AIRY SHEAVES OF LAURENT TYPE: AN INTRODUCTION

NICHOLAS M. KATZ AND PHAM HUU TIEP

To Enrico Bombieri, with the utmost admiration

Abstract. We develop the general theory of Airy sheaves of Laurent type, the local systems whose trace functions have a particular "Airy-Laurent" shape. The main goal is to provide tools for the later determination of their monodromy groups. See [KRLT5] for instances of such determinations.

Contents

1. Introduction

In classical analysis, Airy functions are the Fourier transforms of exponentials $e^{g(x)}$ of polynomials, (originally for the polynomial $g(x) := x^3/3$) and Airy differential equations are the linear differential equations $g'(d/dt)y + ty = 0$ they satisfy. These differential equations have an irregular singularity at ∞, and have quite interesting differential galois groups. In the seminal paper [Such] of Such, he introduces their ℓ-adic finite field analogues, the local systems whose trace functions are of the form

$$t \mapsto -\sum_x \psi(g(x) + tx).$$

The local systems we are concerned with here are generalizations of these Airy local systems in several ways. We allow the "t term" tx to be replaced by tx^a, we allow the polynomial $g(x)$ to be replaced by a Laurent polynomial $f(1/x) + g(x)$, and we allow an "outside factor" $\chi(x)$ in the sum. Here is a more detailed discussion.

We work in characteristic $p > 0$, and denote by $\overline{\mathbb{F}_p}$ an algebraic closure of $\mathbb{F}_p$. We also fix a prime $\ell \neq p$ to be able to speak of $\overline{\mathbb{Q}_\ell}$-adic cohomology. We fix integers

$$A \geq 1, B \geq 1, a > B$$

about which we assume

$$p \nmid ABa.$$

We fix polynomials

$$f(x) \in k[x],\ \deg(f) = A,\ k \text{ some finite subfield of } \overline{\mathbb{F}_p},$$

2010 *Mathematics Subject Classification.* 11T23, 20C15, 20D06, 20G40, 22E46.

Key words and phrases. Exponential sums, Local systems, Monodromy groups.

The second author gratefully acknowledges the support of the NSF (grant DMS-2200850) and the Joshua Barlaz Chair in Mathematics.

$$g(x) \in k[x],\ \deg(f) = B,\ k \text{ some finite subfield of } \overline{\mathbb{F}_p},$$

We make the assumption that both $f(x)$ and $g(x)$ are Artin-Schreier reduced: this means that in the expression $f(x) = \sum_i c_i x^i$, $g(x) = \sum_i d_i x^i$we have $c_i = 0$, $d_i = 0$ if $p|i$. We define

$$\mathrm{gcd}_{\mathrm{deg}}(f) := \gcd(\{i|c_i \neq 0\}),\ \mathrm{gcd}_{\mathrm{deg}}(g) := \gcd(\{i|d_i \neq 0\})$$

the greatest common divisor of the degrees of the monomials appearing in f, respectively in g. We suppose

$$\gcd(a, \mathrm{gcd}_{\mathrm{deg}}(f)) = 1,\ \gcd(a, \mathrm{gcd}_{\mathrm{deg}}(g)) = 1.$$

We also fix a (possibly trivial) multiplicative character χ of $k^\times$, with the convention that for $\chi \neq \mathbb{1}$, we have $\chi(0) = 0$, but $\mathbb{1}(0) = 1$. We denote by $\mathcal{G}(f, g, a, \chi)$ the lisse sheaf on $\mathbb{G}_m/k$ whose trace function at time $t \in L^\times$, for L/k a finite extension, is

$$t \mapsto -\sum_{x \in L^\times} \psi_L(f(1/x) + g(x) + tx^a)\chi_L(x).$$

2. Basic facts about $\mathcal{G}(f, g, a, \chi)$

The local system $\mathcal{G}(f, g, a, \chi)$ is lisse of rank $D = A + a$ on $\mathbb{G}_m$, and pure of weight one. We view it as being the Fourier transform

$$\mathrm{FT}_\psi([a]_\star(\mathcal{L}_{\psi(f(1/x)+g(x))} \otimes \mathcal{L}_\chi(x))).$$

Lemma 2.1. *Given $A \geq 1, B \geq 1, a > B$, $p \nmid ABa$, f, g both Artin-Schreier reduced, and $\gcd(a, \mathrm{gcd}_{\mathrm{deg}}(f)) = 1, \gcd(a, \mathrm{gcd}_{\mathrm{deg}}(g)) = 1$. Then the following statements hold for $\mathcal{G}(f, g, a, \chi)$.*

(i) *The $I(\infty)$-representation of $\mathcal{G}(f, g, a, \chi)$ is irreducible. It has rank $A + a$ and all slopes $A/(A + a)$.*

(ii) *The $I(0)$-representation of $\mathcal{G}(f, g, a, \chi)$ is the direct sum*
$$W(B, a - B) \oplus (\overline{\mathbb{Q}_\ell})^{A+B},$$
with $W(B, a-B)$ an irreducible $I(0)$-representation of rank $a-B$ with all slopes $B/(a-B)$.

Proof. This is a straightforward application of Laumon's results on the local monodromy of FT_ψ. The input sheaf to FT_ψ is lisse on $\mathbb{G}_m$ of rank a, with $I(0)$-slopes A/a and $I(\infty)$ slopes B/a. The hypotheses $\gcd(a, \mathrm{gcd}_{\mathrm{deg}}(g)) = 1, \gcd(a, \mathrm{gcd}_{\mathrm{deg}}(f)) = 1$ imply respectively that the $I(0)$- and $I(\infty)$-representations of the input sheaf are irreducible, cf. the proof of Lemma 2.1 in kt30.

Then the $I(\infty)$-representation of $\mathcal{G}(f, g, a, \chi)$ is $\mathrm{FTloc}(0, \infty)(\text{rank } a, \text{slopes } A/a)$, which has rank $A + a$ and all slopes $A/(A + a)$, cf. [Ka-ESDE, 7.4.4(4)]. The $I(0)$-representation of $\mathcal{G}(f, g, a, \chi)$, modulo its subspace of $I(0)$-invariants, cf. [Ka-ESDE, 7.4.3.1], is $\mathrm{FTloc}(\infty)(\text{rank } a, \text{slopes } B/a)$, which is the asserted $W(B, a-B)$. The asserted irreducibilities result from the the irreducibilities of the input and the fact that $\mathrm{FTloc}(0, \infty)$ and $\mathrm{FTloc}(\infty, 0)$ are suitable equivalences of categories. ☐

Corollary 2.2. *Hypotheses as in Lemma 2.1, suppose in addition that $a > 2B$. Then the determinant of $\mathcal{G}(f, g, a, \chi)$ is tame, so geometrically some Kummer sheaf $\mathcal{L}_\Lambda$. Moreover, if χ has odd order N, then Λ has order dividing $2N$, while if χ has even order N, then Λ has order dividing N.*

Proof. The slopes of $\mathcal{G}(f, g, a, \chi)$ at ∞ are all < 1, and the slopes at 0 are $B/(a-B) < 1$. Therefore the determinant of $\mathcal{G}(f, g, a, \chi)$, a priori of finite order by Grothendieck's local monodromy theorem, is tame, hence geometrically some Kummer sheaf $\mathcal{L}_\Lambda$. Then the arithmetic determinant of $\mathcal{G}(f, g, a, \chi)$ is some constant field twist $\mathcal{L}_\Lambda \otimes \alpha^{\deg}$ of $\mathcal{L}_\Lambda$. Denote by M the order of Λ. Over any finite extension L/k containing μ_M, the trace of $\mathsf{Frob}_{t,L}|\mathcal{L}_\Lambda$, as t runs over $L^\times$, attains all values in μ_M. Then we recover M as the ratios of these Frobenius traces at various points $s, t \in L^\times$. Thus we also recover M as the same ratios of Frobenius trace on the constant field twist $\mathcal{L}_\Lambda \otimes \alpha^{\deg}$ of $\mathcal{L}_\Lambda$.

