ON SOME AIRY SHEAVES OF LAURENT TYPE

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To the memory of Nikolai Aleksandrovich Vavilov

Abstract. We study some one-parameter families of exponential sums of Airy-Laurent type. Their
general theory was developed in [KT6]. In the present paper, we make use of that general theory
to compute monodromy groups in some particularly simple families (in the sense of “simple to
remember”), realizing Weyl groups of type $E_6$ and $E_8$.

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1. Introduction

In classical analysis, Airy functions are the Fourier transforms of exponentials $e^{\theta(x)}$ of polynomials,
(originally for the polynomial $g(x) := x^3/3$) and Airy differential equations are the linear differential
equations $g'(d/dt)y + ty = 0$ they satisfy. These differential equations have an irregular singularity
at $\infty$, and have quite interesting differential galois groups. In the seminal paper [Such] of Such, he
introduces their $\ell$-adic finite field analogues, the local systems whose trace functions are of the form

$$t \mapsto - \sum_x \psi(g(x) + tx).$$

The local systems we are concerned with here are generalizations of these Airy local systems in
several ways. We allow the “t term” $tx$ to be replaced by $tx^a$, we allow the polynomial $g(x)$ to be
replaced by a Laurent polynomial $f(1/x) + g(x)$, and we allow an “outside factor” $\chi(x)$ in the sum.
Here is a more detailed discussion.

We work in odd characteristic $p > 0$, and denote by $\overline{\mathbb{F}}_p$ an algebraic closure of $\mathbb{F}_p$. We also fix a
prime $\ell \neq p$ to be able to speak of $\overline{\mathbb{Q}}_\ell$-adic cohomology. We fix integers

$$A \geq 1, \ a \geq 1$$

2010 Mathematics Subject Classification. 11T23, 20C15, 20D06, 20C33, 20G40.

Key words and phrases. Local systems, Airy sheaves, Monodromy groups, Weyl groups.

The third author gratefully acknowledges the support of the NSF (grant DMS-2200850) and the Joshua Barlaz
Chair in Mathematics.
Moreover, as a \[1.14\], the \[85x126\]

\[
Suppose that \( f(x) \) is Artin-Schreier reduced: this means that in the expression \( f(x) = \sum_i c_i x^i \), we have \( c_i = 0 \), if \( p \nmid i \). We define

\[
gcd_{\text{deg}}(f) := \gcd(\{i | c_i \neq 0\})
\]

the greatest common divisor of the degrees of the monomials appearing in \( f \). We suppose

\[
gcd(a, gcd_{\text{deg}}(f)) = 1.
\]

We fix \( \chi \) a (possibly trivial) multiplicative character of a finite extension \( k/F_p \) containing the coefficients of \( f \) and \( g \). We denote by \( \psi \) a chosen nontrivial additive character of \( F_p \). For \( L/k \) a finite extension, we denote by \( \chi_L \), respectively by \( \psi_L \), the composition of \( \chi \), respectively of \( \psi \), with \( \text{Norm}_{L/k} \), respectively with \( \text{Tr}_{L/F_p} \).

We denote by \( G(f, g, a, \chi) \) the lisse sheaf on \( G_m/k \) whose trace function at time \( t \in L^\times \), for \( L/k \) a finite extension, is

\[
t \mapsto - \sum_{x \in L^\times} \psi_L(f(1/x) + g(x) + tx^a)\chi_L(x).
\]

We will mostly be concerned with the case when \( \chi = \chi_2 \), the quadratic character.

2. **Basic facts about \( G(f, g, a, \chi) \)**

The local system \( G(f, g, a, \chi) \) is lisse of rank \( D = A + a \) on \( G_m \), and pure of weight one. We view it as being the Fourier transform

\[
\text{FT}_\psi([a]_\ast (L_{\psi(f(1/x)+g(x))} \otimes L_{\chi}(x))).
\]

**Lemma 2.1.** Let \( A \geq 1, a \geq 1, p \nmid Aa, f \) be Artin-Schreier reduced, and \( \gcd(a, \gcd_{\text{deg}}(f)) = 1 \). Then the \( I(\infty) \)-representation of \( G(f, g, a, \chi) \) is irreducible. It has rank \( A + a \) and all slopes \( A/(A + a) \).

**Proof.** This is a straightforward application of Laumon’s results on the local monodromy of \( \text{FT}_\psi \). The input sheaf to \( \text{FT}_\psi \) is lisse on \( G_m \) of rank \( a \), with \( I(0) \)-slopes \( A/a \). The hypothesis

\[
\gcd(a, \gcd_{\text{deg}}(f)) = 1
\]

implies that the \( I(0) \)-representation of the input sheaf is irreducible, cf. the proof of \([KT6]\) Lemma 2.1.

Then the \( I(\infty) \)-representation of \( G(f, g, a, \chi) \) is \( \text{FT}\text{loc}(0, \infty)(\text{rank } a, \text{slopes } A/a) \), which has rank \( A + a \) and all slopes \( A/(A + a) \), cf. \([Ka3]\) 7.4.4(4)]. The asserted irreducibility result from the irreducibility of the input and the fact that \( \text{FT}\text{loc}(0, \infty) \) is a suitable equivalence of categories. \( \square \)

**Lemma 2.2.** Suppose that \( A = 1 \) and that \( p \nmid (a + 1) \). Then the image of \( P(\infty) \) in the the representation attached to \( G(f, g, a, \chi) \) is isomorphic to the the additive group of the field \( \overline{\mathbb{F}}_p(\mu_{a+1}) \).

**Proof.** The \( I(\infty) \)-representation is irreducible of prime to \( p \) rank \( a + 1 \) and has \( \text{Swan} = 1 \). By \([Ka2]\) 1.14], the \( I(\infty) \)-representation is the Kummer direct image \([a + 1]_\ast \mathcal{L} \) of some \( \mathcal{L} \) with \( \text{Swan} = 1 \). Moreover, as a \( P(\infty) \)-representation, the \( I(\infty) \) representation is the direct sum of the multiplicative translates, by elements of \( \mu_{a+1} \), of \( \mathcal{L} \). Because \( \mathcal{L} \) has \( \text{Swan} = 1 \), it is of the form \( \mathcal{L}_{\psi(ax)} \) for some \( a \neq 0 \) in \( \overline{\mathbb{F}}_p \). Now repeat the (end of) the proof of \([KRLT]\) Lemma 1.2]. \( \square \)
Lemma 2.3. If both $f, g$ are odd polynomials, the local system $G(f, g, a, \chi_2)$ is geometrically self-dual. Indeed, its constant field twist by $1/Gauss(\psi, \chi_2)$ is arithmetically self-dual.

Proof. The local system $G(f, g, a, \chi_2)$ is geometrically irreducible. The oddness of $f, g, a$ insures that its constant field twist by $1/Gauss(\psi, \chi_2)$, which is pure of weight zero, has real traces, hence the asserted autoduality.

Theorem 2.4. If both $f, g$ are odd polynomials, the geometric determinant of $G(f, g, a, \chi_2)$ is $L_{\chi_2}$.

Proof. We first explain the idea. For fixed data $f, a$, the local system $G(f, g, a, \chi_2)$ makes sense for any odd polynomial $g$ of degree $< a$. Such $g$ form an affine space $\Lambda^{(a-1)/2}$, and indeed there is a local system on $\Lambda^{(a-1)/2} \times G_m$, call it $G_{\text{univ}}(f, a, \chi)$, whose trace function at a point $(g, t)$ is

$$(g, t) \mapsto -\sum_x \psi(f(1/x) + g(x) + tx^a)\chi_2(x).$$

Because this local system is self-dual, its determinant, call it $L_{\text{univ}}$ is either trivial or is nontrivial of order 2. Viewing $\mu_2$ as $\mathbb{Z}/2\mathbb{Z}$, we view $L_{\text{univ}}$ as an element of $H^1((\Lambda^{(a-1)/2} \times G_m)/\mathbb{F}_p, \mathbb{Z}/2\mathbb{Z})$. Because we are in odd characteristic $p$, the groups $H^i(\Lambda^{(a-1)/2}/\mathbb{F}_p, \mathbb{Z}/2\mathbb{Z}) = 0$ for all $i > 0$, and $H^0(\Lambda^{(a-1)/2}/\mathbb{F}_p, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

By the Kunneth formula [Del Cor. 1.11], the map

$$\text{pr}_2 : \Lambda^{(a-1)/2} \times G_m \to G_m, \quad (g, t) \mapsto t,$$

induces by pullback an isomorphism

$$H^1(G_m/\mathbb{F}_p, \mathbb{Z}/2\mathbb{Z}) \cong H^1((\Lambda^{(a-1)/2} \times G_m)/\mathbb{F}_p, \mathbb{Z}/2\mathbb{Z}).$$

For any fixed $g_0 \in \Lambda^{(a-1)/2}$, pullback by the inclusion

$$\text{incl}_{g_0} : G_m \subset \Lambda^{(a-1)/2} \times G_m, \quad t \mapsto (g_0, t)$$

induces an isomorphism

$$H^1((\Lambda^{(a-1)/2} \times G_m)/\mathbb{F}_p, \mathbb{Z}/2\mathbb{Z}) \cong H^1(G_m/\mathbb{F}_p, \mathbb{Z}/2\mathbb{Z}).$$

The composition $\text{pr}_2 \circ \text{incl}_{g_0}$ is the identity map of $G_m$ to itself. Therefore the composition of their pullbacks, $\text{incl}_{g_0}^* \circ \text{pr}_2^*$ is the identity on $H^1(G_m/\mathbb{F}_p, \mathbb{Z}/2\mathbb{Z})$.

On the one hand, if we view $G_{\text{univ}}$ as the pullback by $\text{pr}_2$ of a class $L_0$ on $G_m$, then for any $g_0$ we have

$$\text{incl}_{g_0}^* L_{\text{univ}} = L_0,$$

The key point is that this pullback $\text{incl}_{g_0}^* L_{\text{univ}}$ is the determinant of the local system $G(f, g_0, a, \chi_2)$ on $G_m/\mathbb{F}_p$. What we must show is that $L_0 = L_{\chi_2}$. For this, it suffices to check at the single point $g_0 = 0$.

The local system $G(f, g_0 = 0, a, \chi_2)$ is the local system denoted $G(f, a, \chi_2)$ in [KT5]. This local system has all $\infty$-slopes $A/(A + a) < 1$, and hence its entire $G_{\text{geom}}$ is the Zariski closure of all conjugates of the image of $I(0)$. So its determinant is $L_{\chi_2}$ if and only if the determinant of its $I(0)$-representation is $L_{\chi_2}$. But its $I(0)$-representation is the direct sum

$$\bigoplus_{\rho : \rho^a = \chi_2} L_{\rho} \oplus (\mathbb{Q}^*)^A.$$

Because $a$ is odd, its determinant is $L_{\chi_2}$. Indeed, if we fix one $a^\text{th}$ root $\rho_0$ of $\chi_2$, then all the $a^\text{th}$ roots are $\rho_0 \Lambda$ as $\Lambda$ runs over the characters of order dividing $a$. So we have

$$\prod_{\rho : \rho^a = \chi_2} \rho = \rho_0^a \times \prod_{\Lambda \in \text{Char}(a)} \Lambda = \chi_2 \times 1 = \chi_2.$$
Theorem 2.5. If $f$ is an odd polynomial, and $a > 2 \deg(g)$, the geometric determinant of $G(f, g, a, \chi_2)$ is $L_{\chi_2}$.

Proof. By [KT6 Corollary 2.2], the geometric determinant is either trivial or is $L_{\chi_2}$. Using the argument above, it first suffices to check for the specialization $g = 0$, and then to observe that, again by the previous argument, this specialization has the asserted determinant $L_{\chi_2}$. □

We will now consider the following local systems on $G_m^3$ with trace functions

$$(r, s, t) \mapsto - \sum_{x \in L^\times} \psi_L(r/x + sx^B + tx^a)\chi_{2L}(x),$$

where

(2.5.1) \quad $(B, a, p) = (3, 7, 5), (5, 7, 3), \text{ or } (1, 7, 3)$,

or

(2.5.2) \quad $(B, a, p) = (1, 5, 3) \text{ or } (2, 5, 3)$.

