_m/\overline{\mathbb{F}_p}, \mathcal{L}_\psi(f(1/x)+g(x))\otimes\mathcal{L}_{\chi(x)}\big)$$

vanish for $i\neq 1$, the $H^1_c$ has dimension $a+b$ and is pure of weight one, and the sum in question is $-\mathrm{Trace}(\mathsf{Frob}_k|H^1_c)$. There is no loss of generality in assuming that both $f$ and $g$ are Artin–Schreier reduced (i.e. the powers $x^n$ of $x$ that occur with nonzero coefficients in the polynomial in question all have exponent $n$ prime to $p$), and we will assume that from here on.

If we specify the degrees of the nonzero monomials occurring in $f$ and in $g$, we get two finite sets of prime to $p$ integers, say $D$ and $E$. We then consider the universal Laurent polynomial of this type: we look at the Laurent polynomial

$$\sum_{i\in D} y_i/x^i + \sum_{j\in E} z_j x^j,$$

with coefficients the $y_i$ and the $z_j$ which are independent variables, subject only to the condition that $y_{\rm highest}z_{\rm highest}$ is invertible. More precisely, over the parameter space

$$\mathcal{M} := (\mathbb{G}_m\times\mathbb{A}^{\#D-1}\times\mathbb{A}^{\#E-1}\times\mathbb{G}_m)/\mathbb{F}_p,$$

with coordinates $(y_{\rm highest}, \text{lower } y_i, \text{lower } z_j, z_{\rm highest})$, we have the universally Laurent polynomial of prescribed type.

Suppose we also specify a finite extension $k/\mathbb{F}_p$ and a (possibly trivial) $\overline{\mathbb{Q}_\ell}$-valued multiplicative character of $k^\times$. For $L/k$ a finite extension, we denote by $\chi_L$ the multiplicative character of $L^\times$ obtained from $\chi$ by composition with the norm $\mathsf{Norm}_{L/k}$. Then on $\mathcal{M}/k$, we have the local system

$$\mathcal{F}(D,E,\chi)$$

of rank $\#D+\#E$ which is pure of weight one and whose trace function at "time" $(\{y_i\},\{z_j\})\in\mathcal{M}(L)$, for $L/k$ a finite extension, is

$$(\{y_i\},\{z_j\})\mapsto -\sum_{x\in L^\times}\psi_L\big(f_{\{y_i\}}(1/x)+g_{\{z_j\}}(x)\big)\chi_L(x).$$

For any local system $\mathcal{F}$ on any (smooth, geometrically connected) $\mathcal{M}/k$, a natural question is to determine its geometric monodromy group $G_{\rm geom}$. The first step of any such determination is to decide if $G_{\rm geom}$ is finite or infinite. In this paper, we determine exactly which local systems $\mathcal{F}(D,E,\chi)$ have finite $G_{\rm geom}$, and for each we determine the finite group $G_{\rm geom}$. The main results of the paper are Theorem 3.5 and Theorem 7.2.

We first treat the case when the rank $\#D+\#E$ of $\mathcal{F}(D,E,\chi)$ is $\ge 11$. Consider the case when, in addition, $\#D=\#E=1$, say $D=\{a\}$, $E=\{b\}$. We relate each $\mathcal{F}(a,b,\chi)$ to a Kloosterman sheaf of the same rank, and show that $\mathcal{F}(a,b,\chi)$ has finite $G_{\rm geom}$ if and only if the corresponding Kloosterman sheaf has finite $G_{\rm geom}$. To apply this last result, we determine in Theorem 3.5 all (primitive) Kloosterman sheaves of rank $\ge 11$ with finite $G_{\rm geom}$, and describe $G=G_{\rm geom}$ (up to its center $\mathbf{Z}(G)$ in each case. (Note that this slight indeterminacy of $G_{\rm geom}$ is natural, because tensoring a given local system with a Kummer sheaf $\mathcal{L}_\chi$ preserves finiteness of $G_{\rm geom}$ but may change the structure of $G_{\rm geom}$, see the discussion preceding Theorem 3.5.) We then show, in Theorems 4.8 and 4.10, that, still in rank $\ge 11$, the only $\mathcal{F}(D,E,\chi)$ with finite $G_{\rm geom}$ are the $\mathcal{F}(a,b,\chi)$, with $\{a,b\}=\{1,p^f-2\}$ and $\chi=\chi_2$, and determine $G_{\rm geom}$ for each of these.

For $\mathcal{F}(D,E,\chi)$ of rank $\le 10$, the situation is more complicated. There are cases when one or both of $D,E$ is no longer a singleton, and we lose the relation to Kloosterman sheaves in those cases. There are are also cases where $\chi$ is not $\chi_2$. Nonetheless, we succeed in determining all cases of $\mathcal{F}(D,E,\chi)$ with finite $G_{\mathrm{geom}}$ in all ranks $\ge 2$, see Theorem 7.2, and we determine their $G_{\mathrm{geom}}$ in §8. Along the way, we are able to realize several of the Shepard–Todd complex reflection groups.

## 2. Preliminary results

**Lemma 2.1.** *Given two non-empty finite sets $D,E$ of prime to $p$ integers, the local system $\mathcal{F}(D,E,\chi)$ with data $(D,E,\chi)$ is geometrically irreducible if and only if $\gcd(D\cup E)=1$.*

*Proof.* Suppose first that $\gcd(D\cup E)=d>1$. Choose a multiplicative character $\rho$ (over some extension of $k$) with $\rho^d=\chi$. Then the universal Laurent polynomial in question is a Laurent polynomial in $x^d$. Thus $D=dD_0, E=dE_0$, and $\mathcal{F}(D,E,\chi)$ is $\mathcal{F}(dD_0,dE_0,\rho^d)$, so its trace function is the sum of the trace functions of the various $\mathcal{F}(D_0,E_0,\rho\Lambda)$ as $\Lambda$ runs over the characters of order dividing $d$. By Chebotarev, and the geometric semisimplicity of pure local systems, by Deligne [De2, Thm 3.4.1 (iii)], we have a geometric isomorphism

$$\mathcal{F}(D,E,\chi)\cong\bigoplus_{\rho:\rho^d=\chi}\mathcal{F}(D_0,E_0,\rho). \tag{2.1.1}$$

It remains to show that if $\gcd(D\cup E)=1$, then $\mathcal{F}(D,E,\chi)$ is geometrically irreducible. Pass to the Tate twisted local system $\mathcal{G}(D,E,\chi):=\mathcal{F}(D,E,\chi)(1/2)$ in which we divide traces over $L$ by $\sqrt{L}$. Thus $\mathcal{G}(D,E,\chi)$ is pure of weight zero, hence geometrically semisimple by Deligne [De2, Thm 3.4.1 (iii)]. As explained in [KT3, Prop. 2.1], this amounts to showing that the second moment $M_{1,1}$ (denoted $M_2$ in [KT3]) is 1. This is the limsup, over finite extensions $L/k$, of

$$\frac{1}{\#\mathcal{M}(L)}\sum_{(c_i,d_j)\in\mathcal{M}(L)}(1/\#L)\sum_{x,y\in L^\times}\chi_L(x/y)\psi_L(f_{c_i}(1/x)-f_{c_i}(1/y)+g_{d_j}(x)-g_{d_j}(y)).$$

We interchange the order of summation, and for each fixed $x,y\in L^\times$, we average over $\mathcal{M}(L)$. The coefficient of $\chi_L(x,y)$ is a huge product of individual sums, namely it is $(1/\#L)$ times

$$\frac{1}{\#\mathcal{M}(L)}\Big(\prod_{i\in D}\sum_{c_i}\psi_L\big(c_i(1/x^i-1/y^i)\big)\Big)\Big(\prod_{j\in E}\sum_{d_j}\psi_L(d_j(x^j-y^j))\Big)$$
$$=\Big(\prod_{i\in D}\sum_{c_i}\frac{\psi_L\big(c_i(1/x^i-1/y^i)\big)}{\#L\text{ or, for the highest }c_i,\#L-1}\Big)\Big(\prod_{j\in E}\sum_{d_j}\frac{\psi_L(d_j(x^j-y^j))}{\#L\text{ or, for the highest }d_j,\#L-1}\Big).$$

If $x=y$, every individual product is 1, and the sum over $x=y$ is then $(\#L-1)/\#L$, whose large $L$ limit is 1.

Suppose now that $x\ne y$. Because $\gcd(D\cup E)=1$, it cannot be the case both that $1/x^i=1/y^i$ for every $i\in D$ and that $x^j=y^j$ for every $j\in E$. If $1/x^i\ne 1/y^i$ for some $i\in D$ other than the highest, then the $\sum_{c_i}$ vanishes, and so this contribution is 0. Similarly, if $x^j\ne y^j$ for some $j\in E$ other than the highest, then the $\sum_{d_j}$ vanishes, and so this contribution is 0. What happens if both $1/x^i=1/y^i$ for every non-highest $i\in D$ and $x^j=y^j$ for every non-highest $j\in E$?

If $1/x^i\ne 1/y^i$ for the highest $i\in D$, then the $c_i$ raw sum is $-1$, and the normalized factor is $1/(\#L-1)$. The normalized factor for the highest $d_j$ will be either 1, if $x^j=y^j$, or it will be $1/(\#L-1)$ if $x^j\ne y^j$. So the coefficient of $\chi_L(x/y)$ is either $(1/\#L)(1/(\#L-1))$ or it is $(1/\#L)(1/(\#L-1))^2$. But the number of pairs $x,y\in L$ with $1/x^{c_{\mathrm{highest}}}=1/y^{c_{\mathrm{highest}}}$ is at most $(\#L-1)c_{\mathrm{highest}}$. Similarly with $d_{\mathrm{highest}}$. So the total contribution of terms $x\ne y\in L^\times$ is $O(1/\#L)$, with vanishing large $L$ limit. Thus the $M_{1,1}$ moment is 1. ∎

**Lemma 2.2.** *Given two non-empty finite sets $D, E$ of prime to $p$ integers, the local system $\mathcal{F}(D, E, \mathbb{1})$ with data $(D, E, \mathbb{1})$ does not have finite $G_{\mathrm{geom}}$.*

*Proof.* We may reduce to the case when $\gcd(D \cup E) = 1$, because in the reduction to this case in Lemma 2.1, one of the direct factors will be $\mathcal{F}(D_0, E_0, \mathbb{1})$. The trace function of $\mathcal{F}(D, E, \mathbb{1})$ is ($-1$ times) a sum of values of $\psi_L$ over $L^\times$, so in $\mathbb{Z}[\zeta_p]$ it is 1 mod the unique prime $\mathcal{P}$ lying over $p$. So no trace of $\mathcal{G}(D, E, \mathbb{1}) := \mathcal{F}(D, E, \mathbb{1})(1/2)$ is integral at $p$, so in particular no trace is an algebraic integer. □

**Remark 2.3.** Suppose we fix a **nontrivial** multiplicative character $\chi$ of some $k^\times$. Then for $f \in k[x]$ which is Artin-Schreier reduced of degree $a > 0$, the cohomology groups $H^i_c(\mathbb{G}_m/\overline{\mathbb{F}_p}, \mathcal{L}_{\psi}(f(1/x)) \otimes \mathcal{L}_{\chi(x)})$, respectively $H^i_c(\mathbb{G}_m/\overline{\mathbb{F}_p}, \mathcal{L}_{\psi}(f(x)) \otimes \mathcal{L}_{\chi(x)})$ vanish for $i \neq 1$, and the $H^1_c$ has dimension $a$ and is pure of weight one. The trace of $\mathsf{Frob}_L$, for $L/k$ a finite extension, on this $H^1_c$ is

$$-\sum_{x \in L^\times} \psi_L(f(1/x))\chi_L(x),$$

respectively

$$-\sum_{x \in L^\times} \psi_L(f(x))\chi_L(x).$$

This allows us to speak of the local systems $\mathcal{F}(D, \varnothing, \chi)$ and $\mathcal{F}(\varnothing, D, \chi)$, for any non-empty finite set $D$ of prime to $p$ integers, so long as $\chi$ is nontrivial.

The proof of Lemma 2.1 goes through without change to yield

**Lemma 2.4.** *For $\chi$ nontrivial and $D$ a finite set of prime to $p$ integers, each of the local systems $\mathcal{F}(D, \varnothing, \chi)$ , $\mathcal{F}(\varnothing, D, \chi)$ is geometrically irreducible if and only if $\gcd(D) = 1$.*

Next we prove a semicontinuity result.

**Theorem 2.5.** *Suppose $\chi$ is nontrivial. Let $D, E$ be finite sets of prime to $p$ integers, with $D \cup E$ non-empty. Denote by $G_{D,E,\chi}$ the geometric monodromy group of $\mathcal{F}(D, E, \chi)$. Let $D_0 \subset D$ and $E_0 \subset E$ be subsets, with $D_0 \cup E_0$ non-empty. Then the group $G_{D_0,E_0,\chi}$ is a subquotient of the group $G_{D,E,\chi}$. In particular, if $G_{D,E,\chi}$ is finite, then $G_{D_0,E_0,\chi}$ is finite.*

*Proof.* Because a subquotient of a subquotient is a subquotient, it suffices to treat universally the case in which we remove a single element from either $D$ or from $E$. If what we remove is not the largest element, then $\mathcal{F}(D_0, E_0, \chi)$ is simply a pullback of $\mathcal{F}(D, E, \chi)$. In this case, $G_{D_0,E_0,\chi} < G_{D,E,\chi}$ is a subgroup.

Suppose we remove the highest element, of say $D$. Call this highest element $i_0$. The local system $\mathcal{F}(D, E, \chi)$ lives on $\mathcal{M} := (\mathbb{G}_m \times \mathbb{A}^{\#D-1} \times \mathbb{A}^{\#D-1} \times \mathbb{G}_m)/\mathbb{F}_p$, with coordinates $y_i, i \in D$ and $z_j, j \in E$. On $\mathcal{M}$, $\mathcal{F}(D, E, \chi)$ is obtained as

$$R^1\pi_!(\mathcal{L}_{\psi(\sum_{i \in D} y_i/x^i + \sum_{j \in E} z_j x^j)} \otimes \mathcal{L}_{\chi(x)})$$

for $\pi : \mathbb{G}_m \times \mathcal{M} \to \mathcal{M}$ the projection onto the second factor. We may consider instead the larger parameter space $\overline{\mathcal{M}} := \mathbb{A}^{\#D} \times \mathbb{A}^{\#E-1} \times \mathbb{G}_m$, where we forget the requirement that the highest $y_i$ is required to be invertible. We may then consider the sheaf

$$\mathcal{F}(\overline{D}, E, \chi) := R^1\pi_!(\mathcal{L}_{\psi(\sum_{i \in D} y_i/x^i + \sum_{j \in E} z_j x^j)} \otimes \mathcal{L}_{\chi(x)})$$

for $\pi : \mathbb{G}_m \times \overline{\mathcal{M}} \to \overline{\mathcal{M}}$ the projection onto the second factor. This is then a sheaf of perverse origin in the sense of [Ka4] on $\overline{\mathcal{M}}$, which is lisse on the open set $\mathcal{M}$. According to [Ka4], the restriction of $\mathcal{F}(\overline{D}, E, \chi)$ to any dense open set $V$ of $\overline{\mathcal{M}} \setminus \mathcal{M}$ on which it is lisse has $G_{\mathrm{geom},\mathcal{F}(\overline{D},E,\chi)|V}$ a subquotient of $G_{\mathrm{geom},\mathcal{F}(D,E,\chi)}$.

Consider first the case in which $D$ is the singleton $\{i_0\}$. In this case, $\overline{\mathcal{M}} \smallsetminus \mathcal{M}$ is the parameter space for $\mathcal{F}(\varnothing, E, \chi)$, and the restriction of $\mathcal{F}(\overline{D}, E, \chi)$ is precisely the lisse sheaf $\mathcal{F}(\varnothing, E, \chi)$.

If $\#D \geq 2$, with $i_1 < i_0$ the second largest element of $D$, consider the dense open set $V$ of $\overline{\mathcal{M}} \smallsetminus \mathcal{M}$ on which $y_{i_1}$ is invertible. Then $V$ is the parameter space for $\mathcal{F}(D \smallsetminus \{i_0\}, E, \chi)$, and the restriction of $\mathcal{F}(\overline{D}, E, \chi)$ to $V$ is precisely the lisse sheaf $\mathcal{F}(D \smallsetminus \{i_0\}, E, \chi)$. □

Next we determine the $M_{2,2}$ moment for certain local systems (see e.g. [KT5, §2.6] for the definition of the $M_{a,b}$ moment and a lim sup formula for it).

**Lemma 2.6.** *Let $k/\mathbb{F}_p$ be a finite extension, $\chi$ a (possibly trivial) character of $k^\times$, $a > 1$ a prime to $p$ integer, $f(x) \in k[x]$ a polynomial which is either $0$ or Artin-Schreier reduced of prime to $p$ degree $d$. If $f = 0$, or if $\deg(f) < a$, consider the local system on $(\mathbb{G}_m \times \mathbb{A}^1 \times \mathbb{G}_m)/k$ whose trace function is*

$$(r, s, t) \mapsto -\sum_x \psi(r/x + sx + tx^a + f(x))\chi(x).$$

*If $\deg(f) > a$, consider the local system, but on $(\mathbb{G}_m \times \mathbb{A}^2)$, with the same trace function. Then $M_{2,2} \leq 3$, with equality only when each of the following conditions is satisfied:*

(i) *$f(x)$ is odd, meaning that $f(x) + f(-x) = 0$.*
(ii) *$a$ is odd.*
(iii) *$\chi^2 = \mathbb{1}$.*

*In all other cases, $M_{2,2} = 2$.*

*Proof.* It suffices to prove this after freezing $t = 1$ and replacing $f(x)$ by $f(x) + x^a$. Thus we are reduced to the following statement. □

**Lemma 2.7.** *Let $k/\mathbb{F}_p$ be a finite extension, $\chi$ a (possibly trivial) character of $k^\times$, $f(x) \in k[x]$ a polynomial which is Artin-Schreier reduced of degree $d > 1$. Consider the local system on $(\mathbb{G}_m \times \mathbb{A}^1)/k$ whose trace function is*

$$(r, s) \mapsto -\sum_x \psi(r/x + sx + f(x))\chi(x).$$

*Then $M_{2,2} \leq 3$, with equality only when both of the following conditions are satisfied:*

(i) *$f(x)$ is odd, meaning that $f(x) + f(-x) = 0$.*
(ii) *$\chi^2 = \mathbb{1}$.*

*In all other cases, $M_{2,2} = 2$.*

*Proof.* By [KT6, 2.1], we may calculate its $M_{2,2}$ as the limsup, over finite extensions $\mathbb{F}_q/\mathbb{F}$(values of $\chi$) of

$$\frac{1}{q^3(q-1)} \sum_{\substack{(r,s)\in\mathbb{F}_q^\times\times\mathbb{F}_q \\ x,y,z,w\in\mathbb{F}_q^\times}} \psi_{\mathbb{F}_q}(r(\frac{1}{x}+\frac{1}{y}-\frac{1}{z}-\frac{1}{w})+s(x+y-z-w)+f(x)+f(y)-f(z)-f(w))\chi(xy)\overline{\chi}(zw).$$

By the usual arguments, cf. the proof of [KRLT4, Lemma 2.10], the $M_{2,2}$ is unchanged if we allow all $(r, s) \in \mathbb{A}^2$. Indeed, the difference is the sum

$$\frac{1}{q^3(q-1)} \sum_{s\in\mathbb{F}_q} \sum_{x,y,z,w\in\mathbb{F}_q^\times} \psi_{\mathbb{F}_q}(s(x+y-z-w) + f(x) + f(y) - f(z) - f(w))\chi(xy)\overline{\chi}(zw).$$

For each $s \in \mathbb{F}_q$, the sum

$$\sum_{x,y,z,w \in \mathbb{F}_q^\times} \psi_{\mathbb{F}_q}(s(x+y-z-w)+f(x)+f(y)-f(z)-f(w))\chi(xy)\overline{\chi}(zw)$$

is just

$$\Big|\sum_{x \in \mathbb{F}_q} \psi(sx+f(x))\chi(x)\Big|^4,$$

which (because $f(x)+sx$ is Artin-Schreier reduced) is $\le (\deg(f)\sqrt{q})^4$ by Weil, so $O(q^2)$. The sum over all $s \in F_q$ of these is then $O(q^3)$, but as we divide by $q^3(q-1)$, the difference does not contribute to the limsup.

We now sum over all $(r,s) \in \mathbb{A}^2$. Then $M_{2,2}$ is the limsup, over finite extensions $\mathbb{F}_q/\mathbb{F}(\text{values of } \chi)$, of the absolute values of the sums

$$\frac{1}{q(q-1)} \sum_{\substack{(x,y,z,w) \in (\mathbb{F}_q^\times)^4 \\ 1/x+1/y=1/z+1/w, \\ x+y=z+w, \\ x^a+y^a=z^a+w^a}} \psi_{\mathbb{F}_q}(f(x)+f(y)-f(z)-f(w))\chi_{\mathbb{F}_q}(xy/zw)$$

We rewrite the first two equations defining the domain of summation as

$$(x+y)zw = (z+w)xy, \ \ x+y = z+w.$$

At a point $(x, yz, w)$ where the common value of $x+y=z+w$ is nonzero, we get $xy = zw$. At such a point, we have (equality of elementary symmetric functions) the equality of sets $\{x,y\} = \{z,w\}$. At a point $(x, yz, w)$ where $x+y=z+w=0$, our point is $(x,-x,z,-z)$. When both $f(x)$ and $a$ are odd, the point $(x,y,z,w)$ lies in the domain of summation, and the domain of summation consists of the three planes in $\mathbb{A}^4$ of equations $(x=z, y=z), (x=w, y=z), (x=-y, z=-w)$. On each of these planes, the summand inside the $\psi$ is either $f(x)+f(y)-f(z)-f(w)$, on each of the first two planes, where it vanishes identically, or it is $f(x)+f(-x)-f(z)-f(-z)$, which vanishes on the third plane, provided that $f(x)$ is odd. The factor $\chi(xy/zw)$ is 1 on each of the first two planes, and is $\chi(x^2/z^2) = \chi^2(x/z)$, which is trivial if and only if $\chi^2 = \mathbb{1}$.

If $f(x)$ is a non-odd Artin-Schreier reduced polynomial, the sum over the third plane is

$$\frac{1}{q(q-1)} \sum_{x,z \in \mathbb{F}_q^\times} \psi_{\mathbb{F}_q}\big(f(x)+f(-x)-f(z)-f(-z)\big)\chi_{\mathbb{F}_q}\big(\frac{x^2}{z^2}\big) = \frac{1}{q(q-1)}\Big|\sum_x \psi_{\mathbb{F}_q}(f(x)+f(-x))\chi_{\mathbb{F}_q}(x^2)\Big|^2,$$

in which the sum inside the absolute values, namely $\sum_x \psi_{\mathbb{F}_q}(f(x)+f(-x))\chi_{\mathbb{F}_q}(x^2)$, is $\le d\sqrt{q}$ by Weil, because $f(x)+f(-x)$ is both nonzero and Artin-Schreier reduced, of degree $\le d$. So in this case the sum over the third plane is $O(1/q)$.

Finally, if $f(x)$ is odd but $\chi^2 \ne \mathbb{1}$, the sum over the third plane vanishes identically. □

**Lemma 2.8.** *Let $k/\mathbb{F}_p$ be a finite extension, $\chi$ a (possibly trivial) character of $k^\times$, and $f(x) \in k[x]$ a polynomial which is Artin-Schreier reduced of degree $d \ge 1$. Consider the local system on $(\mathbb{A}^1 \times \mathbb{G}_m)/k$ whose trace function is*

$$(s,t) \mapsto -\sum_x \psi(f(1/x)+sx+tx^2)\chi(x).$$

*Then $M_{2,2} = 2$.*

*Proof.* By [KT6, 2.1], we may calculate its $M_{2,2}$ as the limsup, over finite extensions $\mathbb{F}_q/\mathbb{F}(\text{values of } \chi)$ of

$$\frac{1}{q^3(q-1)}\sum_{\substack{(s,t)\in\mathbb{F}_q\times\mathbb{F}_q^\times\\ x,y,z,w\in\mathbb{F}_q^\times}}\psi_{\mathbb{F}_q}(f(\frac{1}{x})+f(\frac{1}{y})-f(\frac{1}{z})-f(\frac{1}{w})+s(x+y-z-w)+t(x^2+y^2-z^2-w^2))\chi(xy)\overline{\chi}(zw).$$

By the usual arguments, cf. the proof of [KRLT4, Lemma 2.10], the $M_{2,2}$ is unchanged if we allow all $(s,t)\in\mathbb{A}^2$. Indeed, the difference is the sum

$$\frac{1}{q^3(q-1)}\sum_{\substack{s\in\mathbb{F}_q\\ x,y,z,w\in\mathbb{F}_q^\times}}\psi_{\mathbb{F}_q}(f(1/x)+f(1/y)-f(1/z)-f(1/w)+s(x+y-z-w))\chi(xy)\overline{\chi}(zw)$$
$$=\frac{1}{q^3(q-1)}\sum_{s\in\mathbb{F}_q}\Big|\sum_{x\in\mathbb{F}_q^\times}\psi_{\mathbb{F}_q}(f(1/x)+sx)\chi(x)\Big|^4.$$

The sum whose fourth absolute power is taken is, by Weil, of absolute value $\leq(1+\deg(f))\sqrt{q}$. So we have $q$ summands, each $O(q^2)$, but we divide by $q^3(q-1)$. Thus the difference is $O(1/q)$, so does not affect the limsup.

The sum over all $(s,t)$ is then

$$\frac{1}{q^(q-1)}\sum_{\substack{x,y,z,w\in\mathbb{F}_q^\times,\\ x+y=z+w,x^2+y^2=z^2+w^2}}\psi_{\mathbb{F}_q}(f(1/x)+f(1/y)-f(1/z)-f(1/w))\chi(xy)\overline{\chi}(zw).$$

The domain of summation is thus $\{x,y\}=\{z,w\}$. At any point of this domain the sum $f(1/x)+f(1/y)-f(1/z)-f(1/w)$ vanishes, and $)\chi(xy)\overline{\chi}(zw)$ takes the constant value 1. The domain of summation is the union of the two planes $x=z, y=w$ and $x=w, y=z$, with intersection the line $x=y=z=w$, so the domain of summation has $2(q-1)^2-(q-1)$ $\mathbb{F}_q$ points, and we are dividing by $q(q-1)$. So the sum over all $(s,t)\in\mathbb{F}_q^2$ is $(2q-3)/q$, with $\limsup = 2$. ☐

## 3. Primitive Kloosterman sheaves with finite monodromy

The goal of this section is to classify all primitive Kloosterman sheaves of rank $D\geq 11$ which have finite monodromy. [There is no intrinsic reason we do not classify the cases with $D\leq 10$, just the fact that in low rank there will be many many cases and we do not at present have any application for knowing these cases.]

We begin with some auxiliary results.

**Lemma 3.1.** *Let $G$ be a finite almost quasisimple group with $L:=G^{(\infty)}\cong\mathsf{A}_{D+1}$, $D\geq 4$, $D\neq 5$. Suppose $V=\mathbb{C}^D$ is a faithful representation of $G$ such that $L$ acts on $V$ via its deleted natural permutation module. Suppose also that $J<G$ is a solvable subgroup acting irreducibly on $V$. Then the following statements hold.*

(i) *$D=r^n-1$ for some prime $r$ and integer $n\geq 1$.*
(ii) *Suppose that $p$ is a prime such that for $Q:=\mathbf{O}_p(J)$, either $Q\not\leq\mathbf{Z}(G)$ or $J/Q$ is cyclic. Then $p=r$.*

*Proof.* Note that $Z:=\mathbf{Z}(G)$ is cyclic by Schur's lemma, and $L\lhd G/Z\leq\mathrm{Aut}(L)$. If $G/Z\cong L$, then $G=Z\times L$. By hypothesis, $\mathrm{Aut}(L)\cong\mathsf{S}_{D+1}=L\cdot 2$, so the remaining case is $G/Z\cong L\cdot 2$. In this case, there is some $s\in G$ which acts on $L$ via conjugation as a 2-cycle. In particular, $s^2$ acts on $V$ as $\zeta\cdot\mathrm{Id}$, where $\zeta\in\mathbb{C}^\times$ is a root of unity, say of order $N$. Adding $\boldsymbol{z}:=\zeta_{2N}\mathrm{Id}_V$ to $G$ if necessary, we may assume that $\boldsymbol{z}\in Z$, but $Z$ and $G$ are still finite. Replacing $s$ by $s\boldsymbol{z}^j$ with a suitable $j \in \mathbb{Z}$, we achieve that $s^2 = \mathrm{Id}$ on $V$, and hence $H := \langle L, s\rangle \cong \mathsf{S}_{D+1}$. Now, replacing $s$ by $\boldsymbol{z}^N$ if needed, we have that $H$ acts on $V$ via tis deleted natural permutation module, and furthermore $G = ZH = Z \times H$.

Thus we always have $G \le ZH$. Now $ZJ = Z \times J_0$, where $J_0 := ZJ \cap H$. Thus $J_0$ is a solvable subgroup of $H = \mathsf{S}_{D+1}$ that acts irreducibly on $V = \mathbb{C}^D$. It follows that $J_0$ is a subgroup of $H$ acting doubly transitively on $D+1$ points. As $J_0$ is solvable, it follows from the O'Nan-Scott theorem [Cam, Theorem 4.1] that $D + 1 = r^n$ for some prime $r$, proving (i), and furthermore $\mathbf{O}_r(J_0) \cong C_r^n$ is the unique minimal normal subgroup of $J_0$.

For (ii), assume the contrary that $p \ne r$. Since $Q \lhd J$ and $[Z, G] = 1$, we still have $ZQ \lhd ZJ$. Now $ZQ = Z \times Q_0$, where $Q_0 := ZQ \cap H$. Moreover,

$$|Q_0| = \frac{|ZQ|}{|Z|} = \frac{|Z| \cdot |Q|}{|Z| \cdot |Z \cap Q|} = \frac{|Q|}{|Z \cap Q|}, \tag{3.1.1}$$

showing that $Q_0$ is a $p$-subgroup, and hence $Q_0 \le \mathbf{O}_p(J_0)$. But $\mathbf{O}_r(J_0)$ is the unique minimal normal subgroup of $J_0$ and $p \ne r$, so $Q_0 = 1$ and $ZQ = Z$. Thus $Q \le Z$ centralizes $G$. By our assumption, we must have that $J/Q$ is cyclic. As $Q \le \mathbf{Z}(J)$, we see that $J$ is abelian, and hence cannot act irreducibly on $V$, a contradiction. □

**Lemma 3.2.** *Let $n \ge 4$ be an even integer, $q = p^f$ a power of a prime $p$, $(n, q) \ne (4, 2)$. Suppose that $V = \mathbb{C}^D$ with $D = (q^n + q)/(q + 1)$, and $G < \mathrm{GL}(V)$ is a finite almost quasisimple group with $G/\mathbf{Z}(G) \cong \mathrm{PGU}_n(q)$. If $(n, q) = (4, 3)$, assume in addition that $S := G^{(\infty)} \cong \mathrm{PSU}_n(q)$. Suppose $J < G$ is such that $J/Q$ is cyclic $p'$-group for $Q := \mathbf{O}_p(J)$. Then $J$ cannot act irreducibly on $V$.*

*Proof.* (a) Assume the contrary and write $Z := \mathbf{Z}(G)$. Our assumptions and [TZ, Theorem 4.1] imply that $S := G^{(\infty)} \cong \mathrm{PSU}_n(q)$. Also, we can write $J = \langle Q, g\rangle$, where $g$ is a $p'$-element. If $V_1$ is a simple summand of $V|_Q$ and $J_1$ is the corresponding inertia subgroup, then by Clifford's theorem $V = \mathrm{Ind}_{J_1}^J(U_1)$ where $U_1 \in \mathrm{Irr}(J_1)$ lies above $V_1$. Since $V_1$ is $J_1$-invariant and $J_1/Q$ is cyclic, $V_1$ extends to $J_1$ and $\dim(U_1) = \dim(V_1)$ by [Is, (6.17) and (11.22)]. Also, $[J : J_1]$ is coprime to $p$, and $\dim(V_1)$ is a $p$-power. It follows that $\dim(V_1) = q$, and $g$ permutes the $D/q$ simple summands $V_i$, $1 \le i \le D/q$, of $V|_Q$ transitively. Since $ZS = Z \times S$ has index dividing $q + 1$ in $G$, we have $Q \le ZS$, and hence $ZQ = Z \times Q_0$, where $Q_0 := ZQ \cap S$. The computation in (3.1.1) shows that $Q_0$ is a $p$-subgroup normalized by $g$. Each $V_i$ is still a simple module over $ZQ$, hence also over $Q_0$, of dimension $q$. This shows that $Q_0$ is non-abelian and $|Q_0| \ge pq^2$.