Each $\mathsf{Frob}_{t,L}|\mathcal{G}(f,g,a,\chi)$ and all its powers have traces in $\mathbb{Q}(\zeta_p,\zeta_N)$ for N the order of χ. So each Frobenius determinant lies in $\mathbb{Q}(\zeta_p,\zeta_N)$. Therefore the geometric determinant takes values both in μ_M and in $\mathbb{Q}(\zeta_p,\zeta_N)$. But the only roots of unity in $\mathbb{Q}(\zeta_p,\zeta_N)$ lie in $\pm\mu_{pN}$. Thus $\mu_M \le \pm\mu_N$. □

Lemma 2.3. *Given $A \ge 1, B \ge 1, a > B$, $p \nmid ABa$, f,g both Artin-Schreier reduced, and $\gcd(a,\gcd_{\deg}(f)) = 1, \gcd(a,\gcd_{\deg}(g)) = 1$. Then $\mathcal{G}(f,g,a,\chi)$ is geometrically selfdual if and only if ABa is odd, both $f(x)$ and $g(x)$ are odd polynomials, and $\chi^2 = \mathbb{1}$.*

Proof. The oddness conditions, and χ^2 trivial, imply autoduality. For $p = 2$, over even degree extensions $k/\mathbb{F}_2(\text{coef's of } f,g)$, after the constant field twist by $1/\sqrt{\#k}$, the traces are real (in fact in $\mathbb{Q}$). And when p is odd, after the constant field twist by $1/\mathsf{Gauss}(\psi,\chi_2)$ and over even degree extensions of $\mathbb{F}_p(\text{coef's of } f,g)$, the traces are real.

To prove the converse, we argue as follows. Since $\mathcal{G} := \mathcal{G}(f,g,a,\chi)$ is geometrically irreducible, it is self dual if and only if $H^2_c(\mathbb{G}_m/\overline{\mathbb{F}_p},\mathcal{G}\otimes\mathcal{G})$ is nonzero (and in fact has dimension 1). We compute this dimension as the limsup, over extensions $k/\mathbb{F}_p(\text{coef's of } f,g)$, of the sums

$$1/(\#k(\#k-1)) \sum_{t\in k^\times, x,y\in k} \psi_k(f(1/x)+f(1/y)+g(x)+g(y)+t(x^a+y^a))\chi_k(xy).$$

The $t = 0$ "missing" summand is

$$1/(\#k(\#k-1)) \sum_{x,y\in k} \psi_k(f(1/x)+f(1/y)+g(x)+g(y))\chi_k(xy) =$$

$$= 1/(\#k((\#k-1))(\sum_{x,y\in k} \psi_k(f(1/x)+g(x))\chi_k(x)^2,$$

which is $O(1/(\#k-1)$, because the sum being squared is, by the Weil bound, of absolute value $\le (A+a)\sqrt{\#k}$.

So the limsup doesn't change if we add this term.Then we have the limsup of

$$1/(\#k-1) \sum_{x,y\in k, x^a+y^a=0} \psi_k(f(1/x)+f(1/y)+g(x)+g(y))\chi_k(xy).$$

This is then the limsup of the sum of the a sums, one for each ζ with $\zeta^a = -1$,

$$S_\zeta := 1/(\#k-1)\sum_{x\in k} \psi_k(f(1/x)+f(1/\zeta x)+g(x)+g(\zeta x))\chi_k(\zeta x^2).$$

Both $f(x)$ and $g(x)$ are Artin-Schreier reduced and $\gcd(a,\gcd_{\deg}(f)) = \gcd(a,\gcd_{\deg}(g)) = 1$. Then the two sums $f(x)+f(x/\zeta)$ and $g(x)+g(\zeta x)$ are each Artin-Schreier reduced. Unless both sum vanish, the S_ζ summand is $O(1/\sqrt{\#k})$, again by the Weil estimate. And even if both sums do vanish, then the sum still vanishes unless χ^2 is the trivial character. Thus in all cases, we must have χ^2 trivial if we are to have self duality.

Suppose now that χ^2 is trivial and $\mathcal{G}$ is self dual. Then for at least one ζ with $\zeta^a = -1$, both $f(x)+f(x/\zeta) = 0$ and $g(x)+g(\zeta x) = 0$. In the case $p = 2$, both f,g are odd polynomials (because both are Artin-Schreier reduced), so there is nothing to prove.

Thus it remains to treat the case when p is odd. Suppose first that a is even. Then we claim that for any ζ with $\zeta^a = -1$, $f(x)+f(x/\zeta) \ne 0$. To see this, write $f(x) = \sum_n a_n x^n$, and define $\mathcal{E}_f := \{n | a_n \ne 0\}$, the set of exponents which occur in f. By hypothesis, we have

$$\gcd(a,\ \text{all } n \in \mathcal{E}_f) = 1.$$

We rewrite this as

$$\gcd(a,\ \text{all } n-a \text{ with } n \in \mathcal{E}_f) = 1.$$

If $f(x)+f(x/\zeta)=0$, then $a_n(1+1/\zeta^n)=0$ for all $n\in\mathcal{E}_f$, i.e., $\zeta^n=-1$ for all $n\in\mathcal{E}_f$, i.e., $\zeta^n=\zeta^a$ for all $n\in\mathcal{E}_f$, and finally $\zeta^{n-a}=1$ for all $n\in\mathcal{E}_f$. Define

$$D:=\gcd(\text{ all } n-a \text{ with } n\in\mathcal{E}_f).$$

Then $\zeta^D=1$. But $\gcd(a,D)=1$, so there exist integers u,v with $au+Dv=1$. Then $\zeta=(\zeta^a)^u(\zeta^D)^v=(-1)^u$. Thus ζ is ±1, neither of which has $\zeta^a=-1$ if a is even.

Suppose next that a is odd. Then the above argument shows that ζ is ±1. But of these two choices, only $\zeta=-1$ has $\zeta^a=-1$. For this $\zeta=-1$, we have $f(x)+f(-x)=0$, which means precisely that f is an odd polynomial. The same argument applied to g, using the fact that $\gcd(a,\text{ all } n\in\mathcal{E}_g)=1$, shows that $\zeta=-1$, hence that g is an odd polynomial. ☐

Lemma 2.4. *Given $A\geq1, B\geq1, a>B$, $p\nmid ABa$, f,g both Artin-Schreier reduced, and $\gcd(a,\gcd_{\deg}(f))=1, \gcd(a,\gcd_{\deg}(g))=1$. Suppose that χ and ρ are multiplicative characters of $k^\times$ for $k/\mathbb{F}_p$ a finite extension containing the coefficients of both f and g. If $\chi\neq\rho$, then $\mathcal{G}(f,g,a,\chi)$ and $\mathcal{G}(f,g,a,\rho)$ are not geometrically isomorphic.*

Proof. We argue by contradiction. Suppose that $\mathcal{G}(f,g,a,\chi)$ and $\mathcal{G}(f,g,a,\rho)$ are geometrically isomorphic. As each is geometrically irreducible, the cohomology group

$$H^2_c(G_m/\overline{\mathbb{F}_p},\mathcal{G}(f,g,a,\chi)\otimes\mathcal{G}(f,g,a,\rho)^\vee)$$

is pure of weight 2 and of dimension one. The dual $\mathcal{G}(f,g,a,\rho)^\vee$ is the (-1)-Tate twist of its complex conjugate: its trace function at $t\in L^\times$ is

$$t\in L^\times\mapsto\frac{-1}{\#L}\sum_{x\in L^\times}\psi_L(-f(1/x)-g(x)-tx^a)\overline{\rho}(x).$$

So the trace function of $\mathcal{G}(f,g,a,\chi)\otimes\mathcal{G}(f,g,a,\rho)^\vee$ is

$$t\in L^\times\mapsto\frac{-1}{\#L}\sum_{x,y\in L^\times}\psi_L(f(1/x)-f(1/y)+g(x)-g(y)+t(x^a-y^a))\chi(x)\overline{\rho}(x).$$

The sum over t of this trace is, by the Lefschetz trace formula,

$$\mathrm{Trace}(\mathsf{Frob}_L|H^2_c)-\mathrm{Trace}(\mathsf{Frob}_L|H^1_c),$$

with H^2_c pure of weight 2, and H^1_c mixed of weight ≤1. Thus the dimension, namely 1, of the relevant H^2_c is the limsup, as L/k grows,

$$\frac{1}{\#L}\sum_{t\in L^\times}\frac{1}{\#L}\sum_{x,y\in L^\times}\psi_L(f(1/x)-f(1/y)+g(x)-g(y)+t(x^a-y^a))\chi(x)\overline{\rho}(x).$$

So far as the limsup is concerned, we may replace this sum over $t\in L^\times$ by the sum over all $t\in L$: indeed the $t=0$ summand is

$$\frac{1}{(\#L)^2}\Big(-\sum_{x\in L^\times}\psi_L(f(1/x)+g(x))\chi(x)\Big)\Big(-\sum_{y\in L^\times}\psi_L(-f(1/y-g(y))\overline{\rho}(y)\Big).$$

Each of the factors is $O(\sqrt{\#L})$, so this $t=0$ term is $O(1/\#L)$, and hence does not affect the limsup.