We will deviate from our previous notation, and denote this system as $G_{r,s}$ as we always let $t$ vary. In particular, $G_{r_0,s_0}$ is the pullback of $G_{r,s}$ by $r = r_0$ and $s = s_0$ for any $(r_0, s_0) \in G_m^2$.

Recall the conditions $(S+)$ and $(S-)$ defined in [KT3 §1.1]. A basic fact about $G_{r,s}$ is the following

Proposition 2.6. For the three choices of $(B, a, p)$ as in (2.5.1), and for any $(r_0, s_0) \in G_m^2$, $G_{r_0,s_0}$, and hence $G_{r,s}$, satisfies $(S+)$.  

Proof. We already proved in Lemma 2.1 that the underlying representation $V$ is $I(\infty)$-irreducible. Next, primitivity of $G_{r_0,s_0}$ is proved for $(B, a, p) = (3, 7, 5), (5, 7, 3)$ in [KT6 Theorem 2.10(d)], and for $(B, a, p) = (1, 7, 3)$ in [KT6 Theorem 2.11]. An application of [KT6 Proposition 2.8] shows that $V$ is tensor indecomposable over $I(0)$. Finally, for the first two choices of $(B, a, p), V$ is not tensor induced by [KT6 Proposition 2.9(d)]. The same conclusion holds for the third choice by [KT6 Lemma 3.9], for in this case, $D = 8 = 2^3$, so the only possible $n$ is $3$, which is not prime to $p = 3$. □

Proposition 2.7. For the two choices of $(B, a, p)$ as in (2.5.2), and for any $(r_0, s_0) \in G_m^2$, $G_{r_0,s_0}$, and hence $G_{r,s}$, satisfies $(S+)$.  

Proof. This is [KT6 Corollary 2.13]. □

A natural question is what we can say about these local systems after specializing $s = 0$.

Proposition 2.8. In any odd characteristic $p$, for any integer $a \geq 4$ with $p \nmid a$, and any multiplicative character $\Lambda$ of order $N$ prime to $p$, the local system on $(G_m \times G_m)/\mathbb{F}_p(\mu_{aN})$ whose trace function is

$$(r, t) \mapsto - \sum_{x \neq 0} \psi(r/x + tx^a)\Lambda(x)$$

satisfies $(S+)$. In fact, its specialization $r = 1$ satisfies $(S+)$ and has $G_{geom}$ containing a scalar multiple of a complex reflection if $\Lambda \neq 1$.

Proof. Choose an $a^{th}$ root $\chi$ of $\Lambda$, i.e., a character $\chi$ of $\mathbb{F}_p(\mu_{aN})^\times$ with $\chi^a = \Lambda$. [Concretely, in terms of a generator $\omega$ of $\mathbb{F}_p(\mu_{aN})^\times$, $\Lambda(\omega)$ has order $N$, and the choice of $\chi$ is the choice of an $a^{th}$
root of $\Lambda(\omega)$ in $\mu_{aN}$. One then takes $\chi(\omega)$ to be this choice of $a$th root.] We first prove that the $r = 1$ specialization, namely the local system on $\mathbb{G}_m/\mathbb{F}_p(\mu_{aN})$ whose trace function is

$$
t \mapsto - \sum_{x \neq 0} \psi(1/x + tx^a)\chi^a(x)
$$

is geometrically isomorphic to the Kloosterman sheaf $Kl(1, \{\overline{\chi}\}_{\rho \in \text{Char}(a)})$. Indeed, this $r = 1, s = 0$ local system is geometrically calculated in terms of hypergeometric sheaves $H$ as being

$$
\text{FT}_\psi([a]_*(L_\psi(1/x) \otimes L_{\chi^a(x)})) = \text{FT}_\psi(L_\chi \otimes [a]_*H(\mathbb{O}; 1))
\cong \text{FT}_\psi(L_\chi \otimes H(\mathbb{O}; \text{Char}(a))
= \text{FT}_\psi(H(\mathbb{O}; \chi \text{Char}(a))) = Kl_\psi(1, \chi \text{Char}(a)),
$$
cf. [Ka3] 8.1.12 & 8.4.2]. Now, if $\Lambda$ has order $e > 1$ then the $a$th power of a generator $g_0$ of $I(0)$ of this Kloosterman sheaf has spectrum $\{1, \zeta, \ldots, \zeta_e\}$ and thus it is the $\zeta_e$-multiple of a complex reflection.

We next check that this Kloosterman sheaf is primitive. It suffices to show it is not Kummer induced, by Pink’s result [Ka1, Lemmas 11, 12]. We argue by contradiction. Suppose it is Kummer induced of some degree $r$, and its characters are cosets of $\text{Char}(r)$. If $r = a + 1$, then its characters are precisely $\text{Char}(r)$. But if we take two distinct elements of $\overline{\chi} \text{Char}(a)$, their ratio is a nontrivial element of $\text{Char}(a)$, so has order dividing $\text{Char}(r)$ (simply because gcd($r, a$) = 1). If $r < a + 1$, then $r \leq (a + 1)/2$, hence outside the $\text{Char}(r)$ coset of $1$, there are two distinct elements of $\overline{\chi} \text{Char}(a)$, and their ratio again gives a contradiction.

Because this Kloosterman sheaf is primitive, it has $(S^+)$ by [KT2 1.7]. Once this $r = 1$ pullback local system has $(S^+)$, then so does the $(r, t)$ local system.

**Corollary 2.9.** For any odd prime $p$, any integer $a \geq 4$ with $p \nmid a$, any integer $1 \leq B < a$, and any multiplicative character $\Lambda$ of order $N$ prime to $p$, the local system on $(\mathbb{G}_m \times \mathbb{A}^1 \times \mathbb{G}_m)/\mathbb{F}_p(\mu_{aN})$ whose trace function is

$$(r, s, t) \mapsto - \sum_{x \neq 0} \psi(r/x + sx^B + tx^a)\Lambda(x)$$

satisfies $(S^+)$. 

**Proof.** Indeed, its $s = 0$ pullback satisfies $(S^+)$. \hfill \square

Next we study $M_{2,2}$ of $G(1/x, x^B, a, \chi)$ for $B = 1, 2$.

**Lemma 2.10.** For any prime $p$, any integer $a \geq 2$ with $p \nmid a$, and any multiplicative character $\chi$ of order prime to $p$, consider the local system $G(1/x, x^a, \chi)$ on $\mathbb{G}_m^3/\mathbb{F}$(values of $\chi$) whose trace function at points of $\mathbb{G}_m(L)^3$, for $L/\mathbb{F}$(values of $\chi$) a finite extension, is

$$(r, s, t) \mapsto \frac{-1}{\sqrt{\#L}} \sum_{x \in L^x} \psi_L(r/x + sx + tx^a)\chi_L(x).$$

Then $M_{2,2} \leq 3$, with equality precisely when $a$ is odd and $\chi$ is either $1$ or $\chi_2$, this second case allowed only for $p$ odd.

**Proof.** Since $M_{2,2}$ only decreases as $G_{geom}$ grows, it suffices to prove $M_{2,2} \leq 3$ when we freeze $t = 1$, and consider the two parameter local system on $\mathbb{G}_m \times \mathbb{G}_m$ whose trace function is

$$(r, s) \mapsto \frac{-1}{\sqrt{\#L}} \sum_{x \in L^x} \psi_L(r/x + sx + x^a)\chi_L(x).$$
By \cite[2.1]{KT}, we may calculate its $M_{2, 2}$ as the limsup, over finite extensions $\mathbb{F}_q/\mathbb{F}$ (values of $\chi$) of
\[
\frac{1}{q^2(q-1)^2} \sum_{r, s \in \mathbb{F}_q^*} \sum_{x, y, z, w \in \mathbb{F}_q^*} \psi_{\mathbb{F}_q}(r \left( \frac{1}{x} + \frac{1}{y} - \frac{1}{z} - \frac{1}{w} \right) + s(x + y - z - w) + x^a + y^a - z^a - w^a) \chi(xy)\overline{\chi}(zw).
\]

We first show that this limsup does not change if, instead of summing over $(r, s) \in (\mathbb{F}_q^*)^2$, we sum over $(r, s) \in (\mathbb{F}_q)^2$. An individual summand with $r = 0$, any $s$, is
\[
\sum_{x, y, z, w \in \mathbb{F}_q^*} \psi_{\mathbb{F}_q}(s(x + y - z - w) + x^a + y^a - z^a - w^a) \chi(xy)\overline{\chi}(zw) = \sum_{x \in \mathbb{F}_q^*} \bigg| \psi_{\mathbb{F}_q}(sx + x^a) \chi(x) \bigg|^4,
\]
which is $\leq (a/\sqrt{q})^4$, hence is $O(q^2)$. Similarly, an individual summand with $s = 0$, any $r$, is $\leq (a/\sqrt{q})^4$, hence is $O(q^2)$. The total number of $(r, s) \in (\mathbb{F}_q^*)^2$ with $rs = 0$ is $2q - 1$, so we are only changing the inner sum by $O(q^3)$, while we are dividing by $1/q^2(q-1)^2$.

We now examine
\[
\frac{1}{q^2(q-1)^2} \sum_{r, s \in \mathbb{F}_q} \sum_{x, y, z, w \in \mathbb{F}_q^*} \psi_{\mathbb{F}_q}(r \left( \frac{1}{x} + \frac{1}{y} - \frac{1}{z} - \frac{1}{w} \right) + s(x + y - z - w) + x^a + y^a - z^a - w^a) \chi(xy)\overline{\chi}(zw)
\]
\[
= \frac{1}{q^2(q-1)^4} \sum_{x, y, z, w \in \mathbb{F}_q} \psi_{\mathbb{F}_q}(x^a + y^a - z^a - w^a) \chi(xy)\overline{\chi}(zw) \times
\]
\[
\sum_{r, s \in \mathbb{F}_q} \sum_{x, y, z, w \in \mathbb{F}_q^*} \psi_{\mathbb{F}_q}(r \left( \frac{1}{x} + \frac{1}{y} - \frac{1}{z} - \frac{1}{w} \right) + s(x + y - z - w))
\]
\[
= \frac{1}{(q-1)^2} \sum_{x, y, z, w \in \mathbb{F}_q^*} \psi_{\mathbb{F}_q}(x^a + y^a - z^a - w^a) \chi(xy)\overline{\chi}(zw).
\]

We now examine the two equations in $x, y, z, w$ with $xyzw \neq 0$ given by
\[
1/x + 1/y = 1/z + 1/w, \quad x + y = z + w,
\]
which we rewrite as
\[
(x + y)/xy = (z + w)/zw, \quad x + y = z + w.
\]

If $x + y = z + w = 0$, we have the plane $y = -x, z = -w$. If $x + y = z + w \neq 0$, then we divide by them and get $1/xy = 1/zw$. Thus we have $x + y = z + w$ and $xy = zw$, and hence the two sets $\{x, y\}$ and $\{z, w\}$ coincide. So have the two planes $x = z, y = w$ and $x = w, y = z$. On each of these last two planes, the function $x^a + y^a - z^a - w^a$ vanishes, so $\psi(x^a + y^a - z^a - w^a) = 1$, and $\chi(xy)\overline{\chi}(zw) = 1$.

On the first plane $y = -x, z = -w$, the sum $x^a + y^a - z^a - w^a$ vanishes precisely when $a$ is odd, and $\chi(xy)\overline{\chi}(zw) = \chi(x^2/w^2)$.

Thus if both $a$ is odd and $\chi^2 = 1$, then the first plane also contributes 1 to $M_{2, 2}$. In general its contribution is
\[
\frac{1}{(q-1)^2} \sum_{x, w \in \mathbb{F}_q^*} \psi_{\mathbb{F}_q}(x^a + (-x^a) - w^a - (-w)^a) \chi(-x^2)\overline{\chi}(-w^2) =
\]
\[
\frac{1}{(q-1)^2} \sum_{x \in \mathbb{F}_q^*} \psi_{\mathbb{F}_q}(x^a + (-x^a)) \chi(-x^2) \bigg| \chi(-w^2) \bigg|^2,
\]
which is $O(1/q)$ unless both $a$ is odd and $\chi^2 = 1$.