(b) Recall that $|g| = D/q = (q^{n-1} + 1)/(q + 1)$. As $n \ge 4$ and $(n, q) \ne (4, 2)$, by [Zs] we can find a primitive prime divisor $\ell$ of $p^{2(n-1)f} - 1$ that divides $|g|$ but not $q + 1$. So the $\ell$-part $g_1$ of $g$ lies in $ZS$ and acts on $\{V_1, \ldots, V_{D/q}\}$ with orbits of the same length $\ell^a$, the $\ell$-part of $D/q$ (so $a \ge 1$). Writing $g_1 = zh_1$ with $h_1 \in S$, we see that $h_1$ normalizes $Q_0$ and still acts on $\{V_1, \ldots, V_{D/q}\}$ with all orbits of length $\ell^a$. The same conclusions hold for the $\ell$-part $h$ of $h_1$; in particular, $h \ne 1$.

(c) Thus we have found a nontrivial $p$-subgroup $Q_0 < S$ that is normalized by a nontrivial $\ell$-element $h \in S$. Since $S$ is the quotient of $L := \mathrm{SU}_n(q)$ by its center of order $\gcd(n, q + 1)$ which is coprime to $p\ell$, without any loss we may assume that $Q_0$ and $h$ both lie in $L$. Now view $L$ as $\mathrm{SU}(N)$, where $N = \mathbb{F}_{q^2}^n$ is the natural Hermitian space for $L$. The choice of $\ell$ implies that the $h$-module $N$ decomposes as the orthogonal sum $N_1 \oplus N_2$, where $h$ acts irreducibly on $N_1$ of dimension $n - 1$ and trivially on $N_2 = N_1^\perp$. Since $Q_0 \ne 1$ is a $p$-group, its fixed point subspace $M$ on $N$ satisfies $0 \ne M \ne N$. Note that $Q_0$ acts on $M^\perp \ne 0$ and hence has nonzero fixed points on $M^\perp$ as well. Thus $M^\perp \cap M \ne 0$. As $h$ normalizes $Q_0$, $M$ is $h$-invariant. The given structure of the $h$-module $N$ shows that $M = N_1$ or $N_2$, whence $M$ is non-degenerate, a contradiction. □

We also need the following analogues of Lemma 3.2.

**Lemma 3.3.** *Let $n \geq 1$ be any integer, $q = p^f$ a power of an odd prime $p$, and $(n,q) \neq (1,3)$. Suppose that $V = \mathbb{C}^D$ with $D = (q^n+1)/2$, and $G < \mathrm{GL}(V)$ is a finite group with $G/\mathbf{Z}(G) \hookrightarrow \mathrm{PSp}_{2n}(q)$. Suppose $J < G$ is such that $J/Q$ is cyclic $p'$-group for $Q := \mathbf{O}_p(J)$. Then $J$ cannot act irreducibly on $V$.*

*Proof.* Assume the contrary and write $Z := \mathbf{Z}(G)$. Arguing as in part (a) of the proof of Lemma 3.2, we can write $J = \langle Q, g\rangle$, where $g$ is a $p'$-element which permutes the $D$ simple summands $V_i$, $1 \leq i \leq D$, of $V|_Q$ transitively. It follows that $\bar{\mathsf{o}}(g)$, which is the order of the $p'$-element $gZ$ in $G/Z \leq \mathrm{PSp}_{2n}(q)$, is divisible by $D$. Since $D > 1$ and $J$ is irreducible on $V$, $Q \neq 1$, which implies by [KT2, Proposition 4.8(ii)] that the $p$-subgroup $QZ/Z$ of $\mathrm{PSp}_{2n}(q)$ which is normalized by $gZ$ is nontrivial.

Working in $L := \mathrm{Sp}_{2n}(q)$, we now have a $p'$-element $h$ of central order divisible by $D = (q^n+1)/2$ which normalizes a nontrivial $p$-subgroup $R$. Note that $h$ acts irreducibly on the natural module $N = \mathbb{F}_q^{2n}$ of $L$. (Indeed, if $n = 1$, then any reducible semisimple element of $L$ has order dividing $q-1$ which is not divisible by $(q+1)/2$ since $(n,q) \neq (1,3)$. If $n \geq 2$, then $p^{2nf}-1$ admits a primitive prime divisor $\ell$ by [Zs], which divides $|h|$, and this implies that $h$ is irreducible on $N$.) On the other hand, the fixed point subspace $M$ of $R$ on $N$ is nonzero and proper, and $M$ is $h$-invariant, a contradiction. □

**Lemma 3.4.** *Let $n \geq 2$ be an integer, $q = p^f$ a power of a prime $p$, and $(n,q) \neq (2,2)$, $(2,3)$, $(3,2)$. Suppose that $V = \mathbb{C}^D$, where $D = (q^n-1)/(q-1)$, or $D = (q^n-q)/(q-1)$ with $n \geq 3$, and $G < \mathrm{GL}(V)$ is a finite almost quasisimple group with $\mathrm{PSL}_n(q) \lhd G/\mathbf{Z}(G) \leq \mathrm{PGL}_n(q)$. If $D = (q^n-q)/(q-1)$ and $(n,q) = (3,4)$, assume in addition that $G^{(\infty)} \cong \mathrm{PSL}_n(q)$. Suppose $J < G$ is such that $J/Q$ is cyclic $p'$-group for $Q := \mathbf{O}_p(J)$. Then $J$ cannot act irreducibly on $V$.*

*Proof.* Assume the contrary and write $Z := \mathbf{Z}(G)$.

(a) First we consider the case $D = (q^n-q)/(q-1)$ with $n \geq 3$. Then our assumptions and [TZ, Theorem 3.1] imply that $G^{(\infty)} \cong \mathrm{PSL}_n(q)$, and it acts on $V$ via its unipotent Weil representation. Arguing as in part (a) of the proof of Lemma 3.2, we can write $J = \langle Q, g\rangle$, where $g$ is a $p'$-element which permutes the $D$ simple summands $V_i$, $1 \leq i \leq D/q$, of $V|_Q$ transitively. It follows that $\bar{\mathsf{o}}(g)$, which is the order of the $p'$-element $gZ$ in $G/Z \leq \mathrm{PGL}_n(q)$, is divisible by $D/q$. Furthermore, $ZQ = Z \times Q_0$, where $Q_0 := ZQ \times S$, and $Q_0$ is a non-abelian $p$-subgroup normalized by $g$.

Working in $L := \mathrm{GL}_n(q)$, we now have a $p'$-element $h$ of central order divisible by $D/q = (q^{n-1}-1)(q-1)$ which normalizes a non-abelian $p$-subgroup $R$. Clearly, the fixed point subspace $M$ of $R$ on the natural module $N = \mathbb{F}_q^n$ for $L$ is nonzero and proper, and $M$ is $h$-invariant; in particular, the semisimple $\langle h\rangle$-module $N$ is reducible. We claim that this latter module decomposes as a direct sum of two irreducible submodules, one of dimension 1 and another of dimension $n-1$. Indeed, assume the contrary. If $n = 3$, then $N$ decomposes as a direct sum of three 1-dimensional submodules, and hence the order of $h$ divides $q-1$, which is not divisible by $D/q = q+1$. a contradiction. If $(n,q) = (7,2)$, then using [GAP] we can check that $h$ has order 63 and then check the claim directly. If $n \geq 4$ and $(n,q) \geq (7,2)$, then $q^{n-1}-1$ admits a primitive prime divisor $\ell$ by [Zs], which divides $|h|$, but does not divide any $|\mathrm{GL}_j(q)|$ with $1 \leq j \leq n-2$, again a contradiction.

It follows that $\dim_{\mathbb{F}_q} M = n-1$ or 1. In the former case, $R$ is abelian, a contradiction. So $\dim(M) = 1$, and $h$ stabilizes $M$ and acts irreducibly on $N/M$. If $R$ acts trivially on $N/M$, then $R$ is again abelian. Hence $R$ has a nonzero proper fixed point subspace on $N/M$, contrary to the irreducibility of $h$ on $N/M$.

(b) Now let $D = (q^n-1)/(q-1)$. Arguing as in part (a) of the proof of Lemma 3.2, we can write $J = \langle Q, g\rangle$, where $g$ is a $p'$-element which permutes the $D$ simple summands $V_i$, $1 \leq i \leq D$, of $V|_Q$ transitively. It follows that $\bar{\mathsf{o}}(g)$, which is the order of the $p'$-element $gZ$ in $G/Z \leq \mathrm{PGL}_n(q)$, is divisible by $D$. Since $D > 1$ and $J$ is irreducible on $V$, $Q \neq 1$, which implies by [KT2, Proposition 4.8(ii)] that the $p$-subgroup $QZ/Z$ of $\mathrm{PGL}_n(q)$ which is normalized by $gZ$ is nontrivial.

Working in $L = \mathrm{GL}_n(q)$, we now have a $p'$-element $h$ of central order divisible by $D = (q^n - 1)/(q-1)$ which normalizes a nontrivial $p$-subgroup $R$. Note that $h$ acts irreducibly on the natural module $N = \mathbb{F}_q^n$ of $L$. (Indeed, if $n = 2$, then any reducible semisimple element of $L$ has order dividing $q-1$ which is not divisible by $q+1$. If $(n,q) = (6,2)$, then using [GAP] we can check that $h$ has order 63 and then check the claim directly. If $n \geq 3$ and $(n,q) \neq (6,2)$, then $q^n - 1$ admits a primitive prime divisor $\ell$ by [Zs], which divides $|h|$, and this implies that $h$ is irreducible on $N$.) On the other hand, the fixed point subspace $M$ of $R$ on $N$ is nonzero and proper, and $M$ is $h$-invariant, a contradiction. □

Now we are ready to prove the first main result of the paper, which classifies all Kloosterman sheaves of rank $D \geq 11$ that have finite $G = G_{\mathrm{geom}}$, up to tensoring with a Kummer sheaf $\mathcal{L}_\gamma$. By [KRLT3, Lemma 5.12], this tensoring operation preserves finiteness of $G$, as well as isomorphism types of $G/\mathbf{Z}(G)$, $[G,G]$, and $G^{(\infty)}$ (the last term of the derived series of $G$), but it may change $G$ itself.

**Theorem 3.5.** *Let $\mathcal{K} = \mathcal{K}l_\psi(\chi_1, \dots, \chi_D)$ be a primitive Kloosterman sheaf of rank $D \geq 11$ in characteristic $p$. Then $\mathcal{K}$ has finite geometric monodromy group $G = G_{\mathrm{geom}}$ if and only if one of the following holds for $\mathcal{K}$, up to tensoring with a Kummer sheaf $\mathcal{L}_\gamma$ and changing $\psi$ suitably, and for $G$ and $L := G^{(\infty)}$.*

(i) *$D = p^f$ for some $f \geq 1$, $\mathcal{K}$ is the Pink–Sawin sheaf $\mathcal{K}l(\mathsf{Char}_{\mathrm{ntriv}}(D+1))$, $L = 1$, and $G/\mathbf{Z}(G) \cong C_p^{2f} \rtimes C_{p^f+1}$.*

(ii) *$D = p^f - 1$ for some $f \geq 1$, $\mathcal{K}$ is the Sawin sheaf $\mathcal{K}l(\mathsf{Char}(p^f - k) \sqcup \mathsf{Char}_{\mathrm{ntriv}}(k))$ for some $1 < k < p^f - 1$ coprime to $p$, and $L = \mathsf{A}_{D+1}$. Furthermore, $G/\mathbf{Z}(G) \cong \mathsf{A}_{D+1}$ if $2 \nmid D$ and $G/\mathbf{Z}(G) \cong \mathsf{S}_{D+1}$ if $2|D$.*

(iii) *$D = (q-1)/2$ with $q = p^f$ for some $f \geq 1$ and $p > 2$, and $\mathcal{K} = \mathcal{K}l(\mathsf{Char}_{\mathrm{ntriv}}(D+1))$. Furthermore, $L = \mathrm{PSL}_2(q)$ if $2 \nmid D$ and $L = \mathrm{SL}_2(q)$ if $2|D$. Moreover, $G/\mathbf{Z}(G) \cong L/\mathbf{Z}(L) \cong \mathrm{PSL}_2(q)$.*

(iv) *$D = (q^n - q)/(q+1)$, $q = p^f$ for some $f \geq 1$ and $2 \nmid n \in \mathbb{Z}_{\geq 3}$, $\mathcal{K} = \mathcal{K}l(\mathsf{Char}_{\mathrm{ntriv}}(D+1))$, $L = \mathrm{PSU}_n(q)$, and $G/\mathbf{Z}(G) \cong \mathrm{PGU}_n(q)$.*

(v) *$D = (q^{a+b} - 1)/(q+1)$, with $q = p^f$ for some $f \geq 1$, $a, b \geq 1$, $\gcd(a,b) = 1$, $2 \nmid ab$, and*

$$\mathcal{K} = \mathcal{K}l\Big(\mathsf{Char}(AB(q+1)) \smallsetminus \big(\mathsf{Char}(A, \sigma^{-\beta}) \cup \mathsf{Char}(B, \sigma^{-\alpha})\big)\Big),$$

*where $A := (q^a+1)/(q+1)$, $B := (q^b+1)/(q+1)$, $\mathbb{1} \neq \sigma \in \mathsf{Char}(q+1)$, and $\alpha, \beta \in \mathbb{Z}$ are such that $\alpha A - \beta B = 1$ and $\gcd(\alpha+\beta, q+1) = 1$. Furthermore, $L/\mathbf{Z}(L) = \mathrm{PSU}_{a+b}(q)$ and $G/\mathbf{Z}(G) \cong \mathrm{PGU}_{a+b}(q)$.*

(vi) *$D = (q^{a+b} - 1)/2$, with $q = p^f$ for some $f \geq 1$, $p > 2$, $a, b \geq 1$, $\gcd(a,b) = 1$, $2|ab$, and*

$$\mathcal{K} = \mathcal{K}l\Big(\mathsf{Char}(2AB) \smallsetminus \big(\mathsf{Char}(A, \chi_2^\beta) \cup \mathsf{Char}(B, \chi_2^\alpha)\big)\Big),$$

*where $A := (q^a+1)/2$, $B := (q^b+1)/2$, and $\alpha, \beta \in \mathbb{Z}$ are such that $\alpha A - \beta B = 1$ and $2 \nmid (\alpha + \beta)$. Furthermore, $L = \mathrm{PSp}_{2(a+b)}(q)$ if $2 \nmid D$ and $L = \mathrm{Sp}_{2(a+b)}(q)$ if $2|D$. Moreover, $G/\mathbf{Z}(G) \cong L/\mathbf{Z}(L) = \mathrm{PSp}_{2(a+b)}(q)$.*

(vii) *$D = 12$, $\mathcal{K} = \mathcal{K}l\big(\mathsf{Char}(16) \smallsetminus \{\chi_8^{0,1,4,7}\}\big)$, $p = 5, 13$, $L = \mathrm{SU}_3(4)$, and $G/\mathbf{Z}(G) \cong \mathrm{Aut}(L) \cong L \cdot 4$.*

(viii) *$D = 12$, and either*

$$(p, \mathcal{K}, L, G/\mathbf{Z}(G)) = \Big(2, \mathcal{K}l\big(\mathsf{Char}(21)^\times\big), 2 \cdot G_2(4), G_2(4)\Big),$$

*or*

$$(p, \mathcal{K}, L, G/\mathbf{Z}(G)) = \Big(5, \mathcal{K}l\big(\mathsf{Char}(18) \smallsetminus \{\chi_{18}^{2,6,8,14,15,18}\}\big), 6\mathrm{Suz}, \mathrm{Suz}\Big).$$

*Proof.* Recall [KT2, Theorem 1.7] that any primitive Kloosterman sheaf of rank $D \geq 9$, always satisfies ($\mathbf{S}+$). Hence $G := G_{\mathrm{geom}}$ satisfies ($\mathbf{S}+$), and its structure is prescribed by [KT2, Lemma 1.1]. Working on the "only if" part of the theorem, we will now assume that $G$ is finite. In particular, the $D$ characters $\chi_1, \ldots, \chi_D$ are pairwise distinct, and hence a generator $g_0$ of the image of $I(0)$ in $G$ is a $p'$-element which has simple spectrum on the underlying representation space $V$.

First suppose that $G$ is an extraspecial normalizer. Since $D \geq 11$, we can apply [KT5, Theorem 1.2.6] to see that $D = p^f$, and arrive at conclusion (i) if in addition $p > 2$. If $p = 2$, then, after tensoring $\mathcal{K}$ with a suitable Kummer sheaf $\mathcal{L}_\chi$, we have $\mathbf{Z}(G) = C_2$ and $E \lhd G \leq \mathbf{N}_{\mathrm{O}^\epsilon_{2f}(\mathbb{C})}(E)$ where $E = 2^{1+2f}_\epsilon$ is an extraspecial 2-group of type $\epsilon = \pm$. Applying [TY, Theorem 5.7] we again arrive at (i). Conversely, in the case of (i) the $[D+1]_\star$ pullback of $\mathcal{K}$ to $\mathbb{A}^1$ has finite geometric monodromy group $H$, an extraspecial $p$-group of order $p^{1+2f}$, see [KT5, Theorem 7.3.8]. In particular, $G$ is finite, $\mathbf{Z}(H) \leq \mathbf{Z}(G)$, $H/\mathbf{Z}(H) \cong C_p^{2f}$, $G/H \hookrightarrow C_{D+1}$, and $L = 1$. Next, the shape of $\mathcal{K}$ implies that a generator of the image of $I(0)$ in $G$ has central order $D+1$. It follows that $G/\mathbf{Z}(H) \cong C_p^{2f} \rtimes C_{D+1}$.

From now on we may assume that $G$ is almost quasisimple, i.e. $S \lhd G/\mathbf{Z}(G) \leq \mathrm{Aut}(S)$ for some non-abelian finite simple group $S$. As $g_0$ is an $\mathsf{ssp}$-element on $V$, its central order satisfies

$$D < \bar{\mathsf{o}}(g_0) \leq \mathrm{meo}(\mathrm{Aut}(S)), \tag{3.5.1}$$

where $\mathrm{meo}(X)$ denotes the largest order of elements in a finite group $X$. Here, the strict inequality $D < \bar{\mathsf{o}}(g_0)$ follows from the primitivity assumption on $\mathcal{K}$; indeed, if $D = \bar{\mathsf{o}}(g_0)$, then the spectrum of $g_0$ on $V$ consists of all $D^{\mathrm{th}}$ roots of some root of unity, and hence $\mathcal{K}$ would be Kummer induced. Moreover, condition ($\mathbf{S}+$) for $(G, V)$ implies that the quasisimple subgroup $L = G^{(\infty)}$ acts irreducibly (and faithfully) on $V$. Now we can apply results of [KT2, §6] to find possible candidates for $(L, V, g_0)$, and the rest of the proof is to analyze these possibilities.

(a) Assume first that $S$ is one of the 26 sporadic simple groups. Then triples $(L, V, g_0)$ are listed in [KT2, Table 1], which also indicates which triples can give rise to hypergeometric sheaves. Using the condition $D \geq 11$ and (3.5.1) we arrive at the following cases (where we use the notation in [GAP] for conjugacy classes). We also let $Q$, respectively $J$, denote the image of $P(\infty)$, respectively $I(\infty)$, in $G$. Then $Q = \mathbf{O}_p(J)$ and $J/Q$ is cyclic of $p'$-order; furthermore, $J$ acts irreducibly on $V$, and $Q \not\leq \mathbf{Z}(G)$ by [KT2, Proposition 4.8(ii)].

- $L = \mathrm{M}_{23}$, $D = 22$, and $g_0$ comes from classes $23AB$ in $L$.
- $L = 3\mathrm{J}_3$, $D = 18$, and $g_0$ comes from classes $19AB$, $57ABCD$ in $L$.
- $L = 2\mathrm{Ru}$, $D = 28$, and $g_0$ comes from classes $29AB$, $58AB$ in $L$.
- $L = 2\mathrm{J}_2$, $D = 14$, and $g_0$ comes from classes $28AB$, $24CDEF$ in $L \cdot 2$. The classes $28AB$ are ruled out since they have central order 14.
- $L = \mathrm{McL}$, $D = 22$, and $g_0$ comes from the class $30A$ in $L \cdot 2$.
- $L = \mathrm{Co}_2$, $D = 23$, and $g_0$ comes from classes $30BC$ in $L$.
- $L = \mathrm{Co}_3$, $D = 23$, and $g_0$ comes from the class $30A$ in $L$.
- $L = 2\mathrm{Co}_1$, $D = 24$, with 17 possible classes for $g_0$ in $L$ (namely, $52A$, $56AB$, $60BCD$, $35A$, $70A$, $36AB$, $39AB$, $78AB$, $40AB$, $84A$, which are numbered 140, 143, 144, 148, and 155–167 in [GAP]). The first three classes are ruled out since the corresponding sheaves are Kummer induced.
- $L = 6\mathrm{Suz}$, $D = 12$, with 57 possible classes for $g_0$ in $L$.

In all the above cases, the shape of $\mathcal{K}$ (up to tensoring with some $\mathcal{L}_\chi$), is determined by the spectrum of $g_0$ on $V$. The first three cases are ruled out, since in each case a pullback to $\mathbb{A}^1$, namely the sheaf $\mathcal{F}(D+1,1,\mathbb{1})$, has infinite monodromy by [KT5, Theorem 10.2.6]. The next four remaining cases also lead to infinite $G_{\rm geom}$, as one can show using the $V$-test. In the case of $L = 2{\rm Co}_1$, the fourteen classes with primitive $\mathcal{K}$ are also ruled out by the $V$-test. In the final case of $L = 6{\rm Suz}$, when $\bar{\mathsf{o}}(g_0) = 13$ we arrive at (iii) when $p = 5$ and at (iv) when $p = 2$. If $\bar{\mathsf{o}}(g_0) = 15$, 24, or 28, then $\mathcal{K}$ is Kummer induced. When $\bar{\mathsf{o}}(g_0) = 20$, $\mathcal{K}$ fails the $V$-test. If $\bar{\mathsf{o}}(g_0) = 18$ then we arrive at (viii) with $p = 5$. If $\bar{\mathsf{o}}(g_0) = 21$ then we arrive at (viii) with $p = 2$; however, in this case $\mathcal{K}$ is self-dual by [KRLT3, Proposition 6.1(i)] but $V$ is not, ruling out this possibility.

(b) Assume now that $S = \mathsf{A}_n$ for some $n \geq 5$. Since ${\rm meo}({\rm Aut}(S)) > D \geq 11$ by (3.5.1), we in fact have $n \geq 7$. The triples $(L, V, g_0)$ are classified by [KT2, Theorem 6.2] for $n \geq 8$ and listed in [KT2, Table 1] for $n = 7$. Using the assumption $D > 8$ and [KT2, Lemma 9.1], we arrive at one of the following two possibilities.

In the first case, $L = \mathsf{A}_8$, $D = 14$, and $\bar{\mathsf{o}}(g_0) = 15$. Now we can appeal to [KT5, Theorem 10.2.6], which shows that the pullback $\mathcal{F}(15,1,\mathbb{1})$ of $\mathcal{K}$ to $\mathbb{A}^1$ has infinite monodromy.

In the second, main case, $D = n-1$, $L = \mathsf{A}_n$ acts on $V$ via its deleted permutation module, and, up to a scalar, $g_0$ is either an $n$-cycle or the disjoint product of a $k$-cycle and an $(n-k)$-cycle in $\mathsf{S}_n$, where $1 \leq k \leq n-1$ and $\gcd(k, n) = 1$. By Lemma 3.1, $n = p^f$ for some $f \geq 1$. Since $g_0$ is a $p'$-element, this rules out the former case where $g_0$ is an $n$-cycle. So we are in the latter case, and $p \nmid k$. Now (3.5.1) rules out the case $k = 1$ or $k = n-1$. Thus we arrive at conclusion (ii). Conversely, in the case of (ii) we have by [KT2, Theorem 9.3(ii)] that $G$ is finite, with $L = \mathsf{A}_{D+1}$, and $G/\mathbf{Z}(G) \cong \mathsf{A}_{D+1}$ if $p = 2$ and $G/\mathbf{Z}(G) \cong \mathsf{S}_{D+1}$ if $p > 2$.

(c) From now on, we may assume that $S$ is a finite simple group of Lie type in some characteristic $r$. By [KT5, Theorem 3.1.10], we have $p = r$, unless one of the following cases occurs.

- $D = 12$, $L = {\rm SU}_3(4)$, and $p = 2, 5, 13$. Here the possible classes for $g_0$ are $13A$, $15A$, $12AB$, $16AB$, $16CD$ in $L \cdot 4$, and the classes $12AB$ are ruled out by (3.5.1). Furthermore, for the class $13A$, $p$ can be 2 or 5, and both cases are recorded in conclusion (iv), resp. (iii). The case of the classes $16ABCD$ when $p = 5$ or 13 is recorded in (vii) (and proved to be having finite $G_{\rm geom}$ with $L = {\rm SU}_3(4)$ in [KRLT3, Theorem 25.5]).
- $D = 14$, $L = G_2(3)$ and $p = 3, 13$. Here the possible classes for $g_0$ are $14A$, and $18ABC$ in $L \cdot 2$, and class $14A$ is ruled out by (3.5.1).
- $D = 14$, $L = {}^2B_2(8)$ and $p = 2, 13$. Here the possible classes for $g_0$ are $15AB$ in $L \cdot 3$, and we may assume $\mathcal{K} = \mathcal{K}l(\mathsf{Char}_{\rm ntriv}(15))$.

In the remaining possibilities of these exceptional cases, the shape of $\mathcal{K}$ (up to tensoring with some $\mathcal{L}_\chi$), is determined by the spectrum of $g_0$ on $V$, and they are either Kummer induced or ruled out by the $V$-test.

We will now work with the generic case $p = r$ and $S \not\cong G_2(3)$, ${}^2B_2(8)$. Applying [KT2, Theorem 6.6] and [KT2, Table 1], we see that either

- (The generic case) $S = {\rm PSL}_n(q)$ with $n \geq 3$, ${\rm PSU}_n(q)$ with $n \geq 3$, ${\rm PSp}_{2n}(q)$ with $n \geq 2$, and $V|_L$ is a Weil module, or
- $S \cong {\rm PSL}_2(q)$ and $D = \dim(V) < \bar{\mathsf{o}}(g_0) \leq q+1$, or
- $p = 2$, $D = 12$, $S = G_2(4)$, $L = 2G_2(4)$.

Assume we are in the last possibility. As $\bar{\mathsf{o}}(g_0)$ is odd, using [GAP] we arrive at three classes for $g_0$ in $L$. The first class is $13A$, with spectrum $\mu_{13} \smallsetminus \{1\}$, recorded in (iv). The second one is $15B$, with spectrum $\mu_{15} \smallsetminus \mu_3$, which means $\mathcal{K}$ is 5-induced. The third one is $21A$, with spectrum $\mu_{21} \smallsetminus (\mu_3 \cup \mu_7)$, recorded in (viii).

We will now treat the generic case of Weil modules of classical groups (with $S \not\cong {\rm PSL}_2(q)$).

(c1) First assume that $S = \mathrm{PSL}_n(q)$ with $q = p^f$ and $n \geq 3$. By (3.5.1), $D \geq 11$ rules out the case $(n, q) = (3, 2)$; also the case $(n, q) = (4, 2)$ has been handled in (b) as $\mathrm{SL}_4(2) \cong \mathsf{A}_8$. Suppose $S = \mathrm{PSL}_3(4)$. Then $\mathrm{Out}(S) = C_2 \times \mathsf{S}_3$ [CCNPW], but $g_0$ is a $2'$-element, so $g_0\mathbf{Z}(G) \in S \cdot 3 = \mathrm{PGL}_3(4)$. It follows from [KT2, Theorem 4.1] that $G/\mathbf{Z}(G) \leq \mathrm{PGL}_3(4)$, and moreover $L \cong \mathrm{PSL}_3(4)$ if $D = 20$ as $V|_L$ is irreducible. If $S = \mathrm{SL}_3(3)$, then since $\bar{\mathsf{o}}(g_0) \geq 13$ by (3.5.1), $g_0 Z \in S$ and so $G/\mathbf{Z}(G) = S$ by [KT2, Theorem 4.1]. In all other cases we have $G/\mathbf{Z}(G) = \mathrm{PGL}_n(q)$ by [KT2, Corollary 8.4]. Now we can apply Lemma 3.4 to reach the contradiction that $J$ is reducible on $V$.

(c2) Next assume that $S = \mathrm{PSU}_n(q)$ with $q = p^f$ and $n \geq 3$, $(n, q) \neq (3, 2)$. The cases $(n, q) = (3, 3)$, $(4, 2)$ are ruled out since $D \geq 11$. If $(n, q) = (3, 4)$, $(5, 2)$, then since $\mathrm{Out}(S) \hookrightarrow C_4$ [CCNPW] and $g_0$ is a $2'$-element, $g_0\mathbf{Z}(G) \in \mathrm{PGU}_n(q) \cong S$. If $(n, q) = (6, 2)$, then since $\mathrm{Out}(S) = \mathsf{S}_3$ [CCNPW] and $g_0$ is a $2'$-element, we again have $g_0\mathbf{Z}(G) \in \mathrm{PGU}_n(q) \cong S \cdot 3$. If $(n, q) = (4, 3)$, then $D = 20$ or 21, and the $3'$-elements of order $\geq 20$ in $\mathrm{Aut}(S) \cong S \cdot D_8$ all belong to classes $28AB$, in the notation of [GAP], and they are as described in [KT2, Theorem 8.3(iii)]. Thus in all cases [KT2, Theorem 8.3] applies, in particular showing that $g_0\mathbf{Z}(G) \in \mathrm{PGU}_n(q)$ and hence $G/\mathbf{Z}(G) \cong \mathrm{PGU}_n(q)$ by [KT2, Corollary 8.4].