Thus the dimension, 1, of the H^2_c is the limsup of

$$\frac{1}{\#L}\sum_{t\in L}\frac{1}{\#L}\sum_{x,y\in L^\times}\psi_L(f(1/x)-f(1/y)+g(x)-g(y)+t(x^a-y^a))\chi(x)\overline{\rho}(x)$$

$$=\frac{1}{\#L}\sum_{x,y\in L^\times, x^a=y^a}\psi_L(f(1/x)-f(1/y)+g(x)-g(y)+t(x^a-y^a))\chi(x)\overline{\rho}(x)$$

$$=\sum_{\zeta\in\mu_a(\overline{\mathbb{F}_p}}\frac{1}{\#L}\sum_{x,L^\times}\psi_L(f(1/x)-f(1/\zeta x)+g(x)-g(\zeta x))\chi(x)\overline{\rho}(\zeta x).$$

The $\zeta=1$ summand is

$$\frac{1}{\#L}\sum_{x,L^\times}\chi(x)\overline{\rho}(\zeta x)=\frac{\overline{\rho}(\zeta x)}{\#L}\sum_{x,L^\times}(\chi/\overline{\rho})(x)=0,$$

simply because $\chi/\overline{\rho}$ is nontrivial. In each of the remaining summands, the Laurent polynomial $f(1/x)-f(1/\zeta x)+g(x)-g(\zeta x)$ inside the ψ is itself nonzero (because $\gcd(a,\gcd_{\deg}(f))=1$ and $\gcd(a,\gcd_{\deg}(g))=1$) and Artin-Schreier reduced, so each of these summands is $O(1/\sqrt{\#L})$. Thus the limsup vanishes, the desired contradiction. □

Lemma 2.5. *Let $X/\mathbb{F}_q$ be smooth and geometrically connected of dimension $d\geq 1$, $\ell\neq p$, $K/\mathbb{Q}$ a finite extension, and L/K a finite Galois extension. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are nonzero lisse $\overline{\mathbb{Q}_\ell}$-sheaves on X. Suppose that both $\mathcal{F}$ and $\mathcal{G}$ are geometrically irreducible, and have all their Frobenius traces in L. Suppose further that for every $\sigma\in\mathrm{Gal}(L/K)$, there exist lisse sheaves $\mathcal{F}^\sigma$ and $\mathcal{G}^\sigma$ on X whose trace functions are the σ-conjugates of those of $\mathcal{F}$ and of $\mathcal{G}$. If $\mathcal{F}$ and $\mathcal{G}$ are geometrically isomorphic, then $\mathcal{F}^\sigma$ and $\mathcal{G}^\sigma$ are geometrically isomorphic.*

Proof. If $\mathcal{F}$ and $\mathcal{G}$ are geometrically isomorphic, then because each is geometrically isomorphic, there exists an $\alpha^{\deg}$ twist such we have an arithmetic isomorphism

$$\mathcal{F}\cong\mathcal{G}\otimes\alpha^{\deg}.$$

This implies that for every finite extension $k/\mathbb{F}_q$, and every $t\in X(k)$, we have an equality

$$\mathrm{Trace}(\mathsf{Frob}_{t,k}|\mathcal{F})=\alpha^{\deg(k/\mathbb{F}_q)}\mathrm{Trace}(\mathsf{Frob}_{t,k}|\mathcal{G}).$$

Because $\mathcal{F}$ is not the zero sheaf, for some $k_0/\mathbb{F}_q$ and some $t\in X(k_0)$, $\mathrm{Trace}(\mathsf{Frob}_{t,k_0}|\mathcal{F})$ is nonzero. Then $\mathrm{Trace}(\mathsf{Frob}_{t,k_0}|\mathcal{G})$ must also be nonzero, and we recover $\alpha^{\deg(k/\mathbb{F}_q)}$ as the ratio of nonzero traces of $\mathcal{F}$ and of $\mathcal{G}$. As these traces lie in L, it follows that $\alpha^{\deg(k_0/\mathbb{F}_q)}$ lies in L. Extending scalars from $\mathbb{F}_q$ to k_0, we reduce to the case when $\alpha\in L^\times$. Then we simply apply σ to the above equality of traces to obtain

$$\mathrm{Trace}(\mathsf{Frob}_{t,k}|\mathcal{F}^\sigma)=\sigma(\alpha)^{\deg(k/\mathbb{F}_q)}\mathrm{Trace}(\mathsf{Frob}_{t,k}|\mathcal{G}^\sigma),$$

i.e., we have an arithmetic isomorphism

$$\mathcal{F}^\sigma\cong\mathcal{G}^\sigma\otimes\sigma(\alpha)^{\deg},$$

and hence the desired geometric isomorphism. □

Proposition 2.6. *Given $A\geq 1, B\geq 1, a>B$, $p\nmid ABa$, let $\mathcal{F}$ and $\mathcal{G}$ be local systems on $\mathbb{G}_m/\overline{\mathbb{F}_p}$, whose local monodromies are of the form*

$$\mathcal{F}_{I(0)}\cong(A+B)\mathbb{1}\oplus W_{\mathcal{F}},\quad \mathcal{G}_{I(0)}\cong(A+B)\mathbb{1}\oplus W_{\mathcal{G}},$$

with $W_{\mathcal{F}}, W_{\mathcal{G}}$ both irreducible of rank $a-B$ and totally wild, and

$$\mathcal{F}_{I(\infty)}\cong V_{\mathcal{F}},\ \mathcal{G}_{I(\infty)}\cong V_{\mathcal{G}},$$

with $V_{\mathcal{F}}, V_{\mathcal{G}}$ both of rank $A+a$ with all slopes < 1. Let $\mathcal{L}$ be a rank one local system on $\mathbb{G}_m/\overline{\mathbb{F}_p}$, such that $\mathcal{F} \cong \mathcal{G} \otimes \mathcal{L}$. Then $\mathcal{L} \cong \overline{\mathbb{Q}_\ell}$.

Proof. We first show that $\mathcal{L}$ is tame at ∞. Indeed, it if were not, then $V_{\mathcal{G}} \otimes \mathcal{L}$ has all slopes equal to $\mathsf{Swan}_\infty(\mathcal{L}) \geq 1$, while $V_{\mathcal{F}}$ has all slopes < 1. Once we have this, it suffices to show that $\mathcal{L}_{I(0)}$ is trivial. Suppose first that $a - B > 1$. Then

$$\mathcal{G}_{I(0)} \otimes \mathcal{L}_{I(0)} \cong (A+B)\mathcal{L}_{I(0)} \oplus (\text{irreducible of rank } > 1).$$

So the one-dimensional constituents are each $\mathcal{L}_{I(0)}$. But the the one-dimensional constituents of $\mathcal{F}_{I(0)}$ are each $\mathbb{1}$. Suppose next that $a-B = 1$. Then in both $\mathcal{F}_{I(0)}$ and $\mathcal{G}_{I(0)}$, the trivial constituents are in the majority. But after tensoring with $\mathcal{L}_{I(0)}$, the $\mathcal{L}_{I(0)}$ constituents are in the majority. Hence $\mathcal{L}_{I(0)}$ is trivial. ☐

Corollary 2.7. *For $\mathcal{F}$ as in Proposition 2.6 above, suppose $\mathcal{L}$ is a rank one local system on $\mathbb{G}_m/\overline{\mathbb{F}_p}$, such that $\mathcal{F}^\vee \cong \mathcal{F} \otimes \mathcal{L}$. Then $\mathcal{L} \cong \overline{\mathbb{Q}_\ell}$.*

Proof. Indeed, both $\mathcal{F}$ and $\mathcal{F}^\vee$ have the shapes of local monodromies of the Proposition. ☐

Proposition 2.8. *Suppose $p \nmid ABa$, f, g are both Artin-Schreier reduced, and* $\gcd(a, \gcd_{\deg}(f)) = 1$, $\gcd(a, \gcd_{\deg}(g)) = 1$. *Then the $I(0)$-representation of $\mathcal{G}(f, g, a, \chi)$ is tensor indecomposable under each of the following conditions.*

(a) *The rank $A + a \neq 4$.*
(b) *$A + a = 4$ and $p = 2$.*
(c) *$A + a = 4$, $p \neq 2$, and $(A, B, a) \neq (1, 1, 3)$.*