Now we return to $G(1/x, a, \chi)$ on $\mathbb{G}_m^3$. We have proven that for its $t = 1$ specialization, we have $M_{2, 2} \leq 3$, with equality precisely when both $a$ is odd and $\chi^2 = 1$. When we do not have equality,
Lemma 2.11. For any odd prime $p$, any integer $a \geq 3$ with $p \nmid a$, and any multiplicative character $\chi$ of order prime to $p$, consider the local system $S'(1/x, x^2, a, \chi)$ on $G_m^3/F_p$ (values of $\chi$) whose trace function at points of $G_m^3/F_p$, for $L/F_p$ (values of $\chi$) a finite extension, is

$$(r, s, t) \mapsto \frac{-1}{\#L} \sum_{x \in L^x} \psi_L(r/x + sx^2 + tx^a)\chi_L(x).$$

Then $M_{2,2} = 2$.

Proof. By [KT4, 2.1], we may calculate the $M_{2,2}$ as the limsup, over finite extensions $F_q/F$ (values of $\chi$) of

$$\frac{1}{q^2(q-1)^3} \sum_{r,s,t \in F_q^x} \sum_{x,y,z,w \in F_q^x} \psi_{F_q}(r(\frac{1}{x} + \frac{1}{y} - \frac{1}{z} - \frac{1}{w}) + s(x^2 + y^2 - z^2 - w^2) + t(x^a + y^a - z^a - w^a))\chi(xy)\chi(zw).$$

We will refer to

$$\sum_{r,s,t \in F_q^x} \sum_{x,y,z,w \in F_q^x} \psi_{F_q}(r(\frac{1}{x} + \frac{1}{y} - \frac{1}{z} - \frac{1}{w}) + s(x^2 + y^2 - z^2 - w^2) + t(x^a + y^a - z^a - w^a))\chi(xy)\chi(zw)$$

as “a summand”.

We first show that this limsup does not change if, instead of summing over $(r, s, t) \in (F_q^x)^3$, we sum over $(r, s, t) \in (F_q)^3$. The summand with $(r, s, t) = (0, 0, 0)$ is

$$\sum_{x,y,z,w \in F_q^x} \chi(xy)\chi(zw) = |\sum_{x \in F_q^x} \chi(x)|^4,$$

which vanishes if $\chi \neq 1$, and is $(q-1)^4$ if $\chi = 1$.

An individual summand with $r = 0$, any $(s, t) \neq (0, 0)$, is

$$\sum_{x,y,z,w \in F_q^x} \psi_{F_q}(s(x^2 + y^2 - z^2 - w^2) + t(x^a + y^a - z^a - w^a))\chi(xy)\chi(zw) = |\sum_{x \in F_q^x} \psi_{F_q}(sx^2 + tx^a)\chi(x)|^4,$$

which is $\leq ((a+1)\sqrt{q})^4$, hence is $O(q^2)$. Similarly, an individual summand with $s = 0$, any $(r, t) \neq (0, 0)$, is $\leq ((a+1)\sqrt{q})^4$, hence is $O(q^2)$. Finally, an individual summand with $t = 0$, any $(r, s) \neq (0, 0)$, is $\leq 3\sqrt{q}^4$, hence is $O(q^2)$. The total number of $(r, s, t) \in (F_q)^3$ with $(r, s, t) \neq (0, 0, 0)$ but $rst = 0$ is $O(q^3)$, so those terms are only changing the inner sum by $O(q^4)$, and the $(0, 0, 0)$ term is either 0 or $(q-1)^4$, while we are dividing by $q^2(q-1)^3$.

We now examine

$$\frac{1}{q^2(q-1)^3} \sum_{r,s,t \in F_q, x,y,z,w \in F_q^x} \psi_{F_q}(r(\frac{1}{x} + \frac{1}{y} - \frac{1}{z} - \frac{1}{w}) + s(x^2 + y^2 - z^2 - w^2) + t(x^a + y^a - z^a - w^a))\chi(xy)\chi(zw)$$

We have $M_{2,2} = 2$. As $M_{2,2}$ can only decrease for a bigger group, we certainly have $M_{2,2} \leq 2$, and hence $M_{2,2} = 2$, for $G(1, x, x, a, \chi)$ unless both $a$ is add and $\chi^2 = 1$. In the case when $a$ is odd and $\chi^2 = 1$, then $G(1, x, x, a, \chi)$ is self dual, hence has $M_{2,2} \geq 3$, and thus has the asserted $M_{2,2} = 3$. \qed
with \((p, u, t, a)\) a homogeneous polynomial in \((u, t)\) and so that
\[\text{we first look at the intersection of } \Sigma \text{ with } x + y = z + w.\]

If \(x + y = z + w = 0\), then as \(p \neq 2\) the second equation \(x^2 + y^2 = z^2 + w^2\) gives the the two lines \(y = -x, z = -w = \pm x\). If \(x + y = z + w \neq 0\), then we divide the first equation by them and get \(1/xy = 1/zw\). Thus we have \(x + y = z + w\) and \(xy = zw\), and hence the two sets \(\{x, y}\) and \(\{z, w\}\) coincide. So we have the two planes \(x = z, y = w\) and \(x = w, y = z\). On each of these two planes, \(x^a + y^a - z^a - w^a = 0\) and \(\chi(xy)\chi(zw) = 1\). So the contribution of this intersection to the limsup is \(2 + O(1/q)\).

It remains to show that the number of points \((x, y, z, w) \in \Sigma(F_q^*)\) with \(x + y \neq z + w\) is \(O(q)\).

Note that if \(x + y = 0\) then \(1/z + 1/w = 1/x + 1/y = 0\) and hence \(z + w = 0 = x + y\). So we may assume \(x + y \neq 0, z + w \neq 0\), and introduce new variables
\[u = x + y, \quad v = z + w, \quad t = xy/(x + y),\]
so that
\[s := xy = tu, \quad zw = tv, \quad \text{but } u \neq v.\]

Now
\[u^2 - 2tu = x^2 + y^2 = z^2 + w^2 = v^2 - 2tv,\]
and so
\[u + v = 2t, \text{ i.e. } v = 2t - u.\]

An easy induction on odd \(a \geq 3\) shows that there are some integers \(c_0 = 1, c_1, \ldots, c_{(a-1)/2}\) such that
\[x^a + y^a = \sum_{i=0}^{(a-1)/2} c_i u^{n-2i}s^i;\]
in fact, \(c_i = \frac{n}{n-i} \binom{n-i}{i}\). It follows that \(x^a + y^a = \sum_{i=0}^{(a-1)/2} c_i u^{n-i}v^i,\)
and so the condition \(x^a + y^a - z^a - w^a = 0\) is equivalent to the vanishing of
\[\sum_{i=0}^{(a-1)/2} c_i u^{n-i}v^i - \sum_{i=0}^{(a-1)/2} c_i (2t - u)^{n-i}v^i,\]
a homogeneous polynomial in \(u, t\) of degree \(a\) with the coefficient for \(u^a\) being \(1 - (-1)^a = 2\). As \(p \neq 2\), given any \(t \in F_q\) there are at most \(a\) values for \(u\) that satisfies this last condition. For each \((u, t)\), there are at most \(a^2\) pairs \((x, y)\) with \(x + y = u, xy = tu,\) and there are at most \(a^2\) pairs \((z, w)\) with \(z + w = 2t - u, xy = t(2t - u)\). It follows that the number of points \((x, y, z, w) \in \Sigma(F_q^*)\) with \(x + y \neq z + w\) is at most \(4aq\), as stated.

3. Preliminaries on specializations of \(G_{\text{geom}}\)

We first quote verbatim from [KT4 §11 and Theorem 11.1], cf. [Ka3 8.17, 8.18].

“The situation we consider is the following. We are given a normal connected affine noetherian scheme \(S = \text{Spec}(A)\) with \(A\) a noetherian normal integral domain with fraction field \(K\), and a chosen algebraic closure \(\overline{K}\) of \(K\). Thus \(\text{Spec}(K)\) is a generic point \(\eta\) of \(S\), and \(\text{Spec}(\overline{K})\) is a geometric generic point \(\overline{\eta}\) of \(S\). We are given \(X/S\) a smooth \(S\)-scheme of relative dimension \(D\), with

\[\frac{1}{q^2(q - 1)^3} \sum_{x, y, z, w \in F_q^*} \chi(xy)\chi(zw) \sum_{t \in F_q} \psi_{F_q}(t(x^a + y^a - z^a - w^a)) \times\]
\[\sum_{r \in F_q} \psi_{F_q}(r(x + 1/y - 1/z - 1/w)) \sum_{s \in F_q} \psi_{F_q}(s(x^2 + y^2 - z^2 - w^2)) = \frac{q}{(q - 1)^3} \sum_{x, y, z, w \in F_q^*} \chi(xy)\chi(zw),\]
where the locus \(\Sigma\) is defined by the equations
\[1/x + 1/y = 1/z + 1/w, \quad x^2 + y^2 = z^2 + w^2, \quad x^a + y^a = z^a + w^a.\]
geometrically connected fibres, and $\phi \in X(S)$ a section of $X/S$. Then $\phi(\eta)$ is a geometric generic point of $X$. We are given a finite group $G$ and a surjective homomorphism

$$\pi_1(X, \phi(\eta)) \to G.$$ 

For each geometric point $s$ of $S$, $\phi(s)$ is a geometric point of $X_s$ (and also of $X$). We have a continuous group homomorphism

$$\pi_1(X_s, \phi(s)) \to \pi_1(X, \phi(s)) \cong \pi_1(X, \phi(\eta)).$$

This last isomorphism is only canonical up to inner automorphism of the target group $\pi_1(X, \phi(\eta))$. By composition, we get a group homomorphism

$$\pi_1(X_s, \phi(s)) \to G$$

which is well defined up to inner automorphism of $G$. This applies in particular with $s$ taken to be $\eta$. We are interested in how the image of $\pi_1(X_s, \phi(s))$ in $G$ compares with the image of $\pi_1(X_\eta, \phi(\eta))$ in $G$: when are these two subgroups of $G$ conjugate in $G$? Let us denote these image groups $G_s$ and $G_\eta$.

**Theorem 3.1.** [KT4, 11.1] There exists a dense open set $U \subset S$ such that for any geometric point $s \in U$, $G_s$ and $G_\eta$ are conjugate subgroups of $G$. Moreover, for any geometric point $s \in S$, $G_s$ is conjugate to a subgroup of $G_\eta$.

Because $G$ is a finite group, this theorem has the following more precise corollary.

**Corollary 3.2.** The set of points $s \in S$ at which $G_s$ is conjugate to $G_\eta$ in $G$ is open in $S$.

**Proof.** The group $G_\eta$ is finite (because it is a subgroup of the finite group $G$). Therefore it has only finitely many subgroups, say $G_\eta = H_0, H_1, \ldots, H_r$. The proof of [Ka3, 8.17, 8.18] shows that each of the sets

$$Z_i := \{s \in S \mid G_s \text{ is conjugate to } H_i\}$$

and each of the sets

$$W_i := \{s \in S \mid G_s \text{ is conjugate to a subgroup of } H_i\}$$

is constructible (a finite union of sets of the form (open set) \(\cap\) (closed set)). But each set $W_i$ is stable by specialization, hence is closed, cf. [Hart], Ch. II, 3.18. Therefore $\cup_{i \geq 1} W_i$ is closed. Then its open complement is precisely the set $Z_0$. \(\square\)

4. Finiteness theorems

We denote by $q$ the cardinality of $k$, and by $k_r$ the unique extension of $k$ of degree $r$ in $F_q$. $V : (\mathbb{Q}/\mathbb{Z})_{\text{prime to } p} \to [0, 1)$ will denote Kubert’s $V$ function for the prime $p$ (cf. [KRLT1]).