Assume that $2 \nmid n$. Then we are in cases (i) or (ii) of [KT2, Theorem 8.3]. In the former case, $\bar{\mathsf{o}}(g_0) = (q^n + 1)/(q + 1)$, so by (3.5.1) we must have that $D = (q^n - q)/(q + 1)$ and $L = \mathrm{PSU}_n(q)$ (because the Weil representation of degree $D$ of $\mathrm{SU}_n(q)$ is trivial at its center), arriving at conclusion (iv). Conversely, in the case of conclusion (iv), $\mathcal{K}$ is the sheaf $\mathcal{H}^{n,1,0}$ in [KT3, Theorem 10.3] and hence has finite $G_{\mathrm{geom}}$ with $L = \mathrm{PSU}_n(q)$. Suppose we are in the case of [KT2, Theorem 8.3(ii)], so that $\bar{\mathsf{o}}(g_0) = q^{n-1} - 1$ and $D = (q^n - q)/(q + 1)$. Since $D \geq 11$, $(n, q) \neq (3, 3)$, $(5, 2)$. If in addition $(n, q) \neq (3, 4)$, $(3, 5)$, then no such $\mathcal{K}$ (and no hypergeometric sheaf with this kind of $G_{\mathrm{geom}}$) can exist by Theorems 9.13 and 9.14 of [KT2]. In the remaining cases $(n, q) = (3, 4)$, $(3, 5)$, the shape of $\mathcal{K}$ (up to tensoring with some $\mathcal{L}_\chi$), is determined by the spectrum of $g_0$ on $V$, and one can check using [GAP] that $\mathcal{K}$ is $(q - 1)$-induced in these two cases.

Assume now that $2 \mid n$. Then we are in cases (i) or (iiii) of [KT2, Theorem 8.3]. In the former case, $D = (q^n - 1)/(q + 1)$. Then no such $\mathcal{K}$ (in fact, no hypergeometric sheaf of rank $D$, with this kind of $G_{\mathrm{geom}}$ and with wild part of dimension at least two) can exist by [KT2, Theorem 9.17] and its proof. Suppose we are in the case of [KT2, Theorem 8.3(iii)], so that $n = a + b$ with $\gcd(a, b) = 1$ and $2 \nmid ab$, and $\bar{\mathsf{o}}(g_0) = (q^a + 1)(q^b + 1)/(q + 1)$. Again $(n, q) \neq (4, 2)$ as $D \geq 11$. If $D = (q^n + q)/(q + 1)$, then since $L$ acts faithfully on $V$ as on a Weil module, we see that $L = \mathrm{PSU}_n(q)$. Applying Lemma 3.2 to $J$, we arrive at a contradiction. Thus $D = (q^n - 1)/(q + 1)$, and, as the shape of $\mathcal{K}$ is determined by the spectrum of $g_0$ on $V$ and $\langle g_0 Z \rangle$ is unique up to conjugacy in $\mathrm{PGU}_n(q)$, $\mathcal{K}$ is one of the $q$ Kloosterman sheaves in conclusion (v), up to tensoring with some $\mathcal{L}_\chi$. All these sheaves indeed have finite $G_{\mathrm{geom}}$ with $L/\mathbf{Z}(L) = \mathrm{PSU}_n(q)$ by [KT4, Theorem 16.11].

(c3) Next assume that $S = \mathrm{PSp}_{2n}(q)$ with $q = p^f$ and $p \geq 3$. Note that $(n, q) \neq (2, 3)$ since $D \geq 11$, so [KT2, Theorem 8.2(i)] applies, and furthermore $G/\mathbf{Z}(G) \cong \mathrm{PSp}_{2n}(q)$ by [KT2, Corollary 8.4]. By Lemma 3.3 we have $D = (q^n - 1)/2$, and so $\bar{\mathsf{o}}(g_0) \geq (q^n + 1)/2$ by (3.5.1). (Note that the formulation of part $(\alpha)$ of [KT2, Theorem 8.2(i)] has a typo: $\bar{\mathsf{o}}(g_0)$ should be $(q^n \pm 1)/2$, but not $(q^a + 1)(q^b + 1)/2$ as printed.) Suppose we are in case $(\alpha)$ of [KT2, Theorem 8.2(i)], Then (3.5.1) implies that $\bar{\mathsf{o}}(g_0) = (q^n + 1)/2$, and $\mathcal{K}$ is the one described in conclusion (iii) up to tensoring with a suitable $\mathcal{L}_\chi$ (but with $q$ replaced by $q^n$); in this case $G_{\mathrm{geom}}$ is finite but with $L/\mathbf{Z}(L) \cong \mathrm{PSL}_2(q^n)$ by [KT1, Theorem 17.2]. Assume we are in case $(\beta)$ of [KT2, Theorem 8.2(i)]. Again, the shape of $\mathcal{K}$ is determined by the spectrum of $g_0$ on $V$ and $\langle g_0 Z \rangle$ is unique up to conjugacy in $\mathrm{PSp}_{2n}(q)$. So $\mathcal{K}$ is the Kloosterman sheaf in conclusion (vi), up to tensoring with some $\mathcal{L}_\chi$. The latter sheaf indeed has finite $G_{\mathrm{geom}}$ with $L/\mathbf{Z}(L) = \mathrm{PSp}_{2n}(q)$ by [KT4, Theorem 15.7] (also note that changing $\psi$ may give rise to another Weil representation of degree $D$ of $L$).

(d) Now we handle the case $S = \mathrm{PSL}_2(q)$ and $D < \bar{\mathsf{o}}(g_0) \le q+1$, whence $q \ge D \ge 11$. In this case, $L = G^{(\infty)}$ is either $\mathrm{PSL}_2(q)$ or $\mathrm{SL}_2(q)$, and $D = (q\pm 1)/2$, $q-1$, or $q$, since $V|_L$ is irreducible. Now if $D = q$, then $\bar{\mathsf{o}}(g_0) = q+1$ by (3.5.1), so, up to tensoring with some $\mathcal{L}_\chi$, $\mathcal{L}$ is just the Pink–Sawin sheaf in conclusion (i).

Next assume that $D = q-1$. Since $g_0$ is a $p'$-element, (3.5.1) again implies that $\bar{\mathsf{o}}(g_0) = q+1$. Hence, up to tensoring with a suitable $\mathcal{L}_\chi$, $\mathcal{K}$ is $\mathcal{K}l(\mathsf{Char}(q+1) \smallsetminus \{\mathbb{1}, \sigma\})$ for some $\mathbb{1} \ne \sigma \in \mathsf{Char}(q+1)$, which is the one described in conclusion (v) when $(a, b, A, B, \alpha, \beta) = (1, 1, 1, 1, 1, 0)$. Conversely, this latter sheaf has finite $G = G_{\mathrm{geom}}$ with $L/\mathbf{Z}(L) = \mathrm{PSU}_2(q)$ and $G/\mathbf{Z}(G) \cong \mathrm{PGU}_2(q)$ by [KT4, Theorem 17.5].

Now let $D = (q+1)/2$, so $2 \nmid q = p^f \ge 23$. Then $V|_L$ is a Weil representation of degree $D$, and the (non-inner) diagonal automorphism of $L$ fuses the two Weil representations of the same degree $D$. Hence $G$ can only induce field automorphisms of $L$, and so $G/\mathbf{Z}(G) \le \mathrm{PSp}_2(q) \cdot C_f \le \mathrm{PSp}_{2f}(p)$. This case can then be ruled out using Lemma 3.3.

Finally, let $D = (q-1)/2$, so again $2 \nmid q = p^f \ge 23$. Then $G/\mathbf{Z}(G) \le \mathrm{PSp}_2(q) \cdot C_f$, again since the (non-inner) diagonal automorphism of $L$ fuses the two Weil representations of the same degree $D$. We also know that $L \cong \mathrm{PSL}_2(q)$ if $2 \nmid D$, and $L \cong \mathrm{SL}_2(q)$ if $2|D$. Suppose $g_0\mathbf{Z}(G) \notin \mathrm{PSp}_2(q)$, say $g_0$ induces a field automorphism of order $1 < e|f$ of $L$ modulo inner automorphisms. As shown in the proof of [GMPS, Theorem 2.16], in this case $g_0^e$ is conjugate to some $p'$-element in $\mathrm{SL}_2(q^{1/e})$ (modulo $\mathbf{Z}(G)$), and so

$$\bar{\mathsf{o}}(g_0) \le e(q^{1/e} + 1) \le (q-1)/2.$$

Hence, using (3.5.1) we conclude that $g_0\mathbf{Z}(G) \in \mathrm{PSL}_2(q)$, whence $\bar{\mathsf{o}}(g_0) = (q+1)/2$ since the largest element order in $\mathrm{PSL}_2(q)$ is $(q+1)/2$. As $G$ equals the normal closure of $\langle g_0 \rangle$, we must have $G/\mathbf{Z}(G) \cong \mathrm{PSL}_2(q)$ and arrive at conclusion (iii). Conversely, in the case of (iii) the sheaf $\mathcal{K}$ has finite $G_{\mathrm{geom}}$; indeed a Kummer pullback of it to $\mathbb{A}^1$ has $L$ as its geometric monodromy group by [KT1, Theorem 17.2].

(e) The finiteness of $G_{\mathrm{geom}}$ in (viii) will be proved in Theorems 6.15, 6.16. To conclude the proof, we need to identify $L$ and $G/\mathbf{Z}(G)$ for the two cases listed in (viii). The determination of $L$ follows from the above analysis, which gives only one possibility for $L$ for the given $(p, D, \mathcal{K})$. Assume $(p, \mathcal{K}, L) = \big(2, \mathcal{K}l\big(\mathsf{Char}(21)^\times\big), 2 \cdot G_2(4)\big)$; in particular, $G/\mathbf{Z}(G) \hookrightarrow \mathrm{Aut}(L) = L/\mathbf{Z}(L) \cdot 2$ for $G = G_{\mathrm{geom}}$. Then a generator $g_0$ of the image of $I(0)$ in $G$ has odd central order 21 and hence can induce only an inner automorphism of $L$ (see [CCNPW]), showing that $G/\mathbf{Z}(G) \cong L/\mathbf{Z}(L)$. Finally, assume that $(p, \mathcal{K}, L) = \big(5, \mathcal{K}l\big(\mathsf{Char}(18) \smallsetminus \{\chi_{18}^{2,6,8,14,15,18}\}\big)\big)$. Then again $G/\mathbf{Z}(G) \hookrightarrow \mathrm{Aut}(L) = L/\mathbf{Z}(L) \cdot 2$ for $G = G_{\mathrm{geom}}$, and a generator $g_0$ of the image of $I(0)$ in $G$ has central order 18 and hence can induce only an inner automorphism of $L$ (see [CCNPW]), yielding $G/\mathbf{Z}(G) \cong L/\mathbf{Z}(L)$. □

**Remark 3.6.** In the situation of Theorem 3.5, once we have $G/\mathbf{Z}(G) \cong L/\mathbf{Z}(L)$ we also know that

$$G = \mathbf{Z}(G)L.$$

Indeed, the isomorphism $G/\mathbf{Z}(G) \cong L/\mathbf{Z}(L)$ implies that the conjugation action of $G$ on $L$ induces only inner automorphisms of $L$. Since $L$ acts irreducibly on $\mathcal{K}$, the kernel of this action is $\mathbf{C}_G(L) = \mathbf{Z}(G)$ by Schur's lemma, and so $G = \mathbf{Z}(G)L$.

## 4. Relation with Kloosterman sums

In this section, we consider the simplest sort of $\mathcal{F}(D, E, \chi)$, namely the case in which $\#D = \#E = 1$, say $D = \{a\}, E = \{b\}$. With these choices, we denote

$$\mathcal{F}(a, b, \chi) := \mathcal{F}(D, E, \chi).$$

Thus $\mathcal{F}(a,b,\chi)$ is the local system on $(\mathbb{G}_m \times \mathbb{G}_m)/k$, with coordinates $(s,t)$ (the previous $(y_a, z_b)$) is given as follows: for $L/k$ a finite extension and $(s_0,t_0) \in L^\times \times L^\times$, the trace of $\mathsf{Frob}_{L,(s_0,t_0)}$ is

$$-\sum_{x\in L^\times} \psi_L(s/x^a + tx^b)\chi_L(x). \tag{4.0.1}$$

In view of Lemma 2.1, this local system is geometrically irreducible if and only if $\gcd(a,b) = 1$. (This same lemma gives its decomposition into the sum of $\gcd(a,b)$ irreducible summands in general.)

For the rest of this section, we assume that $\gcd(a,b) = 1$. We fix a choice of integers $(A,B)$ with

$$bB - aA = 1.$$

With this choice of $(A,B)$, we define two auxiliary multiplicative characters

$$\Lambda := \chi^A, \rho = \chi^B.$$

In [Ka2, 4.1.1, 5.1 and bottom of p. 60], one finds the Kloosterman sheaf $\mathcal{K}l_\psi(\Lambda,\rho;b,a)$, a lisse sheaf on $\mathbb{G}_m/k$ whose trace function is given as follows. For $L/k$ a finite extension, and $t \in L^\times$, it is

$$-\sum_{Y,Z\in L: Y^bZ^a=t} \psi_L(Y+Z)\Lambda_L(Y)\rho_L(Z).$$

[We will later give the expression of $\mathcal{K}l_\psi(\Lambda,\rho;b,a)$ as a "usual" Kloosterman sheaf, cf. Lemma 4.2.]

**Theorem 4.1.** *Suppose* $\gcd(a,b) = 1$. *Then* $\mathcal{F}(a,b,\chi)$ *has finite* $G_{\mathrm{geom}}$ *if and only if* $\mathcal{K}l_\psi(\Lambda,\rho;b,a)$ *has finite* $G_{\mathrm{geom}}$.

*Proof.* The proof is based on the observation that Kummer pullback (or indeed any pullback by a finite morphism) does not change the identity component $G^\circ_{\mathrm{geom}}$ of $G_{\mathrm{geom}}$. Consider first the partial Kummer pullback $(s,t) \mapsto (s^a,t)$ of $\mathcal{F}(a,b,\chi)$; its trace function is

$$(s,t) \mapsto -\sum_x \psi((s/x)^a + tx^b)\chi(x) = -\sum_x \psi(1/x^a + s^atx^b)\chi(sx) = -\chi(s)\sum_x \psi(1/x^a + s^atx^b)\chi(x),$$

by the substitution $x \mapsto sx$. This is the pullback by the automorphism $(s,t) \mapsto (s, s^at)$ of the local system whose trace function is

$$(s,t) \mapsto -\chi(s)\sum_x \psi(1/x^a + tx^b)\chi(x).$$

The factor $\chi(s)$ does not change $G^\circ_{\mathrm{geom}}$, so we are dealing with the local system on $\mathbb{G}_m$ whose trace function is

$$t \mapsto -\sum_x \psi(1/x^a + tx^b)\chi(x).$$

Because $\gcd(a,b) = 1$, with $bB - aA = 1$, we introduce $Y := 1/x^a, Z := tx^b$. Then

$$x = x^{bB-aA} = (Z/t)^B Y^A,$$

and the trace becomes

$$t \mapsto -\sum_{Y,Z\in L: Y^bZ^a=t^a} \psi(Y+Z)\chi((Z/t)^BY^A) = -\sum_{Y,Z\in L: Y^bZ^a=t^a} \psi(Y+Z)\rho(Z/t)\Lambda((Y).$$

The $\rho(1/t)$ factor does not change $G^\circ_{\mathrm{geom}}$, we we are dealing with the Kummer pullback $t \mapsto t^a$ of the local system whose trace function is

$$t \mapsto -\sum_{Y,Z\in L: Y^bZ^a=t} \psi(Y+Z)\chi((Z)^BY^A) = -\sum_{Y,Z\in L: Y^bZ^a=t} \psi(Y+Z)\Lambda(Y)\rho(Z),$$

which is precisely the trace function of $\mathcal{K}l_\psi(\Lambda, \rho; b, a)$. $\square$

**Lemma 4.2.** *The geometric expression of $\mathcal{K}l_\psi(\Lambda, \rho; b, a)$ as a "usual" Kloosterman sheaf is given as follows. It is the multiplicative translate by $t \mapsto a^a b^b t$ of the Kloosterman sheaf*

$$\mathcal{K} := \mathcal{K}l_\psi\Big(\mathsf{Char}(b, \Lambda) \sqcup \mathsf{Char}(a, \rho)\Big) = \mathcal{K}l_\psi(\text{all } b^{\text{th}} \text{ roots of } \Lambda, \text{all } a^{\text{th}} \text{ roots of } \rho).$$

*If furthermore $a = b = 1$ and $\chi \neq \chi_2$, then $\mathcal{K}$ is primitive. In general, for any choice of $\Lambda$ and $\rho$, if $\gcd(a, b) = 1$ and $a + b \geq 3$, then $\mathcal{K}l_\psi\Big(\mathsf{Char}(b, \Lambda) \sqcup \mathsf{Char}(a, \rho)\Big)$ is primitive.*

*Proof.* For the first statement, by [Ka2, bottom of page 60 and 5.5], $\mathcal{K}l_\psi(\Lambda, \rho; b, a)$ is the ! multiplicative convolution the two Kummer direct images, $[b]_\star(\mathcal{L}_\psi \otimes \mathcal{L}_\Lambda)$ and $[a]_\star(\mathcal{L}_\psi \otimes \mathcal{L}_\rho)$. By the Hasse Davenport theorem for Kummer direct images [Ka2, 5.6.2], we have geometric isomorphisms

$$[b]_\star(\mathcal{L}_\psi \otimes \mathcal{L}_\Lambda) \cong \mathcal{K}l_{\psi(bx)}(\text{all } b^{\text{th}} \text{ roots of } \Lambda),$$
$$[a]_\star(\mathcal{L}_\psi \otimes \mathcal{L}_\rho) \cong \mathcal{K}l_{\psi(ax)}(\text{all } a^{\text{th}} \text{ roots of } \rho).$$

The first has trace function

$$t \mapsto (-1)^{b-1} \sum_{x_1 \ldots x_b = t} \psi(\sum_i bx_i) \prod_{\sigma : \sigma^b = \Lambda} \sigma(x_i) =$$

(via $x_i \mapsto bx_i$)

$$= (-1)^{b-1} \sum_{x_1 \ldots x_b = tb^b} \psi(\sum_i x_i) \prod_{\sigma : \sigma^b = \Lambda} \sigma(x_i/b),$$

which, geometrically, is the trace function of the $t \mapsto tb^b$ pullback of $\mathcal{K}l_\psi(\text{all } b^{\text{th}} \text{ roots of } \Lambda)$. Similarly for $[a]_\star(\mathcal{L}_\psi \otimes \mathcal{L}_\rho)$. Forming their ! multiplicative convolution, and using the fact that for "usual" $\mathcal{K}l_\psi$, the lists of characters concatenate, by [Ka2, 7.3.2, 7.4.1], we get the assertion.

For the second statement, our $\mathcal{K}$ is $\mathcal{K}l_\psi(\Lambda, \rho) = \mathcal{K}l_\psi(\chi^A, \chi^B) = \mathcal{K}l_\psi(\chi^A, \chi^{A+1})$, the last equality because $B - A = 1$. If this $\mathcal{K}$ is induced, then the ratio (in either order) of the characters must be the quadratic character $\chi_2$. But this ratio is $\chi$ or $1/\chi$, hence $\chi = \chi_2$, and our $\mathcal{K}$ is $\mathcal{K}l_\psi(\mathbb{1}, \chi_2)$, the excluded case.

For the third statement, let $\{\alpha_1, \ldots, \alpha_a\}$ be the set of all $a^{\text{th}}$ roots of $\rho$, and let $\{\beta_1, \ldots, \beta_b\}$ be the set of all $b^{\text{th}}$ roots of $\Lambda$; note that we do not assume here that these two sets are disjoint. Assume the contrary that $\mathcal{K}$ is imprimitive. Then, by a result of Pink [Ka1, Lemmas 11, 12], it is Kummer induced, i.e. the multiset $X := \{\alpha_1, \ldots, \alpha_a\} \cup \{\beta_1, \ldots, \beta_b\}$ is stable under multiplication by some nontrivial character $\gamma$. Without any loss we may assume that $\gamma$ has prime order $r$. Thus $X$ is $\mu_r$-stable and hence it is a union of say $c \geq 1$ orbits $\mathcal{O}_1, \ldots, \mathcal{O}_c$ (where each orbit has size $r$ but some of these orbits may be repeated in the case the two sets $\{\alpha_1, \ldots, \alpha_a\}$ and $\{\beta_1, \ldots, \beta_b\}$ are not disjoint).

Suppose $c = 1$. We claim that $a \in \{1, r\}$. Indeed, if $a > 1$, then we can choose $\alpha_1$ and $\alpha_2$ such that the ratio $\alpha_2/\alpha_1$ has order $a$. But $\alpha_1, \alpha_2 \in \mathcal{O}_1$ implies that this ratio has order $r$, whence $a = r$. Similarly, $b \in \{1, r\}$. As $a + b \geq 3$ and $\gcd(a, b) = 1$ we have $\{a, b\} = \{r, 1\}$, and hence

$$\#X = a + b = r + 1 > r = \#\mathcal{O}_1,$$

a contradiction.

Next consider the case $c \geq 2$. Without loss we may assume that $a > b$ (note that $a = b$ would imply $(a, b) = (1, 1)$ by coprimality). Now if each orbit $\mathcal{O}_i$, $1 \leq i \leq c$, intersects $\{\alpha_1, \ldots, \alpha_a\}$ at most once, then $a \leq c = \#X/r \leq (a + b)/2$ and thus $a \leq b$, a contradiction. Hence some orbit $\mathcal{O}_i$ intersects $\{\alpha_1, \ldots, \alpha_a\}$ at least at two characters, say $\alpha_1$ and $\alpha_2$. Then the ratio $\alpha_2/\alpha_1$ has order dividing $a$, on the one hand, and equal $r$, on the other hand, which implies $r \mid a$. Thus the set

$\{\alpha_1, \ldots, \alpha_a\}$ is $\mu_r$-stable. As $X$ is $\mu_r$-stable, the set $\{\beta_1, \ldots, \beta_b\}$ is also $\mu_r$-stable. In particular, $b > 1$, and the ratio $\beta_2/\beta_1$ has order $r$. But this ratio has order dividing $b$, so $r \mid b$. Thus $\gcd(a, b) \geq r$, a contradiction. ☐

Next we discuss $\mathcal{F}(1, b, \chi)$, with $b \geq 1$ prime to $p$, the local system on $(\mathbb{G}_m \times \mathbb{G}_m)/\mathbb{F}_p(\chi)$ whose trace function is given as follows. For $k/\mathbb{F}_p(\chi)$ a finite extension, and $(r, s) \in (k^\times)^2$, the trace is

$$(r, s) \mapsto -\sum_{x \in k^\times} \psi_k(r/x + sx^b)\chi_k(x).$$

Recall that $\mathsf{Char}(b, \overline{\chi})$ denotes the set of characters $\sigma$ with $\sigma^b = \overline{\chi}$.

**Lemma 4.3.** *Denote by $\mathcal{K}$ the $r = 1$ pullback of $\mathcal{F}(1, b, \chi)$, i.e., the local system on $\mathbb{G}_m/\mathbb{F}_p(\chi)$ whose trace function is*

$$s \mapsto -\sum_{x \neq 0} \psi(1/x + sx^b)\chi(x).$$

*Consider the local system $\mathcal{H}$ on $(\mathbb{G}_m \times \mathbb{G}_m)/\mathbb{F}_p(\chi)$, with coordinates $(r, s)$, which is the tensor product*

$$\mathcal{H} := \mathcal{L}_\chi(r) \otimes \mathcal{K}(s).$$

*Then by means of the automorphism $\Phi : (r, s) \mapsto (r, sr^b)$ of $(\mathbb{G}_m \times \mathbb{G}_m)/\mathbb{F}_p(\chi)$, we have an isomorphism of local systems*

$$\mathcal{F}(1, b, \chi) \cong \Phi^\star \mathcal{H}.$$

*Moreover, $\mathcal{K}$ is geometrically isomorphic to (a multiplicative translate of) the Kloosterman sheaf $\mathcal{K}l(\mathbb{1}, \mathsf{Char}(b, \overline{\chi}))$.*

*Proof.* The source $\mathcal{F}(1, b, \chi)$ is geometrically irreducible, by Lemma 2.1, hence is arithmetically irreducible. So it suffices to show that source and target have the same trace function. The trace function of $\mathcal{K}(s)$ is

$$s \mapsto -\sum_{x} \psi(1/x + sx^b)\chi(x).$$

So $\Phi^\star \mathcal{H}$ has trace function

$$(r, s) \mapsto \chi(r)\Big(-\sum_{x} \psi(1/x + sr^b x)\chi(x)\Big),$$

which, by the substitution $x \mapsto x/r$ is $(-\sum_x \psi(r/x + sx^b)\chi(x))$, the trace function of $\mathcal{F}(1, b, \chi)$.

The "moreover" statement is proven inside the proof of) [KRLT4, Propostion 2.8]. It may be extracted from a careful reading of the proofs of Theorem 4.1 and of Lemma 4.2, remembering that $a = 1$ there. ☐

**Corollary 4.4.** *The geometric determinant of $\mathcal{F}(1, b, \chi)$ is given as follows.*

(i) *If $b$ is odd, the determinant is $\mathcal{L}_{\chi(r/s)}$.*
(ii) *If $b$ is even, the determinant is $\mathcal{L}_{\chi(r/s)}\mathcal{L}_{\chi_2(s)}$.*

*Proof.* The geometric determinant of any Kloosterman sheaf is the product of its "upstairs" characters. For $\mathcal{K}l(\mathbb{1}, \mathsf{Char}(b, \overline{\chi}))$, this product is $\overline{\chi}$ if $b$ is odd, and is $\chi_2\overline{\chi}$ if $b$ is even. Thus

$$\Phi^\star(\mathcal{L}_\chi(r) \otimes \mathcal{K}l(\mathbb{1}, \mathsf{Char}(b, \overline{\chi}))(s)$$

has determinant $\mathcal{L}^{b+1}_{\chi(r)}\mathcal{L}_{\overline{\chi}(sr^b)}$ if $b$ is odd, and $\mathcal{L}^{b+1}_{\chi(r)}\mathcal{L}_{(\chi_2\overline{\chi})(sr^b)}$ if $b$ is even. ☐

**Corollary 4.5.** *The order of the geometric determinant of $\mathcal{F}(1, b, \chi)$ is equal to the order of $\chi$ if $b$ is odd or if $\chi$ has even order. Otherwise the order of this geometric determinant is twice the order of $\chi$.*

We will now study the finiteness of $G_{\rm geom}$ for $\mathcal{F}(a,b,\chi)$.

**Proposition 4.6.** *Suppose* $\gcd(a,b)=1$. *If* $\chi$ *has order* $d \geq 6$, *then* $\mathcal{F}(a,b,\chi)$ *has infinite* $G_{\rm geom}$. *Moreover, if* $\mathcal{F}(a,b,\chi)$ *has finite* $G_{\rm geom}$ *with* $\chi$ *of order* $3 \leq d \leq 5$, *then we have* $a=b$ *when* $d=4$ *or* 5, *and* $a+b \leq \min(4a,4b)$ *when* $d=3$.

*Proof.* We argue by contradiction. If $\chi$ has order $\geq 6$, and $\mathcal{F}(a,b,\chi)$ has finite $G_{\rm geom}$, then

$$\mathcal{K} := \mathcal{K}l_\psi(\text{all } b^{\rm th} \text{ roots of } \Lambda, \text{all } a^{\rm th} \text{ roots of } \rho)$$

has finite $G_{\rm geom}$, and is primitive by Lemma 4.2. Let $\gamma \in I(0)$ be a generator of $I(0)/P(0)$. Then $\gamma^{ab}$ in the representation $V$ attached to $\mathcal{K}$ is the diagonalizable element with eigenvalues

$$(\chi^{Aa}(\gamma) \text{ repeated } b \text{ times}, \chi^{Bb}(\gamma) \text{ repeated } a \text{ times}).$$

The ratio of these eigenvalues, namely $\chi^{Bb}(\gamma)/\chi^{Aa}(\gamma) = \chi(\gamma)$, is thus a root of unity of order $d \geq 6$. After any complex embedding, a correctly chosen power $\gamma^\nu$ of $\gamma$ has $\chi(\gamma^\nu) = \exp(2\pi i/d)$. Then $\gamma^{ab\nu}|_V$ has all eigenvalues within 60 degrees of either of its two eigenvalues. But in a primitive finite group, any such element is a scalar, by Blichfeldt's theorem [Bl, Theorem 8, p. 96]. But $\gamma^{ab\nu}|_V$ is not a scalar, as it two eigenvalues have ratio $\exp(2\pi i/d)$.

For the "moreover" statement, notice that after multiplication by a $d^{\rm th}$ root of unity (so inside the primitive finite group $\mu_d G_{\rm geom}$), we have the two elements

$$(1 \text{ repeated } b \text{ times}, \exp(2\pi i/d) \text{ repeated } a \text{ times})$$

and

$$(1 \text{ repeated } a \text{ times}, \exp(-2\pi i/d) \text{ repeated } b \text{ times}),$$

whose "drops" are respectively $a$ and $b$. Then $a=b$ for $\chi$ of order 4 or 5, cf. [Zal, 11.2] and [Wa, Theorem 1]. If $\chi$ has order 3, then $a+b \leq 4 \times \text{drop}$, cf. [Wa, §5, p. 606]. ☐

**Lemma 4.7.** *Let* $\gcd(a,b)=1$ *and* $D := a+b \geq 9$. *Suppose that*

$$\mathcal{K} := \mathcal{K}l_\psi\Big(\mathsf{Char}(b,\Lambda) \sqcup \mathsf{Char}(a,\rho)\Big)$$

*have finite* $G_{\rm geom}$, *and, after multiplying by some character* $\gamma$, *all the characters in*

$$\mathsf{Char}(b,\Lambda) \sqcup \mathsf{Char}(a,\rho)$$

*have order dividing some integer* $N$. *Then either* $D \leq N/2$, *or* $2|N$ *and* $D \leq N/2+2$.

*Proof.* Since $\gcd(a,b)=1$ and $a+b>2$, we may assume that $a>b$. By assumption, all characters in $\mathsf{Char}(b,\Lambda) \sqcup \mathsf{Char}(a,\rho)$ belong to $\overline{\gamma}\mathsf{Char}(N)$. The ratio between any two characters in $\overline{\gamma}\mathsf{Char}(N)$ have order dividing $N$. As $a \geq 2$, $\mathsf{Char}(a,\rho)$ contains two characters $\alpha_1, \alpha_2$ with ratio of order exactly $a$, whence $a|N$. The same argument applied to $\mathsf{Char}(b,\Lambda)$ shows that $b|N$ if $b \geq 2$, and this conclusion is automatic if $b=1$. We have shown that $a|N$ and $b|N$, and hence

$$ab|N \tag{4.7.1}$$

since $\gcd(a,b)=1$.

Now, if $a=N$, then $\mathsf{Char}(b,\Lambda) \sqcup \mathsf{Char}(a,\rho)$ contains $a+b>N$ characters, contrary to the assumption that it is contained in $\overline{\gamma}\mathsf{Char}(N)$. Hence $a$ is a proper divisor of $N$ by (4.7.1). If in addition $a \leq N/4$, then $D=a+b<2a \leq N/2$, and so we are done.

It remains to consider the cases where $a=N/2$ and $2|N$, or $a=N/3$. Suppose $a=N/2$ and $2|N$. Then $b \leq 2$ by (4.7.1), and so $D=a+b \leq N/2+2$.

Suppose that $a=N/3$ but $D>N/2$. Then $b \leq 3$ by (4.7.1), and so

$$N/2 < D = a+b \leq N/3+3,$$

i.e. $N<18$, $a<6$, and $D \leq 5+3=8$, again a contradiction. ☐

**Theorem 4.8.** *Let* $\gcd(c,d)=1$ *and* $D := c+d \geq 11$. *Then the sheaf*

$$\mathcal{F} := \mathcal{F}(c,d,\chi)$$

*has finite* $G_{\rm geom}$ *if and only if* $p>2$, $D=p^f-1$ *for some integer* $f\geq 1$, $\{c,d\}=\{D-1,1\}$, *and* $\chi=\chi_2$.

*Proof.* By Theorem 4.1 and Lemma 4.2, $\mathcal{F}$ has finite monodromy if and only if

$$\mathcal{K} := \mathcal{K}l_\psi\Big(\mathsf{Char}(d,\Lambda)\sqcup\mathsf{Char}(c,\rho)\Big)$$

does; moreover, $\mathcal{K}$ is primitive. Now, the "if" part of the statement follows from the case $k=2$ of Theorem 3.5(ii).