Proof. Indeed, the $I(0)$-representation is the direct sum $T \oplus W$ of a nonzero tame part and an irreducible wild part. In rank $\neq 4$, the result follows from [KRLT3, 10.4]. In the case of rank 4, the tame part has rank $A + B \geq 2$, so in characteristic $p = 2$ we may again apply [KRLT3, 10.4]. To apply [KRLT3, 10.4] with p odd and rank 4, we must avoid the case $A + B = 2$, i.e., the case $A = 1 = B$ and $a = 3$. ☐

Proposition 2.9. *Suppose $p \nmid ABa$, f, g are both Artin-Schreier reduced, and* $\gcd(a, \gcd_{\deg}(f)) = 1$, $\gcd(a, \gcd_{\deg}(g)) = 1$. *Suppose that $\mathcal{G}(f, g, a, \chi)$ is tensor indecomposable for $I(0)$. Let us denote*

$$D := A + a,\ t := A + B,\ w := a - B,$$

the rank, the dimension of the tame part T, and the dimension of the wild part W of the $I(0)$-representation $V = T \oplus W$. Then $\mathcal{G}(f, g, a, \chi)$ is not tensor induced over $I(0)$ under each of the following conditions.

(a) *D is not a power.*
(b) *$w = 1$.*
(c) *$t - w > \sqrt{D}$.*
(d) *$p \nmid w$ and $w < D - \sqrt{D}$.*

Proof. Case (a) is trivial.

To treat case (b), suppose $w = 1$, and V is tensor induced: $V = U_1 \otimes \ldots \otimes U_n$ with $n \geq 2$, $\dim(U_i) = d \geq 2$, and $I(0)$ acts through $\mathrm{GL}_d(\mathbb{C}) \wr \mathsf{S}_n$. As $p \nmid aB$ and $a - B = w = 1$, $p > 2$. Since W has dimension $w = 1$, some element $\gamma \in P(0)$ must act on W as a scalar $\zeta \neq 1$, an N^{th} root of unity with $N > 1$ a p-power. By Lemma 2.2(ii) of KT30, γ is tensor indecomposable, so it must induce an n-cycle while permuting the n tensor factors of V. By the formula for tensor induction [GI],

$$|\mathrm{Trace}(\gamma|_V)| \leq d \leq D/2 \leq D - 2$$

since $D = d^n \geq 4$. On the other hand,

$$|\mathrm{Trace}(\gamma|_V)| = |D - 1 + \zeta| \geq D - 2,$$

with equality only when $\zeta = -1$, which is impossible since $p > 2$.

To treat case (c), use the $I(0)$-tensor indecomposability of V to apply (ii) of Lemma 2.4 of kt30. It shows the existence of an element $h \in I(0)$ with $|\mathrm{Trace}(h|_V)| \leq \sqrt{D}$ if V is tensor induced. But $V = t\mathbb{1} + W$, and any $h \in I(0)$, being of finite order, has $|\mathrm{Trace}(h|_W)| \leq w$. Hence $\sqrt{D} \geq |\mathrm{Trace}(h|_V)| \geq t - w$, a contradiction.

For case (d), we may assume $w > 1$ by (b), and then use the fact that an element $\gamma \in I(0)$ which is generator of $I(0)/P(0)$ has spectrum on W consisting of all the w^{th} roots of some root of unity ρ (because when $p \nmid w$, W is the Kummer induction $[w]_\star\mathcal{L}$ of some rank one $\mathcal{L}$, and γ acts by cyclically permuting the w factors of the induction: because γ has finite order on V, ρ is itself a root of unity). Then we apply Lemma 2.2 (i) of kt30 (with its $a = w$) to see that γ is tensor indecomposable in the $I(0)$-representation if $w < D - \sqrt{D}$. Then we repeat the argument of case (b): if V is n-tensor induced, then

$$|\mathrm{Trace}(\gamma|_V)| \leq d = D^{1/n} \leq \sqrt{D}.$$

But $\mathrm{Trace}(\gamma|_W) = 0$ since $w > 1$, and hence $\sqrt{D} \geq |\mathrm{Trace}(\gamma|_V) = t = D - w$, i.e $w \geq D - \sqrt{D}$, a contradiction. □

Theorem 2.10. *Suppose that $p \nmid ABa(A+a)(a-B)$, f and g are both Artin-Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = \gcd(a, \gcd_{\deg}(g)) = 1$. Then $\mathcal{G}(f, g, a, \chi)$ is primitive on $\mathbb{G}_m/\overline{\mathbb{F}_p}$, under each of the following conditions.*

(a) *$w := a - B$ is not of the form $p^s - 1$ for any $s \geq 1$.*
(b) *$w = p^s - 1$ and $A \neq 1$.*
(c) *$w = p^s - 1$, $A = 1$, and $\chi \neq \chi_2$ (for χ_2 the quadratic character).*
(d) *$w = p^s - 1$, $A = 1$, $\chi = \chi_2$, ABa is odd, each of f, g is an odd polynomial, and $B < 2p$.*
(e) *$w = p^s - 1$, $A = 1$, $\chi = \chi_2$, and $\mathcal{G}(f, g, a, \chi)$ has infinite G_{geom}.*
(f) *$w = p^s - 1$, $A = 1$, $\chi = \chi_2$, $p \geq 5$, each of $f, g \in \mathbb{F}_p[x]$, with $g(x) = \sum_{i=0}^{B} a_i x^i$, and either*

$$B \equiv \frac{p-1}{2} \bmod (p-1)$$

or

$$\sum_{i:\ i \equiv \frac{p-1}{2} \bmod (p-1)} a_i \neq 0.$$

Proof. We argue by contradiction. Suppose $\mathcal{G}(f, g, a, \chi) = \pi_\star\mathcal{H}$ for some finite etale $\pi : U \to \mathbb{G}_m$ of degree $d > 1$ and some local system $\mathcal{H}$ on U. Then $d \times \mathrm{rank}(\mathcal{H}) = \mathrm{rank}(\mathcal{G}(f, g, a, \chi)) = A + a$ is prime to p. Also U is geometrically connected, otherwise $\pi_\star\mathcal{H}$ is not irreducible. Denote by X the complete nonsingular model of U, and denote by $\pi : X \to \mathbb{P}^1$ the finite flat map on the complete curves. Let

$$C := \pi^{-1}(0), \quad E = \pi^{-1}(\infty),$$

of cardinalities c, e respectively.

For each point $x \in E$, denote by

$$\pi_x : \mathrm{Spec}\,((K_{X,x})^\wedge) \to \mathrm{Spec}\,((K_{\mathbb{P}^1,\infty})^\wedge)$$

the induced map of the spec's of completed function fields. Then for $\mathcal{G} := \mathcal{G}(f, g, a, \chi)$, we have

$$\mathcal{G}|_{I(\infty)} = \oplus_{x \in E} \pi_{x\star}(\mathcal{H}_{I(x)}).$$

But $\mathcal{G}|_{I(\infty)}$ is irreducible, hence there is precisely one point in E, call it ∞up, and

$$\mathcal{G}|_{I(\infty)} = \pi_{\infty up_\star}(\mathcal{H}_{I(\infty up)}),$$

with $\mathcal{H}_{I(\infty up)}$ irreducible (because its direct image is irreducible). Because ∞up is the unique point lying over ∞, the degree of $\pi_{\infty up}$ is precisely $d := \deg(\pi)$, which is a divisor of $A + a$. Looking at degrees, we thus have

$$d \times \mathrm{rank}(\mathcal{H}) = \mathrm{rank}(\mathcal{G}).$$

Therefore $\deg(\pi_{\infty up}) = d$ is prime to p, hence $\pi_{\infty up}$ is tame. By [Ka-TLFM, 1.6.4.1], it follows that

$$\mathsf{Swan}_{\infty up}(\mathcal{H}) = \mathsf{Swan}_\infty(\mathcal{G}).$$

Similarly, we have

$$\mathcal{G}|_{I(0)} = \oplus_{x \in C} \pi_{x\star}(\mathcal{H}_{I(x)}),$$

while

$$\mathcal{G}|_{I(0)} = W_{B,a-B} \oplus (\overline{\mathbb{Q}_\ell})^{A+B},$$

with $W_{B,a-B}$ irreducible of rank $w := a - B$ with all slopes $B/(a - B)$. There is precisely one point $x_0 \in C$ whose $\pi_{x_0\star}(\mathcal{H}_{I(x)})$ contains $W_{B,a-B}$ as a summand. More precisely, we have

$$\pi_{x_0\star}(\mathcal{H}_{I(x_0)}) = W_{B,a-B} \oplus (\overline{\mathbb{Q}_\ell})^n, \text{ for some } n \geq 0.$$