**Theorem 4.1.** Let $d_1 > d_2 > \ldots > d_n > 0$ be prime to $p$ integers, $d = (d_1, \ldots, d_n)$ and $F$ the local system on $\mathbb{G}_{m,k}^{n+1}$ whose trace function is given by

$$F(L; s, t_1, \ldots, t_n) = -\frac{1}{|L|} \sum_{x \in L^*} \psi_L(s/x + t_1x^{d_1} + \cdots + t_nx^{d_n})\chi_{2,L}(x).$$

Then $F$ has finite (geometric and arithmetic) monodromy group if and only if

$$V \left( d_1x_1 + \cdots + d_nx_n + \frac{1}{2} \right) + V(x_1) + \cdots + V(x_n) \geq \frac{1}{2}$$

for every $(x_1, \ldots, x_n) \in (\mathbb{Q}/\mathbb{Z})^{n}_{\text{prime to } p}$. 


Proof. By [KRL1] Proposition 2.1, we need to show that $F(L; s, t_1, \ldots, t_n)$ is an algebraic integer or, equivalently, that

$$\text{ord}_{q^r} \left( \sum_{x \in k_r^\times} \psi_{kr}(s/x + t_1 x^{d_1} + \cdots + t_n x^{d_n}) \chi_{2, kr}(x) \right) \geq \frac{1}{2}$$

for every $r \geq 1$ and $(s, t_1, \ldots, t_n) \in (k_r)^n$. Taking Mellin transform on $\mathbb{G}_m^{n+1}$, this is equivalent to

$$\sum_{s, t_1, \ldots, t_n \in k_r^\times} \eta(s) \xi_1(t_1) \cdots \xi_n(t_n) \sum_{x \in k_r^\times} \psi_{kr}(s/x + t_1 x^{d_1} + \cdots + t_n x^{d_n}) \chi_{2, kr}(x) =$$

$$= \sum_{x \in k_r^\times} \chi_{2, kr}(x) \left( \sum_{s \in k_r^\times} \psi_{kr}(s/x) \eta(s) \right) \left( \sum_{t_1 \in k_r^\times} \psi_{kr}(t_1 x^{d_1}) \xi_1(t_1) \right) \cdots \left( \sum_{t_n \in k_r^\times} \psi_{kr}(t_n x^{d_n}) \xi_n(t_n) \right) =$$

$$= \sum_{x \in k_r^\times} \chi_{2, kr}(x) \eta(x) G_r(\eta) \tilde{\xi}_1^{d_1}(x) G_r(\xi_1) \cdots \tilde{\xi}_n^{d_n}(x) G_r(\xi_n) =$$

$$= G_r(\eta) G_r(\xi_1) \cdots G_r(\xi_n) \sum_{x \in k_r^\times} (\chi_{2, kr} \eta \tilde{\xi}_1^{d_1} \cdots \tilde{\xi}_n^{d_n})(x) =$$

$$= \begin{cases} 
0 & \text{if } \chi_{2, kr} \eta \tilde{\xi}_1^{d_1} \cdots \tilde{\xi}_n^{d_n} \neq 1 \\
(q^r - 1) G_r(\eta) G_r(\xi_1) \cdots G_r(\xi_n) & \text{if } \chi_{2, kr} \eta \tilde{\xi}_1^{d_1} \cdots \tilde{\xi}_n^{d_n} = 1 
\end{cases}$$

has $\text{ord}_{q^r} \geq \frac{1}{2}$ for every $\eta, \xi_1, \ldots, \xi_n \in \hat{k}_r^\times$, where $G_r(\chi)$ denotes the Gauss sum associated to the multiplicative character $\chi$ on $k_r$. This reduces to

$$\text{ord}_{q^r}(G_r(\chi_{2, kr} \xi_1^{d_1} \cdots \xi_n^{d_n}) G_r(\xi_1) \cdots G_r(\xi_n)) \geq \frac{1}{2}$$

for every $\xi_1, \ldots, \xi_n \in \hat{k}_r^\times$ which, by Stickelberger, is equivalent to the given condition. \hfill \Box

If $d_1 x_1 + \cdots + d_n x_n + \frac{1}{2} \neq 0$, using that $V(y) + V(-y) = 1$ for $y \neq 0$, we can rewrite the condition as

$$V \left( d_1 x_1 + \cdots + d_n x_n + \frac{1}{2} \right) \leq V(-x_1) + \cdots + V(-x_n) + \frac{1}{2},$$

which trivially holds for $d_1 x_1 + \cdots + d_n x_n + \frac{1}{2} = 0$. So we have

**Corollary 4.2.** The local system $\mathcal{F}$ has finite monodromy if and only if the following two conditions hold:

(i) $V \left( d_1 x_1 + \cdots + d_n x_n + \frac{1}{2} \right) \leq V(-x_1) + \cdots + V(-x_n) + \frac{1}{2}$ for every $(x_1, \ldots, x_n) \in (\mathbb{Q}/\mathbb{Z})^n$ prime to $p$.

(ii) $V(x_1) + V(x_2) + \cdots + V(x_n) \geq \frac{1}{2}$ for every $(x_1, \ldots, x_n) \in (\mathbb{Q}/\mathbb{Z})^n$ prime to $p$ with $\sum_{i=1}^n d_i x_i = \frac{1}{2}$.

Note that the second condition is the criterion for the local system on $\mathbb{G}_m^\times$ with trace function

$$F(L; t_1, \ldots, t_n) = -\frac{1}{\sqrt{|L|}} \sum_{x \in L^\times} \psi_L(t_1 x^{d_1} + \cdots + t_n x^{d_n}) \chi_{2, L}(x)$$

to have finite monodromy. In terms of the sum-of-digits function $[\cdot]_{p, r, -}$ defined in [KRL] Appendix, the first condition becomes

$$\left[ d_1 x_1 + \cdots + d_n x_n + \frac{p^r - 1}{2} \right]_{p, r, -} \leq [p^r - 1 - x_1]_{p, r, -} + \cdots + [p^r - 1 - x_n]_{p, r, -} + \frac{r(p - 1)}{2}$$
Proposition 4.4. Suppose that there exists some real $A \geq 0$ such that
\[
\left[d_1x_1 + \cdots + d_nx_n + \frac{p^r - 1}{2}\right]_p \leq [p^r - 1 - x_1]_p + \cdots + [p^r - 1 - x_n]_p + \frac{r(p-1)}{2} + A
\]
for every $r \geq 1$ and every $0 \leq x_1, \ldots, x_n \leq p^r - 1$, where $[x]_p$ denotes the sum of the $p$-adic digits of $x$. Then condition (i) in Corollary 4.2 holds.

For $r \geq 1$ and an $n$-tuple $(x_1, \ldots, x_n)$ with $0 \leq x_i \leq p^r - 1$, let
\[
C(r; x_1, \ldots, x_n) = \left[\sum_{i=1}^{n} d_ix_i + \frac{p^r - 1}{2}\right]_p - \sum_{i=1}^{n} [p^r - 1 - x_i]_p - \frac{r(p-1)}{2}.
\]
For $s \geq 1$ we say that the $n$-tuple $(z_1, \ldots, z_n)$ with $0 \leq z_1, \ldots, z_n \leq p^s - 1$ is $s$-good if one of these conditions hold:

a) $C(s; z_1, \ldots, z_n) \leq 0$.  
b) There exists an $s' < s$ and an $n$-tuple $(z'_1, \ldots, z'_n)$ with $0 \leq z'_i \leq p^{s'} - 1$ such that $C(s'; z'_1, \ldots, z'_n) \geq C(s; z_1, \ldots, z_n)$ and for every $j > 0$ the $(s+j)$-th digit in the $p$-adic expansion of $\sum_{i=1}^{n} d_i z_i + \frac{p^j - 1}{p^{j+1}}$ is greater than or equal to the $(s'+j)$-th digit in the $p$-adic expansion of $\sum_{i=1}^{n} d_i z'_i + \frac{p^j - 1}{p^{j+1}}$ (counting the digits from right to left).

We say that the $n$ tuple $(x_1, \ldots, x_n)$ with $0 \leq x_1, \ldots, x_n \leq p^r - 1$ has good termination if, for some $1 \leq s < r$, the $n$-tuple $(z_1, \ldots, z_n)$ whose $i$-th coordinate is the number formed by the last $s$ $p$-adic digits of $x_i$ (i.e. the remainder of the division of $x_i$ by $p^s$) is $s$-good.

Proposition 4.4. Suppose that there exists some $r_0 \geq 1$ such that all $n$-tuples $(x_1, \ldots, x_n)$ with $0 \leq x_1, \ldots, x_n \leq p^{r_0} - 1$ have good termination. Then the hypothesis of Proposition 4.3 holds.

Proof. Let
\[
A = \max_{1 \leq r \leq r_0} \max_{1 \leq i \leq n} C(r; x_1, \ldots, x_n).
\]
We will prove by induction on $r$ that
\[
C(r; x_1, \ldots, x_n) \leq A
\]
for every $r \geq 1$ and every $0 \leq x_1, \ldots, x_n \leq p^r - 1$. For $r \leq r_0$ this is obvious by definition of $A$. Fix $r > r_0$ and assume that the inequality holds for all smaller $r$. Let $(x_1, \ldots, x_n)$ be an $n$-tuple with $0 \leq x_1, \ldots, x_n \leq p^r - 1$. By hypothesis it has good termination, since every $s < r_0$ is also $< r$. Hence there is some $1 \leq s < r_0$ such that the $n$-tuple $(z_1, \ldots, z_n)$ whose $i$-th coordinate is the number formed by the last $s$ $p$-adic digits of $x_i$ is $s$-good. Let $y_i = p^{-s}(x_i - z_i)$, that is, the number obtained from $x_i$ by removing its last $s$ $p$-adic digits. Then $0 \leq y_i \leq p^{r-s} - 1$, $[p^r - 1 - x_i] = [p^{r-s} - 1 - y_i] + [p^s - 1 - z_i]$ and
\[
\left[\sum_{i=1}^{n} d_i x_i + \frac{p^r - 1}{2}\right] = \left[\sum_{i=1}^{n} d_i y_i + \frac{p^{r-s} - 1}{2}\right] + \left[\sum_{i=1}^{n} d_i z_i + \frac{p^s - 1}{2}\right] - \delta (p-1)
\]
where $\delta$ is the number of digit carries in the (outer) sum
\[
\sum_{i=1}^{n} d_i x_i + \frac{p^r - 1}{2} = p^s \left(\sum_{i=1}^{n} d_i y_i + \frac{p^{r-s} - 1}{2}\right) + \left(\sum_{i=1}^{n} d_i z_i + \frac{p^s - 1}{2}\right).
\]
In particular, \( C(r; x_1, \ldots, x_n) = C(r - s; y_1, \ldots, y_n) + C(s; z_1, \ldots, z_n) - \delta(p - 1) \). We now have two options according to the definition of \( s \)-good.

(a) \( C(s; z_1, \ldots, z_n) \leq 0 \). Then

\[
C(r; x_1, \ldots, x_n) \leq C(r - s; y_1, \ldots, y_n) + C(s; z_1, \ldots, z_n) \leq C(r - s; y_1, \ldots, y_n) \leq A
\]

by induction.

(b) There exists an \( s' < s \) and an \( n \)-tuple \( (z'_1, \ldots, z'_n) \) with \( 0 \leq z'_i \leq p^{s'} - 1 \) such that \( C(s'; z'_1, \ldots, z'_n) \geq C(s; z_1, \ldots, z_n) \) and for every \( j > 0 \) the \((s + j)\)-th digit in the \( p \)-adic expansion of \( \sum_{i=1}^n d_i z_i + \frac{p^{s'-1}}{2} \) is greater than or equal to the \((s' + j)\)-th digit in the \( p \)-adic expansion of \( \sum_{i=1}^n d_i z'_i + \frac{p^{s'-1}}{2} \).