To prove the "only if" part of the statement, we apply Theorem 3.5 to $\mathcal{K}$ to see that, up to tensoring with a Kummer sheaf $\mathcal{L}_\gamma$, which preserves finiteness of $G_{\rm geom}$, $\mathcal{K}$ is one of the local systems described in Theorem 3.5. We will use this description to see that $\mathsf{Char}(d,\Lambda)\sqcup\mathsf{Char}(c,\rho)$ is contained in $\overline{\gamma}\mathsf{Char}(N)$ for a suitable integer $N$, and then apply Lemma 4.7 to deduce the desired conclusion.

In the cases (i), (iii), and (iv) of Theorem 3.5, we have $N=D+1\geq 10$, so $D=N-1>N/2+2$, contrary to Lemma 4.7.

Suppose we are in the case of 3.5(vi). Then $N=(q^a+1)(q^b+1)/2$, and so

$$4D-2N = 2(q^{a+b}-1)-(q^a+1)(q^b+1) = (q^a-1)(q^b-1)-4 \geq 8\cdot 2-4 = 12$$

(since $q\geq 3$ and $2|ab$), and so $D\geq N/2+3$, contrary to Lemma 4.7.

Suppose we are in the case of 3.5(v). If $a=b=1$, then $N\geq D+1\geq 10$, so $D\geq N-2>N/2+2$, contrary to Lemma 4.7. We may now assume that $a\geq 3$ and $a>b$; furthermore, $N=(q^a+1)(q^b+1)/(q+1)$ and $D=(q^{a+b}-1)/(q+1)$. If in addition $q\geq 3$, then

$$2D(q+1)-N(q+1) = 2(q^{a+b}-1)-(q^a+1)(q^b+1) = (q^a-1)(q^b-1)-4 \geq 2(q^3-1)-4 > 6(q+1),$$

and so $D>N/2+3$, again contradicting Lemma 4.7. If $q=2$, then $2\nmid N$, and

$$2D(q+1)-N(q+1) = 2(q^{a+b}-1)-(q^a+1)(q^b+1) = (q^a-1)(q^b-1)-4 \geq (q^3-1)-4 > 0,$$

and so $D>N/2$, also contradicting Lemma 4.7.

Suppose we are in the case of 3.5(vii), (viii). Then $D=12$, and $N\in\{16,18,21\}$, all contradicting Lemma 4.7.

Thus we are left with the case (ii) of Theorem 3.5; in particular, $D=p^f-1$. Then $k>1$ is coprime to $p$, and without loss we may assume that $k<p^f/2$, $c>d$; furthermore,

$$\mathsf{Char}(c,\rho)\sqcup\mathsf{Char}(d,\Lambda) = \overline{\gamma}\big(\mathsf{Char}(p^f-k)\cup\mathsf{Char}(k)\big) \tag{4.8.1}$$

for some character $\gamma$. Note that $2k\leq p^f-1=D<2c$, so $c>k$ and $c\geq 5$. The equality $c>k$ implies that some character $\alpha\in\mathsf{Char}(c,\rho)$ must belong to $\overline{\gamma}\mathsf{Char}(p^f-k)$. Next we show that

$$c|(p^f-k). \tag{4.8.2}$$

Assume the contrary: $c\nmid(p^f-k)$. Fix a character $\beta$ of order $c$, so that all the $c$ characters

$$\alpha,\beta\alpha,\beta^2\alpha,\ldots,\beta^{c-1}\alpha$$

belong to $\mathsf{Char}(c,\rho)$. By (4.8.1), we can color each of them in red if it belongs to $\overline{\gamma}\mathsf{Char}(p^f-k)$ and blue if it belongs to $\overline{\gamma}\mathsf{Char}(k)$. Since $\beta$ has order $c>1$ which is coprime to $p^f-k$, no two consecutive characters in this sequence can be both in red. Next, as $c>k$, no two consecutive characters in the sequence can be both in blue. We have shown that the colors of the $c$ characters in the sequence are alternating, starting from red. Thus the two character $\beta^3\alpha$ and $\beta\alpha$ are both in blue, and so their ratio $\beta^2$ belongs to $\mathsf{Char}(k)$. If in addition $2 \nmid c$, then $\beta^2$ has order $c > k$, a contradiction. So $2|c \geq 6$, and $k$ is divisible by the order $c/2$ of $\beta^2$. Now, the two characters $\beta^2\alpha$ and $\alpha$ are both in red, and so their ratio $\beta^2$ belongs to $\mathsf{Char}(p^f - k)$, whence $p^f - k$ is also divisible by the order $c/2$ of $\beta^2$. It follows that $\gcd(k, p^f - k) > 1$, a contradiction.

Thus we have proved (4.8.2). In particular, $\mathsf{Char}(c, \rho) \subseteq \overline{\gamma}\mathsf{Char}(p^f - k)$. Assume in addition that $c < p^f - k$. Since $p^f - k > d$, we can find a character $\alpha_1$ in $\overline{\gamma}\mathsf{Char}(p^f - k)$ which belongs to $\mathsf{Char}(c, \rho)$, and fix a character $\eta$ of order $p^f - k$, so that all the $p^f - k$ characters

$$\alpha_1, \eta\alpha_1, \eta^2\alpha_1, \ldots, \eta^{p^f-k-1}\alpha_1$$

belong to $\overline{\gamma}\mathsf{Char}(p^f - k)$. By (4.8.1), we can again color each of them in red if it belongs to $\mathsf{Char}(c, \rho)$ and blue if it belongs to $\mathsf{Char}(d, \Lambda)$. Since $\eta$ has order $p^f - k > \max(c, d)$, no two consecutive characters in this sequence can be in the same color. We have shown that the colors of the characters in the sequence are alternating, starting from red. Thus the two character $\eta^3\alpha_1$ and $\eta\alpha_1$ are both in blue, and so their ratio $\eta^2$ belongs to $\mathsf{Char}(d)$. If in addition $2 \nmid (p^f - k)$, then $\eta^2$ has order $p^f - k > c > d$, a contradiction. So $2|(p^f - k) \geq 6$, and $d$ is divisible by the order $(p^f - k)/2$ of $\eta^2$. Now, the two characters $\eta^2\alpha_1$ and $\alpha_1$ are both in red, and so their ratio $\eta^2$ belongs to $\mathsf{Char}(c)$, whence $c$ is also divisible by the order $(p^f - k)/2$ of $\eta^2$. It follows that $\gcd(c, d) > 1$, a contradiction.

We have therefore shown that $c = p^f - k$, and so $\mathsf{Char}(c, \rho) = \overline{\gamma}\mathsf{Char}(p^f - k)$. It follows that $d = k - 1$ and

$$\mathsf{Char}(d, \Lambda) = \overline{\gamma}\big(\mathsf{Char}(k) \smallsetminus \{\mathbb{1}\}\big).$$

If furthermore $d > 1$, then we can find two characters in $\mathsf{Char}(d, \Lambda)$ with ratio of order $d$, and this ratio belongs to $\mathsf{Char}(d+1)$, so $d|(d+1)$, a contradiction. Hence $d = 1$, $k = 2$, and $p > 2$. Recalling the relationship between $\mathcal{F}$ and $\mathcal{K}$, we have that $dB - cA = 1$ and

$$(\delta_1/\delta_2)^{cd} = \rho^d/\Lambda^c = \chi^{dB-cA} = \chi$$

whenever $\delta_1 \in \mathsf{Char}(c, \rho)$ and $\delta_2 \in \mathsf{Char}(d, \Lambda)$. As mentioned above, $\delta_2 = \overline{\gamma}\chi_2$, and $\delta_1 \in \overline{\gamma}\mathsf{Char}(c)$. Since $c = p^f - 2$ is odd, it follows that

$$\chi = (\delta_1/\delta_2)^{cd} = \chi_2^{-c} = \chi_2,$$

as desired. □

Now we will remove the coprimality condition in Theorem 4.8:

**Theorem 4.9.** *Let $a, b \in \mathbb{Z}_{\geq 1}$. If the sheaf*

$$\mathcal{F} := \mathcal{F}(a, b, \chi)$$

*has finite $G_{\mathrm{geom}}$, then either $\gcd(a, b) = 1$, or $a = b = 2$ and $\chi = \chi_2$. If $D \geq 11$ in addition, then $\mathcal{F}$ has finite $G_{\mathrm{geom}}$ if and only if $p > 2$, $D = p^f - 1$ for some integer $f \geq 1$, $\{a, b\} = \{D - 1, 1\}$, and $\chi = \chi_2$.*

*Proof.* For the first statement, assume by way of contradiction that

$$d := \gcd(a, b) > 1.$$

Setting $a_0 := a/d$ and $b_0 := b/d$, by (2.1.1) we have

$$\mathcal{F} \cong \bigoplus_{\rho^d = \chi} \mathcal{F}(a_0, b_0, \rho).$$

Since $\mathcal{F}$ has finite $G_{\mathrm{geom}}$, the same holds for each summand $\mathcal{F}(a_0, b_0, \rho)$. Choosing $\rho$ with largest possible order $de$, where $e := \mathsf{o}(\chi)$, by Proposition 4.6 we get

$$de \leq 5.$$

Moreover, $e > 1$ by Lemma 2.2. As $d \geq 2$, we actually have $d = e = 2$ and $\chi = \chi_2$. Again choosing $\rho$ of largest possible order 4 and applying Proposition 4.6, we have $a_0 = b_0$, and so $a_0 = b_0 = 1$ by coprimality. Thus $a = b = 2$ and $D = 4$.

The second statement now follows from the first statement and Theorem 4.8. $\square$

Now we can handle the universal Laurent sheaves of rank $\geq 11$:

**Theorem 4.10.** *Let $k, l \in \mathbb{Z}_{\geq 1}$, $D := \{b_1 > b_2 > \ldots > b_k \geq 1\}$, and $E := \{a_1 > a_2 > \ldots > a_l \geq 1\}$. Suppose that $a_1 + b_1 \geq 11$. Then the sheaf*

$$\mathcal{F} := \mathcal{F}(D, E, \chi)$$

*has finite $G_{\mathrm{geom}}$ if and only if $p > 2$, $k = l = 1$, $a_1 + b_1 = p^e - 1$ for some integer $e \geq 1$, $\{a_1, b_1\} = \{p^e - 2, 1\}$, and $\chi = \chi_2$.*

*Proof.* By Theorem 4.8, it suffices to prove the "only if" part of the statement. The finiteness of $G_{\mathrm{geom}}$ for $\mathcal{F}$ implies $\chi \neq \mathbb{1}$ by Lemma 2.2, and we can now freely use the semicontinuity Theorem 2.5. In particular, Theorem 4.9 applied to $\mathcal{F}(a_1, b_1, \chi)$ implies that, up to the multiplicative inversion $x \mapsto x^{-1}$,

$$p > 2,\ \chi = \chi_2,\ k = 1,\ (a_1, b_1) = (p^e - 2, 1) \text{ for some } e \in \mathbb{Z}_{\geq 1}, \text{ and } a_1 \geq 10.$$

If $l = 1$, then we are done. It remains to consider the following two cases, where we set

$$d := \gcd(a_1, \ldots, a_l).$$

We will apply results in [KT5] to the specialization $z_{a_1} = 1$ of the local system $\mathcal{F}(\varnothing, E, \chi)$.

(a) $l \geq 3$. First suppose that $d \geq 2$. Since $\chi = \chi_2$, we are in the case (iii) of [KT5, Theorem 11.2.4]. Hence

$$p^e - 2 = a_1 = d\frac{q^n + 1}{q + 1}$$

for some $q = p^f$ with $f \in \mathbb{Z}_{\geq 1}$ and odd integer $n \geq 3$; moreover, $d|(q + 1)$ and $d$ is odd. The latter condition implies that $q > 3$. In this case we also have $a_1 \geq q^2 - q + 1 > q - 2$, so $p^e > q$ and thus $a_1 \equiv -2 \pmod q$. As $(q^n + 1)/(q + 1) \equiv 1 \pmod q$, it follows that $d \equiv -2 \pmod q$. Next, $d|(q + 1)$ and $q > 3$ imply that $d < 2q - 2$, whence $d = q - 2$; furthermore $(q - 2)|((q + 1) - (q - 2)) = 3$, i.e. $q = 5$, $d = 3$. Now

$$5^e = a_1 + 2 = 3(5^{n-1} - 5^{n-2} + \ldots - 5 + 1) + 2$$

with $e \geq 2$ and $n \geq 3$, which is absurd since $a_1 + 2 \equiv -10 \pmod{25}$.

We have shown that $d = 1$. Now we can apply [KT5, Theorem 11.2.3]. Relaxing the condition $a_1 \geq 10$ to $a_1 \geq 7$ (for later use), we arrive at the following possibilities.

- $2p^e - 4 = 2a_1 = p^{nf} + 1$ for some $n, f \in \mathbb{Z}_{\geq 1}$. In this case, $\min(p^e, p^{nf})$ divides 5, and hence $a_1 \leq 3$, a contradiction.
- $p^e = a_1 + 2 = (q^n + 1)/(q + 1) + 2 \equiv -q + 3 \pmod{p^{f+1}}$ with $q = p^f$ and $2 \nmid n \geq 3$. In this case we have $e \geq f + 1$, $q = 3$. Now, $a_1 + 2 \equiv 3^2 \pmod{3^3}$ and $a_1 + 2 = 3^e \geq 3^2$, so $e = 2$. Thus $E = \{7, 1\}$, contrary to the assumption $l \geq 3$.
- $p = 3$, $a_1 = 7$, and $E = \{7, 5, 1\}$.

In particular, we have ruled out the case $l \geq 2$ when $a_1 \geq 10$.

(b) $l = 2$. First suppose that $d \geq 2$. Since $\chi = \chi_2$, we are in the case (b) of [KT5, Theorem 10.3.14]. Hence

$$p^e - 2 = a_1 = d\frac{q^n+1}{q+1}$$

for some $q = p^f$ with $f \in \mathbb{Z}_{\geq 1}$ and odd integer $n \geq 3$; moreover, $d|(q+1)$ and $d$ is odd. This possibility is ruled out by the arguments in (a).

Therefore $d = 1$. Assume in addition that $a_2 > 1$. Then we can apply [KT5, Theorem 10.3.13] when $a_2 > 1$ and [KT5, Theorem 10.2.6] when $a_2 = 1$. Relaxing the condition $a_1 \geq 10$ to $a_1 \geq 7$ (for later use), we arrive at the following possibilities.

- $2p^e - 4 = 2a_1 = p^{nf} + 1$ for some $n, f \in \mathbb{Z}_{\geq 1}$. As shown in (a), in this case we have $a_1 \leq 3$, a contradiction.
- $p^e = a_1 + 2 = (q^n+1)/(q+1) + 2 \equiv -q + 3 \pmod{p^{f+1}}$ with $q = p^f$ and $2 \nmid n \geq 3$. As shown in (a), this leads to $p = 3$ and $E = \{7, 1\}$.
- $p^e = 2p^f + 1$ for some $f \in \mathbb{Z}_{\geq 1}$, which is impossible.
- $p = 3$, $a_1 = 7$, and $E = \{7, 5\}$.

Thus we have also ruled out the case $l = 2$ when $a_1 \geq 10$. □

## 5. Finiteness theorems: Preliminaries on the $V$-test

We denote by $q$ the cardinality of $k$, and by $k_r$ the unique extension of $k$ of degree $r$ in $\overline{\mathbb{F}_q}$. $V : (\mathbb{Q}/\mathbb{Z})_{\text{prime to } p} \to [0, 1)$ will denote Kubert's $V$ function for the prime $p$ (cf. [KRLT1]).

We generalize here the results of [KRLT4, §4] to local systems twisted by multiplicative characters of arbitrary order. Let $\chi_m$ be a multiplicative character of $k$ of order $m|(q-1)$, and denote by $\chi_{m,L}$ its pull-back to the finite extension $L$ of $k$.

**Theorem 5.1.** *Let $d_1 > d_2 > \ldots > d_n > 0$ be prime to $p$ integers, $\mathbf{d} = (d_1, \ldots, d_n)$ and $\mathcal{F}$ the local system on $\mathbb{G}_{m,k}^{n+1}$ whose trace function is given by*

$$F(L; s, t_1, \ldots, t_n) = -\frac{1}{\sqrt{|L|}} \sum_{x \in L^\times} \psi_L(s/x + t_1 x^{d_1} + \cdots + t_n x^{d_n}) \chi_{m,L}(x).$$

*Then $\mathcal{F}$ has finite (geometric and arithmetic) monodromy group if and only if*

$$V\left(d_1 x_1 + \cdots + d_n x_n + \frac{a}{m}\right) + V(x_1) + \cdots + V(x_n) \geq \frac{1}{2}$$

*for every $(x_1, \ldots, x_n) \in (\mathbb{Q}/\mathbb{Z})^n_{\text{prime to } p}$ and every $a \in \{1, \ldots, m-1\}$ prime to $m$.*

*Proof.* By [KRLT1, Proposition 2.1], we need to show that $F(L; s, t_1, \ldots, t_n)$ is an algebraic integer or, equivalently, that

$$\mathrm{ord}_{\mathfrak{q}}\left(\sum_{x \in k_r^\times} \psi_{k_r}(s/x + t_1 x^{d_1} + \cdots + t_n x^{d_n}) \chi_{m,k_r}(x)\right) \geq \frac{r}{2}$$

for every $r \geq 1$, every prime $\mathfrak{q}$ of $\mathbb{Q}(\mu_p, \mu_m)$ over $p$ and every $(s, t_1, \ldots, t_n) \in (k_r^\times)^{n+1}$, where $\mathrm{ord}_{\mathfrak{q}}$ is normalized so that $\mathrm{ord}_{\mathfrak{q}}(q) = 1$. Taking Mellin transform on $\mathbb{G}_m^{n+1}$ and computing as in [KRLT4, Theorem 4.1], this is equivalent to

$$\sum_{s, t_1, \ldots, t_n \in k_r^\times} \eta(s)\xi_1(t_1)\cdots\xi_n(t_n) \sum_{x \in k_r^\times} \psi_{k_r}(s/x + t_1 x^{d_1} + \cdots + t_n x^{d_n}) \chi_{m,k_r}(x) =$$

$$= \begin{cases} 0 & \text{if } \chi_{m,k_r}\eta\bar{\xi}_1^{d_1}\cdots\bar{\xi}_n^{d_n} \neq \mathbf{1} \\ (q^r-1)G_r(\eta)G_r(\xi_1)\cdots G_r(\xi_n) & \text{if } \chi_{m,k_r}\eta\bar{\xi}_1^{d_1}\cdots\bar{\xi}_n^{d_n} = \mathbf{1} \end{cases}$$

has $\mathrm{ord}_{\mathfrak{q}} \geq \frac{r}{2}$ for every $\eta, \xi_1, \ldots, \xi_n \in \widehat{k_r^\times}$ and every $\mathfrak{q}$, where $G_r(\chi)$ denotes the Gauss sum associated to the multiplicative character $\chi$ on $k_r$. This reduces to

$$\mathrm{ord}_{\mathfrak{q}}(G_r(\chi_{m,k_r}\xi_1^{d_1}\cdots\xi_n^{d_n})G_r(\xi_1)\cdots G_r(\xi_n)) \geq \frac{1}{2}$$

for every $\mathfrak{q}$ over $p$ and every $\xi_1, \ldots, \xi_n \in \widehat{k_r^\times}$. Now, since all such $\mathfrak{q}$ are Galois conjugated by $\mathrm{Gal}(\mathbb{Q}(\mu_{\mathrm{p}}, \mu_{\mathrm{m}})/\mathbb{Q}(\mu_{\mathrm{p}}))$, and the Galois orbit of $\chi_m$ consists of all characters of order $m$ (that is, all $\chi_m^a$ for $a$ prime to $m$), this is equivalent to

$$\mathrm{ord}_{\mathfrak{q}}(G_r(\chi_{m,k_r}^a\xi_1^{d_1}\cdots\xi_n^{d_n})G_r(\xi_1)\cdots G_r(\xi_n)) \geq \frac{1}{2}$$

for a fixed $\mathfrak{q}$, every $a \in \{1, \ldots, m-1\}$ prime to $p$ and every $\xi_1, \ldots, \xi_n \in \widehat{k_r^\times}$. By Stickelberger, this is equivalent to the given condition. □

If $d_1x_1 + \cdots + d_nx_n + \frac{a}{m} \neq 0$, using that $V(y) + V(-y) = 1$ for $y \neq 0$, we can rewrite the condition as

$$V\left(d_1x_1 + \cdots + d_nx_n + \frac{a}{m}\right) \leq V(-x_1) + \cdots + V(-x_n) + \frac{1}{2},$$

which trivially holds for $d_1x_1 + \cdots + d_nx_n + \frac{a}{m} = 0$. So we have

**Corollary 5.2.** *The local system $\mathcal{F}$ has finite monodromy if and only if the following two conditions hold for every $a \in \{1, \ldots, m-1\}$ prime to $m$:*

(i) $V\left(d_1x_1 + \cdots + d_nx_n + \frac{a}{m}\right) \leq V(-x_1) + \cdots + V(-x_n) + \frac{1}{2}$ *for every* $(x_1, \ldots, x_n) \in (\mathbb{Q}/\mathbb{Z})^n_{\text{prime to } p}$.
(ii) $V(x_1) + V(x_2) + \cdots + V(x_n) \geq \frac{1}{2}$ *for every* $(x_1, \ldots, x_n) \in (\mathbb{Q}/\mathbb{Z})^n_{\text{prime to } p}$ *with* $\sum_{i=1}^n d_ix_i = \frac{a}{m}$.

Note that the second condition is the criterion for the local system on $\mathbb{G}_m^n$ with trace function

$$F(L; t_1, \ldots, t_n) = -\frac{1}{\sqrt{|L|}} \sum_{x \in L^\times} \psi_L(t_1x^{d_1} + \cdots + t_nx^{d_n})\chi_{m,L}(x)$$

to have finite monodromy. In terms of the sum-of-digits function $[-]_{p,r,-}$ defined in [KRL, Appendix], the first condition becomes

$$\left[d_1x_1 + \cdots + d_nx_n + \frac{a(p^r-1)}{m}\right]_{p,r,-} \leq [p^r - 1 - x_1]_{p,r,-} + \cdots + [p^r - 1 - x_n]_{p,r,-} + \frac{r(p-1)}{2}$$

for every $r \geq 1$ multiple of $r_0$ (defined as the multiplicative order of $p$ modulo $m$), every $a \in \{1, \ldots, m-1\}$ prime to $m$ and every $0 < x_1, \ldots, x_n < p^r$ such that $p^r - 1$ does not divide $d_1x_1 + \cdots + d_nx_n + \frac{p^r-1}{2}$. An argument similar to [KRLT1, Theorem 2.12] then shows

**Proposition 5.3.** *Suppose that there exists some real $A \geq 0$ such that*

$$\left[d_1x_1 + \cdots + d_nx_n + \frac{a(p^r-1)}{m}\right]_p \leq [p^r - 1 - x_1]_p + \cdots + [p^r - 1 - x_n]_p + \frac{r(p-1)}{2} + A$$

*for every $r \geq 1$ multiple of $r_0$, every $a \in \{1, \ldots, m-1\}$ prime to $m$ and every $0 \leq x_1, \ldots, x_n \leq p^r - 1$, where $[x]_p$ denotes the sum of the $p$-adic digits of $x$. Then condition* (i) *in Corollary 5.2 holds.*

Recall from [KRLT4] the definition of good termination. Fix $a \in \{1,\dots,m-1\}$ prime to $m$ and, for $r \geq 1$ multiple of $r_0$ and an $n$-tuple $(x_1,\dots,x_n)$ with $0 \leq x_i \leq p^r-1$, let

$$C_a(r;x_1,\dots,x_n) = \left[\sum_{i=1}^n d_i x_i + \frac{a(p^r-1)}{m}\right]_p - \sum_{i=1}^n [p^r-1-x_i]_p - \frac{r(p-1)}{2}.$$

For $s \geq 1$ multiple of $r_0$ we say that the $n$-tuple $(z_1,\dots,z_n)$ with $0 \leq z_1,\dots,z_n \leq p^s-1$ is *s-good* if one of these conditions hold:

a) $C_a(s;z_1,\dots,z_n) \leq 0$.
b) There exists an $s' < s$ multiple of $r_0$ and an $n$-tuple $(z'_1,\dots,z'_n)$ with $0 \leq z'_i \leq p^{s'}-1$ such that $C_a(s';z'_1,\dots,z'_n) \geq C_a(s;z_1,\dots,z_n)$ and for every $j > 0$ the $(s+j)$-th digit in the $p$-adic expansion of $\sum_{i=1}^n d_i z_i + \frac{a(p^s-1)}{m}$ is greater than or equal to the $(s'+j)$-th digit in the $p$-adic expansion of $\sum_{i=1}^n d_i z'_i + \frac{a(p^{s'}-1)}{m}$ (counting the digits from right to left).

We say that the $n$ tuple $(x_1,\dots,x_n)$ with $0 \leq x_1,\dots,x_n \leq p^r-1$ has *good termination* if, for some $1 \leq s < r$ multiple of $r_0$, the $n$-tuple $(z_1,\dots,z_n)$ whose $i$-th coordinate is the number formed by the last $s$ $p$-adic digits of $x_i$ (i.e. the remainder of the division of $x_i$ by $p^s$) is $s$-good. Exactly as in [KRLT4, Proposition 4.4] one can show

**Proposition 5.4.** *Suppose that there exists some $r_1 \geq 1$ multiple of $r_0$ such that all $n$-tuples $(x_1,\dots,x_n)$ with $0 \leq x_1,\dots,x_n \leq p^{r_1}-1$ have good termination. Then the hypothesis of Proposition 5.3 holds for the chosen $a$.*

## 6. Finiteness theorems for some small-rank local systems

**Theorem 6.1.** *The local system on $\mathbb{G}^2_{m,\mathbb{F}_{16}}$ whose trace function is given by*

$$(\mathbb{F}_{16^r};s,t) \mapsto -\frac{1}{16^{r/2}} \sum_{x \in \mathbb{F}^\times_{16^r}} \psi_{\mathbb{F}_{16^r}}(s/x+tx)\chi_{5,\mathbb{F}_{16^r}}(x)$$

*has finite monodromy.*

*Proof.* By Corollary 5.2, we need to show

(i) $V\left(x+\frac{a}{5}\right) \leq V(-x)+\frac{1}{2}$ for every $x \in (\mathbb{Q}/\mathbb{Z})_{\text{prime to }2}$ and every $a = 1,2,3,4$.
(ii) $V(\frac{a}{5}) \geq \frac{1}{2}$ for every $a = 1,2,3,4$.

For condition (ii) one easily checks that $V(\frac{a}{5}) = \frac{1}{2}$ for $i = 1,2,3,4$. For the first condition, by the change of variable $x \mapsto 2x$ it suffices to prove it for $a = 1$, since $V(2x) = V(x+1) = V(x)$ and 2 is a primitive root modulo 5. A computer check shows that all $x$ with $0 \leq x \leq 2^4-1$ already have $C(4,x) \leq 0$, so they are 4-good. This finishes the proof. □

**Theorem 6.2.** *The local system on $\mathbb{G}^2_{m,\mathbb{F}_{81}}$ whose trace function is given by*

$$(\mathbb{F}_{81^r};s,t) \mapsto -\frac{1}{81^{r/2}} \sum_{x \in \mathbb{F}^\times_{81^r}} \psi_{\mathbb{F}_{81^r}}(s/x+tx)\chi_{5,\mathbb{F}_{81^r}}(x)$$

*has finite monodromy.*

*Proof.* By Corollary 5.2, we need to show

(i) $V\left(x+\frac{a}{5}\right) \leq V(-x)+\frac{1}{2}$ for every $x \in (\mathbb{Q}/\mathbb{Z})_{\text{prime to }3}$ and every $a = 1,2,3,4$.
(ii) $V(\frac{a}{5}) \geq \frac{1}{2}$ for every $a = 1,2,3,4$.

For condition (ii) one easily checks that $V(\frac{a}{5}) = \frac{1}{2}$ for $i = 1, 2, 3, 4$. For the first condition, by the change of variable $x \mapsto 3x$ it suffices to prove it for $a = 1$, since $V(3x) = V(x + 1) = V(x)$ and 3 is a primitive root modulo 5. A computer check shows that all $x$ with $0 \le x \le 3^4 - 1$ already have $C(4, x) \le 0$, so they are 4-good. This finishes the proof. □

**Theorem 6.3.** *The local system on $\mathbb{G}^2_{m,\mathbb{F}_9}$ whose trace function is given by*

$$(\mathbb{F}_{9^r}; s, t) \mapsto -\frac{1}{9^{r/2}} \sum_{x \in \mathbb{F}_{9^r}^\times} \psi_{\mathbb{F}_{9^r}}(s/x + tx)\chi_{4,\mathbb{F}_{9^r}}(x)$$

*has finite monodromy.*

*Proof.* By Corollary 5.2, we need to show

(i) $V\left(x \pm \frac{1}{4}\right) \le V(-x) + \frac{1}{2}$ for every $x \in (\mathbb{Q}/\mathbb{Z})_{\text{prime to }3}$.

(ii) $V(\pm\frac{1}{4}) \ge \frac{1}{2}$.

For condition (ii) one easily checks that $V(\pm\frac{1}{4}) = \frac{1}{2}$. For the first condition, by the change of variable $x \mapsto 3x$ it suffices to prove the '+' case, since $V(3x) = V(x+1) = V(x)$. And for this case a quick check shows that all $x$ with $0 \le x \le 3^2 - 1$ already have $C(2, x) \le 0$, so they are 2-good. This finishes the proof. □

Next we record three elementary results.

**Lemma 6.4.** *Given a prime $p$ and multiplicative character $\chi$ of order prime to $p$, the local system $\mathcal{F}(1, 1, \chi)$ as defined in* (4.0.1) *has finite $G_{\text{geom}}$ if and only if the Kloosterman sheaf $\mathcal{K}l(\mathbb{1}, \chi)$ has finite $G_{\text{geom}}$.*

*Proof.* The trace function of $\mathcal{F}(1, 1, \chi)$ is the sum

$$-\sum_{x \ne 0} \psi(s/x + tx)\chi(x).$$

After the change of variable $x \mapsto sx$, it becomes

$$-\sum_{x \ne 0} \psi(1/x + stx)\chi(sx) = \chi(s)\mathcal{K}l(\mathbb{1}, \chi)(st).$$

Let $\chi$ have order $N$. After the partial Kummer pullback $(s, t) \mapsto (s^N, t)$, the $\chi(s)$ factor disappears, and we are left with $\mathcal{K}l(\mathbb{1}, \chi)(s^N t)$, which, after the automorphism $(s, t) \mapsto (s, t/s^N)$ is just $\mathcal{K}l(\mathbb{1}, \chi)(t)$. □

**Corollary 6.5.** *In characteristic $p = 3$, $\mathcal{K}l(\mathbb{1}, \chi_4)$ has finite $G_{\text{geom}}$.*

*Proof.* Immediate from Theorem 6.3. □

Note that Corollary 6.5 is a $q = 3$ analogue of Theorem 3.5(v), where we take $a = b = A = B = 1$, $\sigma = \chi_4$, and $(\alpha, \beta) = (2, 1)$.