We first consider the case $n = 0$. Then $\mathcal{H}_{I(x_0)}$ is irreducible. Moreover, it cannot be tame, i.e., it cannot be a Kummer sheaf $\mathcal{L}_\chi$: if it were, then by Frobenius reciprocity its direct image contains all $\mathcal{L}_\rho$ with $\rho^{\deg(\pi_{x_0})} = \chi$, whereas its direct image is totally wild. Looking at degrees, we have

$$\deg(\pi_{x_0}) \times \mathrm{rank}(\mathcal{H}) = \mathrm{rank}(W_{B,a-B}) = a - B.$$

As $p \nmid (a - B)$, we see that π_{x_0} has degree prime to p. Again by [Ka-TLFM, 1.6.4.1], it follows that

$$\mathsf{Swan}_{x_0}(\mathcal{H}) = \mathsf{Swan}_(\mathcal{G}) = B.$$

In this $n = 0$ case, we now argue as follows. On the one hand, for $\mathcal{G} := \mathcal{G}(f, g, a, \chi)$, for the Euler–Poincaré characteristic we have

$$\mathsf{EP}(U, \mathcal{H}) = \mathsf{EP}(\mathbb{G}_m, \mathcal{G}) = -\mathsf{Swan}_0(\mathcal{G}) - \mathsf{Swan}_\infty(\mathcal{G}) = -B - A.$$

But

$$\begin{aligned}\mathsf{EP}(U, \mathcal{H}) &= \mathsf{EP}(U)\mathrm{rank}(\mathcal{H}) - \sum_{x \in C} \mathsf{Swan}_x(\mathcal{H}) - \mathsf{Swan}_{\infty up}(\mathcal{H}) \\ &= \mathsf{EP}(U)\mathrm{rank}(\mathcal{H}) - B - \sum_{x \in C, x \neq x_0} \mathsf{Swan}_x(\mathcal{H}) - A.\end{aligned}$$

Subtracting these two expressions for $\mathsf{EP}(U, \mathcal{H})$, we find that

$$\mathsf{EP}(U)\mathrm{rank}(\mathcal{H}) = \sum_{x \in C, x \neq x_0} \mathsf{Swan}_x(\mathcal{H}).$$

In particular, $\mathsf{EP}(U)\mathrm{rank}(\mathcal{H}) \geq 0$, and hence $\mathsf{EP}(U) \geq 0$. As U is the complement of at least two points (one in D and at least one in C) in a complete nonsingular curve, call it X, we have $\mathsf{EP}(U) = 2 - 2g_X - 1 - \#C \geq 0$. On the other hand $2 - 2g_X - 1 - \#C \leq 0$, with equality only if $g_X = 0$ and $\#C = 1$. Because $\#C = 1$, $\deg(\pi_{x_0})$ must be $d = \deg(\pi)$. Then the entire $I(0)$-representation is wild, a contradiction, since the $I(0)$-representation has an $A + B \geq 2$ dimensional trivial part.

Suppose next that $n \geq 1$. Then $\pi_{x_0\star}(\mathcal{H}_{I(x_0)})$ contains $\overline{\mathbb{Q}_\ell}$, which we write as $\mathcal{L}_{\mathbb{1}}$. Then by Frobenius reciprocity, $\mathcal{H}_{I(x_0)}$ contains $\mathcal{L}_{\mathbb{1}}$. Then $\deg(\pi_{x_0})$ cannot be divisible by any prime to p integer $r > 1$, for otherwise $\pi_{x_0\star}(\mathcal{H}_{I(x_0)})$ contains $\pi_{x_0\star}(\mathcal{L}_{\mathbb{1}})$, which contains all $\mathcal{L}_\rho$ with $\rho^r = \mathbb{1}$. This is impossible, because the entire tame part of the $I(0)$-representation of $\mathcal{G}$ is copies of $\mathcal{L}_{\mathbb{1}}$. Thus

$\deg(\pi_{x_0}) = p^s$ for some $s \geq 0$. If $s = 0$, i.e. if $\deg(\pi_{x_0}) = 1$ is prime to p, then $\mathsf{Swan}_{x_0}(\mathcal{H}) = \mathsf{Swan}_(\mathcal{G}) = B$, and we conclude as in the $n = 0$ case above.

Suppose next that $n \geq 1$ and $\deg(\pi_{x_0}) = p^s$ with $s \geq 1$. Then by Frobenius reciprocity, $\pi_{x_0\star}(\mathcal{L}_{\mathbb{1}})$ contains $\mathcal{L}_{\mathbb{1}}$ just once, and contains no $\mathcal{L}_\rho$ for any nontrivial ρ (because it only contains $\mathcal{L}_\rho$ if $\rho^{\deg(\pi_{x_0})} = \mathbb{1}$). Therefore

$$\pi_{x_0\star}(\mathcal{L}_{\mathbb{1}}) = \mathcal{L}_{\mathbb{1}} \oplus (\text{totally wild of rank } p^s - 1).$$

If $\mathcal{H}_{I(x_0)}$ were not simply $\mathcal{L}_{\mathbb{1}}$, any other irreducible constituent would either be tame (in which case its direct image would also have a wild part of rank $p^s - 1$, or would be wild, in which case its direct image would be totally wild.

Thus in this $n \geq 1$ case, we have $n = 1$, $\mathrm{rank}(\mathcal{H}) = 1, \mathcal{H}_{I(x_0)} = \mathcal{L}_{\mathbb{1}}$, and

$$\pi_{x_0\star}(\mathcal{H}_{I(x_0)}) = \pi_{x_0\star}(\mathcal{L}_{\mathbb{1}}) = \mathcal{L}_{\mathbb{1}} \oplus (\text{totally wild of rank } p^s - 1).$$

So the wild part of the $I(0)$-representation of $\mathcal{G}$ has dimension $w = p^s - 1$.

We now continue with the analysis of the case when $w = p^s - 1$. Looking at what remains of the $I(0)$-representation, we find

$$\mathcal{L}_{\mathbb{1}}^{A+B-1} = \bigoplus_{x \in C, x \neq x_0} \pi_{x\star}(\mathcal{H}_{I(x)}).$$

Each individual direct image $\pi_{x\star}(\mathcal{H}_{I(x)})$ is then a sum of $\mathcal{L}_{\mathbb{1}}$. Being tame, it follows that $\mathcal{H}$ is tame at each $x \neq x_0$ in C. Then $\mathcal{H}$ must be $I(x)$-trivial at each such x, otherwise its direct image contains various $\mathcal{L}_\rho$ with nontrivial ρ. Then each π_x for $x \neq x_0$ must have degree 1: it cannot have degree divisible by a prime to p integer $r > 1$ because that introduces nontrivial tame pieces in the direct image, and it cannot have degree a strictly positive power of p, because that introduces nonzero wild parts in the direct image. Thus at each $x \neq x_0$ in C, the degree of π_x is 1. From the above displayed equation

$$\mathcal{L}_{\mathbb{1}}^{A+B-1} = \bigoplus_{x \in C, x \neq x_0} \pi_{x\star}(\mathcal{H}_{I(x)}),$$

we then see that

$$\#C = A + B,$$

and that $\mathcal{H}$ is lisse of rank one outside of the single point ∞up. Thus

$$\mathsf{EP}(U, \mathcal{H}) = \mathsf{EP}(\mathbb{G}_m, \mathcal{G}) = -\mathsf{Swan}_0(\mathcal{G}) - \mathsf{Swan}_\infty(\mathcal{G}) = -B - A.$$

At the same time, remembering that $\mathcal{H}$ has rank one, we have

$$\mathsf{EP}(U, \mathcal{H}) = \mathsf{EP}(U) - \mathsf{Swan}_{\infty up}(\mathcal{H}) = \mathsf{EP}(U) - A.$$