Let \( x'_i = p^{s'} y_i + z'_i \) for \( i = 1, \ldots, n \). Then

\[
C(r - s + s'; x'_1, \ldots, x'_n) = C(r - s; y_1, \ldots, y_n) + C(s'; z'_1, \ldots, z'_n) - \epsilon(p - 1),
\]

where \( \epsilon \) is the number of digit carries in the (outer) sum

\[
\sum_{i=1}^n d_i x'_i + \frac{p^{s'+s'-1}}{2} = p^{s'} \left( \sum_{i=1}^n d_i y_i + \frac{p^{s-s'-1}}{2} \right) + \left( \sum_{i=1}^n d_i z'_i + \frac{p^{s'-1}}{2} \right).
\]

The hypothesis on the digits of \( \sum_{i=1}^n d_i z_i + \frac{p^{s'-1}}{2} \) and \( \sum_{i=1}^n d_i z'_i + \frac{p^{s'-1}}{2} \) implies that \( \epsilon \leq \delta \). Therefore

\[
C(r; x_1, \ldots, x_n) = C(r - s; y_1, \ldots, y_n) + C(s; z_1, \ldots, z_n) - \delta(p - 1)
\]

\[
\leq C(r - s; y_1, \ldots, y_n) + C(s'; z'_1, \ldots, z'_n) - \epsilon(p - 1) = C(r - s + s'; x'_1, \ldots, x'_n)
\]

\[
\leq A
\]

by induction.

\[\square\]

**Theorem 4.5.** The local system on \( \mathbb{G}^3_{m, \mathbb{F}_3} \) whose trace function is given by

\[
(\mathbb{F}_3^*; s, t, u) \mapsto -\frac{1}{3^{r/2}} \sum_{x \in \mathbb{F}_3^*} \psi_{\mathbb{F}_3^*}(s/x + tx + ux^T) \chi_{2, \mathbb{F}_3^*}(x)
\]

has finite monodromy.

**Proof.** By Corollary 4.2 we need to show

(i) \( V(x_1 + 7x_2 + \frac{1}{2}) \leq V(-x_1) + V(-x_2) + \frac{1}{2} \) for every \((x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^2_{\text{prime to } 3}\).

(ii) \( V(x) + V(-7x + \frac{1}{2}) \geq \frac{1}{2} \) for every \( x \in (\mathbb{Q}/\mathbb{Z})_{\text{prime to } 3}\).

Condition (ii) is the criterion for finite monodromy of the local system with trace function

\[
F(\mathbb{F}_3^*; t, u) \mapsto -\frac{1}{3^{r/2}} \sum_{x \in \mathbb{F}_3^*} \psi_{\mathbb{F}_3^*}(tx + ux^T) \chi_{2, \mathbb{F}_3^*}(x)
\]

which holds by [KTI] Theorem 4.3] since \( 7 = \frac{3^3 + 1}{3+1} \). For the first condition, following Proposition 4.4 we check by a computer search that all pairs \((x_1, x_2)\) with \( 0 \leq x_1, x_2 \leq 3^3 - 1 \) have good termination.

For each \( s = 1, 2, 3 \), the following tables show the list of all pairs \((z_1, z_2)\) with \( 0 \leq z_1, z_2 \leq 3^s - 1 \) such that

(a) \( C(s; z_1, z_2) > 0 \)

(b) do not have good termination (ie. all their last-digits truncations appear in the previous tables marked with \( \bullet \)).
If the condition (b) for being $s$-good can be applied to them, we show the possible values of $s', z'_1, z'_2$ on the table, otherwise we mark it with $\bullet$ and move it on to the next $s$. All values of $z_i$ and $z'_i$ are shown as their 3-adic expansion. The columns $D$ and $D'$ show the result of removing the last $s$ (respectively $s'$) digits of $z_1 + 7z_2 + \frac{3^{s-1}}{2}$ (resp. of $z'_1 + 7z'_2 + \frac{3^{s'-1}}{2}$). Each digit of the number in column $D$ must be greater than or equal to the corresponding digit of the number in column $D'$.

<table>
<thead>
<tr>
<th>$s = 1$</th>
<th>$z_1$</th>
<th>$z_2$</th>
<th>$C(s; z_1, z_2)$</th>
<th>$D$</th>
<th>$\bullet$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>12</td>
<td>$\bullet$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>12</td>
<td>$\bullet$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$s = 2$</th>
<th>$z_1$</th>
<th>$z_2$</th>
<th>$C(s; z_1, z_2)$</th>
<th>$D$</th>
<th>$s'$</th>
<th>$z'_1$</th>
<th>$z'_2$</th>
<th>$C(s'; z'_1, z'_2)$</th>
<th>$D'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>22</td>
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</table>

In the last table (for $s = 3$) there are no remaining cases left with $\bullet$, so this finishes the proof. $\square$

Theorem 4.6. The local system on $\mathbb{G}^3_{m, \mathbb{F}_3}$ whose trace function is given by

$$\langle \mathbb{F}_{3^r}; s, t, u \rangle \mapsto -\frac{1}{3^{r/2}} \sum_{x \in \mathbb{F}_{3^r}^\times} \psi_{\mathbb{F}_{3^r}}(s/x + tx^5 + ux^7)\chi_{2, \mathbb{F}_{3^r}}(x)$$
has finite monodromy.

Proof. By Corollary 4.2, we need to show

(ii) \( V(5x_1 + 7x_2 + \frac{1}{2}) \leq V(-x_1) + V(-x_2) + \frac{1}{2} \) for every \((x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^2_{\text{prime to } 3}\).

(iii) \( V(x_1) + V(x_2) \geq \frac{1}{2} \) for every \((x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^2_{\text{prime to } 3}\) such that \(5x_1 + 7x_2 = \frac{1}{2}\).

Condition (ii) is the criterion for finite monodromy of the local system with trace function

\[
F(F_{3r}; t, u) \mapsto -\frac{1}{3r/2} \sum_{x \in F_{3r}} \psi_{F_{3r}}(tx^5 + ux^7)\chi_{2F_{3r}}(x)
\]

which holds by [KT3, Theorem 10.3.13(vi)]. For the first condition, following Proposition 4.4, we check by a computer search that all pairs \((x_1, x_2)\) with \(0 \leq x_1, x_2 \leq 3^6 - 1\) have good termination.

For each \(s = 1, 2, 3, 4, 5\), the following tables show the list of all pairs \((z_1, z_2)\) with \(0 \leq z_1, z_2 \leq 3^s - 1\) such that

(a) \(C(s; z_1, z_2) > 0\) and

(b) do not have good termination (i.e. all their last-digits truncations appear in the previous tables marked with •).

If the condition (b) for being \(s\)-good can be applied to them, we show the possible values of \(s', z_1', z_2'\) on the table, otherwise we mark it with • and move it on to the next \(s\). All values of \(z_i\) and \(z_i'\) are shown as their 3-adic expansion. The columns \(D\) and \(D'\) show the result of removing the last \(s\) (respectively \(s'\)) digits of \(5z_1 + 7z_2 + \frac{3^s - 1}{2}\) (resp. of \(5z_1' + 7z_2' + \frac{3^{s'} - 1}{2}\)). Each digit of the number in column \(D\) must be greater than or equal to the corresponding digit of the number in column \(D'\).

\[
\begin{array}{|c|c|c|c|}
\hline
s & z_1 & z_2 & C(s; z_1, z_2) \\hline
1 & 1 & 2 & 2 & 20 & • \\hline
2 & 1 & 2 & 2 & 22 & • \\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
s & z_1 & z_2 & C(s; z_1, z_2) & D & s' & z_1' & z_2' & C(s'; z_1', z_2') \\hline
1 & 11 & 22 & 4 & 22 & 1 & 2 & 2 & 4 & 22 \\hline
& 21 & 02 & 2 & 12 & • & & & & \\hline
& 21 & 12 & 2 & 22 & 1 & 2 & 2 & 4 & 22 \\hline
& 21 & 22 & 2 & 101 & • & & & & \\hline
& 02 & 22 & 2 & 21 & 1 & 1 & 2 & 2 & 20 \\hline
& 22 & 12 & 4 & 22 & 1 & 2 & 2 & 4 & 22 \\hline
& 22 & 22 & 2 & 102 & • & & & & \\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
s & z_1 & z_2 & C(s; z_1, z_2) & D & s' & z_1' & z_2' & C(s'; z_1', z_2') \\hline
1 & 121 & 202 & 2 & 22 & 1 & 2 & 2 & 4 & 22 \\hline
& 221 & 102 & 2 & 21 & 1 & 1 & 2 & 2 & 20 \\hline
& 021 & 222 & 2 & 22 & 1 & 2 & 2 & 4 & 22 \\hline
& 121 & 122 & 2 & 21 & 1 & 1 & 2 & 2 & 20 \\hline
& 221 & 222 & 4 & 102 & • & & & & \\hline
& 022 & 222 & 2 & 22 & 1 & 2 & 2 & 4 & 22 \\hline
\end{array}
\]
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<th>$z'_1$</th>
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<th>$z'_1$</th>
<th>$z'_2$</th>
<th>$C(s'; z'_1, z'_2)$</th>
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In the last table (for $s = 5$) there are no remaining cases left with $\bullet$, so this finishes the proof. □

Theorem 4.7. The local system on $\mathbb{G}^3_{m,F_5}$ whose trace function is given by

$$F(F_{5r}; s, t, u) \mapsto -\frac{1}{5^{r/2}} \sum_{x \in F_{5r}^s} \psi_{F_{5r}}(s/x + tx^3 + ux^7)\chi_{2,F_{5r}}(x)$$

has finite monodromy.

Proof. By Corollary 4.2, we need to show

(i) $V(3x_1 + 7x_2 + \frac{1}{2}) \leq V(-x_1) + V(-x_2) + \frac{1}{2}$ for every $(x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^2_{\text{prime to } 5}$.

(ii) $V(x_1) + V(x_2) \geq \frac{1}{2}$ for every $(x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^2_{\text{prime to } 5}$ such that $3x_1 + 7x_2 = \frac{1}{2}$.

Condition (ii) is the criterion for finite monodromy of the local system with trace function

$$F(F_{5r}; t, u) \mapsto -\frac{1}{5^{r/2}} \sum_{x \in F_{5r}^s} \psi_{F_{5r}}(tx^3 + ux^7)\chi_{2,F_{5r}}(x)$$

which holds by [KT3, Theorem 10.3.13(ix)]. For the first condition, following Proposition 4.4, we check by a computer search that all pairs $(x_1, x_2)$ with $0 \leq x_1, x_2 \leq 5^6 - 1$ have good termination.

For each $s = 1, 2, 3, 4, 5$, the following tables show the list of all pairs $(z_1, z_2)$ with

$$0 \leq z_1, z_2 \leq 5^s - 1$$

such that

(a) $C(s; z_1, z_2) > 0$ and

(b) do not have good termination (ie. all their last-digits truncations appear in the previous tables marked with $\bullet$).

If the condition (b) for being $s$-good can be applied to them, we show the possible values of $s', z'_1, z'_2$ on the table, otherwise we mark it with $\bullet$ and move it on to the next $s$. All values of $z_i$ and
$z'_i$ are shown as their 5-adic expansion. The columns $D$ and $D'$ show the result of removing the last $s$ (respectively $s'$) digits of $3z_1 + 7z_2 + \frac{5^s - 1}{2}$ (resp. of $3z'_1 + 7z'_2 + \frac{5's' - 1}{2}$). Each digit of the number in column $D$ must be greater than or equal to the corresponding digit of the number in column $D'$.

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<th>$z'_1$</th>
<th>$C(s'; z'_1, z'_2)$</th>
<th>$D'$</th>
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<th>$z_2$</th>
<th>$C(s; z_1, z_2)$</th>
<th>$D$</th>
<th>$s'$</th>
<th>$z'_1$</th>
<th>$C(s'; z'_1, z'_2)$</th>
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</tr>
</tbody>
</table>
\[
\begin{array}{cccccccc}
\text{z}_1 & \text{z}_2 & C'(s; \text{z}_1, \text{z}_2) & D & \text{s}' & \text{z}'_1 & \text{z}'_2 & C'(s'; \text{z}'_1, \text{z}'_2) & D' \\
3243 & 4144 & 4 & 13 & 1 & 4 & 4 & 4 & 13 \\
4243 & 3144 & 4 & 12 & 1 & 3 & 4 & 4 & 12 \\
4243 & 4144 & 4 & 14 & 1 & 4 & 4 & 4 & 13 \\
1343 & 4244 & 4 & 12 & 1 & 3 & 4 & 4 & 12 \\
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1443 & 4344 & 4 & 13 & 1 & 4 & 4 & 4 & 13 \\
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4443 & 3444 & 4 & 14 & 1 & 4 & 4 & 4 & 13 \\
4443 & 4444 & 4 & 20 & 3 & 443 & 444 & 4 & 20 \\
\end{array}
\]
In the last table (for $s = 5$) there are no remaining cases left with $\bullet$, so this finishes the proof. □

**Theorem 4.8.** The local system on $\mathbb{G}_{m, \mathbb{F}_3}$ whose trace function is given by

$$(\mathbb{F}_{3^r}; s, t, u) \mapsto -\frac{1}{3^{r/2}} \sum_{x \in \mathbb{F}_{3^r}^\times} \psi_{\mathbb{F}_{3^r}}(s/x + tx + ux^5)\chi_{2, \mathbb{F}_{3^r}}(x)$$

has finite monodromy.