**Corollary 6.6.** *In both characteristics $p = 2$ and $p = 5$, the local system $\mathcal{F}(1, 1, \chi_3)$ as defined in* (4.0.1) *has finite $G_{\text{geom}}$.*

*Proof.* It is equivalent to prove that $\mathcal{K}l(\mathbb{1}, \chi_3)$ has finite $G_{\text{geom}}$ in the named characteristics. Tensoring with $\mathcal{L}_{\chi_3}$, it is equivalent to prove that $\mathcal{K}l(\chi_3, \chi_3^2)$ has finite $G_{\text{geom}}$. The $[3]^\star$ Kummer pullback of $\mathcal{K}l(\chi_3, \chi_3^2)$ is a multiplicative translate of the local system whose trace function is

$$t \mapsto -\sum_x \psi(tx + x^3),$$

cf. [Ka3, 9.2.3] or [KRLT2, Lemma 1.2]. For $p = 2$, we have $3 = p + 1$, and for $p = 5$ we have $3 = (p+1)/2$, so in both cases the finiteness is a special case of [KT5, 10.2.6]. ☐

**Theorem 6.7.** *The local system on $\mathbb{G}^2_{m,\mathbb{F}_7}$ whose trace function is given by*

$$(\mathbb{F}_{7^r}; s, t) \mapsto -\frac{1}{7^{r/2}} \sum_{x \in \mathbb{F}^\times_{7^r}} \psi_{\mathbb{F}_{7^r}}(s/x + tx^2)\chi_{2,\mathbb{F}_{7^r}}(x)$$

*has finite monodromy.*

*Proof.* By Corollary 5.2, we need to show

(i) $V\left(2x + \frac{1}{2}\right) \le V(-x) + \frac{1}{2}$ for every $x \in (\mathbb{Q}/\mathbb{Z})_{\text{prime to } 7}$.

(ii) $V(x) \ge \frac{1}{2}$ for every $x \in (\mathbb{Q}/\mathbb{Z})_{\text{prime to } 7}$ such that $2x = \frac{1}{2}$.

For condition (ii) one easily checks that $V(\frac{i}{4}) = \frac{1}{2}$ for $i = 1, 2, 3$. For the first condition, following Proposition 5.4, we check by a computer search that all $x$ with $0 \le x \le 7^3 - 1$ have good termination.

For each $s = 1, 2$, the following tables show the list of all $z$ with $0 \le z \le 7^s - 1$ such that

(a) $C(s; z) > 0$ and

(b) do not have good termination (i.e. all their last-digits truncations appear in the previous tables marked with •).

If the condition (b) for being $s$-good can be applied to them, we show the possible values of $s', z'$ on the table, otherwise we mark it with • and move it on to the next $s$. All values of $z$ and $z'$ are shown as their 7-adic expansion. The columns $D$ and $D'$ show the result of removing the last $s$ (respectively $s'$) digits of $2z + \frac{7^s-1}{2}$ (resp. of $2z' + \frac{7^{s'}-1}{2}$). Each digit of the number in column $D$ must be greater than or equal to the corresponding digit of the number in column $D'$.

| $s=1$ | | | |
|---|---|---|---|
| $z$ | $C(s;z)$ | $D$ | |
| 5 | 3 | 1 | • |

| $s=2$ | | | | | | |
|---|---|---|---|---|---|---|
| $z$ | $C(s;z)$ | $D$ | $s'$ | $z'$ | $C(s';z')$ | $D'$ |
| 45 | 3 | 1 | 1 | 5 | 3 | 1 |
| 65 | 3 | 2 | 1 | 5 | 3 | 1 |

In the last table (for $s = 2$) there are no remaining cases left with •, so this finishes the proof. ☐

**Theorem 6.8.** *The local system on $\mathbb{G}^3_{m,\mathbb{F}_5}$ whose trace function is given by*

$$(\mathbb{F}_{5^r}; s, t, u) \mapsto -\frac{1}{5^{r/2}} \sum_{x \in \mathbb{F}^\times_{5^r}} \psi_{\mathbb{F}_{5^r}}(s/x + tx + ux^3)\chi_{2,\mathbb{F}_{5^r}}(x)$$

*has finite monodromy.*

*Proof.* By Corollary 5.2, we need to show

(i) $V\left(x_1 + 3x_2 + \frac{1}{2}\right) \le V(-x_1) + V(-x_2) + \frac{1}{2}$ for every $(x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^2_{\text{prime to } 5}$.

(ii) $V(x) + V(-3x + \frac{1}{2}) \ge \frac{1}{2}$ for every $x \in (\mathbb{Q}/\mathbb{Z})_{\text{prime to } 5}$.

Condition (ii) is the criterion for finite monodromy of the local system with trace function

$$F(\mathbb{F}_{5^r}; t, u) \mapsto -\frac{1}{5^{r/2}} \sum_{x \in \mathbb{F}^\times_{5^r}} \psi_{\mathbb{F}_{5^r}}(tx + ux^3)\chi_{2,\mathbb{F}_{5^r}}(x)$$

which holds by [KT5, Theorem 10.2.6(i)] since $3 = \frac{5+1}{2}$. For the first condition, following Proposition 5.4, we check by a computer search that all pairs $(x_1, x_2)$ with $0 \leq x_1, x_2 \leq 5^3 - 1$ have good termination.

For each $s = 1, 2$, the following tables show the list of all pairs $(z_1, z_2)$ with $0 \leq z_1, z_2 \leq 5^s - 1$ such that

(a) $C(s; z_1, z_2) > 0$ and

(b) do not have good termination (i.e. all their last-digits truncations appear in the previous tables marked with $\bullet$).

If the condition (b) for being $s$-good can be applied to them, we show the possible values of $s', z_1', z_2'$ on the table, otherwise we mark it with $\bullet$ and move it on to the next $s$. All values of $z_i$ and $z_i'$ are shown as their 5-adic expansion. The columns $D$ and $D'$ show the result of removing the last $s$ (respectively $s'$) digits of $z_1 + 3z_2 + \frac{5^s-1}{2}$ (resp. of $z_1' + 3z_2' + \frac{5^{s'}-1}{2}$). Each digit of the number in column $D$ must be greater than or equal to the corresponding digit of the number in column $D'$.

| $s = 1$ | | | | |
|---|---|---|---|---|
| $z_1$ | $z_2$ | $C(s; z_1, z_2)$ | $D$ | |
| 3 | 3 | 2 | 2 | $\bullet$ |
| 3 | 4 | 2 | 3 | $\bullet$ |
| 4 | 4 | 4 | 3 | $\bullet$ |

| $s = 2$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $z_1$ | $z_2$ | $C(s; z_1, z_2)$ | $D$ | $s'$ | $z_1'$ | $z_2'$ | $C(s'; z_1', z_2')$ | $D'$ |
| 23 | 43 | 2 | 3 | 1 | 3 | 3 | 2 | 2 |
| 33 | 43 | 4 | 3 | 1 | 4 | 4 | 4 | 3 |
| 43 | 23 | 2 | 2 | 1 | 3 | 3 | 2 | 2 |
| 43 | 33 | 2 | 3 | 1 | 3 | 3 | 2 | 2 |
| 43 | 43 | 2 | 4 | 1 | 3 | 3 | 2 | 2 |
| 23 | 44 | 2 | 3 | 1 | 3 | 3 | 2 | 2 |
| 43 | 34 | 2 | 3 | 1 | 3 | 3 | 2 | 2 |
| 43 | 44 | 2 | 4 | 1 | 3 | 3 | 2 | 2 |
| 14 | 44 | 2 | 3 | 1 | 3 | 3 | 2 | 2 |
| 24 | 44 | 4 | 3 | 1 | 4 | 4 | 4 | 3 |
| 34 | 24 | 2 | 2 | 1 | 3 | 3 | 2 | 2 |
| 34 | 34 | 2 | 3 | 1 | 3 | 3 | 2 | 2 |
| 34 | 44 | 2 | 4 | 1 | 3 | 3 | 2 | 2 |
| 44 | 34 | 4 | 3 | 1 | 4 | 4 | 4 | 3 |
| 44 | 44 | 4 | 4 | 1 | 4 | 4 | 4 | 3 |

In the last table (for $s = 2$) there are no remaining cases left with $\bullet$, so this finishes the proof. ☐

**Theorem 6.9.** *The local system on $\mathbb{G}^3_{m,\mathbb{F}_4}$ whose trace function is given by*

$$(\mathbb{F}_{4^r}; s, t, u) \mapsto -\frac{1}{4^{r/2}} \sum_{x \in \mathbb{F}_{4^r}^\times} \psi_{\mathbb{F}_{4^r}}(s/x + tx + ux^3)\chi_{3,\mathbb{F}_{4^r}}(x)$$

*has finite monodromy.*

*Proof.* By Corollary 5.2, we need to show

(i) $V\left(x_1 + 3x_2 \pm \frac{1}{3}\right) \leq V(-x_1) + V(-x_2) + \frac{1}{2}$ for every $(x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^2_{\text{prime to } 2}$.

(ii) $V(x) + V(-3x \pm \frac{1}{3}) \geq \frac{1}{2}$ for every $x \in (\mathbb{Q}/\mathbb{Z})_{\text{prime to } 2}$.

Condition (ii) is the criterion for finite monodromy of the local system with trace function

$$F(\mathbb{F}_{4^r}; t, u) \mapsto -\frac{1}{4^{r/2}} \sum_{x \in \mathbb{F}_{4^r}^\times} \psi_{\mathbb{F}_{4^r}}(tx + ux^3)\chi_{3,\mathbb{F}_{4^r}}(x)$$

which holds by [KT5, Theorem 10.2.6(ii)] since $3 = \frac{2^3+1}{2+1}$. For the first condition, by the change of variable $(x_1, x_2) \mapsto (2x_1, 2x_2)$ it suffices to prove the '+' case, since $V(2x) = V(x+1) = V(x)$. Following Proposition 5.4, we check by a computer search that all pairs $(x_1, x_2)$ with $0 \le x_1, x_2 \le 2^{10} - 1$ have good termination.

For each $s = 2, 4, 6, 8$, the following tables show the list of all pairs $(z_1, z_2)$ with $0 \le z_1, z_2 \le 5^s - 1$ such that

(a) $C(s; z_1, z_2) > 0$ and

(b) do not have good termination (i.e. all their last-digits truncations appear in the previous tables marked with $\bullet$).

If the condition (b) for being $s$-good can be applied to them, we show the possible values of $s', z_1', z_2'$ on the table, otherwise we mark it with $\bullet$ and move it on to the next $s$. All values of $z_i$ and $z_i'$ are shown as their 2-adic expansion. The columns $D$ and $D'$ show the result of removing the last $s$ (respectively $s'$) digits of $z_1 + 3z_2 + \frac{2^s-1}{3}$ (resp. of $z_1' + 3z_2' + \frac{2^{s'}-1}{3}$). Each digit of the number in column $D$ must be greater than or equal to the corresponding digit of the number in column $D'$.

$s = 2$

| $z_1$ | $z_2$ | $C(s; z_1, z_2)$ | $D$ | |
|---|---|---|---|---|
| 01 | 11 | 1 | 10 | $\bullet$ |
| 11 | 01 | 1 | 1 | $\bullet$ |
| 11 | 11 | 2 | 11 | $\bullet$ |

$s = 4$

| $z_1$ | $z_2$ | $C(s; z_1, z_2)$ | $D$ | $s'$ | $z_1'$ | $z_2'$ | $C(s'; z_1', z_2')$ | $D'$ |
|---|---|---|---|---|---|---|---|---|
| 0101 | 1111 | 1 | 11 | 2 | 01 | 11 | 1 | 10 |
| 1001 | 1111 | 1 | 11 | 2 | 01 | 11 | 1 | 10 |
| 1101 | 1111 | 3 | 11 | $\bullet$ | | | | |
| 1011 | 1101 | 1 | 11 | 2 | 01 | 11 | 1 | 10 |
| 1111 | 1001 | 1 | 10 | 2 | 01 | 11 | 1 | 10 |
| 1111 | 1101 | 2 | 11 | 2 | 11 | 11 | 2 | 11 |
| 0111 | 1111 | 1 | 11 | 2 | 01 | 11 | 1 | 10 |
| 1011 | 1111 | 2 | 11 | 2 | 11 | 11 | 2 | 11 |
| 1111 | 1011 | 1 | 11 | 2 | 01 | 11 | 1 | 10 |

$s = 6$

| $z_1$ | $z_2$ | $C(s; z_1, z_2)$ | $D$ | $s'$ | $z_1'$ | $z_2'$ | $C(s'; z_1', z_2')$ | $D'$ |
|---|---|---|---|---|---|---|---|---|
| 001101 | 111111 | 1 | 11 | 2 | 01 | 11 | 1 | 10 |
| 011101 | 101111 | 1 | 10 | 2 | 01 | 11 | 1 | 10 |
| 011101 | 111111 | 2 | 11 | 2 | 11 | 11 | 2 | 11 |
| 101101 | 111111 | 3 | 11 | 4 | 1101 | 1111 | 3 | 11 |
| 111101 | 001111 | 1 | 1 | 2 | 11 | 01 | 1 | 1 |
| 111101 | 011111 | 1 | 10 | 2 | 01 | 11 | 1 | 10 |
| 111101 | 101111 | 2 | 11 | 2 | 11 | 11 | 2 | 11 |
| 111101 | 111111 | 1 | 100 | $\bullet$ | | | | |

| $s=8$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $z_1$ | $z_2$ | $C(s;z_1,z_2)$ | $D$ | $s'$ | $z'_1$ | $z'_2$ | $C(s';z'_1,z'_2)$ | $D'$ |
| 01111101 | 11111111 | 2 | 11 | 2 | 11 | 11 | 2 | 11 |
| 11111101 | 01111111 | 1 | 10 | 2 | 01 | 11 | 1 | 10 |
| 11111101 | 10111111 | 1 | 11 | 2 | 01 | 11 | 1 | 10 |
| 11111101 | 11111111 | 1 | 100 | 6 | 111101 | 111111 | 1 | 100 |

In the last table (for $s = 8$) there are no remaining cases left with $\bullet$, so this finishes the proof. $\square$

**Theorem 6.10.** *The local system on $\mathbb{G}^2_{m,\mathbb{F}_{25}}$ whose trace function is given by*

$$(\mathbb{F}_{25^r}; s, t) \mapsto -\frac{1}{25^{r/2}} \sum_{x\in\mathbb{F}^\times_{25^r}} \psi_{\mathbb{F}_{25^r}}(s/x + tx^3)\chi_{3,\mathbb{F}_{25^r}}(x)$$

*has finite monodromy.*

*Proof.* By Corollary 5.2, we need to show

(i) $V\left(3x \pm \frac{1}{3}\right) \leq V(-x) + \frac{1}{2}$ for every $x \in (\mathbb{Q}/\mathbb{Z})_{\text{prime to }5}$.
(ii) $V(x) \geq \frac{1}{2}$ for every $x \in (\mathbb{Q}/\mathbb{Z})_{\text{prime to }5}$ such that $3x = \pm\frac{1}{3}$.

For condition (ii) one easily checks that $V(\frac{i}{9}) = \frac{1}{2}$ for $i = 1, \ldots, 8$. For the first condition, by the change of variable $(x_1, x_2) \mapsto (5x_1, 5x_2)$ it suffices to prove the '+' case, since $V(5x) = V(x+1) = V(x)$. Following Proposition 5.4, we check by a computer search that all $x$ with $0 \leq x \leq 5^8 - 1$ have good termination.

For each $s = 2, 4, 6$, the following tables show the list of all $z$ with $0 \leq z \leq 5^s - 1$ such that
(a) $C(s; z) > 0$ and
(b) do not have good termination (i.e. all their last-digits truncations appear in the previous tables marked with $\bullet$).

If the condition (b) for being $s$-good can be applied to them, we show the possible values of $s', z'$ on the table, otherwise we mark it with $\bullet$ and move it on to the next $s$. All values of $z$ and $z'$ are shown as their 5-adic expansion. The columns $D$ and $D'$ show the result of removing the last $s$ (respectively $s'$) digits of $3z + \frac{5^s-1}{3}$ (resp. of $3z' + \frac{5^{s'}-1}{3}$). Each digit of the number in column $D$ must be greater than or equal to the corresponding digit of the number in column $D'$.

| $s=2$ | | | |
|---|---|---|---|
| $z$ | $C(s;z)$ | $D$ | |
| 42 | 4 | 2 | $\bullet$ |

| $s=4$ | | | | | | |
|---|---|---|---|---|---|---|
| $z$ | $C(s;z)$ | $D$ | $s'$ | $z'$ | $C(s';z')$ | $D'$ |
| 2342 | 4 | 1 | $\bullet$ | | | |
| 3342 | 4 | 2 | 2 | 42 | 4 | 2 |
| 3442 | 4 | 2 | 2 | 42 | 4 | 2 |
| 4142 | 4 | 2 | 2 | 42 | 4 | 2 |
| 4342 | 4 | 3 | 2 | 42 | 4 | 2 |
| 4442 | 4 | 3 | 2 | 42 | 4 | 2 |

| $s=6$ | | | | | | |
|---|---|---|---|---|---|---|
| $z$ | $C(s;z)$ | $D$ | $s'$ | $z'$ | $C(s';z')$ | $D'$ |
| 232342 | 4 | 1 | 4 | 2342 | 4 | 1 |
| 332342 | 4 | 2 | 2 | 42 | 4 | 2 |
| 342342 | 4 | 2 | 2 | 42 | 4 | 2 |
| 402342 | 4 | 2 | 2 | 42 | 4 | 2 |
| 412342 | 4 | 2 | 2 | 42 | 4 | 2 |
| 432342 | 4 | 3 | 2 | 42 | 4 | 2 |
| 442342 | 4 | 3 | 2 | 42 | 4 | 2 |

In the last table (for $s=6$) there are no remaining cases left with $\bullet$, so this finishes the proof. □

**Theorem 6.11.** *The local system on $\mathbb{G}^4_{m,\mathbb{F}_3}$ whose trace function is given by*

$$(\mathbb{F}_{3^r};s,t,u,v)\mapsto -\frac{1}{3^{r/2}}\sum_{x\in\mathbb{F}_{3^r}^\times}\psi_{\mathbb{F}_{3^r}}(s/x+tx+ux^2+vx^5)\chi_{2,\mathbb{F}_{3^r}}(x)$$

*has finite monodromy.*

*Proof.* By Corollary 5.2, we need to show

(i) $V\left(x_1+2x_2+5x_3+\frac{1}{2}\right)\leq V(-x_1)+V(-x_2)+V(-x_3)+\frac{1}{2}$ for every $(x_1,x_2)\in(\mathbb{Q}/\mathbb{Z})^2_{\text{prime to }3}$.
(ii) $V(x_1)+V(x_2)+V(-2x_1-5x_2+\frac{1}{2})\geq\frac{1}{2}$ for every $x\in(\mathbb{Q}/\mathbb{Z})_{\text{prime to }3}$.

Condition (ii) is the criterion for finite monodromy of the local system with trace function

$$F(\mathbb{F}_{3^r};t,u)\mapsto -\frac{1}{3^{r/2}}\sum_{x\in\mathbb{F}_{3^r}^\times}\psi_{\mathbb{F}_{3^r}}(tx+ux^2+vx^5)\chi_{2,\mathbb{F}_{3^r}}(x)$$

which holds by [KT5, Theorem 11.2.3(i)]. For the first condition, following Proposition 5.4, we check by a computer search that all triples $(x_1,x_2,x_3)$ with $0\leq x_1,x_2,x_3\leq 3^7-1$ have good termination.

For each $s\leq 6$, the following tables show the list of all triples $(z_1,z_2,z_3)$ with $0\leq z_1,z_2,z_3\leq 3^s-1$ such that

(a) $C(s;z_1,z_2,z_3)>0$ and

(b) do not have good termination (i.e. all their last-digits truncations appear in the previous tables marked with $\bullet$).

If the condition (b) for being $s$-good can be applied to them, we show the possible values of $s',z_1',z_2',z_3'$ on the table, otherwise we mark it with $\bullet$ and move it on to the next $s$. All values of $z_i$ and $z_i'$ are shown as their 3-adic expansion. The columns $D$ and $D'$ show the result of removing the last $s$ (respectively $s'$) digits of $z_1+2z_2+5z_2+\frac{3^s-1}{2}$ (resp. of $z_1'+2z_2'+5z_3'+\frac{3^{s'}-1}{2}$). Each digit of the number in column $D$ must be greater than or equal to the corresponding digit of the number in column $D'$.

| $s=1$ | | | | | |
|---|---|---|---|---|---|
| $z_1$ | $z_2$ | $z_3$ | $C(s;z_1,z_2,z_3)$ | $D$ | |
| 1 | 1 | 2 | 1 | 11 | $\bullet$ |
| 1 | 2 | 2 | 2 | 12 | $\bullet$ |
| 2 | 1 | 2 | 1 | 12 | $\bullet$ |
| 2 | 2 | 2 | 4 | 12 | $\bullet$ |

| $s=2$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $z_1$ | $z_2$ | $z_3$ | $C(s;z_1,z_2,z_3)$ | $D$ | $s'$ | $z'_1$ | $z'_2$ | $z'_3$ | $C(s';z'_1,z'_2,z'_3)$ | $D'$ |
| 11 | 21 | 22 | 1 | 20 | • | | | | | |
| 21 | 01 | 22 | 1 | 12 | 1 | 1 | 1 | 2 | 1 | 11 |
| 21 | 21 | 12 | 1 | 12 | 1 | 1 | 1 | 2 | 1 | 11 |
| 21 | 21 | 22 | 1 | 21 | 1 | 1 | 1 | 2 | 1 | 11 |
| 21 | 12 | 22 | 1 | 20 | • | | | | | |
| 21 | 22 | 12 | 2 | 12 | 1 | 1 | 2 | 2 | 2 | 12 |
| 21 | 22 | 22 | 2 | 21 | • | | | | | |
| 22 | 21 | 12 | 1 | 12 | 1 | 1 | 1 | 2 | 1 | 11 |
| 22 | 21 | 22 | 1 | 21 | 1 | 1 | 1 | 2 | 1 | 11 |
| 02 | 22 | 22 | 2 | 20 | • | | | | | |
| 12 | 02 | 22 | 2 | 12 | 1 | 1 | 2 | 2 | 2 | 12 |
| 12 | 12 | 12 | 1 | 11 | 1 | 1 | 1 | 2 | 1 | 11 |
| 12 | 12 | 22 | 1 | 20 | • | | | | | |
| 12 | 22 | 12 | 2 | 12 | 1 | 1 | 2 | 2 | 2 | 12 |
| 12 | 22 | 22 | 2 | 21 | • | | | | | |
| 22 | 12 | 12 | 1 | 12 | 1 | 1 | 1 | 2 | 1 | 11 |
| 22 | 12 | 22 | 3 | 20 | • | | | | | |
| 22 | 22 | 12 | 4 | 12 | 1 | 2 | 2 | 2 | 4 | 12 |
| 22 | 22 | 22 | 4 | 21 | • | | | | | |

| $s=3$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $z_1$ | $z_2$ | $z_3$ | $C(s;z_1,z_2,z_3)$ | $D$ | $s'$ | $z'_1$ | $z'_2$ | $z'_3$ | $C(s';z'_1,z'_2,z'_3)$ | $D'$ |
| 111 | 221 | 122 | 1 | 12 | 1 | 1 | 1 | 2 | 1 | 11 |
| 111 | 221 | 222 | 1 | 21 | 1 | 1 | 1 | 2 | 1 | 11 |
| 211 | 221 | 222 | 3 | 21 | 2 | 22 | 12 | 22 | 3 | 20 |
| 121 | 212 | 122 | 1 | 12 | 1 | 1 | 1 | 2 | 1 | 11 |
| 121 | 212 | 222 | 1 | 21 | 1 | 1 | 1 | 2 | 1 | 11 |
| 221 | 212 | 222 | 3 | 21 | 2 | 22 | 12 | 22 | 3 | 20 |
| 121 | 222 | 222 | 2 | 21 | 2 | 21 | 22 | 22 | 2 | 21 |
| 221 | 122 | 122 | 1 | 12 | 1 | 1 | 1 | 2 | 1 | 11 |
| 221 | 122 | 222 | 1 | 21 | 1 | 1 | 1 | 2 | 1 | 11 |
| 221 | 222 | 222 | 2 | 22 | 1 | 1 | 2 | 2 | 2 | 12 |
| 102 | 122 | 222 | 1 | 20 | 2 | 11 | 21 | 22 | 1 | 20 |
| 102 | 222 | 122 | 2 | 12 | 1 | 1 | 2 | 2 | 2 | 12 |
| 102 | 222 | 222 | 2 | 21 | 2 | 21 | 22 | 22 | 2 | 21 |
| 202 | 122 | 122 | 1 | 12 | 1 | 1 | 1 | 2 | 1 | 11 |
| 202 | 122 | 222 | 1 | 21 | 1 | 1 | 1 | 2 | 1 | 11 |
| 202 | 222 | 222 | 4 | 21 | 2 | 22 | 22 | 22 | 4 | 21 |
| 112 | 212 | 122 | 1 | 12 | 1 | 1 | 1 | 2 | 1 | 11 |
| 112 | 212 | 222 | 1 | 21 | 1 | 1 | 1 | 2 | 1 | 11 |
| 212 | 212 | 222 | 3 | 21 | 2 | 22 | 12 | 22 | 3 | 20 |
| 112 | 222 | 222 | 2 | 21 | 2 | 21 | 22 | 22 | 2 | 21 |
| 212 | 122 | 122 | 1 | 12 | 1 | 1 | 1 | 2 | 1 | 11 |
| 212 | 122 | 222 | 1 | 21 | 1 | 1 | 1 | 2 | 1 | 11 |
| 212 | 222 | 222 | 2 | 22 | 1 | 1 | 2 | 2 | 2 | 12 |
| 022 | 012 | 222 | 1 | 12 | 1 | 1 | 1 | 2 | 1 | 11 |
| 022 | 212 | 122 | 1 | 12 | 1 | 1 | 1 | 2 | 1 | 11 |
| 022 | 212 | 222 | 1 | 21 | 1 | 1 | 1 | 2 | 1 | 11 |
| 122 | 112 | 222 | 2 | 20 | 2 | 02 | 22 | 22 | 2 | 20 |
| 122 | 212 | 122 | 3 | 12 | 1 | 2 | 2 | 2 | 4 | 12 |
| 122 | 212 | 222 | 3 | 21 | 2 | 22 | 12 | 22 | 3 | 20 |
| 222 | 012 | 122 | 1 | 11 | 1 | 1 | 1 | 2 | 1 | 11 |
| 222 | 012 | 222 | 1 | 20 | 2 | 11 | 21 | 22 | 1 | 20 |
| 222 | 112 | 122 | 2 | 12 | 1 | 1 | 2 | 2 | 2 | 12 |
| 222 | 112 | 222 | 2 | 21 | 2 | 21 | 22 | 22 | 2 | 21 |
| 222 | 212 | 022 | 1 | 11 | 1 | 1 | 1 | 2 | 1 | 11 |
| 222 | 212 | 122 | 1 | 20 | 2 | 11 | 21 | 22 | 1 | 20 |
| 222 | 212 | 222 | 5 | 21 | • | | | | | |
| 022 | 122 | 222 | 1 | 20 | 2 | 11 | 21 | 22 | 1 | 20 |
| 022 | 222 | 122 | 2 | 12 | 1 | 1 | 2 | 2 | 2 | 12 |
| 022 | 222 | 222 | 2 | 21 | 2 | 21 | 22 | 22 | 2 | 21 |
| 122 | 122 | 122 | 1 | 12 | 1 | 1 | 1 | 2 | 1 | 11 |
| 122 | 122 | 222 | 1 | 21 | 1 | 1 | 1 | 2 | 1 | 11 |
| 122 | 222 | 222 | 4 | 21 | 2 | 22 | 22 | 22 | 4 | 21 |
| 222 | 022 | 222 | 2 | 20 | 2 | 02 | 22 | 22 | 2 | 20 |
| 222 | 122 | 122 | 3 | 12 | 1 | 2 | 2 | 2 | 4 | 12 |
| 222 | 122 | 222 | 3 | 21 | 2 | 22 | 12 | 22 | 3 | 20 |
| 222 | 222 | 022 | 2 | 11 | • | | | | | |
| 222 | 222 | 122 | 2 | 20 | 2 | 02 | 22 | 22 | 2 | 20 |
| 222 | 222 | 222 | 4 | 22 | 1 | 2 | 2 | 2 | 4 | 12 |

| $s=4$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $z_1$ | $z_2$ | $z_3$ | $C(s;z_1,z_2,z_3)$ | $D$ | $s'$ | $z'_1$ | $z'_2$ | $z'_3$ | $C(s';z'_1,z'_2,z'_3)$ | $D'$ |
| 0222 | 1212 | 2222 | 2 | 20 | 2 | 02 | 22 | 22 | 2 | 20 |
| 0222 | 2212 | 1222 | 3 | 12 | 1 | 2 | 2 | 2 | 4 | 12 |
| 0222 | 2212 | 2222 | 3 | 21 | 2 | 22 | 12 | 22 | 3 | 20 |
| 1222 | 0212 | 1222 | 1 | 11 | 1 | 1 | 1 | 2 | 1 | 11 |
| 1222 | 0212 | 2222 | 1 | 20 | 2 | 11 | 21 | 22 | 1 | 20 |
| 1222 | 1212 | 1222 | 2 | 12 | 1 | 1 | 2 | 2 | 2 | 12 |
| 1222 | 1212 | 2222 | 2 | 21 | 2 | 21 | 22 | 22 | 2 | 21 |
| 1222 | 2212 | 0222 | 1 | 11 | 1 | 1 | 1 | 2 | 1 | 11 |
| 1222 | 2212 | 1222 | 1 | 20 | 2 | 11 | 21 | 22 | 1 | 20 |
| 1222 | 2212 | 2222 | 5 | 21 | 3 | 222 | 212 | 222 | 5 | 21 |
| 2222 | 0212 | 1222 | 1 | 12 | 1 | 1 | 1 | 2 | 1 | 11 |
| 2222 | 0212 | 2222 | 3 | 20 | 2 | 22 | 12 | 22 | 3 | 20 |
| 2222 | 1212 | 1222 | 4 | 12 | 1 | 2 | 2 | 2 | 4 | 12 |
| 2222 | 1212 | 2222 | 4 | 21 | 2 | 22 | 22 | 22 | 4 | 21 |
| 2222 | 2212 | 0222 | 3 | 11 | • | | | | | |
| 2222 | 2212 | 1222 | 3 | 20 | 2 | 22 | 12 | 22 | 3 | 20 |
| 2222 | 2212 | 2222 | 5 | 22 | 3 | 222 | 212 | 222 | 5 | 21 |
| 0222 | 1222 | 2022 | 1 | 12 | 1 | 1 | 1 | 2 | 1 | 11 |
| 1222 | 2222 | 2022 | 2 | 20 | 2 | 02 | 22 | 22 | 2 | 20 |
| 2222 | 0222 | 2022 | 2 | 12 | 1 | 1 | 2 | 2 | 2 | 12 |
| 2222 | 1222 | 1022 | 1 | 11 | 1 | 1 | 1 | 2 | 1 | 11 |
| 2222 | 1222 | 2022 | 1 | 20 | 2 | 11 | 21 | 22 | 1 | 20 |
| 2222 | 2222 | 1022 | 2 | 12 | 1 | 1 | 2 | 2 | 2 | 12 |
| 2222 | 2222 | 2022 | 2 | 21 | 2 | 21 | 22 | 22 | 2 | 21 |

| $s=5$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $z_1$ | $z_2$ | $z_3$ | $C(s;z_1,z_2,z_3)$ | $D$ | $s'$ | $z'_1$ | $z'_2$ | $z'_3$ | $C(s';z'_1,z'_2,z'_3)$ | $D'$ |
| 02222 | 12212 | 20222 | 2 | 12 | 1 | 1 | 2 | 2 | 2 | 12 |
| 02222 | 22212 | 10222 | 1 | 11 | 1 | 1 | 1 | 2 | 1 | 11 |
| 02222 | 22212 | 20222 | 1 | 20 | 2 | 11 | 21 | 22 | 1 | 20 |
| 12222 | 02212 | 20222 | 1 | 12 | 1 | 1 | 1 | 2 | 1 | 11 |
| 12222 | 22212 | 10222 | 1 | 12 | 1 | 1 | 1 | 2 | 1 | 11 |
| 12222 | 22212 | 20222 | 3 | 20 | 2 | 22 | 12 | 22 | 3 | 20 |
| 22222 | 02212 | 20222 | 3 | 12 | 1 | 2 | 2 | 2 | 4 | 12 |
| 22222 | 12212 | 10222 | 2 | 11 | 3 | 222 | 222 | 022 | 2 | 11 |
| 22222 | 12212 | 20222 | 2 | 20 | 2 | 02 | 22 | 22 | 2 | 20 |
| 22222 | 22212 | 00222 | 1 | 10 | • | | | | | |
| 22222 | 22212 | 10222 | 3 | 12 | 1 | 2 | 2 | 2 | 4 | 12 |
| 22222 | 22212 | 20222 | 3 | 21 | 2 | 22 | 12 | 22 | 3 | 20 |

| $s=6$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $z_1$ | $z_2$ | $z_3$ | $C(s;z_1,z_2,z_3)$ | $D$ | $s'$ | $z'_1$ | $z'_2$ | $z'_3$ | $C(s';z'_1,z'_2,z'_3)$ | $D'$ |
| 122222 | 122212 | 200222 | 2 | 12 | 1 | 1 | 2 | 2 | 2 | 12 |
| 122222 | 222212 | 100222 | 1 | 11 | 1 | 1 | 1 | 2 | 1 | 11 |
| 122222 | 222212 | 200222 | 1 | 20 | 2 | 11 | 21 | 22 | 1 | 20 |
| 222222 | 022212 | 200222 | 1 | 12 | 1 | 1 | 1 | 2 | 1 | 11 |
| 222222 | 222212 | 100222 | 1 | 12 | 1 | 1 | 1 | 2 | 1 | 11 |
| 222222 | 222212 | 200222 | 3 | 20 | 2 | 22 | 12 | 22 | 3 | 20 |