Thus $\mathsf{EP}(U) - A = -B - A$, and hence

$$\mathsf{EP}(U) = -B.$$

In terms of the complete nonsingular model X of U, this gives

$$-B = \mathsf{EP}(U) = 2 - 2g_X - \#D - \#C = 2 - 2g_X - 1 - (A + B),$$

hence $2 - 2g_X - 1 - A = 0$, i.e., $-2g_X = A - 1$. This can only hold if $g_X = 0$ and $A = 1$. Putting ∞up at ∞, $\mathcal{H}$ is lisse of rank one on $\mathbb{A}^1$, with $\mathsf{Swan}_\infty(\mathcal{H}) = 1$. Thus $\mathcal{H}$ is $\mathcal{L}_{\psi(\alpha x)}$ for some $\alpha \neq 0$ in $\overline{\mathbb{F}_p}$. Putting x_0 at 0, the morphism π is a polynomial $H(x) \in \overline{\mathbb{F}_p}[x]$ which has degree $A + a = 1 + a = 1 + B + w = p^s + B$, which has 0 as a root of multiplicity p^s, and which has B simple zeros, each of which is nonzero. Thus we obtain a geometric isomorphism

$$\mathcal{G}(f, g, a, \chi) \cong [H(x)]_\star \mathcal{L}_{\psi(\alpha x)}.$$

Over a large enough finite extension $k/\mathbb{F}_p$ (namely one which contains α and the coefficients of each of the polynomials f, g, H, and with $\chi^{\#k-1} = \mathbb{1}$) both of $\mathcal{G}(f,g,a,\chi)$ and $[H(x)]_\star\mathcal{L}_{\psi(\alpha x)}$ are geometrically irreducible and geometrically isomorphic local systems on $\mathbb{G}_m/k$. Therefore there exists some $\gamma \in \overline{\mathbb{Q}_\ell}^\times$ for which we have an **arithmetic** isomorphism

$$\mathcal{G}(f,g,a,\chi) \otimes \gamma^{\deg} \cong [H(x)]_\star\mathcal{L}_{\psi(\alpha x)}. \tag{2.10.1}$$

Recall that $\mathcal{G}(f,g,a,\chi)$ is, arithmetically, the Fourier transform

$$\mathcal{G}(f,g,a,\chi) := \mathrm{FT}_\psi([a]_\star(\mathcal{L}_{\psi(f(1/x)+g(x))} \otimes \mathcal{L}_\chi(x))),$$

and hence

$$\mathcal{G}(f,g,a,\chi) \otimes \gamma^{\deg} := \mathrm{FT}_\psi([a]_\star(\mathcal{L}_{\psi(f(1/x)+g(x))} \otimes \mathcal{L}_\chi(x) \otimes \gamma^{\deg})).$$

Applying the inverse Fourier transform $\mathrm{FT}_{\overline{\psi}}$ to equation 2.10.1, we get an arithmetic isomorphism

$$[a]_\star(\mathcal{L}_{\psi(f(1/x)+g(x))} \otimes \mathcal{L}_\chi(x) \otimes \gamma^{\deg}) \cong \mathrm{FT}_{\overline{\psi}}([H(x)]_\star\mathcal{L}_{\psi(\alpha x)}).$$

We next prove that this cannot happen if χ has order ≥ 3. The key point is that

$$\mathrm{Gal}(\mathbb{Q}(\chi,\zeta_p)/\mathbb{Q}(\zeta_p)) = \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}).$$

So we may choose $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi,\zeta_p)/\mathbb{Q}(\zeta_p))$ so that the σ-conjugate system to $\mathcal{G}(f,g,a,\chi)$ is

$$\mathcal{G}(f,g,a,\chi)^\sigma = \mathcal{G}(f,g,a,\chi^\sigma),$$

while

$$([H(x)]_\star\mathcal{L}_{\psi(\alpha x)})^\sigma = [H(x)]_\star\mathcal{L}_{\psi(\alpha x)}.$$

Applying Lemma 2.5, we find that $[H(x)]_\star\mathcal{L}_{\psi(\alpha x)}$ is isomorphic to both $\mathcal{G}(f,g,a,\chi)$ and to $\mathcal{G}(f,g,a,\chi^\sigma)$. But if χ has order ≥ 3, there exists σ for which χ^σ is any character of the same order as χ, and in particular there exists σ for which $\chi^\sigma \neq \chi$. For such a σ, $\mathcal{G}(f,g,a,\chi)$ and to $\mathcal{G}(f,g,a,\chi^\sigma)$ are not geometrically isomorphic, by Lemma 2.4.

For $\chi = \mathbb{1}$, the "traces nowhere vanishing" argument of the proof of Proposition 3.6 of kt30 shows that $\mathcal{G}(f,g,a,\mathbb{1})$ is always primitive.

We now deal with the case $A = 1$, $w = p^s - 1$, $\chi = \chi_2$ has order 2, and $\mathcal{G}(f,g,a,\chi_2) \cong [H(x)]_\star\mathcal{L}_{\psi(\alpha x)}$ with H a polynomial of degree $B + p^s$, with 0 as a root of multiplicity p^s and with B simple roots, each nonzero. For an $I(0)$ representation V, $[H(x)]_\star V$ as $I(0)$ representation is the induction through H viewed as lying in $\overline{\mathbb{F}_p}[[x]]$, call it H^{fml}. We apply this to $V := \mathcal{L}_{\psi(\alpha x)}$, which is trivial as $I(0)$ representation We recall from [Ka-MMP, 6.4.5, 2)] that we may compute $\mathsf{Swan}_0([H^{fml}(x)]_\star\mathcal{L}_{\psi(\alpha x)}) = \mathsf{Swan}_0([H^{fml}(x)]_\star\mathcal{L}_{\mathbb{1}})$ as follows. Expand $H^{fml}(x)$:

$$H^{fml}(x) = x^{p^s}(\sum_{m\geq 0} \alpha_m x^m).$$

Then $\mathsf{Swan}_0([H^{fml}(x)]_\star\mathcal{L}_{\mathbb{1}})$ is the least prime to p integer m with $\alpha_m \neq 0$.

On the other hand, the $I(0)$ representation of $[H(x)]_\star\mathcal{L}_{\psi(\alpha x)}$ is the direct sum of $[H^{fml}(x)]_\star\mathcal{L}_{\psi(\alpha x)}$ with B copies of $\mathcal{L}_{\mathbb{1}}$, so

$$\mathsf{Swan}_0([H(x)]_\star\mathcal{L}_{\psi(\alpha x)}) = \mathsf{Swan}_0([H^{fml}(x)]_\star\mathcal{L}_{\psi(\alpha x)})\ (= \mathsf{Swan}_0([H^{fml}(x)]_\star\mathcal{L}_{\mathbb{1}})).$$

But H is a polynomial of degree $B + p^s$. Thus in $\overline{\mathbb{F}_p}[[x]]$, $H = H^{fml}$, and its expansion is

$$H(x) = x^{p^s}(\sum_{m=0}^{B} a_m x^m),$$

with a_0, a_B both nonzero. Moreover, if $1 \leq m \leq B-1$ is nonzero, then $a_m = 0$ (otherwise Swan_0 would be this lower m).

Suppose now that ABa is odd and that both f, g are odd polynomials. Then $\mathcal{G}(f, g, a, \chi_2)$ is self dual (in fact orthogonally self dual). So if $\mathcal{G}(f, g, a, \chi_2) \cong [H(x)]_\star \mathcal{L}_{\psi(\alpha x)}$, then

$$\mathcal{H} := [H(x)]_\star \mathcal{L}_{\psi(\alpha x)}$$

is self dual. As $\mathcal{H}$ is pure of weight zero and geometrically irreducible, its autoduality is equivalent to having $\dim(H^2_c(\mathbb{G}_m/\overline{\mathbb{F}_p}, \mathcal{H}^{\otimes 2})) = 1$. This dimension is the limsup over larger and larger extension L of any chosen finite extension $k/\mathbb{F}_p$ which contains the coefficients of f, g, H, of the complex absolute value of

$$(1/\#L) \sum_{t \in L^\times} (\mathrm{Trace}(\mathsf{Frob}_{t,L}|\mathcal{H}))^2 = (1/\#L) \sum_{t \in L^\times} \sum_{x,y \in L : H(x) = t = H(y)} \psi_L(\alpha x) \psi_L(\alpha y)$$
$$= (1/\#L) \sum_{x,y \in L : H(x) = H(y) \neq 0} \psi_L(\alpha(x+y)).$$

The "missing" term with $t = 0$ is $(1/\#L)(\sum_{x \in L : H(x) = 0} \psi_L(\alpha x)))^2$, which is at most $\deg(H)^2/\#L$, so does not affect the limsup. So the dimension of this H^2_c is the limsup of

$$(1/\#L) \sum_{x,y \in L : H(x) = H(y)} \psi_L(\alpha(x+y)).$$

The affine curve $H(x) = H(y)$ is smooth outside the point $(0,0)$. Indeed, its singularities are the points on the curve where $dH(x)/dx = 0 = dH(y)/dy$. From the explicit form of H above, we see that $dH(x)/dx = a_B(p^s + B)x^{p^s + B - 1}, dH(y)/dy = a_B(p^s + B)y^{p^s + B - 1}$.