**Proof.** By Corollary 4.2 we need to show

(i) $V(x_1 + 5x_2 + \frac{1}{2}) \leq V(-x_1) + V(-x_2) + \frac{1}{2}$ for every $(x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^2$ prime to 3.

(ii) $V(x) + V(-5x + \frac{1}{2}) \geq \frac{1}{2}$ for every $x \in (\mathbb{Q}/\mathbb{Z})$ prime to 3.

Condition (ii) is the criterion for finite monodromy of the local system with trace function

$F(\mathbb{F}_{3^r}; t, u) \mapsto -\frac{1}{3^{r/2}} \sum_{x \in \mathbb{F}_{3^r}^\times} \psi_{\mathbb{F}_{3^r}}(tx + ux^5)\chi_{2, \mathbb{F}_{3^r}}(x)$

which holds by [KT1 Theorem 4.2] since $5 = \frac{3^2 + 1}{2}$. For the first condition, following Proposition 4.4 we check by a computer search that all pairs $(x_1, x_2)$ with $0 \leq x_1, x_2 \leq 3^5 - 1$ have good termination.

For each $s = 1, 2, 3, 4$, the following tables show the list of all pairs $(z_1, z_2)$ with $0 \leq z_1, z_2 \leq 3^s - 1$ such that

(a) $C(s; z_1, z_2) > 0$ and

(b) do not have good termination (ie. all their last-digits truncations appear in the previous tables marked with $\bullet$).

If the condition (b) for being $s$-good can be applied to them, we show the possible values of $s', z'_1, z'_2$ on the table, otherwise we mark it with $\bullet$ and move it on to the next $s$. All values of $z_i$ and $z'_i$ are shown as their 3-adic expansion. The columns $D$ and $D'$ show the result of removing the last $s$ (respectively $s'$) digits of $z_1 + 5z_2 + \frac{3^{s-1}}{2}$ (resp. of $z'_1 + 5z'_2 + \frac{3^{s'-1}}{2}$). Each digit of the number in column $D$ must be greater than or equal to the corresponding digit of the number in column $D'$. 

<table>
<thead>
<tr>
<th>$z_1$</th>
<th>$z_2$</th>
<th>$C(s; z_1, z_2)$</th>
<th>$D$</th>
<th>$s'$</th>
<th>$z'_1$</th>
<th>$z'_2$</th>
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</thead>
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<tr>
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<td>8</td>
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ON SOME AIRY SHEAVES OF LAURENT TYPE

\[ s = 1 \]

<table>
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<tr>
<th>( z_1 )</th>
<th>( z_2 )</th>
<th>( C(s; z_1, z_2) )</th>
<th>( D )</th>
</tr>
</thead>
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</table>

\[ s = 2 \]

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<th>( C(s; z_1, z_2) )</th>
<th>( s' )</th>
<th>( z'_1 )</th>
<th>( z'_2 )</th>
<th>( C(s'; z'_1, z'_2) )</th>
<th>( D' )</th>
</tr>
</thead>
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\[ s = 3 \]

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<th>( C(s; z_1, z_2) )</th>
<th>( s' )</th>
<th>( z'_1 )</th>
<th>( z'_2 )</th>
<th>( C(s'; z'_1, z'_2) )</th>
<th>( D' )</th>
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<tbody>
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<td>222</td>
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<tr>
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<td>022</td>
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\[ s = 4 \]

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<th>( s' )</th>
<th>( z'_1 )</th>
<th>( z'_2 )</th>
<th>( C(s'; z'_1, z'_2) )</th>
<th>( D' )</th>
</tr>
</thead>
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<td>2</td>
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<td>11</td>
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<tr>
<td>2222</td>
<td>1222</td>
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<td>1</td>
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<td>11</td>
</tr>
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<td>20</td>
<td>3</td>
<td>22</td>
<td>2</td>
<td>20</td>
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</tbody>
</table>

In the last table (for \( s = 4 \)) there are no remaining cases left with \( \bullet \), so this finishes the proof. \( \square \)

**Theorem 4.9.** The local system on \( \mathbb{G}^3_{m,F_3} \) whose trace function is given by

\[
(F_{3r}; s, t, u) \mapsto -\frac{1}{3^{r/2}} \sum_{x \in \mathbb{F}_{3r}} \psi_{3r}(s/x + tx^2 + ux^5)\chi_{2,\mathbb{F}_{3r}}(x)
\]

has finite monodromy.

**Proof.** By Corollary 4.2, we need to show

(i) \( V(2x_1 + 5x_2 + \frac{1}{2}) \leq V(-x_1 + V(-x_2) + \frac{1}{2} \) for every \( (x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^2 \) prime to 3.

(ii) \( V(x_1) + V(x_2) \geq \frac{1}{2} \) for every \( (x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^2 \) prime to 3 such that \( 2x_1 + 5x_2 = \frac{1}{2} \).

Condition (ii) is the criterion for finite monodromy of the local system with trace function

\[
F(F_{3r}; t, u) \mapsto -\frac{1}{3^{r/2}} \sum_{x \in \mathbb{F}_{3r}} \psi_{3r}(tx^2 + ux^5)\chi_{2,\mathbb{F}_{3r}}(x)
\]

which holds by [KT3, Theorem 10.3.13(i)]. For the first condition, following Proposition 4.4, we check by a computer search that all pairs \( (x_1, x_2) \) with \( 0 \leq x_1, x_2 \leq 3^6 - 1 \) have good termination.

For each \( s = 1, 2, 3, 4, 5 \), the following tables show the list of all pairs \( (z_1, z_2) \) with \( 0 \leq z_1, z_2 \leq 3^s - 1 \) such that

(a) \( C(s; z_1, z_2) > 0 \) and

(b) do not have good termination (ie. all their last-digits truncations appear in the previous tables marked with \( \bullet \)).

If the condition (b) for being \( s \)-good can be applied to them, we show the possible values of \( s', z'_1, z'_2 \) on the table, otherwise we mark it with \( \bullet \) and move it on to the next \( s \). All values of \( z_i \) and...
$z_1'$ are shown as their 3-adic expansion. The columns $D$ and $D'$ show the result of removing the last $s$ (respectively $s'$) digits of $2z_1 + 5z_2 + \frac{3s-1}{2}$ (resp. of $2z_1' + 5z_2' + \frac{3s'-1}{2}$). Each digit of the number in column $D$ must be greater than or equal to the corresponding digit of the number in column $D'$.

<table>
<thead>
<tr>
<th>$s = 1$</th>
<th>$z_1$</th>
<th>$z_2$</th>
<th>$C(s; z_1, z_2)$</th>
<th>$D$</th>
<th>$s'$</th>
<th>$z_1'$</th>
<th>$z_2'$</th>
<th>$C(s'; z_1', z_2')$</th>
<th>$D'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s = 2$</td>
<td>$z_1$</td>
<td>$z_2$</td>
<td>$C(s; z_1, z_2)$</td>
<td>$D$</td>
<td>$s'$</td>
<td>$z_1'$</td>
<td>$z_2'$</td>
<td>$C(s'; z_1', z_2')$</td>
<td>$D'$</td>
</tr>
<tr>
<td>$s = 3$</td>
<td>$z_1$</td>
<td>$z_2$</td>
<td>$C(s; z_1, z_2)$</td>
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<td>$z_1'$</td>
<td>$z_2'$</td>
<td>$C(s'; z_1', z_2')$</td>
<td>$D'$</td>
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<tr>
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<td>$C(s; z_1, z_2)$</td>
<td>$D$</td>
<td>$s'$</td>
<td>$z_1'$</td>
<td>$z_2'$</td>
<td>$C(s'; z_1', z_2')$</td>
<td>$D'$</td>
</tr>
</tbody>
</table>
Theorem 5.1. Let $\pi$ the geometric monodromy group of $F$ for a local system $\pi$. By Theorem 4.7, its subgroup $H$ of $G$ satisfies $(2.5.1)$, it follows from Theorems 4.5, 4.7, and 4.6 that $G$ is finite. [Those results show that the restriction of $G$ to the open dense subset $X$ of $\mathbb{G}_{m} \times \mathbb{A}^{1} \times \mathbb{Z}$ has finite $G_{\text{geom}}$. In general, for a local system $F$ on some smooth, geometrically connected variety $X$, and $U \subset X$ a dense open set, $\pi_{1}(U)$ maps onto $\pi_{1}(X)$, so $F$ on $X$ and $F|_{U}$ on $U$ have the same $G_{\text{geom}}$ (indeed have the same image of $\pi_{1}$).]

**Theorem 5.1.** Let $(B,a,p) = (3,7,5)$. Then both $G_{r,s}$, and $G_{r_{0},s_{0}}$ for any $(r_{0},s_{0}) \in \mathbb{G}_{m}^{2}$, have $G_{\text{geom}} = W(E_{8})$, the Weyl group of type $E_{8}$.

**Proof.** By Theorem 4.7, $G := G_{\text{geom},r,s}$ is a finite subgroup of $\text{GL}_{8}(\mathbb{C})$, whence the same holds for its subgroup $H := G_{\text{geom},r_{0},s_{0}}$. Next, $H$ satisfies (S+) by Proposition 2.6, whence the same holds for $G$.

Let $\varphi$ denote the $G$-character afforded by the underlying representation. By Lemma 2.3, $\varphi$ takes real values for any specializations of $(r,s)$, and hence $\varphi$ is real-valued. This implies that

$$Z(H) \leq Z(G) \leq C_{2}. \tag{5.1.1}$$

Next, by Lemma 2.2, the image of $P(\infty)$ in $H$ is isomorphic to the additive group of $\mathbb{F}_{5}(\mu_{8}) = \mathbb{F}_{5^{2}}$, which is elementary abelian of order $5^{2}$, whence (5.1.1) implies that

$$5^{2} \text{ divides } |H/Z(H)|. \tag{5.1.2}$$

Now we can apply [KT2, Lemma 1.1] to both $H$ and $G$. If either of them is an extraspecial normalizer, then there is some $\varepsilon = \pm$ such that

$$H \leq 2_{\varepsilon}^{1+6} \cdot O_{6}^{1}(2),$$

which violates (5.1.2). So both $H$ and $G$ are almost quasisimple; in particular, $L := H^{(\infty)}$ is a quasisimple group with

$$S := L/Z(L)$$

being the unique non-abelian composition factor of $H$. The condition (S+) implies that $\varphi|_{L}$ is irreducible, and so $C_{H}(L) = Z(H)$ by Schur’s lemma. It follows that

$$H/Z(H) \hookrightarrow \text{Aut}(L),$$

and so $5^{2} \nmid |\text{Aut}(L)|$ by (5.1.2). Now we can inspect Table 2 in [HM] to see that $2 \cdot \Omega_{8}^{+}(2)$ is the unique possibility for $L$. Note that $\varphi|_{L}$ is of type $+$, so

$$2 \cdot \Omega_{8}^{+}(2) = L \leq H \leq N_{O_{8}(\mathbb{C})}(L) = W(E_{8}) = L \cdot 2.$$
Also note from Theorem 2.4 that $G_{r_0, s_0}$ has geometric determinant $\chi_2$, which shows that $H$ cannot be perfect. Hence $H = W(E_8)$. □

The preceding arguments can also be repeated to show that $G \cong W(E_8)$. As $H \leq G$, we conclude that $G = H$.