In the last table (for $s=6$) there are no remaining cases left with •, so this finishes the proof. □

**Theorem 6.12.** *Let $k=\mathbb{F}_3$ and $\mathcal{F}$ be the local system on $\mathbb{G}^4_{m,k}$ whose trace function is given by*

$$F(k_r;s,t,u,v)=-\frac{1}{\sqrt{3^r}}\sum_{x\in k_r^\times}\psi_{k_r}(s/x^2+t/x+ux+vx^2)\chi_{2,k_r}(x).$$

*Then $\mathcal{F}$ has finite (geometric and arithmetic) monodromy group if and only if*

$$V(x)+V(y)+V(z)+V\left(x+2y-2z+\frac{1}{2}\right)\geq\frac{1}{2}$$

*for every $(x,y,z)\in(\mathbb{Q}/\mathbb{Z})^3_{prime\ to\ 3}$.*

*Proof.* By [KRLT1, Proposition 2.1], we need to show that $F(k_r;s,t,u,v)$ is an algebraic integer or, equivalently, that

$$\mathrm{ord}_{3^r}\left(\sum_{x\in k_r^\times}\psi_{k_r}(s/x^2+t/x+ux+vx^2)\chi_{2,k_r}(x)\right)\geq\frac{1}{2}$$

for every $r\geq 1$ and $(s,t,u,v)\in(k_r^\times)^4$. Taking Mellin transform on $\mathbb{G}^4_m$, this is equivalent to

$$\sum_{s,t,u,v\in k_r^\times}\xi_1(s)\xi_2(t)\xi_3(u)\xi_4(v)\sum_{x\in k_r^\times}\psi_{k_r}(s/x^2+t/x+ux+vx^2)\chi_{2,k_r}(x)=$$

$$=\sum_{x\in k_r^\times}\chi_{2,k_r}(x)\left(\sum_{s\in k_r^\times}\psi_{k_r}(s/x^2)\xi_1(s)\right)\left(\sum_{t\in k_r^\times}\psi_{k_r}(t/x)\xi_2(t)\right)\left(\sum_{u\in k_r^\times}\psi_{k_r}(ux)\xi_2(u)\right)\left(\sum_{u\in k_r^\times}\psi_{k_r}(vx^2)\xi_2(v)\right)=$$

$$=\sum_{x\in k_r^\times}\chi_{2,k_r}(x)\xi_1(x^2)G_r(\xi_1)\xi_2(x)G_r(\xi_2)\bar\xi_3(x)G_r(\xi_3)\bar\xi_4(x^2)G_r(\xi_4)=$$

$$=G_r(\xi_1)G_r(\xi_2)G_r(\xi_3)G_r(\xi_4)\sum_{x\in k_r^\times}(\chi_{2,k_r}\xi_1^2\xi_2\bar\xi_3\bar\xi_4^2)(x)=$$

$$=\begin{cases}0 & \text{if } \chi_{2,k_r}\xi_1^2\xi_2\bar\xi_3\bar\xi_4^2\neq\mathbf{1}\\ (q^r-1)G_r(\xi_1)G_r(\xi_2)G_r(\xi_3)G_r(\xi_4) & \text{if } \chi_{2,k_r}\xi_1^2\xi_2\bar\xi_3\bar\xi_4^2=\mathbf{1}\end{cases}$$

has $\mathrm{ord}_{3^r}\geq\frac{1}{2}$ for every $\xi_1,\xi_2,\xi_3,\xi_4\in\widehat{k_r^\times}$, where $G_r(\chi)$ denotes the Gauss sum associated to the multiplicative character $\chi$ on $k_r$. This reduces to

$$\mathrm{ord}_{3^r}(G_r(\chi_{2,k_r}\bar\xi_1^2\xi_3\xi_4^2)G_r(\xi_1)G_r(\xi_3)G_r(\xi_4))\geq\frac{1}{2}$$

for every $\xi_1,\xi_3,\xi_4\in\widehat{k_r^\times}$ which, by Stickelberger, is equivalent to the given condition. □

If $y = 0$ the criterion reads $V(x) + V(z) + V(x - 2z + \frac{1}{2}) \geq \frac{1}{2}$ which, after the change of variable $x \mapsto x+2z+\frac{1}{2}$ becomes $V(x+2z+\frac{1}{2})+V(x)+V(z) \geq \frac{1}{2}$, which is the criterion for finite monodromy for the local system with trace function

$$F(k_r; t, u, v) = -\frac{1}{\sqrt{3^r}} \sum_{x \in k_r^\times} \psi_{k_r}(t/x + ux + vx^2)\chi_{2,k_r}(x).$$

so it holds by Theorem 6.11. Similarly for $z = 0$. If $x + 2y - 2z = \frac{1}{2}$ but $y, z \neq 0$, the criterion is

$$V\left(2z - 2y + \frac{1}{2}\right) + V(y) + V(z) \geq \frac{1}{2} \Leftrightarrow V\left(2z + \frac{1}{2}\right) + V(y) + V(y+z) \geq \frac{1}{2}.$$

For $z = \pm\frac{1}{4}$ we get $V(y) + V(y \pm \frac{1}{4}) \geq \frac{1}{2}$, which is the criterion for finite monodromy of the Kloosterman sheaf $\mathcal{K}l(\mathbb{1}, \chi_4)$, and the latter sheaf has finite monodromy by Corollary 6.5. Otherwise it is equivalent to

$$V\left(2z + \frac{1}{2}\right) + V(y+z) \leq V(-y) + \frac{3}{2} \text{ for } y \neq 0, \pm\frac{1}{4}; z \neq 0. \tag{6.12.1}$$

Finally, if $y, z, x + 2y - 2z + \frac{1}{2} \neq 0$, after the change of variable $y \mapsto y + z$ the criterion becomes $V(x) + V(z) + V(y + z) + V(x + 2y + \frac{1}{2}) \geq \frac{1}{2}$, and this is equivalent to

$$V(y+z) + V\left(x + 2y + \frac{1}{2}\right) \leq V(-x) + V(-z) + \frac{3}{2}, \tag{6.12.2}$$

which reduces to 6.12.1 if $x = 0$. So, in order to prove finite monodromy for $\mathcal{F}$ it suffices to prove 6.12.2 for all $x, y, z$.

By an argument similar to [KRLT1, Theorem 2.12] this is a consequence of

**Proposition 6.13.** *The inequality*

$$\left[x + 2y + \frac{3^r - 1}{2}\right]_3 + [y+z]_3 \leq [3^r - 1 - x]_3 + [3^r - 1 - z]_3 + 3r + 3$$

*holds for every $r \geq 1$ and every $0 \leq x, y, z \leq 3^r - 1$, where $[x]_3$ denotes the sum of the 3-adic digits of $x$.*

*Proof of Proposition 6.13.* In order to prove this, we will follow a slight variant of the strategy used in [KRLT4, Section 4].

For $r \geq 1$ and $(x, y, z) \in \{0, \dots, 3^r - 1\}^3$, let

$$C(r; x, y, z) = \left[x + 2y + \frac{3^r - 1}{2}\right]_3 + [y+z]_3 - [3^r - 1 - x]_3 - [3^r - 1 - z]_3 - 3r.$$

We aim to show that $C(r; x, y, z) \leq 3$. For $s \geq 1$ we say that the triple $(x, y, z) \in \{0, \dots, 3^s - 1\}^3$ is *s-good* if one of these conditions hold:

a) $C(s; x, y, z) \leq 0$.
b) There exists an $s' < s$ and a triple $(x', y', z') \in \{0, \dots, 3^{s'} - 1\}^3$ such that $C(s'; x', y', z') \geq C(s; x, y, z)$ and for every $j > 0$ the $(s+j)$-th digit in the 3-adic expansion of $x + 2y + \frac{3^s-1}{2}$ (respectively of $y + z$) is greater than or equal to the $(s'+j)$-th digit in the 3-adic expansion of $x' + 2y' + \frac{3^{s'}-1}{2}$ (resp. of $y' + z'$), counting the digits from right to left and adding leading 0's as needed.

We say that the triple $(x, y, z) \in \{0, \ldots, 3^r - 1\}^3$ has *good termination* if, for some $1 \le s < r$, the triple $(x_s, y_s, z_s) \in \{0, \ldots, 3^s - 1\}^3$ formed by taking the last $s$ 3-adic digits of $x, y, z$ (i.e. their remainders when dividing by $p^s$) is $s$-good. Then, exactly as in [KRLT4, Proposition 4.4], one shows that Proposition 6.13 is a consequence of the fact that all triples $(x, y, z) \in \{0, \ldots, 3^5 - 1\}^3$ have good termination, which we test with a computer check.

For each $s = 1, 2, 3, 4$, the following tables show the list of all triples $(x, y, z) \in \{0, \ldots, 3^s - 1\}^3$ such that

(a) $C(s; x, y, z) > 0$ and

(b) do not have good termination (i.e. all their last-digits truncations appear in the previous tables marked with $\bullet$).

If the condition (b) for being $s$-good can be applied to them, we show the possible values of $s', x', y', z'$ on the table, otherwise we mark it with $\bullet$ and move it on to the next $s$. All values of $x, y, z, x', y', z'$ are shown as their 3-adic expansion. The columns $D, E$ and $D', E'$ show the result of removing the last $s$ (respectively $s'$) digits of $x + 2y + \frac{3^s-1}{2}$ and $y + z$ (resp. of $x' + 2y' + \frac{3^{s'}-1}{2}$) and $y' + z'$). Each digit of the number in columns $D, E$ must be greater than or equal to the corresponding digit of the number in columns $D', E'$.

$s = 1$

| $x$ | $y$ | $z$ | $C(s; x, y, z)$ | $D$ | $E$ | |
|---|---|---|---|---|---|---|
| 2 | 1 | 1 | 1 | 1 | 0 | $\bullet$ |
| 2 | 1 | 2 | 1 | 1 | 1 | $\bullet$ |
| 2 | 2 | 2 | 2 | 2 | 1 | $\bullet$ |

$s = 2$

| $x$ | $y$ | $z$ | $C(s; x, y, z)$ | $D$ | $E$ | $s'$ | $x'$ | $y'$ | $z'$ | $C(s'; x', y', z')$ | $D'$ | $E'$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 12 | 21 | 21 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |
| 22 | 01 | 21 | 1 | 1 | 0 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |
| 22 | 21 | 01 | 1 | 2 | 0 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |
| 22 | 21 | 11 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |
| 22 | 21 | 21 | 3 | 2 | 1 | $\bullet$ | | | | | | |
| 12 | 21 | 22 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |
| 22 | 21 | 12 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |
| 22 | 21 | 22 | 3 | 2 | 1 | $\bullet$ | | | | | | |
| 12 | 22 | 22 | 2 | 2 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 1 |
| 22 | 12 | 22 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |

| $s=3$ | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | $y$ | $z$ | $C(s;x,y,z)$ | $D$ | $E$ | $s'$ | $x'$ | $y'$ | $z'$ | $C(s';x',y',z')$ | $D'$ | $E'$ |
| 022 | 221 | 221 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |
| 122 | 221 | 121 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |
| 122 | 221 | 221 | 3 | 2 | 1 | 2 | 22 | 21 | 21 | 3 | 2 | 1 |
| 222 | 021 | 121 | 1 | 1 | 0 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |
| 222 | 021 | 221 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |
| 222 | 121 | 221 | 2 | 2 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 1 |
| 222 | 221 | 221 | 1 | 10 | 1 | $\bullet$ | | | | | | |
| 022 | 221 | 222 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |
| 122 | 221 | 122 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |
| 122 | 221 | 222 | 3 | 2 | 1 | 2 | 22 | 21 | 21 | 3 | 2 | 1 |
| 222 | 021 | 122 | 1 | 1 | 0 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |
| 222 | 021 | 222 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |
| 222 | 121 | 222 | 2 | 2 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 1 |
| 222 | 221 | 222 | 1 | 10 | 1 | $\bullet$ | | | | | | |

| $s=4$ | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | $y$ | $z$ | $C(s;x,y,z)$ | $D$ | $E$ | $s'$ | $x'$ | $y'$ | $z'$ | $C(s';x',y',z')$ | $D'$ | $E'$ |
| 0222 | 2221 | 2221 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |
| 2222 | 1221 | 2221 | 2 | 2 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 1 |
| 2222 | 2221 | 2221 | 1 | 10 | 1 | 3 | 222 | 221 | 221 | 1 | 10 | 1 |
| 0222 | 2221 | 2222 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |
| 2222 | 1221 | 2222 | 2 | 2 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 1 |
| 2222 | 2221 | 2222 | 1 | 10 | 1 | 3 | 222 | 221 | 221 | 1 | 10 | 1 |

In the last table (for $s = 4$) there are no remaining cases left with $\bullet$, so this finishes the proof of Proposition 6.13, and hence of Theorem 6.12. $\square$

**Theorem 6.14.** *The local system on $\mathbb{G}^4_{m,\mathbb{F}_3}$ whose trace function is given by*

$$(\mathbb{F}_{3^r}; s, t, u, v) \mapsto -\frac{1}{3^{r/2}} \sum_{x \in \mathbb{F}_{3^r}^\times} \psi_{\mathbb{F}_{3^r}}(s/x + tx + ux^5 + vx^7)\chi_{2,\mathbb{F}_{3^r}}(x)$$

*has finite monodromy.*

*Proof.* By Corollary 5.2, we need to show

(i) $V\left(x_1 + 5x_2 + 7x_3 + \frac{1}{2}\right) \le V(-x_1) + V(-x_2) + V(-x_3) + \frac{1}{2}$ for every $(x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^2_{\text{prime to }3}$.

(ii) $V(x_1) + V(x_2) + V(-5x_1 - 7x_2 + \frac{1}{2}) \ge \frac{1}{2}$ for every $x \in (\mathbb{Q}/\mathbb{Z})_{\text{prime to }3}$.

Condition (ii) is the criterion for finite monodromy of the local system with trace function

$$F(\mathbb{F}_{3^r}; t, u) \mapsto -\frac{1}{3^{r/2}} \sum_{x \in \mathbb{F}_{3^r}^\times} \psi_{\mathbb{F}_{3^r}}(tx + ux^5 + vx^7)\chi_{2,\mathbb{F}_{3^r}}(x)$$

which holds by [KT5, Theorem 11.2.3(vi)]. For the first condition, following Proposition 5.4, we check by a computer search that all triples $(x_1, x_2, x_3)$ with $0 \le x_1, x_2, x_3 \le 3^7 - 1$ have good termination.

For each $s \le 6$, the following tables show the list of all triples $(z_1, z_2, z_3)$ with $0 \le z_1, z_2, z_3 \le 3^s - 1$ such that

(a) $C(s; z_1, z_2, z_3) > 0$ and

(b) do not have good termination (i.e. all their last-digits truncations appear in the previous tables marked with •).

If the condition (b) for being $s$-good can be applied to them, we show the possible values of $s', z_1', z_2', z_3'$ on the table, otherwise we mark it with • and move it on to the next $s$. All values of $z_i$ and $z_i'$ are shown as their 3-adic expansion. The columns $D$ and $D'$ show the result of removing the last $s$ (respectively $s'$) digits of $z_1 + 5z_2 + 7z_2 + \frac{3^s-1}{2}$ (resp. of $z_1' + 5z_2' + 7z_3' + \frac{3^{s'}-1}{2}$). Each digit of the number in column $D$ must be greater than or equal to the corresponding digit of the number in column $D'$.

| $s=1$ | | | | | |
|---|---|---|---|---|---|
| $z_1$ | $z_2$ | $z_3$ | $C(s;z_1,z_2,z_3)$ | $D$ | |
| 0 | 2 | 2 | 2 | 22 | • |
| 1 | 2 | 2 | 4 | 22 | • |
| 2 | 0 | 2 | 2 | 12 | • |
| 2 | 1 | 2 | 2 | 21 | • |
| 2 | 2 | 1 | 2 | 20 | • |

| $s=2$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $z_1$ | $z_2$ | $z_3$ | $C(s;z_1,z_2,z_3)$ | $D$ | $s'$ | $z_1'$ | $z_2'$ | $z_3'$ | $C(s';z_1',z_2',z_3')$ | $D'$ |
| 20 | 22 | 22 | 2 | 102 | • | | | | | |
| 01 | 22 | 12 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 11 | 22 | 22 | 2 | 102 | • | | | | | |
| 21 | 02 | 22 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 21 | 12 | 12 | 2 | 21 | 1 | 2 | 1 | 2 | 2 | 21 |
| 21 | 22 | 22 | 4 | 102 | • | | | | | |
| 12 | 10 | 22 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 22 | 20 | 12 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 22 | 20 | 22 | 2 | 101 | • | | | | | |
| 12 | 21 | 12 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 22 | 21 | 22 | 2 | 102 | • | | | | | |
| 02 | 12 | 21 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 12 | 22 | 11 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 12 | 22 | 21 | 2 | 101 | • | | | | | |
| 22 | 02 | 21 | 2 | 21 | 1 | 2 | 1 | 2 | 2 | 21 |
| 22 | 22 | 11 | 4 | 22 | 1 | 1 | 2 | 2 | 4 | 22 |
| 22 | 22 | 21 | 2 | 102 | • | | | | | |

| $s=3$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $z_1$ | $z_2$ | $z_3$ | $C(s;z_1,z_2,z_3)$ | $D$ | $s'$ | $z'_1$ | $z'_2$ | $z'_3$ | $C(s';z'_1,z'_2,z'_3)$ | $D'$ |
| 220 | 122 | 122 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 220 | 222 | 222 | 2 | 110 | • | | | | | |
| 211 | 122 | 122 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 211 | 222 | 222 | 2 | 110 | • | | | | | |
| 021 | 022 | 222 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 121 | 122 | 122 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 121 | 122 | 222 | 2 | 101 | 2 | 22 | 20 | 22 | 2 | 101 |
| 121 | 222 | 022 | 2 | 21 | 1 | 2 | 1 | 2 | 2 | 21 |
| 121 | 222 | 222 | 2 | 110 | • | | | | | |
| 221 | 122 | 122 | 4 | 22 | 1 | 1 | 2 | 2 | 4 | 22 |
| 221 | 122 | 222 | 2 | 102 | 2 | 20 | 22 | 22 | 2 | 102 |
| 221 | 222 | 022 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 221 | 222 | 122 | 2 | 101 | 2 | 22 | 20 | 22 | 2 | 101 |
| 221 | 222 | 222 | 4 | 110 | • | | | | | |
| 022 | 220 | 222 | 2 | 102 | 2 | 20 | 22 | 22 | 2 | 102 |
| 122 | 020 | 222 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 222 | 120 | 122 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 222 | 120 | 222 | 2 | 101 | 2 | 22 | 20 | 22 | 2 | 101 |
| 222 | 220 | 022 | 2 | 21 | 1 | 2 | 1 | 2 | 2 | 21 |
| 222 | 220 | 222 | 2 | 110 | • | | | | | |
| 222 | 121 | 122 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 222 | 221 | 222 | 2 | 110 | • | | | | | |
| 012 | 222 | 221 | 2 | 102 | 2 | 20 | 22 | 22 | 2 | 102 |
| 112 | 022 | 221 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 212 | 122 | 121 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 212 | 122 | 221 | 2 | 101 | 2 | 22 | 20 | 22 | 2 | 101 |
| 212 | 222 | 021 | 2 | 21 | 1 | 2 | 1 | 2 | 2 | 21 |
| 212 | 222 | 221 | 2 | 110 | • | | | | | |
| 222 | 122 | 121 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 222 | 222 | 221 | 2 | 110 | • | | | | | |

| $s = 4$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $z_1$ | $z_2$ | $z_3$ | $C(s; z_1, z_2, z_3)$ | $D$ | $s'$ | $z'_1$ | $z'_2$ | $z'_3$ | $C(s'; z'_1, z'_2, z'_3)$ | $D'$ |
| 1220 | 1222 | 1222 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 1220 | 2222 | 2222 | 2 | 110 | 3 | 220 | 222 | 222 | 2 | 110 |
| 2220 | 1222 | 2222 | 2 | 102 | 2 | 20 | 22 | 22 | 2 | 102 |
| 2220 | 2222 | 0222 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 2220 | 2222 | 1222 | 2 | 101 | 2 | 22 | 20 | 22 | 2 | 101 |
| 2220 | 2222 | 2222 | 2 | 111 | 2 | 22 | 20 | 22 | 2 | 101 |
| 1211 | 1222 | 1222 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 1211 | 2222 | 2222 | 2 | 110 | 3 | 220 | 222 | 222 | 2 | 110 |
| 2211 | 1222 | 2222 | 2 | 102 | 2 | 20 | 22 | 22 | 2 | 102 |
| 2211 | 2222 | 0222 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 2211 | 2222 | 1222 | 2 | 101 | 2 | 22 | 20 | 22 | 2 | 101 |
| 2211 | 2222 | 2222 | 2 | 111 | 2 | 22 | 20 | 22 | 2 | 101 |
| 1121 | 1222 | 1222 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 1121 | 2222 | 2222 | 2 | 110 | 3 | 220 | 222 | 222 | 2 | 110 |
| 2121 | 1222 | 2222 | 2 | 102 | 2 | 20 | 22 | 22 | 2 | 102 |
| 2121 | 2222 | 0222 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 2121 | 2222 | 1222 | 2 | 101 | 2 | 22 | 20 | 22 | 2 | 101 |
| 2121 | 2222 | 2222 | 2 | 111 | 2 | 22 | 20 | 22 | 2 | 101 |
| 0221 | 1222 | 1222 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 0221 | 1222 | 2222 | 2 | 101 | 2 | 22 | 20 | 22 | 2 | 101 |
| 0221 | 2222 | 0222 | 2 | 21 | 1 | 2 | 1 | 2 | 2 | 21 |
| 0221 | 2222 | 2222 | 2 | 110 | 3 | 220 | 222 | 222 | 2 | 110 |
| 1221 | 1222 | 1222 | 4 | 22 | 1 | 1 | 2 | 2 | 4 | 22 |
| 1221 | 1222 | 2222 | 2 | 102 | 2 | 20 | 22 | 22 | 2 | 102 |
| 1221 | 2222 | 0222 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 1221 | 2222 | 1222 | 2 | 101 | 2 | 22 | 20 | 22 | 2 | 101 |
| 1221 | 2222 | 2222 | 4 | 110 | 3 | 221 | 222 | 222 | 4 | 110 |
| 2221 | 0222 | 1222 | 2 | 21 | 1 | 2 | 1 | 2 | 2 | 21 |
| 2221 | 0222 | 2222 | 2 | 100 | • | | | | | |
| 2221 | 1222 | 0222 | 2 | 20 | 1 | 2 | 2 | 1 | 2 | 20 |
| 2221 | 1222 | 2222 | 4 | 102 | 2 | 21 | 22 | 22 | 4 | 102 |
| 2221 | 2222 | 0222 | 4 | 22 | 1 | 1 | 2 | 2 | 4 | 22 |
| 2221 | 2222 | 1222 | 4 | 101 | • | | | | | |
| 2221 | 2222 | 2222 | 4 | 111 | 3 | 221 | 222 | 222 | 4 | 110 |
| 1222 | 1220 | 1222 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 1222 | 2220 | 2222 | 2 | 110 | 3 | 220 | 222 | 222 | 2 | 110 |
| 2222 | 1220 | 2222 | 2 | 102 | 2 | 20 | 22 | 22 | 2 | 102 |
| 2222 | 2220 | 0222 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 2222 | 2220 | 1222 | 2 | 101 | 2 | 22 | 20 | 22 | 2 | 101 |
| 2222 | 2220 | 2222 | 2 | 111 | 2 | 22 | 20 | 22 | 2 | 101 |
| 1222 | 1221 | 1222 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 1222 | 2221 | 2222 | 2 | 110 | 3 | 220 | 222 | 222 | 2 | 110 |
| 2222 | 1221 | 2222 | 2 | 102 | 2 | 20 | 22 | 22 | 2 | 102 |
| 2222 | 2221 | 0222 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 2222 | 2221 | 1222 | 2 | 101 | 2 | 22 | 20 | 22 | 2 | 101 |
| 2222 | 2221 | 2222 | 2 | 111 | 2 | 22 | 20 | 22 | 2 | 101 |
| 1212 | 1222 | 1221 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 1212 | 2222 | 2221 | 2 | 110 | 3 | 220 | 222 | 222 | 2 | 110 |
| 2212 | 1222 | 2221 | 2 | 102 | 2 | 20 | 22 | 22 | 2 | 102 |
| 2212 | 2222 | 0221 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 2212 | 2222 | 1221 | 2 | 101 | 2 | 22 | 20 | 22 | 2 | 101 |
| 2212 | 2222 | 2221 | 2 | 111 | 2 | 22 | 20 | 22 | 2 | 101 |

| $s=5$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $z_1$ | $z_2$ | $z_3$ | $C(s;z_1,z_2,z_3)$ | $D$ | $s'$ | $z_1'$ | $z_2'$ | $z_3'$ | $C(s';z_1',z_2',z_3')$ | $D'$ |
| 02221 | 20222 | 22222 | 2 | 102 | 2 | 20 | 22 | 22 | 2 | 102 |
| 12221 | 00222 | 22222 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 12221 | 10222 | 12222 | 2 | 21 | 1 | 2 | 1 | 2 | 2 | 21 |
| 12221 | 20222 | 22222 | 4 | 102 | 2 | 21 | 22 | 22 | 4 | 102 |
| 22221 | 00222 | 22222 | 4 | 22 | 1 | 1 | 2 | 2 | 4 | 22 |
| 22221 | 10222 | 02222 | 2 | 12 | 1 | 2 | 0 | 2 | 2 | 12 |
| 22221 | 10222 | 12222 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 22221 | 10222 | 22222 | 2 | 101 | 2 | 22 | 20 | 22 | 2 | 101 |
| 22221 | 20222 | 02222 | 2 | 21 | 1 | 2 | 1 | 2 | 2 | 21 |
| 22221 | 20222 | 12222 | 2 | 100 | 4 | 2221 | 0222 | 2222 | 2 | 100 |
| 22221 | 20222 | 22222 | 2 | 110 | 3 | 220 | 222 | 222 | 2 | 110 |
| 02221 | 02222 | 21222 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 02221 | 12222 | 11222 | 2 | 21 | 1 | 2 | 1 | 2 | 2 | 21 |
| 02221 | 22222 | 21222 | 4 | 102 | 2 | 21 | 22 | 22 | 4 | 102 |
| 12221 | 02222 | 21222 | 4 | 22 | 1 | 1 | 2 | 2 | 4 | 22 |
| 12221 | 12222 | 01222 | 2 | 12 | 1 | 2 | 0 | 2 | 2 | 12 |
| 12221 | 12222 | 11222 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 12221 | 12222 | 21222 | 2 | 101 | 2 | 22 | 20 | 22 | 2 | 101 |
| 12221 | 22222 | 01222 | 2 | 21 | 1 | 2 | 1 | 2 | 2 | 21 |
| 12221 | 22222 | 11222 | 2 | 100 | 4 | 2221 | 0222 | 2222 | 2 | 100 |
| 12221 | 22222 | 21222 | 2 | 110 | 3 | 220 | 222 | 222 | 2 | 110 |
| 22221 | 02222 | 11222 | 2 | 20 | 1 | 2 | 2 | 1 | 2 | 20 |
| 22221 | 12222 | 11222 | 4 | 22 | 1 | 1 | 2 | 2 | 4 | 22 |
| 22221 | 12222 | 21222 | 4 | 101 | 4 | 2221 | 2222 | 1222 | 4 | 101 |
| 22221 | 22222 | 01222 | 4 | 21 | • | | | | | |
| 22221 | 22222 | 11222 | 2 | 101 | 2 | 22 | 20 | 22 | 2 | 101 |
| 22221 | 22222 | 21222 | 4 | 110 | 3 | 221 | 222 | 222 | 4 | 110 |

| $s=6$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $z_1$ | $z_2$ | $z_3$ | $C(s;z_1,z_2,z_3)$ | $D$ | $s'$ | $z_1'$ | $z_2'$ | $z_3'$ | $C(s';z_1',z_2',z_3')$ | $D'$ |
| 022221 | 222222 | 101222 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 022221 | 222222 | 201222 | 2 | 101 | 2 | 22 | 20 | 22 | 2 | 101 |
| 122221 | 022222 | 201222 | 2 | 21 | 1 | 2 | 1 | 2 | 2 | 21 |
| 122221 | 222222 | 101222 | 4 | 22 | 1 | 1 | 2 | 2 | 4 | 22 |
| 122221 | 222222 | 201222 | 2 | 102 | 2 | 20 | 22 | 22 | 2 | 102 |
| 222221 | 022222 | 101222 | 2 | 12 | 1 | 2 | 0 | 2 | 2 | 12 |
| 222221 | 022222 | 201222 | 2 | 22 | 1 | 0 | 2 | 2 | 2 | 22 |
| 222221 | 122222 | 101222 | 2 | 21 | 1 | 2 | 1 | 2 | 2 | 21 |
| 222221 | 122222 | 201222 | 2 | 100 | 4 | 2221 | 0222 | 2222 | 2 | 100 |
| 222221 | 222222 | 001222 | 2 | 20 | 1 | 2 | 2 | 1 | 2 | 20 |
| 222221 | 222222 | 201222 | 4 | 102 | 2 | 21 | 22 | 22 | 4 | 102 |

In the last table (for $s=6$) there are no remaining cases left with •, so this finishes the proof. □

**Theorem 6.15.** *The local system $\mathcal{K}l(\mathsf{Char}(21)^\times)$ on $\mathbb{G}_{m,\mathbb{F}_{2^6}}$ has finite monodromy.*

*Proof.* The criterion for finite monodromy in this case is

$$S(x) := \sum_{i\in I} V\left(x + \frac{i}{21}\right) = \sum_{i\in I} V\left(x + \frac{3i}{2^6-1}\right) \geq \frac{11}{2}$$

for $x \in (\mathbb{Q}/\mathbb{Z})_{\text{prime to } 2}$, where $I = \{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$. If $x \in (\frac{1}{21}\mathbb{Z})/\mathbb{Z}$ one checks directly that $S(x) \in \{\frac{11}{2}, 6, \frac{17}{3}, \frac{19}{3}\}$. For the other values of $x$ it suffices to show, by [KRLT3, Lemma 9.1], that

$$\sum_{i\in I}[x + (2^r - 1)3i] - \frac{11}{2}r \geq 0$$

for every $r \in 6\mathbb{Z}_+$ and every $0 \leq x \leq 2^r - 1$. We prove this by induction on $r$ as explained in [KRLT3, Section 9]. Since multiplying by 2 permutes the elements of $I$ modulo 21, we may take $r_1 = 1$ (following the notation of [KRLT3, Section 9]). For $r \leq 7$ the inequality is verified by a computer check. For $r > 7$ we proceed by replacing the last 7 2-adic digits of $x$ with a shorter string of $s' < 7$ digits with smaller $\Delta$ which gives the same "overflow" digits when adding the different $h_{s,i}$ (cf. [KRLT3, Section 9] for details). This reduces the inequality to a similar one for smaller $r$, already proved by induction.