The polynomial $H(x) - H(y)$ has the factorization

$$H(x) - H(y) = (x - y)\Delta_H, \text{ with } \Delta_H := (H(x) - H(y))/(x - y).$$

The polynomial Δ_H is not divisible by $x - y$, indeed its leading term is $\alpha_B \prod_{\zeta \in \mu_{p^s+B}, \zeta \neq 1} (x - \zeta y)$. The intersection of the two loci $x - y = 0$ and $\Delta_H = 0$ is the single point $(0,0)$).Thus the curve $\Delta_H = 0$ is lisse outside the point $(0,0)$ (because this open set of $\Delta_H = 0$ is the complement of $x = y$ in $H(x) = H(y)$).

The sum of $\psi_L(\alpha(x+y))$ over the locus $x = y$ vanishes. So our limsup is the limsup of

$$(1/\#L) \sum_{x,y \in L : \Delta_H = 0} \psi_L(\alpha(x+y)).$$

We will show that this limsup is in fact 0 provided that $B < 2p$. Suppose first that $B < p$. Then in the expansion of H, there can be no middle terms: we must have

$$H(x) = x^{p^s}(\alpha_0 + \alpha_B x^B).$$

Then

$$\Delta_H = a_0(x - y)^{p^s - 1} + a_B \prod_{1 \neq \zeta \in \mu_{p^s+B}} (x - \zeta y).$$

This is the finite part of the projective curve of equation

$$a_0 Z^B (X - Y)^{p^s - 1} + a_B \prod_{1 \neq \zeta \in \mu_{p^s+B}} (X - \zeta Y) = 0,$$

which has $p^s + B - 1$ points at ∞. If we invert $X - Y$, then in coordinates

$$z := Z/(X - Y), x := X/(X - Y), \text{ and thus } Y/(X - Y) = x - 1,$$

this curve becomes

$$a_0 z^B + a_B \prod_{1 \neq \zeta \in \mu_{p^s+B}} (x - \zeta(x - 1)) = 0.$$

This affine curve, call it $\mathcal{C}$, is defined by this polynomial, which is an Eisenstein polynomial in z for any of the factors $(x-\zeta(x-1))$ with $1\neq\zeta\in\mu_{p^s+B}$. In particular, it is Eisenstein for the factor $2x-1$ (present because p^s+B is even, as B was odd). Thus $\mathcal{C}$ is geometrically irreducible, hence $\Delta_H=0$ is geometrically irreducible. The points on $\mathcal{C}$ with $z=0$ are the points at ∞ on $\Delta_H=0$, and z has a simple pole on $\Delta_H=0$ at each of its zeroes in $\mathcal{C}$. In particular, $2x-1$ has a pole of order B at the zero of z over $2x-1=0$. Over $\Delta_H=0$, we are summing $\mathcal{L}_{\psi(\alpha((X+Y)/(X-Y)))}=\mathcal{L}_{\psi(\alpha(2x-1))}$, which has $\mathsf{Swan}=B$ at the zero of z in $\mathcal{C}$ over $2x-1$. In particular, $\mathcal{L}_{\psi(\alpha((X+Y)/(X-Y)))}$ is not geometrically constant. Hence this sum is $O(1/\sqrt{\#L})$, and the limsup is 0.

Suppose now that $2p>B>p$. Then in the expansion of H, there can be a middle term:

$$H(x)=x^{p^s}(a_0+a_px^p+a_Bx^B)=a_0x^{p^s}+a_px^{p+p^s}+a_Bx^{p^s+B}.$$

In this case, we write

$$p+p^s:=pN,$$

and

$$\Delta_H=a_0(x-y)^{p^s-1}+a_p\big((x^N-y^N)^{p-1}((x^N-y^N)/(x-y)\big)+a_B\prod_{1\neq\zeta\in\mu_{p^s+B}}(x-\zeta y).$$

Because $N=1+p^{s-1}$ is even, the factor $(X^N-Y^N)/(X-Y)$ is divisible by $X+Y$. Now we repeat the above argument. The curve $\Delta_H=0$ is the finite part of the projective curve of equation

$$a_0Z^B(X-Y)^{p^s-1}+a_pz^{B-p}\big((X^N-Y^N)^{p-1}((X^N-Y^N)/(X-Y)\big)+a_B\prod_{1\neq\zeta\in\mu_{p^s+B}}(X-\zeta Y)=0,$$

which has p^s+B-1 points at ∞. If we invert $X-Y$, then in coordinates

$$z:=Z/(X-Y),x:=X/(X-Y),\text{ and thus }Y/(X-Y)=x-1,$$

we obtain the affine curve $\mathcal{C}$ of equation

$$a_0z^B+a_pz^{B-p}\big((x^N-(x-1)^N)^{p-1}((x^N-(x-1)^N)\big)+a_B\prod_{1\neq\zeta\in\mu_{p^s+B}}(x-\zeta(x-1))=0.$$

The curve $\mathcal{C}$ is defined by this polynomial, which is (again) an Eisenstein polynomial in z for the for the factor $(2x-1)$. From this point on, we repeat verbatim the proof in the case $B<p$ above.

In this $w=p^s-1, A=1$ case, if $\mathcal{G}(f,g,a,\chi_2)$ is induced, then it is induced from a rank one local system $\mathcal{L}_{\psi(\alpha x)}$, which has finite $G_{\rm geom}$, and hence $\mathcal{G}(f,g,a,\chi)$ itself has finite $G_{\rm geom}$.

In the $w=p^s-1, A=1$ case with $f,g\in\mathbb{F}_p[x]$, $p\geq 5$ and $g(x)=\sum_i a_ix^i$, we will use the hypothesis that either $B\equiv\frac{p-1}{2}\bmod (p-1)$ or

$$\sum_{i:\ i\equiv\frac{p-1}{2}\bmod (p-1)}a_i\neq 0$$

to show that $G_{\rm geom}$ is infinite. For this, it suffices to exhibit a point $t\in\mathbb{F}_p^\times$ where

$$\mathrm{Trace}(\mathsf{Frob}_{t,\mathbb{F}_p}|\mathcal{G}(f,g,a,\chi_2))$$

is not divisible by $\mathsf{Gauss}(\psi,\chi_2)$ as an algebraic integer. [Recall that $\mathcal{G}(f,g,a,\chi_2)$ has Frobenius traces in $\mathbb{Z}[\zeta_p]$, and is pure of weight one. Pass to $\mathcal{G}_0(f,g,a,\chi_2):=\mathcal{G}(f,g,a,\chi_2)\otimes(\mathsf{Gauss}(\psi,\chi_2)^{-deg}$, which is pure of weight zero with traces in $\mathbb{Z}[\zeta_p][1/p]$. This twist $\mathcal{G}_0$ has arithmetic determinant of finite order. Indeed, any Frobenius determinant on $\mathcal{G}_0$ at a point $t\in\mathbb{F}_q^\times$ is an element of $\mathbb{Z}[\zeta_p][1/p]$ which is a unit all all places $\lambda\nmid p$ of $\mathbb{Q}(\zeta_p)$ (because $\mathcal{G}_0$ is part of a compatible system) and has complex absolute value after all complex embeddings. As $\mathbb{Q}(\zeta_p)$ has a unique place over p, it follows (product formula) that this determinant has absolute value 1 everywhere, so is a root of unity in $\mathbb{Q}(\zeta_p)$, so has order dividing $2p$. Thus the arithmetic determinant of $\mathcal{G}_0$ has finite order. Then finiteness of $G_{\rm geom}$ is equivalent to $\mathcal{G}_0$ having all Frobenius traces algebraic integers.]