**Theorem 5.2.** Let $(B, a, p) = (5, 7, 3)$. Then $G_{r, s}$ has $G_{\text{geom}} = W(E_8)$, the Weyl group of type $E_8$. Furthermore, there is a dense open set $U \subseteq S^2_{\text{geom}}$ that contains $(1, 1)$, such that $G_{r_0, s_0}$ has $G_{\text{geom}}$ equal to $W(E_8)$ when $(r_0, s_0) \in U$, and to $S_9$ when $(r_0, s_0) \notin U$.

**Proof.** By Theorem 4.6, $G := G_{\text{geom}, r, s}$ is a finite subgroup of $\text{GL}_8(\mathbb{C})$, whence the same holds for its subgroup $H := G_{\text{geom}, r_0, s_0}$. Next, $H$ satisfies $(S^+)$ by Proposition 2.6, whence the same holds for $G$.

(a) Let $\varphi$ denote the $G$-character afforded by the underlying representation $V$. By Lemma 2.3, $\varphi$ takes real values for any specializations of $(r, s)$, and hence $\varphi$ is real-valued. This implies that

$$Z(H) \leq Z(G) \leq C_2.$$  

(5.2.1)

Also note from Theorem 2.4 that $G_{r_0, s_0}$ has geometric determinant $\chi_2$, which shows that $\varphi|_H$ cannot be of symplectic type, and

$$H \leq O(V) \text{ but } H \nsubseteq SO(V).$$  

(5.2.2)

Next, the wild part of the $I(0)$-representation $V$ has rank 2 and slopes 5/2, so by [Kaz2 1.14] it is the Kummer induction $[2], L$ of some $L$ with $\text{Swan} = 5$, and the $P(0)$-representation is the direct sum $L \oplus [x \mapsto -x]^*L$. Moreover, these two pieces are permuted by any element $g_0 \in I(0)$ which is a generator of $I(0)$ modulo $P(0)$. Thus $g_0$ acts on $V$ with spectrum $(\alpha, -\alpha, 1, \ldots, 1)$ for some $\alpha \in \mathbb{C}^\times$. Since the image $Q$ of $P(0)$ is a 3-group, it is contained in $SO(V)$. Now, if the image of $g_0$ is contained in $SO(V)$, then so is the image $J$ of $I(0)$. By [KT2 2.5.1], the fact that all $\alpha$-slopes are $< 1$ implies that $H$ is the normal closure of $J$, so we get $H \leq SO(V)$, contrary to (5.2.2). Thus $-\alpha^2 = -1$, i.e. $\alpha = \pm 1$ and $g_0$ acts on $V$ as a reflection.

The proof of [KT3 Theorem 4.2.3] now shows that $H = Z(H)H_0$, where $H_0 = W(E_8)$, or $H_0$ is $S_9$ acting in the deleted natural permutation representation. In the former case, (5.2.1) implies that $H = W(E_8)$.

Suppose we are in the latter case. First we consider the case $Z(H) = C_2$. Then note that $L := H(\infty) \cong A_9$ and $H/L \cong C_2^3$. In particular, $Q \leq L$, and $J$ is contained in $\langle L, g_0 \rangle$, a subgroup of index 2, whence normal, in $H$. Hence the normal closure of $J$ in $H$ is contained in $\langle L, g_0 \rangle$, and so cannot be equal to $H$, a contradiction. We have shown that, in the latter case, $H = S_9$ in its deleted natural permutation representation.

Now we apply the above consideration to $(r_0, s_0) = (1, 1)$, and assume that $H = S_9$. We consider the weight zero twist of $G$ by $1/\text{Gauss}(\psi, \chi_2)$, which is orthogonally self-dual with integer Frobenius traces. Let us denote by $H_{\text{arith}}$ its arithmetic monodromy group. Then $H_{\text{arith}}$ normalizes $H$, and hence we have $H \leq H_{\text{arith}} \leq N_{O(V)}(H) = C_2 \times H$. In either case, over any even degree extension $k/\mathbb{F}_3$, we have $H_{\text{arith}, k} = H = S_9$. So over any such $k$, all Frobenius traces lie in $[-1, 8]$. But a Magma calculation shows that over $\mathbb{F}_{31}$, both $-3$ and $-2$ (as well as 2 and 3) occur as Frobenius traces. That $H = W(E_8)$ at $(r_0, s_0) = (1, 1)$.

(b) The preceding arguments can also be repeated to show that either $G \cong W(E_8)$ or $G \leq C_2 \times S_9$. Since $W(E_8) = G_{\text{geom}, 1, 1} \leq G$ and $|C_2 \times S_9| < |W(E_8)|$, we conclude that $G \cong W(E_8)$.

By Corollary 3.3, there is a dense open subset $U$ of $G_{3m}^2$ containing $(1, 1)$ such that $G_{\text{geom}, r_0, s_0}$ equals $W(E_8)$ for $(r_0, s_0) \in U$ and $G_{\text{geom}, r_0, s_0} = S_9$ for $(r_0, s_0) \notin U$. □
Theorem 5.3. Let \((B,a,p) = (1,7,3)\). Then both \(G_{r,s}\), and \(G_{r_0,s_0}\) for any \((r_0, s_0) \in \mathbb{G}_m^2\), have \(G_{\text{geom}} = W(E_8)\), the Weyl group of type \(E_8\).

Proof. (a) By Theorem 1.5, \(G := G_{\text{geom},r,s}\) is a finite subgroup of \(\text{GL}_8(\mathbb{C})\), whence the same holds for its subgroup \(H := G_{\text{geom},r_0,s_0}\). Let \(\varphi\) denote the \(G\)-character afforded by the underlying representation \(V\). By Lemma 2.3 \(\varphi\) takes real values for any specializations of \((r, s)\), and hence \(\varphi\) is real-valued. This implies that

\[
\mathbb{Z}(H) \leq \mathbb{Z}(G) \leq C_2.
\]

Also note from Theorem 2.4 that \(G_{r_0,s_0}\) has geometric determinant \(\chi_2\), which shows that \(\varphi|_H\) cannot be of symplectic type, and

\[
H \leq G \leq \text{O}(V) \text{ but } G, H \not\leq \text{SO}(V).
\]

Next, the wild part of the \((0)\)-representation \(V\) has rank 6, and so the image \(Q\) of \(P(0)\) is non-abelian, and hence is a 3-group of order at least \(3^2\). It follows that \(3^3\) divides \(|H|\) and \(|G|\). On the other hand, \(G\) has \(M_{2,2} = 3\) by Lemma 2.10. It follows from \([\text{GT}]\) Theorem 1.5 and (5.3.2) that either

\[
E = 2^{1+6} \triangleleft G \leq \text{N}_{\text{O}(V)}(E) = E \cdot O_6^+(2),
\]

or \(G = 2 \cdot A_9\), or \(2 \cdot \Omega_8^+(2) \leq G \leq W(E_8)\). The first possibility is ruled out since \(3^3\) divides \(|G|\). Next, \(G\) is not perfect by (5.3.2), ruling out the groups \(2 \cdot A_9\) and \(2 \cdot \Omega_8^+(2)\). Hence we conclude that \(G = W(E_8)\).

(b) It remains to determine \(H = G_{\text{geom},r_0,s_0}\), which is a subgroup of \(G = W(E_8)\). By Proposition 2.6 \(H\) satisfies condition \((S+)\). Hence, by \([\text{KT2}]\) Lemma 1.1, one of the following two cases holds.

(b1) \(H\) is an extraspecial normalizer, i.e. \(R\) contains a normal 2-subgroup \(R = \mathbb{Z}(R)E\), with \(E = 2^{1+6}\) acting irreducibly on \(V = \mathbb{C}^8\), \(\epsilon = \pm\), and \(\mathbb{Z}(R) = \mathbb{Z}(E)\) or \(\mathbb{Z}(R) = C_4\). Now (5.3.2) implies that \(\epsilon = +\) and \(\mathbb{Z}(R) = \mathbb{Z}(E)\). Thus \(R = E = 2^{1+6}\) and

\[
H \leq \text{N}_{\text{O}(V)}(E) = E \cdot O_6^+(2).
\]

This is however impossible since \(3^3\) divides \(|H|\).

(b2) \(H\) is almost quasisimple, i.e. \(S \triangleleft H/\mathbb{Z}(H) \leq \text{Aut}(S)\) for a non-abelian simple group \(S\), and the quasisimple group \(L = E(H)\), with \(S = L/\mathbb{Z}(L)\), acts irreducibly on \(V\). Furthermore, (5.3.1) implies that \(3^3\) divides \(|\text{Aut}(S)|\). Now we analyze the possibilities for \((S, L)\) as listed in \([\text{HM}]\).

\(\bullet\) \(S = L = \text{SL}_2(8)\). In this case, \(H/\mathbb{Z}(H) \leq \text{Aut}(S) = \text{SL}_2(8) \cdot 3\) contains no element of order 4, whereas a generator \(g_{\infty}\) of \(I(\infty)\) modulo \(P(\infty)\) has order 8 in \(H/\mathbb{Z}(H)\), a contradiction.

\(\bullet\) \((S, L) = (\text{Sp}_6(2), 2 \cdot \text{Sp}_6(2))\). Since \(\text{Aut}(S) \cong S\), by (5.3.1) we have \(H = L\). Let \(J\) and \(Q\) denote the image of \(I(\infty)\), respectively of \(P(\infty)\) in \(H\). Then \(Q\) is elementary abelian of order 9 by Lemma 2.2. As \(J\) acts irreducibly on \(V\), \(J \mathbb{Z}(H)\) transitively permutes the 8 nontrivial irreducible characters of \(Q\) which all occur in \(V\). Identifying \(Q\) with \(\mathbb{Q}(\mathbb{Z}(H))/\mathbb{Z}(H)\), we see that the subgroup \(J \mathbb{Z}(H)/\mathbb{Z}(H)\) of \(S\) also permutes the 8 nontrivial irreducible characters of \(Q\) transitively. Note that \(S = \text{Sp}_6(2)\) admits an irreducible complex character \(\theta\) of degree 7, and certainly \(\theta|_Q\) contains some irreducible constituent \(\lambda \neq 1_Q\). But then all 8 nontrivial irreducible characters of \(Q\) must occur in \(\theta|_Q\) of degree 7, a contradiction.

\(\bullet\) \((S, L) = (A_9, 2 \cdot A_9)\). Since \(2 \cdot S_9\) does not act on \(\mathbb{C}^8\), we must have \(H = L\). In particular, \(H/\mathbb{Z}(H)\) contains no element of order 8, whereas \(g_{\infty}\) has order 8 in \(H/\mathbb{Z}(H)\), a contradiction.

\(\bullet\) \(S = L = A_9\). As in the previous case, the fact that \(g_{\infty}\) has central order 8 implies that \(H/\mathbb{Z}(H) \cong S_9\). It follows from (5.3.1) that \(H/L\) is a group of order 2 or 4, whence the image \(P\) of \(P(0)\) in \(H\) is contained in \(L = S\). Note that the restriction of the character \(\varphi\) to \(L\) is just the character of the deleted permutation module of \(A_9\). Now, \(\varphi|_P = 2 \cdot 1_P + \alpha + \beta\) for some irreducible
character α of P of degree 3, using the fact that φ is real-valued. It follows that the 3-subgroup P of A_9 acts on \{1, 2, \ldots, 9\} with exactly three orbits. The length of any of these orbits is a power of 3. So we conclude that each of them has length 3; say they are \{1, 2, 3\}, \{4, 5, 6\}, and \{7, 8, 9\}. Now, as P fixes each of these three subsets, we see that P ≤ A_9^3 and hence abelian, contrary to the fact that α has degree 3.

- (S, L) = (Ω^3_8(2), 2Ω^5_8(2)). Here we have \(Z(H) = Z(L) = Z(G) = C_2\), and \(S ≤ H/Z(H) ≤ S \cdot 2\).