The following table shows the required substitutions according to $z$, the last 7 digits of $x$. To verify that the overflow digits are indeed the same, note that in this case the $h_{s,i}$ are truncations of the first $s$ digits of the following 12 strings of digits (concatenating as many copies as needed): 000011, 000110, 001100, 001111, 011000, 011110, 100001, 100111, 110000, 110011, 111001, 111100.

| $z \in$ | $\Delta(7, z)$ | $s'$ | $z'$ | $\Delta(s', z')$ |
|---|---|---|---|---|
| $[0000000, 0000110]$ | $\geq 5/2$ | 5 | 00000 | 5/2 |
| $[0000111, 0001100]$ | $\geq 3/2$ | 5 | 00010 | 3/2 |
| $[0001101, 0011000]$ | $\geq 3/2$ | 3 | 001 | 3/2 |
| $[0011001, 0011110]$ | $\geq 7/2$ | 6 | 001101 | 3 |
| $[0011111, 0110000]$ | $\geq 7/2$ | 6 | 010001 | 3 |
| $[0110001, 0111100]$ | $\geq 11/2$ | 5 | 01101 | 11/2 |
| $[0111101, 1000011]$ | $\geq 13/2$ | 1 | 1 | 13/2 |
| $[1000100, 1001111]$ | $\geq 15/2$ | 6 | 100010 | 7 |
| $[1010000, 1100001]$ | $\geq 15/2$ | 3 | 101 | 15/2 |
| $[1100010, 1100111]$ | $\geq 21/2$ | 6 | 110001 | 10 |
| $[1101000, 1110011]$ | $\geq 21/2$ | 5 | 11010 | 21/2 |
| $[1110100, 1111001]$ | $\geq 25/2$ | 6 | 111010 | 12 |
| $[1111010, 1111111]$ | $\geq 25/2$ | 6 | 111101 | 12 |

□

**Theorem 6.16.** *The local system $\mathcal{K}l(\mathsf{Char}(18) \smallsetminus \{\chi_{18}^{2,6,8,14,15,18}\})$ on $\mathbb{G}_{m,\mathbb{F}_{5^6}}$ has finite monodromy.*

*Proof.* The criterion for finite monodromy in this case is

$$S_1(x) := \sum_{i\in I_1} V\left(x + \frac{i}{18}\right) = \sum_{i\in I_1} V\left(x + \frac{868i}{5^6-1}\right) \geq \frac{11}{2}$$

and

$$S_2(x) := \sum_{i\in I_2} V\left(x + \frac{i}{18}\right) = \sum_{i\in I} V\left(x + \frac{868i}{5^6-1}\right) \geq \frac{11}{2}$$

for $x \in (\mathbb{Q}/\mathbb{Z})_{\text{prime to } 5}$, where

$$I_1 = \{1, 3, 4, 5, 7, 9, 10, 11, 12, 13, 16, 17\} \text{ and } I_2 = \{1, 2, 5, 6, 7, 8, 9, 11, 13, 14, 15, 17\}.$$

Since $S_2(x) = S_1(5x)$, it suffices to prove the first inequality. If $x \in (\frac{1}{18}\mathbb{Z})/\mathbb{Z}$ one checks directly that $S_1(x) \in \{\frac{11}{2}, 6\}$. For the other values of $x$ it suffices to show, by [KRLT3, Lemma 9.1], that

$$\sum_{i\in I_1}[x + (5^r - 1)868i] - 22r \geq 0$$

for every $r \in 6\mathbb{Z}_+$ and every $0 \leq x \leq 5^r - 1$. We prove this by induction on $r$ as explained in [KRLT3, Section 9]. Since multiplying by $5^2$ permutes the elements of $I_1$ modulo 18, we may take $r_1 = 2$. For $r \leq 6$ the inequality is verified by a computer check. For $r > 6$ we proceed by replacing the last 6 5-adic digits of $x$ with a string of $s' = 4$ digits with smaller $\Delta$ which gives the same "overflow" digits when adding the different $h_{s,i}$ (cf. [KRLT3, Section 9] for details). This reduces the inequality to a similar one for smaller $r$, already proved by induction.

The following table shows the required substitutions according to $z$, the last 6 digits of $x$. To verify that the overflow digits are indeed the same, note that in this case the $h_{s,i}$ are truncations of the first $s$ digits of the following 12 strings of digits (concatenating as many copies as needed): 011433, 040404, 102342, 114330, 143301, 222222, 234210, 301143, 313131, 330114, 421023, 433011.

| $z \in$ | $\Delta(6, z)$ | $s'$ | $z'$ | $\Delta(s', z')$ |
|---|---|---|---|---|
| $[000000, 011433]$ | $\geq 8$ | 4 | 0000 | 8 |
| $[011434, 023421]$ | $\geq 4$ | 4 | 0120 | 4 |
| $[023422, 114330]$ | $\geq 4$ | 4 | 0240 | 4 |
| $[114331, 131313]$ | $\geq 8$ | 4 | 1144 | 8 |
| $[131314, 143301]$ | $\geq 8$ | 4 | 1314 | 8 |
| $[143302, 210234]$ | $\geq 8$ | 4 | 1434 | 8 |
| $[210240, 222222]$ | $\geq 8$ | 4 | 2103 | 8 |
| $[222223, 301143]$ | $\geq 8$ | 4 | 2223 | 8 |
| $[301144, 330114]$ | $\geq 12$ | 4 | 3012 | 12 |
| $[330120, 342102]$ | $\geq 12$ | 4 | 3302 | 12 |
| $[342103, 404040]$ | $\geq 12$ | 4 | 3422 | 12 |
| $[404041, 433011]$ | $\geq 12$ | 4 | 4041 | 12 |
| $[433012, 444444]$ | $\geq 12$ | 4 | 4331 | 12 |

□

## 7. Finiteness of universal Laurent sheaves in any rank

For any prime $p$, and for prime to $p$ integers

$$1 \leq n_1 < \ldots < n_k < d,\ d > m_1 > \ldots > m_l \geq 1,$$

we consider the local system $\mathcal{G}(-m_l, \ldots, -m_1, n_1, \ldots, n_k, d, \chi)$ on $(\mathbb{G}_m/\mathbb{F}_p)^{k+l+1}$ with trace function

(7.0.1) $$(r_1, \ldots, r_l, s_1, \ldots, s_k, t) \mapsto -\sum_x \psi(r_1x^{-m_1} + \ldots + r_lx^{-m_l} + s_1x^{n_1} + \ldots + s_kx^{n_k} + tx^d)\chi(x).$$

First we determine the sheaves $\mathcal{F}(a, b, \chi)$ in rank $D = a + b \leq 10$ which have finite $G_{\rm geom}$.

**7A. The case** $\gcd(a, b) = 1$**.** We have already noted in Proposition 4.6 that if $\mathcal{F}(a, b, \chi)$, as defined in (4.0.1) and with $\gcd(a, b) = 1$, is to have finite $G_{\rm geom}$ in characteristic $p$, then $\chi$ must be nontrivial, of order one of $2, 3, 4, 5$. Moreover, the requirement that $\gcd(a, b) = 1$ rules out $\chi$ of order 4 or 5 except in the case $a = b = 1$. From [KT5, Theorem 2.4.4], combined with Lemma 4.2, we see that $G_{\rm geom}$ will be infinite if $p > 2(a + b) + 1$ and $(D, \chi) \neq (2, \chi_2)$. We also remark that the change of variable $x \mapsto 1/x$ interchanges $\mathcal{F}(a, b, \chi)$ and $\mathcal{F}(b, a, \overline{\chi})$. A given $\mathcal{F}(a, b, \chi)$ fails to have finite $G_{\rm geom}$ in a given characteristic $p$ if it fails the V-test (done in Mathematica) over some finite extension of

$\mathbb{F}_p$(values of $\chi$). This V-test failure eliminates all but the listed finite many cases, and those are proven to be finite in §6 and in [KRLT4, Theorem 4.7], aside from $\mathcal{F}(1, p^f - 2, \chi_2)$ whose finiteness was proved in [KT7, Lemma 5.5] and also follows from Theorem 3.5(ii).

| type $(a, b)$ | $(p; \chi)$ such that $\mathcal{F}(a, b, \chi)$ has finite $G_{\text{geom}}$ in characteristic $p$ | Proof |
|---|---|---|
| $b > a \geq 2$ | none | V-test |
| $(1, 1)$ | (all $p > 2; \chi_2$) | Salié |
| $(1, 1)$ | $(p = 2, 5; \chi_3)$ | Cor. 6.6 |
| $(1, 1)$ | $(p = 3; \chi_4)$ | Thm. 6.3 |
| $(1, 1)$ | $(p = 2, 3; \chi_5)$ | Thms 6.1, 6.2 |
| $(1, 2)$ | $(p = 3, 7; \chi_2)$ | Thms 6.11, 6.7 |
| $(1, 3)$ | $(p = 5; \chi_2)$ | $3 = 5 - 2$ |
| $(1, 3)$ | $(p = 2, 5; \chi_3)$ | Thms 6.9, 6.10 |
| $(1, 4)$ | none | V-test |
| $(1, 5)$ | $(p = 3, 7; \chi_2)$ | Thm. 6.14, $5 = 7 - 2$ |
| $(1, 6)$ | none | V-test |
| $(1, 7)$ | $(p = 3, 5; \chi_2)$ | $3^2 - 2 = 7$, [KRLT4, Thm. 4.7] |
| $(1, 8)$ | none | V-test |
| $(1, 9)$ | $(p = 11; \chi_2)$ | $9 = 11 - 2$ |

TABLE 1. Finite $G_{\text{geom}}$ for $\mathcal{F}(a, b, \chi)$ with $\gcd(a, b) = 1$, $a + b \leq 10$, $a \leq b$

**Corollary 7.1.** *Let $a, b \in \mathbb{Z}_{\geq 1}$ and $D := a + b \leq 10$. Then the local system $\mathcal{F} := \mathcal{F}(a, b, \chi)$ defined in* (4.0.1) *in characteristic p has finite $G_{\text{geom}}$ if and only if either $(a, b, p, \chi) = (2, 2, 3, \chi_2)$, or* $\min(a, b) = 1$ *and $(a, b, p, \chi)$ are listed in Table 1.*

*Proof.* We apply the first statement of Theorem 4.9. If $\gcd(a, b) = 1$, then we are done. Otherwise $a = b = 2$, $\chi = \chi_2$, and then we are again done by using (2.1.1) and the lines of Tables 1 for $\mathcal{F}(1, 1, \rho)$ with $\rho = \chi_2$ and $\rho = \chi_4$. □

**7B. The situation in rank $\leq 10$ with $\chi_2$.** Fix an odd prime $p$. We now determine all the systems $\mathcal{G}(-1, n, d, \chi_2)$ as defined in (7.0.1) which have finite $G_{\text{geom}}$ in characteristic $p$ for $1 \leq n < d$. By the semicontinuity Theorem 2.5, if $\mathcal{G}(-1, n, d, \chi_2)$ is to have finite $G_{\text{geom}}$, then all three systems of $\mathcal{G}(-1, d, \chi_2) := \mathcal{F}(1, d, \chi_2)$, $\mathcal{G}(-1, n, \chi_2) := \mathcal{F}(1, n, \chi_2)$, and $\mathcal{G}(n, d, \chi_2)$ must have finite monodromy in the given characteristic $p$. Looking at Table 1 in §7, we see that $p$ must be one of $3, 5, 7, 11$, and $d$ must be one of $2, 3, 5, 7, 9$.

(a) For $d = 9$, we must have $p = 11$, but then no $\mathcal{G}(-1, n, \chi_2)$ with $2 \leq n < 9$ in characteristic $p = 11$ has finite $G_{\text{geom}}$. And the system $\mathcal{G}(1, 9, \chi_2)$ in characteristic $p = 11$ fails to have finite monodromy by [KT5, Theorem 10.2.6]. So there are no cases with $d = 9$.

(b) For $d = 7$, we must have $p = 3, 5$.

Suppose first $p = 3$. For $1 \leq n < 7$ and $\mathcal{G}(-1, n, \chi_2)$ has finite monodromy in characteristic 3, then $n$ must be one of $1, 2, 5$. But $\mathcal{G}(2, 7, \chi_2)$ in characteristic $p = 3$ does not have finite monodromy by [KT5, Theorem 10.3.13]. So the maximal possible case is $\mathcal{G}(-1, 1, 5, 7, \chi_2)$, which by Theorem 6.14 does in fact have finite $G_{\text{geom}}$.

Assume now that $p = 5$. Then the possible $1 \leq n < 7$ such that $\mathcal{G}(-1, n, \chi_2)$ has finite $G_{\text{geom}}$ in characteristic 5 are 1 and 3. The $n = 1$ case is $\mathcal{G}(-1, 1, 7, \chi_2)$, but in characteristic $p = 5$,

$\mathcal{G}(1,7,\chi_2)$ does not have finite $G_{\rm geom}$ by [KT5, Theorem 10.2.6]. So the maximal possible case is $\mathcal{G}(-1,3,7,\chi_2)$, which does in fact have finite $G_{\rm geom}$, by [KRLT4, Theorem 4.7].

(c) For $d=5$, we must have $p=3,7$.

Suppose first $p=3$. The possible $1\le n\le 4$, $n\neq 3$ for which $\mathcal{G}(-1,n,\chi_2)$ has finite monodromy in characteristic 3 are $n=1,2$. So the maximal possible case is $\mathcal{G}(-1,1,2,5,\chi_2)$, which in characteristic $p=3$ does in fact have finite $G_{\rm geom}$, by Theorem 6.11.

Assume now that $p=7$. The possible $1\le n\le 4$ for which $\mathcal{G}(-1,n,\chi_2)$ has finite monodromy in characteristic 7 are $n=1,2$. But in characteristic $p=7$, neither $\mathcal{G}(1,5,\chi_2)$ nor $\mathcal{G}(2,5,\chi_2)$ has finite $G_{\rm geom}$, cf. [KT5, Theorem 10.2.6, respectively Theorem 10.3.13]. So there are no $d=5,p=7$ cases.

(d) For $d=3$, we must have $p=5$, and the only possible $1\le n\le 2$ for which $\mathcal{G}(-1,n,\chi_2)$ has finite monodromy in characteristic $p=5$ is $n=1$. So the only possibility is $\mathcal{G}(-1,1,3,\chi_2)$, which in characteristic $p=5$ does have finite $G_{\rm geom}$ by Theorem 6.8.

(e) For $d=2$, we must have $p=3,7$, and the only $n$ possible is $n=1$. Here $\mathcal{G}(-1,1,2,\chi_2)$ has finite $G_{\rm geom}$ in characteristic $p=3$, by semicontinuity and the fact that $\mathcal{G}(-1,1,2,5,\chi_2)$ has finite $G_{\rm geom}$ in characteristic $p=3$ by Theorem 6.11. For $p=7$, $\mathcal{G}(1,2,\chi_2)$ does not have finite $G_{\rm geom}$ because it is of the rank $2<(p-1)/2$.

**7C. The situation in rank $\le 10$, with $\mathsf{o}(\chi)\ge 3$.** As one sees from Table 1 in §7, if $\chi$ has order 4 or 5, the only finite cases are $\mathcal{G}(-1,1,\chi)$, in the named characteristics ($p=2,3$ for $\chi_5$, $p=3$ for $\chi_4$).

Suppose now that $\mathcal{G}(-1,n,d,\chi_3)$ has finite $G_{\rm geom}$. Here we must have $d=3$ and $p=2,5$. For both these $p$, the only $1\le n\le 2$, $n\neq p$, for which $\mathcal{G}(-1,n,\chi_3)$ has finite $G_{\rm geom}$ is $n=1$. However, in characteristic $p=5$, $\mathcal{G}(1,3,\chi_3)$ does not have finite monodromy by [KT5, Theorem 10.2.6]. In characteristic $p=2$, $\mathcal{G}(-1,1,3,\chi_3)$ does have finite $G_{\rm geom}$ by Theorem 6.9.

**7D. Universal Laurent sheaves of rank $D\ge 2$ with finite $G_{\rm geom}$.** We are now ready to prove the second main result of the paper, in whose statement the phrase "any $S\subseteq J$" in the notation for a local system $\mathcal{G}(\ldots)$ means we can add any (including the empty subset) $S$ of the given index set $J$.

**Theorem 7.2.** *In rank $D\ge 2$, the universal Laurent sheaves $\mathcal{F}$ with finite $G_{\rm geom}$ in characteristic $p$ are precisely the following local systems, in the notation of* (7.0.1)*.*

(i) $\mathcal{F}=\mathcal{F}(1,p^f-2,\chi_2)=\mathcal{G}(-1,p^f-2,\chi_2)$, *where $p>2$ and $f\in\mathbb{Z}_{\ge 1}$.*
(ii) $\mathcal{F}=\mathcal{G}(-1,1,\chi)$ *of rank $D=2$, with $(a=1,b=1,p,\chi)$ listed in Table 1 in* §7*.*
(iii) $D=8$, $\mathcal{G}(-1,\text{any } S\subseteq\{1,5\},7,\chi_2)$ *in characteristic $p=3$, and* $\mathcal{G}(-1,\text{any } S\subseteq\{3\},7,\chi_2)$ *in characteristic $p=5$.*
(iv) $D=6$, $\mathcal{G}(-1,\text{any } S\subseteq\{1,2\},5,\chi_2)$ *in characteristic $p=3$.*
(v) $D=4$, $\mathcal{G}(-1,\text{any } S\subseteq\{1\},3,\chi_2)$ *in characteristic $p=5$,* $\mathcal{G}(-1,\text{any } S\subseteq\{1\},3,\chi_3)$ *in characteristic $p=2$, and* $\mathcal{G}(-2,\text{any } \varnothing\neq S\subseteq\{-1,1\},2,\chi_2)$ *in characteristic $p=3$.*
(vi) $D=3$, $\mathcal{G}(-1,\text{any } S\subseteq\{1\},2,\chi_2)$ *in characteristic $p=3$, and* $\mathcal{G}(-1,2,\chi_2)$ *in characteristic $p=7$.*

*Proof.* By Theorem 4.10 and the preceding discussion, it suffices to show that if $\mathcal{F}$ of rank $D\le 10$ has finite monodromy then it is one of the listed systems.

Write $\mathcal{F}=\mathcal{F}(B,A,\chi)$, where $B=\{b_1>\ldots>b_k\ge 1\}$ is the set of the "positive" exponents, and $A=\{a_1>\ldots>a_l\ge 1)$ is the set of the "negative" exponents, with $k,l\ge 1$. We may assume that $b_1\ge a_1$. By Theorem 2.5, $\mathcal{F}(a_1,b_1,\chi)$ has finite monodromy. Hence, by Corollary 7.1, either $a_1=1$ and $(a_1,b_1,p,\chi)$ is listed in Table 1, or $(a_1,b_1,p,\chi)=(2,2,3,\chi_2)$. In the former case, we are done by the discussion in §7B and §7C. In the latter case, we apply Theorem 6.12 (and note that the system $\mathcal{G}(-2, 2, \chi_2)$ is reducible). □

## 8. Determination of finite $G_{\rm geom}$

In this section, we determine the geometric monodromy groups $G_{\rm geom}$ for the local systems listed in Theorem 7.2. Along the way, we realize several of the Shephard–Todd complex reflection groups.

**Proposition 8.1.** *The local system $\mathcal{F}(1, p^f - 2, \chi_2)$ in Theorem 7.2(i), where $p \geq 3$ and $q = p^f \geq 5$, has $G_{\rm geom} = 2 \times \mathsf{S}_q$, where $\mathsf{S}_q$ acts via its deleted natural permutation representation.*

*Proof.* By Lemma 4.3, we can decompose the finite group $G = G_{\rm geom}$ as the central product $C * K$, where $C = C_2$ is the geometric monodromy group of $\mathcal{L}_{\chi_2}$, and $K$ is the geometric monodromy group of the local system $\mathcal{K} = \mathcal{K}l(\mathbb{1}, \mathsf{Char}(q-2, \chi_2))$ with trace function

$$s \mapsto -\sum_{x \neq 0} \psi(1/x + sx^{q-2})\chi_2(x).$$

By the proof of [KT2, Theorem 9.3(ii)], the Kloosterman sheaf $\mathcal{K}l(\chi_2, \mathsf{Char}(q-2)) = \mathcal{K} \otimes \mathcal{L}_{\chi_2}(s)$ has geometric monodromy group $H = \mathsf{S}_q$ acting in its its deleted natural permutation representation. Clearly, $CH$ and $CK$ have the same image in $\mathrm{GL}(V)$. It follows that $G = CH = C_2 \times \mathsf{S}_q$, as stated. □

**Proposition 8.2.** *Let $\mathcal{F}$ be any of the rank 8 local systems $\mathcal{G}(-1, \text{any } S \subseteq \{1, 5\}, 7, \chi_2)$ in characteristic $p = 3$ with $S \neq \varnothing$, or $\mathcal{G}(-1, \text{any } S \subseteq \{3\}, 7, \chi_2)$ in characteristic $p = 5$ listed in Theorem 7.2(iii). Then $\mathcal{F}$ has $G_{\rm geom} = W(E_8)$.*

*Proof.* (a) First we consider the case $\mathcal{F} = \mathcal{G}(-1, \text{any } S \subseteq \{1, 5\}, 7, \chi_2)$ in characteristic $p = 3$, with $S \neq \varnothing$ (note that the case $S = \varnothing$ has already been considered in Proposition 8.1). By [KRLT4, Lemma 2.3], $\mathcal{F}$ is self-dual, which implies that the finite group $G = G_{\rm geom}$ satisfies

$$|\mathbf{Z}(G)| \leq 2, \;\; M_{2,2}(G, V) \geq 3, \tag{8.2.1}$$

where $V$ denotes the underlying representation.

By Theorems 5.2 and 5.3 of [KRLT4], some specialization of $\mathcal{F}$ yields a local system with geometric monodromy group $H = W(E_8)$, which shows that $G \geq H$, whence $M_{2,2}(G, V) \leq M_{2,2}(H, V) = 3$ and $V$ is orthogonally self-dual. Together with (8.2.1), this implies that $M_{2,2}(G, V) = 3$. Now we can apply [GT1, Theorem 1.5], and use the containment $H \leq G < \mathrm{O}(V)$ to obtain that

$$2 \cdot \Omega_8^+(2) \cong L \lhd G \leq \mathbf{N}_{\mathrm{O}(V)}(L) \cong L \cdot 2.$$

Since $|H| = 2|L|$, we conclude that $G = H$.

(b) Now we consider the case $\mathcal{F} = \mathcal{G}(-1, \text{any } S \subseteq \{3\}, 7, \chi_2)$ in characteristic $p = 5$. By [KRLT4, Theorem 5.1], $\mathcal{G}(-1, 3, 7, \chi_2)$ has $G_{\rm geom} = W(E_8)$.

It remains to consider the case $S = \varnothing$, in which case $G = G_{\rm geom}$ embeds in $W(E_8)$ and hence satisfies (8.2.1). By Lemma 4.3, we can decompose $G$ as the central product $C * K$, where $C = C_2$ is the geometric monodromy group of $\mathcal{L}_{\chi_2}$, and $K$ is the geometric monodromy group of the local system $\mathcal{K} = \mathcal{K}l(\mathbb{1}, \mathsf{Char}(7, \chi_2))$. Applying Lemma 4.2 to $\mathcal{K}$, we see that $K$, and hence $G$, is primitive. Next, applying [KT7, Corollary 2.6] to the local system $\mathcal{H}$ with trace function

$$t \mapsto -\sum_{x \neq 0} \psi(tx + 1/x^7)\chi_2(x) = -\sum_{x \neq 0} \psi(t/x + x^7)\chi_2(x)$$

which is a specialization of $\mathcal{F}$, we see that $G$ contains a (true) reflection. It now follows from [KT7, Theorem 5.6] that $G = \mathbf{Z}(G)G_0$, where $G_0 = W(E_8)$, or $G_0 = \mathsf{S}_9$ in its deleted permutation representation. In the former case, we get $G = G_0 = W(E_8)$ by (8.2.1). Suppose we are in the latter case. By [KT7, Lemma 2.1(i)], the $I(\infty)$-representation of $\mathcal{H}$ (which is denoted by $\mathcal{G}(1/x^7, 1, \chi_2)$ there) is irreducible. Hence, by Lemma 3.1(ii), $D + 1 = 9$ must be a power of $p = 5$, which is absurd. □

**Proposition 8.3.** *Let $\mathcal{F}$ be any of the rank 6 local systems $\mathcal{G}(-1, \text{any } S \subseteq \{1, 2\}, 5, \chi_2)$ in characteristic $p = 3$ listed in Theorem 7.2(iv). If $2 \in S$, then $\mathcal{F}$ has $G_{\rm geom} = 6_1 \cdot \mathrm{PSU}_4(3) \cdot 2_2$, the Mitchell group. Otherwise $\mathcal{F}$ has $G_{\rm geom} = C_2 \times W(E_6)$.*

*Proof.* (a) First we consider the case $2 \in S$. Since any trace takes value in $\mathbb{Q}(\zeta_3)$, the finite group $G = G_{\rm geom}$ satisfies

$$\mathbf{Z}(G) \hookrightarrow C_6. \tag{8.3.1}$$

By [KRLT4, Theorem 6.2], some specialization of $\mathcal{F}$ yields a local system with geometric monodromy group $H$ the Mitchell group, which shows that $G \geq H$. Moreover, as $\mathbf{Z}(H) \cong C_6$ acts via scalars on the underlying representation $V$, (8.3.1) now implies that $\mathbf{Z}(G) = \mathbf{Z}(H) = C_6$. The inclusion $H \leq G$ also implies that $M_{2,2}(G, V) \leq M_{2,2}(H, V) = 2$. Now we can apply [GT1, Theorem 1.5], and use the containment $H \leq G < \mathrm{GL}(V)$ and [CCNPW] to obtain that

$$6 \cdot \mathrm{PSU}_4(3) \cong L \lhd G \leq (\mathbf{C}_G(L) * L) \cdot 2_2.$$

Since $\mathbf{C}_G(L) = \mathbf{Z}(G) \geq \mathbf{Z}(L)$ and $|\mathbf{Z}(G)| = |\mathbf{Z}(L)| = 6$, we get $L \lhd G \leq L \cdot 2_2$. But $|H| = 2|L|$, so we conclude that $G = H$.

(b) Now we consider the case $2 \notin S$. By [KRLT4, Theorem 5.1], $\mathcal{G}(-1, 1, 5, \chi_2)$ has $G_{\rm geom} = C_2 \times W(E_6)$.

It remains to consider the case $S = \varnothing$, in which case $G = G_{\rm geom}$ embeds in $C_2 \times W(E_6)$; in particular, $|\mathbf{Z}(G)| \leq 2$. By Lemma 4.3, we can decompose $G$ as the central product $C * K$, where $C = C_2$ is the geometric monodromy group of $\mathcal{L}_{\chi_2}$, and $K$ is the geometric monodromy group of the local system $\mathcal{K} = \mathcal{K}l(\mathbb{1}, \mathsf{Char}(5, \chi_2))$. As $C$ acts via scalars on the underlying representation, we see that

$$\mathbf{Z}(G) = C_2. \tag{8.3.2}$$

Applying Lemma 4.2 to $\mathcal{K}$, we see that $K$, and hence $G$, is primitive. Next, applying [KT7, Corollary 2.6] to the local system $\mathcal{H}$ with trace function

$$t \mapsto -\sum_{x \neq 0} \psi(tx + 1/x^5)\chi_2(x) = -\sum_{x \neq 0} \psi(t/x + x^5)\chi_2(x)$$

which is a specialization of $\mathcal{F}$, we see that $G$ contains a (true) reflection. It now follows from [KT7, Theorem 5.6] that $G = \mathbf{Z}(G)G_0$, where $G_0 = W(E_6)$, or $G_0 = H$ the Mitchell group. In the former case, we get $G = C_2 \times W(E_6)$ by (8.3.2). Suppose we are in the latter case. Then $C_6 = \mathbf{Z}(H) \leq \mathbf{Z}(G)$, contradicting (8.3.2). □

**Proposition 8.4.** *Let $\mathcal{F}$ be any of the rank 4 local systems listed in Theorem 7.2(v). Then the following statements hold for its $G_{\rm geom}$.*

(i) *If $\mathcal{F} = \mathcal{G}(-1, \text{any } S \subseteq \{1\}, 3, \chi_3)$ in characteristic $p = 2$, then $G_{\rm geom} = C_3 \times \mathrm{Sp}_4(3)$.*
(ii) *If $\mathcal{F} = \mathcal{G}(-2, \text{any } \varnothing \neq S \subseteq \{-1, 1\}, 2, \chi_2)$ in characteristic $p = 3$, then $G_{\rm geom} = C_3 \times \mathrm{Sp}_4(3)$.*
(iii) *If $\mathcal{F} = \mathcal{G}(-1, \text{any } S \subseteq \{1\}, 3, \chi_2)$ in characteristic $p = 5$, then $G_{\rm geom} = C_2 \times \mathsf{S}_5$ if $S = \varnothing$, and $G_{\rm geom} = W(H_4) = (\mathrm{SL}_2(5) * \mathrm{SL}_2(5)) \cdot 2$, the Shephard–Todd group $ST30$, if $S = \{1\}$.*

*Proof.* (i) It is clear that the trace function of $\mathcal{F}$ take values in $\mathbb{Q}(\zeta_3)$. Hence, if we denote the geometric monodromy group of $\mathcal{F}_1 = \mathcal{G}(-1, 1, 3, \chi_3)$ by $G_1$, and of $\mathcal{F}_0 = \mathcal{G}(-1, 3, \chi_3)$ by $G_0$, then

$$G_0 \hookrightarrow G_1, \ \mathbf{Z}(G_0) \hookrightarrow \mathbf{Z}(G_1) \hookrightarrow C_6. \tag{8.4.1}$$

By Lemma 4.3, we can decompose $G_0$ as the central product $C * K$, where $C = C_3$ is the geometric monodromy group of $\mathcal{L}_{\chi_3}$, and $K$ is the geometric monodromy group of the local system $\mathcal{K} = \mathcal{K}l(\mathbb{1}, \mathsf{Char}(3, \chi_3))$. As a generator $\boldsymbol{z}$ of $C$ acts via the scalar $\zeta_3$ on the underlying representation $V$, we see that

$$C \leq \mathbf{Z}(G_0). \tag{8.4.2}$$

Furthermore, a generator $g_0$ of the image of $I(0)$ in $K$ acts on $V$ with spectrum $\{1, \zeta_9, \zeta_9^4, \zeta_9^7\}$; in particular, $g_0$ has central order 9, and $h := g_0^3 \boldsymbol{z}^2$ is a complex reflection of order 3. Applying Lemma 4.2 to $\mathcal{K}$, we see that $K$ is primitive. Now, if $\mathcal{K}$ is tensor decomposable, or tensor induced, then, since $\dim V = 4$, the element $h$ fixes a tensor decomposition $V = A \otimes B$ with $\dim A = \dim B = 2$, which is impossible. We have therefore shown that $K$, and hence $G_0$ and $G_1$, satisfies the condition ($\mathbf{S}+$) of [KT2, Definition 1.2]. It follows from [KT2, Lemma 1.1] that each $G_i$ with $i = 0, 1$ is either almost quasisimple, or an extraspecial normalizer.

Suppose we are in the latter case. Then $G_i$ contains a normal irreducible subgroup $E = 2^{1+4}_\epsilon$ with $\epsilon = \pm$ or $E = C_4 * 2^{1+4}_+$. If $E = C_4 * 2^{1+4}_+$, then $\mathbf{Z}(G_1) \geq \mathbf{Z}(E) = C_4$, contrary to (8.4.1). So $E = 2^{1+4}_\epsilon$, and hence

$$E \lhd G_1 \leq \mathbf{N}_{\mathrm{GL}(V)}(E) = \mathbf{Z}(\mathrm{GL}(V))E \cdot \mathrm{O}^\epsilon_4(2),$$

since $\mathrm{Aut}(E) \cong (E/\mathbf{Z}(E)) \cdot \mathrm{O}^\epsilon_4(2)$. Recall that $g_0$ has central order 9, and $|E| = 2^5$. It follows that $\mathrm{O}^\epsilon_4(2)$, and hence its normal subgroup $\Omega^\epsilon_4(2)$, contains an element of order 9, which is however impossible: by [KlL, Proposition 2.9.1(iv), (v)] we have

$$\Omega^+_4(2) \cong \mathrm{SL}_2(2) \times \mathrm{SL}_2(2) \cong \mathsf{S}_3 \times \mathsf{S}_3,$$

and

$$\Omega^-_4(2) \cong \mathrm{SL}_2(4) \cong \mathsf{A}_5.$$

We have shown that each of $G_i$ is almost quasisimple: $S \lhd G_i/\mathbf{Z}(G_i) \leq \mathrm{Aut}(S)$ for some finite non-abelian simple group $S$ (which may depend on $i = 0, 1$). As $\bar{\mathsf{o}}(g_0) = 9$, $\mathrm{Aut}(S)$ contains an element of order 9; furthermore, the quasisimple group $L = G_i^{(\infty)}$ which is a cover of $S$ acts irreducibly on $V$ and $\mathbf{Z}(L) \leq \mathbf{Z}(G_1) \leq C_6$ by (8.4.1). Checking the possibilities for $L$ listed in [HM], we see that $L \cong \mathrm{Sp}_4(3)$. Since the representation $V|_L$ is not invariant under the outer automorphism of $L$, using (8.4.1) and (8.4.2) we can conclude that $G_i = CL = C_3 \times \mathrm{Sp}_4(3)$ for each $i = 0, 1$, and thus $G_0 = G_1$.

(ii) Note that $\mathcal{F}$ is not irreducible if $S = \varnothing$. By symmetry we may assume $1 \in S$, and note by [KT8, Lemma 2.3] that (some specialization of, and hence) $\mathcal{F}$ is not self-dual, which implies that the field of traces is $\mathbb{Q}(\zeta_3)$. Next, by Lemma 2.8 we have $M_{2,2} = 2$, which implies that $G_{\mathrm{geom}}$ is irreducible on the symmetric square of the underlying representation $V = \mathbb{C}^4$. In turn, the latter condition implies by [GT2, Lemma 2.1] that $G_{\mathrm{geom}}$ satisfies ($\mathbf{S}+$). Denoting the geometric monodromy group of $\mathcal{F}_1 = \mathcal{G}(-2, -1, 1, 2, \chi_2)$ by $G_1$, and of $\mathcal{F}_0 = \mathcal{G}(-2, 1, 2, \chi_2)$ by $G_0$, we see that (8.4.1) holds. It also follows from [KT2, Lemma 1.1] that each $G_i$ with $i = 0, 1$ is either almost quasisimple, or an extraspecial normalizer. Assume in addition that $G_i$ is an extraspecial normalizer for some $i = 0, 1$. Then $G_i$ contains an irreducible 2-subgroup $E = C_4 * 2^{1+4}_+$ or $E = 2^{1+4}_\epsilon$ with $\epsilon = \pm$. The former case is impossible since the field of traces $\mathbb{Q}(\zeta_3)$ does not contain $\zeta_4$. In the latter case the irreducible action of $E$ on $V = \mathbb{C}^4$ fixes a unique (up to scalar) non-degenerate, symplectic if $\epsilon = -$ and orthogonal if $\epsilon = +$, bilinear form $(\cdot, \cdot)$ on $V$. Since $E \lhd G_i$, $G_i$ preserves this form up to a scalar. Thus $G_i$ fixes a 1-dimensional subspace of $(V^*)^{\otimes 2}$, and hence of $V^{\otimes 2}$, contrary to the equality $M_{2,2} = 2$.

Therefore, $G_i$ is almost quasisimple, and so $L_i := G_i^{(\infty)}$ is quasisimple, acting irreducibly on $V$, whence $\mathbf{C}_{G_i}(L_i) = \mathbf{Z}(G_i) \leq C_6$ by (8.4.1) and $G_i/\mathbf{Z}(G_i)L_i \hookrightarrow \mathrm{Out}(L_i)$. The same arguments as in the extraspecial normalizer case show that $V|_{L_i}$ is not self-dual, and hence the field of traces of $V|_{L_i}$ is $\mathbb{Q}(\zeta_3)$. Checking the possibilities for $L$ listed in [HM], we see that $L_i \cong \mathrm{Sp}_4(3)$. Since the $L_i$-module $V$ is not invariant under outer automorphisms of $L_i$, we have $G_i = \mathbf{Z}(G_i)L_i$; also $\mathbf{Z}(G_i) \geq \mathbf{Z}(L_i) = C_2$. Now, if $\mathbf{Z}(G_i) = C_2$, then $G_i < \mathrm{SL}(V)$. However, the geometric determinant of $\mathcal{F}_0 = \mathcal{G}(-2, 1, 2, \chi_2)$ has order divisible by 3. To see this, we argue as follows. Consider the weight zero twist of $\mathcal{F}_0$ given by

$$(r, s, t) \mapsto (1/\mathsf{Gauss}(\psi, \chi_2)) \sum_{x \neq 0} \psi(r/x^2 + sx + tx^2)\chi_2(x).$$

For a chosen multiplicative generator $w$ of $\mathbb{F}_9^\times$, a $\mathsf{Magma}$ calculation [BCP] shows that

$$\det(\mathsf{Frob}_{(1,w,1),\mathbb{F}_9}) = 1, \;\; \det(\mathsf{Frob}_{(1,w^2,1),\mathbb{F}_9}) = \zeta_3^2 \text{ for } \zeta_3 := \psi_{\mathbb{F}_3}(1) = \psi_{\mathbb{F}_9}(w).$$

The ratio of the determinants of any two Frobenii of the same degree is the determinant of an element of $\pi_1^{\mathrm{geom}}$. Hence $\zeta_3$ is a value assumed by the geometric determinant. It follows from (8.4.1) that $\mathbf{Z}(G_i) \cong C_6$, and so $G_i \cong C_3 \times \mathrm{Sp}_4(3)$.

(iii) Denote the geometric monodromy group of $\mathcal{F}_1 = \mathcal{G}(-1, 1, 3, \chi_2)$ by $G_1$, and that of $\mathcal{F}_0 = \mathcal{G}(-1, 3, \chi_2)$ by $G_0$. Then $G_0 = C_2 \times \mathsf{S}_5$ by Proposition 8.1, and the underlying representation $V$ is an orthogonal $G_0$-module. By [KRLT4, Lemma 2.3], $\mathcal{F}_1$ is self-dual, which implies that the $G_1$-module $V$ is also orthogonal, and $\mathbf{Z}(G_0) \leq \mathbf{Z}(G_1) \hookrightarrow C_2$. But $\mathbf{Z}(G_0) \cong C_2$, so we conclude that

$$\mathbf{Z}(G_1) = \mathbf{Z}(G_0) \cong C_2. \tag{8.4.3}$$

where $V$ denotes the underlying representation. The self-duality also implies that the trace function of $\mathcal{F}_1$ takes values in the real subfield $\mathbb{Q}(\sqrt{5})$ of $\mathbb{Q}(\zeta_5)$.

By Lemma 4.3, we can decompose $G_0$ as the central product $C * K$, where $C = C_2$ is the geometric monodromy group of $\mathcal{L}_{\chi_2}$, and $K$ is the geometric monodromy group of the local system $\mathcal{K} = \mathcal{K}l(\mathbb{1}, \mathsf{Char}(3, \chi_2))$. Furthermore, a generator $g_0$ of the image of $I(0)$ in $K$ acts on $V$ with spectrum $\{1, \zeta_6, \zeta_6^3, \zeta_6^5\}$; in particular, $g_0$ has central order 6, and $h := g_0^3\boldsymbol{z}$ is a (true) reflection. Applying Lemma 4.2 to $\mathcal{K}$, we see that $K$ is primitive. Now, if $\mathcal{K}$ is tensor decomposable then the element $h$ fixes a tensor decomposition $V = A \otimes B$ with $\dim A = \dim B = 2$, which is impossible. We have therefore shown that $K$, and hence $G_1$, satisfies the condition ($\mathbf{S}-$) of [KT5, §1.1.1]. It follows from [KT5, Lemma 1.1.9] that, for $i = 0, 1$, $G_1$ is either an extraspecial normalizer, or the layer $L = E(G_1)$ of $G_1$ is a central product $L_1 * \ldots * L_n$ of $n \leq 2$ quasisimple groups which are transitively permuted by $G_1$ and $V|_L$ is irreducible.

In the former case, $G_1$ contains a normal irreducible subgroup $E = 2_\epsilon^{1+4}$ with $\epsilon = \pm$ or $E = C_4 * 2_+^{1+4}$. Since $V$ is orthogonal, we must have that $E = 2_+^{1+4}$, and, as in (i), we have

$$E \lhd G_1 \leq \mathbf{N}_{\mathrm{O}(V)}(E) \leq E \cdot \mathrm{O}_4^+(2),$$

where $\mathrm{O}_4^+(2)$ is solvable. It follows that $G_1$ is solvable, a contradiction since $\mathsf{S}_5 \leq G_0 \leq G_1$.

Thus we are in the latter case. In what follows we use one more piece of information that two traces of $\mathcal{F}_1$ over $\mathbb{F}_{5^4}$ are $\sqrt{5}$ and 4, which shows in particular that

$$G_1 = G_{\mathrm{arith},\mathbb{F}_{5^4}}. \tag{8.4.4}$$

Assume in addition that $n = 1$, i.e. $L$ is quasisimple. Recalling $V|_L$ is irreducible and orthogonal, we can check using [HM] that $L \cong \mathsf{A}_5$. But $G_0 = \mathbf{Z}(G_1) \times \mathsf{S}_5$ normalizes $L$ and $\mathrm{Aut}(L) \cong \mathsf{S}_5$. Hence $L = [G_0, G_0]$ and $G_1 = \mathbf{Z}(G_1)G_0 = C_2 \times \mathsf{S}_5$. But this implies by (8.4.4) that all traces of $\mathcal{F}_1$ over $\mathbb{F}_{5^4}$ are integers, a contradiction.

We have shown that $L = L_1 * L_2$ where $L_1 \cong L_2$ is quasisimple. As $V|_L$ is irreducible and faithful, its character is $\alpha_1 \otimes \alpha_2$ with $\alpha_j \in \mathrm{Irr}(L_j)$ of degree $> 1$, whence of degree 2. This implies, say again by [HM], that $L_1 \cong L_2 \cong \mathrm{SL}_2(5)$. Recall that $G_1$ acts on the set $\{L_1, L_2\}$ transitively, with kernel say $H$ of index 2. Then $H$ stabilizes the character $2\alpha_1$ of $L_1$, which implies that $H$ cannot induce an outer automorphism of $L_1$. Similarly, $H$ cannot induce an outer automorphism of $L_2$. As

$$G_1/\mathbf{C}_{G_1}(L)L \hookrightarrow \mathrm{Out}(L) = C_2^2 \cdot C_2,$$

it follows that $G_1/\mathbf{C}_{G_1}(L)$ has order 2. Also, $\mathbf{Z}(L) \leq \mathbf{C}_{G_1}(L) = \mathbf{Z}(G_1)$ by Schur's lemma and $\mathbf{Z}(L) \geq \mathbf{Z}(L_1) \cong C_2$. Using (8.4.3), we now obtain that

$$G_1 = (\mathrm{SL}_2(5) * \mathrm{SL}_2(5)) \cdot C_2. \tag{8.4.5}$$

Let $R$ be the (normal) subgroup of $G_1$ generated by all reflections (including $h$). Then $|R|$ is even, but $> 2$ (otherwise $R \leq \mathbf{Z}(G_1)$ and hence cannot contain any reflection. The described structure (8.4.5) of $G_1$ now implies that either $R = L$ or $R = G_1$. In the former case, $V|_R = V|_L$ is tensor decomposable which is impossible since $R \ni h$. So $R = G_1$ and thus $G_1$ is a primitive complex reflection group. Using [ST] and (8.4.5), we readily conclude that $G_1 \cong W(H_4)$.

Note that $W(H_4)$ satisfies (**S**−), but **not** (**S**+). □

**Lemma 8.5.** *The Frobenius traces of the local system $\mathcal{G}(-1, 2, \chi_2)$ in characteristic $p = 7$ lie in the field $\mathbb{Q}(\sqrt{-7})$.*

*Proof.* Denote by Gauss a choice of quadratic Gauss sum over $\mathbb{F}_7$. The (normalized, to be of weight zero) trace, for $L/\mathbb{F}_7$ a finite extension of degree $f$, at $(r, s) \in (L^\times)^2$, is

$$(-1/\mathsf{Gauss})^f \sum_{x \in L^\times} \psi_L(r/x + sx^2)\chi_{2,L}(x).$$

The quantity Gauss itself lies in $\mathbb{Q}(\sqrt{-7})$, indeed it is $\pm\sqrt{-7}$. So it suffices to show that the "numerator"

$$\sum_{x \in L^\times} \psi_L(r/x + sx^2)\chi_{2,L}(x)$$

lies in this field, i.e., to show that it is fixed by the subgroup of squares in $\mathbb{F}_7^\times = \mathrm{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q})$. The subgroup of squares is $\{1, 2, 4\}$, generated by $4 = 2^2$. This automorphism maps the above numerator to

$$\begin{aligned}\sum_{x \in L^\times} \psi_L(4r/x + 4sx^2)\chi_{2,L}(x) &= \sum_{x \in L^\times} \psi_L(8r/(2x) + s(2x)^2)\chi_{2,L}(x) \\ &= \sum_{x \in L^\times} \psi_L(r/x + sx^2)\chi_{2,L}(x/2),\end{aligned}$$

and $\chi_{2,L}(2) = 1$, precisely because 2 is a square (namely $4^2$) in $\mathbb{F}_7$. □

**Proposition 8.6.** *Let $\mathcal{F}$ be any of the rank 3 local systems listed in Theorem 7.2(vi). Then the following statements hold for its $G_{\mathrm{geom}}$.*

(i) *If $\mathcal{F} = \mathcal{G}(-1, 2, \chi_2)$ in characteristic $p = 7$, then $G_{\mathrm{geom}} = C_2 \times \mathrm{PSL}_2(7)$, the Shephard–Todd group $ST24$.*

(ii) *If $\mathcal{F} = \mathcal{G}(-1, \text{any } S \subseteq \{1\}, 2, \chi_2)$ in characteristic $p = 3$, then $G_{\mathrm{geom}} = C_2 \times (3_+^{1+2} \rtimes C_4)$ if $S = \varnothing$, and $G_{\mathrm{geom}} = C_2 \times (3_+^{1+2} \rtimes \mathrm{SL}_2(3))$, the Shephard–Todd group $ST26$, if $S = \{1\}$.*

*Proof.* (i) By Lemma 8.5, the trace function of $\mathcal{F}$ take values in $\mathbb{Q}(\sqrt{-7})$. Hence, for $G = G_{\mathrm{geom}}$ we have $|\mathbf{Z}(G)| \leq 2$. By Lemma 4.3, we can decompose $G$ as the central product $C * K$, where $C = C_2$ is the geometric monodromy group of $\mathcal{L}_{\chi_2}$, and $K$ is the geometric monodromy group of the local system $\mathcal{K} = \mathcal{K}l(\mathbb{1}, \chi_4, \chi_4^3)$. As a generator $\boldsymbol{z}$ of $C$ acts via the scalar $-1$ on the underlying representation $V$, we see that

$$\mathbf{Z}(G) = C_2.$$

Next, the Kummer $[4]_\star$ pullback of $\mathcal{K}$ is the local system on $\mathbb{A}^1$ considered in [KT1, Theorem 11.1], where it was shown to have $L = \mathrm{PSL}_2(7)$ as its geometric monodromy group. Now the simple group $L$ is normal in $K$, and hence in $G$, but its irreducible action on $V$ is not invariant under outer automorphisms of $L$. Since $G/\mathbf{C}_G(L) \hookrightarrow \mathrm{Aut}(L) \cong L \cdot 2$ and $\mathbf{C}_G(L) = \mathbf{Z}(G)$, it follows that $G = C \times L$. Since $G$ is generated by its involutions which act as reflections on $V$, we conclude that $G \cong ST24$.

(ii) Denote the geometric monodromy group of $\mathcal{F}_1 = \mathcal{G}(-1, 1, 2, \chi_2)$ by $G_1$, and that of $\mathcal{F}_0 = \mathcal{G}(-1, 2, \chi_2)$ by $G_0$. By Lemma 4.3, we can decompose $G_0$ as the central product $C * K$, where $C = \langle \boldsymbol{z} \rangle \cong C_2$ is the geometric monodromy group of $\mathcal{L}_{\chi_2}$, and $K$ is the geometric monodromy group of the local system $\mathcal{K} = \mathcal{K}l(\mathbb{1}, \chi_4, \chi_4^3)$. Here, $\boldsymbol{z}$ acts via the scalar $-1$ on the underlying representation $V$. Since the field of traces of $\mathcal{F}_i$, $i = 0, 1$, is contained in $\mathbb{Q}(\zeta_3)$, we have

$$C \leq \mathbf{Z}(G_0) \leq \mathbf{Z}(G_1) \hookrightarrow C_6. \tag{8.6.1}$$

Furthermore, a generator $g_0$ of the image of $I(0)$ in $K$ acts on $V$ with spectrum $\{1, \zeta_4, \zeta_4^3\}$; in particular, $g_0$ has central order 4 and $h := g_0^2 \boldsymbol{z}$ is a (true) reflection. Applying Lemma 4.2 to $\mathcal{K}$, we see that $K$ is primitive of rank 3, hence satisfies ($\mathbf{S}+$). The same is true for $G_i$ with any $i = 0, 1$, so we can apply [KT2, Lemma 1.1] to $G_i$. Since the field of traces is contained in $\mathbb{Q}(\zeta_3)$, $G_i$ cannot be be almost quasisimple by [HM]. Hence it must be an extraspecial normalizer, i.e. $G_i$ contains a normal irreducible extraspecial subgroup $E_i = 3^{1+2}_\epsilon$, where $\epsilon = \pm$. By Schur's lemma, $\mathbf{C}_{G_i}(E_i) = \mathbf{Z}(G_i)$, and so $G_i/\mathbf{Z}(G_i) \hookrightarrow \mathrm{Aut}(E_i)$. Now, if $\epsilon = -$, i.e. $\exp(E_i) = 9$, then $\mathrm{Aut}(E_i)$ has order $6 \cdot |E_i/\mathbf{Z}(E_i)| = 54$. In particular, $G_i$ cannot contain any element of central order 4, contrary to the containment $g_0 \in G_i$. It follows that $\epsilon = +$. Also, $\mathbf{Z}(G_i) \geq \mathbf{Z}(E_i) \cong C_3$, so (8.6.1) implies that

$$\mathbf{Z}(G_i) = C \times \mathbf{Z}(E_i) \cong C_6, \ G_i/\mathbf{Z}(G_i)E_i \hookrightarrow \mathrm{Out}^+(E_i) \cong \mathrm{SL}_2(3), \tag{8.6.2}$$

where $\mathrm{Out}^+(E_i)$ consists of outer automorphisms of $E_i$ that act trivially on $\mathbf{Z}(E_i)$ (note that the action of $G_i$ via conjugation on $\mathbf{Z}(E_i)$ is trivial since $\mathbf{Z}(E_i) \leq \mathbf{Z}(G_i)$).

Suppose $i = 0$. By [KT2, Theorem 4.1], $G_0/\mathbf{Z}(G_0)$ is the normal closure of the image of $I(0)$. Together with (8.6.2), this implies that

$$G_0/\mathbf{Z}(G_0)E_0 \leq \mathbf{O}^3(\mathrm{SL}_2(3)) = Q_8.$$

Note that the Kummer $[4]_\star$ pullback of $\mathcal{F}_0$ is the Pink–Sawin sheaf considered in [KT5, Theorem 7.3.8], where it was shown to have $E_0 = 3^{1+2}_+$ as its geometric monodromy group. It follows that $|K/E_0|$ divides 4. But $K$ contains the element $g_0$ of order 4, so we conclude that $K = 3^{1+2}_+ \rtimes C_4$ and $G_0 = C_2 \times (3^{1+2}_+ \rtimes C_4)$.

Assume now that $i = 1$. Then Lemma 2.6 implies that $M_{2,2} = 2$, which implies that $G_1/\mathbf{Z}(G_1)E_1$ acts transitively on the 8 nontrivial linear characters of $E_1$ that occur in $V \otimes V^*$. Using (8.6.2), we see that $G_1/\mathbf{Z}(G_1)E_1$ is $Q_8$ or $\mathrm{SL}_2(3)$. Assume in addition that we are in the former case. Noting that $E_1 = \mathbf{O}_3(G_1)$ is normal in $H := G_{\mathrm{arith},\mathbb{F}_3}$ and the field of traces of $H$ is still (contained in) $\mathbb{Q}(\zeta_3)$, we see that $\mathbf{Z}(H) = \mathbf{Z}(G_1) \cong C_6$ and $H/\mathbf{Z}(H)E_1 \hookrightarrow \mathrm{Out}^+(E_1)$. This implies that $H/G_1 \hookrightarrow C_3$, and hence $G_{\mathrm{arith},\mathbb{F}_{27}} = G_1 = \mathbf{Z}(G_1)E_1 \rtimes Q_8$, which shows that the arithmetic determinant of $\mathcal{F}_1$ over $\mathbb{F}_{27}$ is $\pm 1$. However, for the weight zero normalization of $\mathcal{G}(-1, 1, 2, \chi_2)$ with trace function on $L := \mathbb{F}_{3^f}$ given by

$$(r, s, t) \in (L^\times)^3 \mapsto (-1/\mathsf{Gauss}^f) \sum_{x \in L^\times} \psi_L(r/x + sx + tx^2)\chi_{2,L}(x),$$

and a chosen generator $w$ of $\mathbb{F}_{27}^\times$, a $\mathsf{Magma}$ calculation shows that $\mathsf{Frob}_{(1,w,1),\mathbb{F}_{27}}$ has determinant $\zeta_3$. Thus we must be in the latter case: $G_1/\mathbf{Z}(G_1)E_1 \cong \mathrm{SL}_2(3)$.

It remains to show that $G_1 \cong H_1$, where $H_1$ is the Hessian complex reflection group $ST26$. First we note that $H_1$ has the same center $C_6$ and the same normal irreducible subgroup $E_1$ (after a suitable identification using their actions on $V = \mathbb{C}^3$). Moreover, $H_1$ acting via conjugation on $E_1$ induces the same subgroup $C_3^2 \cdot \mathrm{SL}_2(3)$ of $\mathrm{Aut}(E_1)$, as of $G_1$. Furthermore, $H_1$ is generated by a set $R$ of complex reflections $c$, of order 2 or 3. Any $c \in R$ has trace $-1$, $1-\zeta_3$, or $1-\zeta_3^2$, which all belong to $\mathbb{Q}(\zeta_3)^\times$. Since $H_1$ and $G_1$ induce the same subgroup of automorphisms of $E_1$, there is some $g \in G_1$ such that the element $cg^{-1} \in \mathrm{GL}(V)$ centralizes the image of $E_1$ in $\mathrm{GL}(V)$. By Schur's lemma, $cg^{-1}$ acts a scalar $\gamma \in \mathbb{C}$. As $c$ and $g$ both have finite order, $\gamma$ is a root of unity. Next,

$$\mathbb{Q}(\zeta_3)^\times \ni \mathrm{Trace}(c) = \gamma \cdot \mathrm{Trace}(g),$$

and $\mathrm{Trace}(g) \in \mathbb{Q}(\zeta_3)$ as mentioned above. It follows that $\gamma$ is a root of unity in $\mathbb{Q}(\zeta_3)$, whence $\gamma^6 = 1$, and so $\gamma \cdot \mathrm{Id}_V = z$ for some $z \in \mathbf{Z}(G_1)$. Thus $c = gz \in G_1$. This is true for all $c \in R$, so $H_1 = \langle R \rangle \leq G_1$. By order consideration, we have $H_1 = G$, as stated. ☐

**Proposition 8.7.** *Let $\mathcal{F} = \mathcal{G}(-1,1,\chi_d)$ be any of the rank 2 local systems listed in Theorem 7.2(ii). Then the following statements hold for its $G_{\mathrm{geom}}$.*

(i) *If $\mathcal{F} = \mathcal{G}(-1,1,\chi_2)$ in characteristic $p > 2$, then $G_{\mathrm{geom}} = C_2 \times D_{2p}$, where $D_{2p}$ denotes the dihedral group of order $2p$.*
(ii) *Suppose $\mathcal{F} = \mathcal{G}(-1,1,\chi_3)$ in characteristic $p = 2, 5$. Then $G = C_3 \times \mathrm{SL}_2(3)$, the Shephard–Todd group $ST5$ if $p = 2$. If $p = 5$, then $G = C_3 \times \mathrm{SL}_2(5)$, the Shephard–Todd group $ST20$.*
(iii) *If $\mathcal{F} = \mathcal{G}(-1,1,\chi_4)$ in characteristic $p = 3$, then $G = C_4 * 2\mathsf{S}_4$, the Shephard–Todd group $ST8$.*
(iv) *If $\mathcal{F} = \mathcal{G}(-1,1,\chi_5)$ in characteristic $p = 2, 3$, then $G = C_5 \times \mathrm{SL}_2(5)$, the Shephard–Todd group $ST16$.*

*Proof.* By Lemma 4.3, we can decompose $G = G_{\mathrm{geom}}$ as the central product $C * K$, where $C = C_d$ is the geometric monodromy group of $\mathcal{L}_{\chi_d}$, and $K$ is the geometric monodromy group of the local system $\mathcal{K} = \mathcal{K}l(\mathbb{1}, \chi_d)$. As a generator $\boldsymbol{z}$ of $C$ acts via the scalar $\zeta_d$ on the underlying representation $V = \mathbb{C}^2$, we see that

$$C = C_d \hookrightarrow \mathbf{Z}(G),\ G = C * K. \tag{8.7.1}$$

Also, by [KT2, Theorem 4.1], $K$ is the normal closure of the image of $I(0)$ which is generated by a complex reflection of order $d$. Thus $K$ is generated by complex reflections of order $d$.

First we consider the case $d = 2$. In this Salie case, $\mathcal{K}$ is the Kummer $[2]_\star$ direct image of $\mathcal{L}_\psi$, cf. [Ka2, 5.6.2], whence $|K|$ is $p$ or $2p$. As mentioned above, $K$ is generated by involutions. It follows that $|K| = 2p$, and in fact $K \cong D_{2p}$. Hence $G = C * K \cong C_2 \times D_{2p}$, as stated in (i).

From now on we will assume that $d > 2$. Then $\mathcal{K}$ is not Kummer induced, and hence it is primitive. Thus $K$ is one of the primitive complex reflection groups in dimension 2, described in [ST].

Suppose that $d = 5$, so $p = 2, 3$. By [ST] we have $K = \mathbf{Z}(K) * \mathrm{SL}_2(5)$, where $\mathbf{Z}(K) \cong C_{10a}$ with $a \in \{1,2,3,6\}$. Moreover, since $K$ is generated by elements of order 5, we have $K = \mathbf{O}^{5'}(K)$, which rules out the values 2, 3, and 6 for $a$. Hence $K = C_{10} * \mathrm{SL}_2(5) = C_5 \times \mathrm{SL}_2(5) = ST16$. Using (8.7.1), we get $G = K$, as stated in (iv).

Suppose that $d = 4$, so $p = 3$. By [ST] we have $K = \mathbf{Z}(K) * 2\mathsf{S}_4$, where $\mathbf{Z}(K) \cong C_{4a}$ with $a \in \{1,2,3,6\}$. Since $K$ is generated by complex reflections of order 4, we see that $\det(x)^4 = 1$ for all $z \in K$, which rules out the values 3 and 6 for $a$. Moreover, the field of traces of $\mathcal{F}$ is contained in $\mathbb{Q}(\zeta_4, \zeta_3)$ which does not contain $\zeta_8$, ruling out the case $a = 2$. Hence $K = C_4 * 2\mathsf{S}_4 = ST8$. Using (8.7.1), we get $G = K$, as stated in (iv).

Suppose that $d = 3$, so $p = 2, 5$. As $K$ is generated by complex reflections of order 3, we see that $\det(x)^3 = 1$ for all $z \in K$, which then implies by (8.7.1) that

$$\det(\mathcal{F}) \text{ has exact order } 3, \text{ and } |\mathbf{Z}(G)| \in \{3, 6\}. \tag{8.7.2}$$

Assume in addition that $p = 5$. Then the image of $P(\infty)$ in $K$ has order divisible by 5. Using [ST] and (8.7.2), we see that $K = C_3 \times \mathrm{SL}_2(5) = ST20$. It follows from (8.7.1), we obtain that $G = K = ST20$. Now assume that $p = 2$. Note that every primitive Shephard–Todd groups in dimension 2 of order divisible by 5 contains $\mathrm{SL}_2(5)$ and hence its field of traces contains $\sqrt{5}$. On the other hand, the field of traces of $\mathcal{F}$ is (contained in) $\mathbb{Q}(\zeta_3)$. It follows that $5 \nmid |K|$. In combination with (8.7.2) and [ST], this implies that $K = \mathrm{SL}_2(3) = ST4$, $C_3 \times \mathrm{SL}_2(3) = ST5$, or $C_6 * 2\mathsf{S}_4 = ST14$. Note, however, that $ST14$ contains a reflection, which has determinant $-1$, and this would violate (8.7.2). Now using (8.7.1), and noting that $\mathrm{SL}_2(3)$ contains the central subgroup $C_2$, we conclude that $G = C * K = C_3 \times \mathrm{SL}_2(3) = ST5$. □

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Department of Mathematics, Princeton University, Princeton, NJ 08544
*E-mail address*: nmk@math.princeton.edu

Departamento de Álgebra, Universidad de Sevilla, c/Tarfia s/n, 41012 Sevilla, Spain
*E-mail address*: arojas@us.es

Department of Mathematics, Rutgers University, Piscataway, NJ 08854
*E-mail address*: tiep@math.rutgers.edu