Again using the fact that \(G_{r_0,s_0}\) having nontrivial determinant we see that \(H > L\). Hence we conclude that \(|H/Z(H)| = |S \cdot 2| = |G/Z(H)|\) and thus \(H = G\). □

**Remark 5.4.** As we mentioned at the beginning of the section, the extension to \(G_{r,s}\) to \(G_m × A^1 × G_m\) does not change its \(G_{geom}\). Now, the specialization \(G_{1,0}\) of \(G_{r,s}\) in both Theorems 5.2 and 5.3 yield the Kloosterman sheaf \(Kl(\chi_2, \text{Char}(7)) ⊗ L_{χ_2}\) by the proof of Proposition 2.8. In particular, its geometric monodromy group \(G_{1,0}\) contains minus a reflection. Applying \([DT]\) Theorem 9.3(b)], we see that \(G_{1,0} = S_9\), acting on the tensor product of the deleted permutation representation with the sign representation, i.e. on the non-reflection representation.

## 6. \(G_{geom}\) For Local Systems of Rank 6

**Theorem 6.1.** Let \((B, a, p) = (1, 5, 3)\). Then \(G_{r,s}\) has \(G_{geom} = W(E_6) × 2\). Moreover, for any \((r_0, s_0) ∈ G_m^2\), the geometric monodromy group of \(G_{r_0,s_0}\) is \(W(E_6)\), acting on the non-reflection representation of degree 6.

**Proof.** (a) By Theorem 4.8 \(G := G_{geom,r,s}\) is a finite subgroup of \(GL_6(\mathbb{C})\), whence the same holds for its subgroup \(H := G_{geom,r_0,s_0}\) for any \((r_0, s_0) ∈ G_m^2\). Let \(φ\) denote the \(G\)-character afforded by the underlying representation \(V\). Then \n
\begin{equation}
\mathbb{Q}(φ) = \mathbb{Q}, \text{ and so } \mathbb{Z}(H) ≤ \mathbb{Z}(G) ≤ C_2
\end{equation}

Also note from Theorem 2.4 that \(G_{r_0,s_0}\) has geometric determinant \(χ_2\), and so it cannot be of symplectic type, and

\begin{equation}
H ≤ G ≤ O(V) \text{ and } H ≤ SO(V).
\end{equation}

Now we apply Lemma 2.10 to get \(M_{2,2}(G) = 3\). Applying \([GT]\) Theorem 1.5 and using (6.1.1), we see that \(G\) is almost quasisimple, and arrive at one of the following cases for \(L = G^{(∞)}\).

- \(L = SU_3(3)\). In this case, \(L ≤ G/Z(G) ≤ L \cdot 2\). Using \([GAP]\) one can check that the rational-valued character \(φ|_L\) does not have rational-valued extensions to \(L \cdot 2\). Hence (6.1.1) implies that \(G = \mathbb{Z}(G) \times L\). But in this case \(G < SO(V)\), contradicting (6.1.2).

- \(L = SU_4(2)\). In this case, \(L ≤ G/Z(G) ≤ L \cdot 2\). Since \(G ≤ SO(V)\) by (6.1.2), \(G\) must induce an outer automorphism of \(L\), i.e. \(G/Z(G) = L \cdot 2 ≅ W(E_6)\). Together with (6.1.1), this implies that \(W(E_6) ≤ G ≤ W(E_6) \times 2\). The same arguments applied to \(G_{arith, \mathbb{F}_3}\) show that \(G_{arith, \mathbb{F}_3} ≤ W(E_6) \times 2\). In particular, \([G_{arith, \mathbb{F}_3} : G] ≤ 2\) and \(G_{arith, \mathbb{F}_3} = G\). Now, a calculation with \(Magma\) \([BCP]\) over \(\mathbb{F}_{3^8}\) shows that the Frobenius at the point \((r, s, t) = (1, 1, w^{437})\) for \(w\) a primitive element in \(\mathbb{F}_{3^8}\) has trace −4. We also note that a change of variable \(x → rx\) in the trace function sends the trace of the Frobenius at \((1, 1, t)\) to \(χ_2(r)\) times the trace of the Frobenius at \((1, r, tr^5)\). Choosing \(r ∈ \mathbb{F}_{3^8}\) with \(χ_2(r) = -1\), we then get a trace 4, namely at \((1, sr, tr^5)\), in addition to trace −4. Since neither of the two 6-dimensional irreducible representations of \(W(E_6)\) possesses both traces 4 and −4, we conclude that \(G = W(E_6) \times 2\).

(b) It remains to determine \(H = G_{geom,r_0,s_0}\) which is a subgroup of \(G = W(E_6) \times 2\). By Proposition 2.7 \(H\) satisfies condition (S+). Hence, by \([KT2]\) Lemma 1.1, \(H\) is almost quasisimple: \(S ≤ H/Z(H) ≤ \text{Aut}(S)\) for a non-abelian simple group \(S\), and the quasisimple group \(K = E(H)\) with \(S = K/Z(K)\), acts irreducibly on \(V\). By Lemma 2.1 the image of \(P(∞)\) in \(H\) is a non-abelian
3-group, so \( (6.1.1) \) implies that \( 3^3 \) divides \( |\text{Aut}(S)| \). Since \( K = K^{(\infty)} \leq G^{(\infty)} = \SU_4(2) \), the list of maximal subgroups of \( \SU_4(2) \) in [Atlas] shows that \( K = \SU_4(2) = L \). Now \( Z(G)L \) has index 2 in \( G \) and \( Z(G)L \leq SO(V) \). Hence \( (6.1.2) \) implies that

\[
L < H \leq G = L \rtimes C_2^2.
\]

Now we look at the image \( J \) of \( I(0) \) in \( H \). Since \( H/L \) is a 2-group, the image \( Q \) of \( P(0) \), a 3-subgroup, is contained in \( L \). Next, \( J = (Q, h) \), where \( h \) is the image in \( H \) of a generator \( g_0 \) of \( I(0) \) modulo \( P(0) \). Since \( H/L \) has exponent 2, we have \( h^2 \in L \), and hence

\[
(6.1.4)
\]

\[
[LP : L] \leq 2.
\]

Also, since \( G/L \) is abelian, \( LJ \) is normal in \( G \) and hence also in \( H \). But \( H \) is the normal closure of \( J \) by \([KT^2]\) Proposition 4.2, so \( H \leq LJ \). Hence \( H = LJ \), and now \( (6.1.3) \) and \( (6.1.4) \) imply that \( [H : L] = 2 = [G : H] \). Among the three subgroups of index 2 in \( G \), \( Z(G) \times L \) is contained in \( SO(V) \), and the other two are isomorphic to \( W(E_6) \), the reflection and the non-reflection representation. Using \( (6.1.2) \), we obtain

\[
(6.1.5)
\]

\[
H \cong W(E_6).
\]

By Theorem 3.1, there is a subgroup \( G_\eta \) of \( G \) and an open dense subset \( U \) of \( \mathbb{G}_m^2 \) such that for all \( (r_1, s_1) \in U \), \( G_{\text{geom},r_1,s_1} \) is conjugate to \( G_\eta \) in \( G \). Now \( (6.1.5) \) implies that \( |G_{\text{geom},r_1,s_1}| = |G_\eta| \). It follows that \( G_\eta \cong W(E_6) \) and hence, being of index 2, that \( G_\eta \cdot G \). Also by Theorem 3.1, for any \( (r_2, s_2) \in \mathbb{G}_m^2 \), \( G_{\text{geom},r_2,s_2} \) is conjugate in \( G \) to a subgroup of \( G_\eta \cdot G \), hence it is a subgroup \( G_\eta \). Again using \( (6.1.5) \), we obtain that \( H = G_\eta \). In particular, \( G_{\text{geom},1,1} = H \), and the calculation in (a) shows that \( H \) acts on \( V \) via the non-reflection representation.

**Theorem 6.2.** Let \( (B, a, p) = (2, 5, 3) \). Then both \( G_{r,s} \), \( G_{r_0,s_0} \) for any \( (r_0, s_0) \in \mathbb{G}_m^2 \), have \( G_{\text{geom}} = 61 \cdot \PSU_4(3) \cdot 2_2 \), the Mitchell group.

**Proof.** (a) By Theorem 4.9, \( G := G_{\text{geom},r,s} \) is a finite subgroup of \( \GL_6(\mathbb{C}) \), whence the same holds for its subgroup \( H := G_{\text{geom},r_0,s_0} \) for any \( (r_0, s_0) \in \mathbb{G}_m^2 \). Let \( \varphi \) denote the \( G \)-character afforded by the underlying representation \( V \). Then

\[
(6.2.1)
\]

\[
\mathbb{Q}(\varphi) \subseteq \mathbb{Q}(\zeta_3), \quad \text{and so } Z(H) \leq Z(G) \leq C_6.
\]

Also note from Theorem 2.5 that \( G_{r,s} \) has geometric determinant \( \chi_2 \), and so

\[
(6.2.2)
\]

\[
G \not\leq \SL(V), \quad H \not\leq \SL(V).
\]

Now we apply Lemma 2.11 to get \( M_{2,2}(G) = 2 \). Applying [GT] Theorem 1.5 and using \( (6.2.1) \), we see that \( G \) is almost quasisimple, and arrive at one of the following cases for \( L = G^{(\infty)} \).

- \( L = \SU_4(2) \) or \( \SU_3(3) \). In this case, \( L \lhd G/Z(G) \leq L \cdot 2 \). Using [GAP] one can check that \( M_{2,2} = 3 \), a contradiction.
- \( L = 6 \cdot \PSL_3(4) \). In this case, \( L \lhd G \leq L \cdot 2_1 \) (in the notation of [GAP]). The condition \( (6.2.1) \) now implies that \( G = L \) is perfect, which contradicts \( (6.2.2) \).
- \( L = 6 \cdot \PSU_4(3) \). In this case, \( L \lhd G \leq L \cdot 2_2 \) (in the notation of [GAP]). Since \( G \) is not perfect by \( (6.2.2) \), we have that \( G = L \cdot 2_2 \), the Mitchell group.

(b) It remains to determine \( H = G_{\text{geom},r_0,s_0} \) which is a subgroup of \( G \), the Mitchell group. By Proposition 2.7, \( H \) satisfies condition \( (S^+) \). Hence, by [KT2] Lemma 1.1, \( H \) is almost quasisimple: \( S \lhd H/Z(H) \leq \text{Aut}(S) \) for a non-abelian simple group \( S \), and the quasisimple group \( K = E(H) \) with \( S = K/Z(K) \) acts irreducibly on \( V \). We next show that

\[
(6.2.3)
\]

\[
\mathbb{Q}(\varphi|K) = \mathbb{Q}(\zeta_3).
\]
By (6.2.1), it suffices to show that \( V|_K \) is not self-dual. Assume the contrary. Then \( V \) and \( V^* \) are two extensions of the absolutely irreducible module \( V|_K \) to \( H \). By Gallagher’s theorem [Is] (6.17), \( V^* \cong V \otimes U \) for some one-dimensional \( H/K \)-module \( U \). Applying [KT6] Corollary 2.7, we see that \( U \cong \mathbb{Q}_6 \) is trivial, and thus \( V \) is self-dual. But this is impossible by [KT6] Lemma 2.3.

Using [IM] and (6.2.3), we arrive at one of the following cases for \( K \).

- \( K = 3 \cdot A_6 \). Since the faithful module \( V|_K \) is invariant only under the outer automorphisms \( 2 \cdot 3 \) of \( K \) (in the notation of [Atlas]), we have \( H = KC_{H}(K) = KZ(H) \) or \( H \leq Z(H)K \cdot 2 \cdot 3 \). In the former case, \( K \) is perfect and \( Z(H) \leq C_6 \) has determinant 1 on \( V \), and so \( H \leq SL(V) \), contrary to (6.2.2).

- \( K = 6 \cdot PSL_3(4) \). As in part (a), this implies \( H = K \) is perfect, again contradicting (6.2.2).

- \( K = 6 \cdot PSU_4(3) \). As in part (a), using (6.2.2) we obtain that \( H = K \cdot 2 \). Since \( H \leq G \) and \( |H| = |G| \), it follows that \( H = G \).

\[ \square \]

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