To show that $\mathrm{Trace}(\mathsf{Frob}_{t,\mathbb{F}_p}|\mathcal{G}(f,g,a,\chi_2))$ is not divisible by $\mathsf{Gauss}(\psi,\chi_2)$ in $Z]\zeta_p]$, we use a p-adic calculation. Define

$$\pi := \zeta_p - 1.$$

Then $\mathrm{ord_p}(\pi) = 1/(\mathrm{p}-1)$, while $\mathrm{ord_p}(\mathsf{Gauss}(\psi,\chi_2)) = 1/2$. For $p \geq 5$, we have $1/2 > 1/(p-1)$. So we need only find a Frobenius trace $\mathrm{Trace}(\mathsf{Frob}_{t,\mathbb{F}_p}|\mathcal{G}(f,g,a,\chi_2))$ which is divisible by π but not by π^2. This amounts to computing this Frobenius trace mod π^2. For any $x \in \mathbb{F}_p$,

$$\psi(x) = \zeta_p^x = (1+\pi)^x \equiv 1 + \pi x \bmod \pi^2,$$

and for any $x \in \mathbb{F}_p^\times$,

$$\chi_2(x) \equiv x^{(p-1)/2} \bmod p.$$

So for any Laurent polynomial $L(x) = \sum_i a_i x^i \in \mathbb{F}_p[x, 1/x]$, and any $x \in \mathbb{F}_p^\times$, we have

$$\chi_2(x)\psi(L(x)) \equiv x^{(p-1)/2}(1+\pi)^{L(x)} \equiv x^{(p-1)/2}(1+\pi L(x)) \bmod \pi^2.$$

Expanding out $L(x) = \sum_i a_i x^i$,

$$\sum_{x\in\mathbb{F}_p^\times} \chi_2(x)\psi(L(x)) \equiv \sum_{x\in\mathbb{F}_p^\times} x^{(p-1)/2}(1+\pi\sum_i a_i x^i) \bmod \pi^2 \equiv$$

$$\equiv \sum_{x\in\mathbb{F}_p^\times}\chi_2(x) + \sum_i a_i \sum_{x\in\mathbb{F}_p^\times} x^{a_i+(p-1)/2} \bmod \pi^2.$$

The sum $\sum_{x\in\mathbb{F}_p^\times}\chi_2(x)$ vanishes. The sum $\sum_{x\in\mathbb{F}_p^\times} x^{a_i+(p-1)/2}$ vanishes mod p unless the exponent $a_i + (p-1)/2$ is a multiple of $p-1$, in which case it is $-1 \bmod p$. Thus

$$-\sum_{x\in\mathbb{F}_p^\times}\chi_2(x)\psi(L(x)) \equiv \pi \sum_{i:\ i\equiv\frac{p-1}{2} \bmod (p-1)} a_i \bmod \pi^2.$$

In $\mathcal{G}(f,g,a,\chi_2)$, the relevant Laurent polynomial is $f(1/x)+g(x)+tx^a$. Here $f(1/x) = a_{-1}/x$, $g(x) = \sum_{i=0}^B a_i x^i$. Because $p \geq 5$, the $1/x$ term contributes 0. If B is not $(p-1)/2 \bmod (p-1)$, the tx^a term contributes 0, no matter what the value of $t \in \mathbb{F}_p^\times$. So for such B, we are done; if $\sum_{i\leq B:\ i\equiv\frac{p-1}{2}} a_i \neq 0$, then we may choose any $t \in \mathbb{F}_p^\times$ at which to take the trace.

If, on the other hand, B is not $(p-1)/2 \bmod (p-1)$, then the exponent a in tx^a, which is $a = B + p^s - 1$, is $(p-1)/2 \bmod (p-1)$. So the tx^a term contributes t, and

$$-\sum_{x\in\mathbb{F}_p^\times}\chi_2(x)\psi(L(x)) \equiv \pi(\sum_{i\leq B:\ i\equiv\frac{p-1}{2} \bmod (p-1)} a_i) + t \bmod \pi^2.$$

We may always choose $t \in \mathbb{F}_p^\times$ so that the the innermost sum is nonzero mod p. ☐

Here is an extension of the previous Theorem 2.10 to the special case $B = 1$, where we drop the hypothesis that $p \nmid (a-B)$.

Theorem 2.11. *Suppose that $p \nmid ABa(A+a)$, f and g are both Artin-Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = \gcd(a, \gcd_{\deg}(g)) = 1$. Suppose that $B = 1$. Then $\mathcal{G}(f,g,a,\chi)$ is primitive on $\mathbb{G}_m/\overline{\mathbb{F}_p}$.*

Proof. Repeat verbatim the first four paragraphs of the proof of Theorem 2.10, down to the point

$$\deg(\pi_{x_0}) \times \mathrm{rank}(\mathcal{H}) = \mathrm{rank}(W_{B,a-B}) = a - B$$

in the discussion of the $n = 0$ case. Because $B = 1$,$W_{B,a-B}$ has $\mathsf{Swan}_0(W_{B,a-B}) = B = 1$. In this $n = 0$ case, $\mathcal{H}$ is totally wild, and

$$\pi_{x_0\star}\mathcal{H} \cong W_{B,a-B}.$$

By [Ka-TLFM, 1.6.4.1], we have

$$\mathsf{Swan}_0(\pi_{x_0\star}\mathcal{H}) = \mathsf{Swan}_{x_0}(\mathcal{H}) + \mathrm{rank}(\mathcal{H})\mathsf{Swan}_0(\pi_{x_0\star}\mathcal{L}_{\mathbb{1}}).$$

In this equality, the left hand side is $\mathsf{Swan}_0(\pi_{x_0\star}\mathcal{H}) = \mathsf{Swan}_0(W_{B,a-B}) = 1$, while on the right $\mathsf{Swan}_{x_0}(\mathcal{H}) \geq 1$ and $\mathsf{Swan}_0(\pi_{x_0\star}\mathcal{L}_{\mathbb{1}}) \geq 0$. Therefore we must have

$$\mathsf{Swan}_{x_0}(\mathcal{H}) = 1 \ \text{ and } \mathsf{Swan}_0(\pi_{x_0\star}\mathcal{L}_{\mathbb{1}}) = 0.$$

Because $\mathsf{Swan}_0(\pi_{x_0\star}\mathcal{L}_{\mathbb{1}}) = 0$, we must have $\deg(\pi_{x_0}) := D$ prime to p. [Indeed, if $\pi_{x_0\star}\mathcal{L}_{\mathbb{1}}$ is tame, then $\pi_{x_0\star}\mathcal{L}_{\mathbb{1}})$ has rank D, and, being tame, is given by

$$\pi_{x_0\star}\mathcal{L}_{\mathbb{1}} = \oplus_{\chi \text{ with } \chi^D=\mathbb{1}}.$$

Thus there are precisely D characters of order dividing D, hence D is prime to p.

Once we have $\deg(\pi_{x_0}) := D$ prime to p, repeat the rest of the $n = 0$ case EP argument to get a contradiction.

Suppose now that $n \geq 1$. Exactly as in the proof of Theorem 2.10, we see that $\deg(\pi_{x_0}) = p^s$ for some $s \geq 0$. If $s = 0$, i.e. if $\deg(\pi_{x_0}) = 1$ is prime to p, then $\mathsf{Swan}_{x_0}(\mathcal{H}) = \mathsf{Swan}_(\mathcal{G}) = B$, and we conclude as in the $n = 0$ case above.

We further see that when $s \geq 1$, we have $n = 1$, $\mathrm{rank}(\mathcal{H}) = 1$, $\mathcal{H}_{I(x_0)} = \mathcal{L}_{\mathbb{1}}$, and

$$\pi_{x_0\star}(\mathcal{H}_{I(x_0)}) = \pi_{x_0\star}(\mathcal{L}_{\mathbb{1}}) = \mathcal{L}_{\mathbb{1}} \oplus (\text{totally wild of rank } p^s - 1).$$

But $\deg(\pi_{x_0})$ divides the rank of $\pi_{x_0\star}(\mathcal{H}_{I(x_0)})$, which is $1 + (a - B) = a$ (because $B = 1$). But $p \nmid a$, so $\deg(\pi_{x_0})$ cannot be p^s with $s \geq 1$. ☐

Here is an extension of Theorem 2.10 to the special case $A = 1$, where we (partially) drop the hypothesis that $p \nmid (A + a)(a - B)$.

Theorem 2.12. *Suppose that* $p \nmid ABa$, f *and* g *are both Artin-Schreier reduced, and* $\gcd(a, \gcd_{\deg}(f)) = \gcd(a, \gcd_{\deg}(g)) = 1$. *Suppose that* $A = 1$ *and that* $A + a = n_0 p^e$ *with* $e \geq 0$ *and* $1 \leq n_0 < p$. *Then* $\mathcal{G}(f, g, a, \chi)$ *is primitive on* $\mathbb{G}_m/\overline{\mathbb{F}_p}$.

Proof. Because $A = 1$, the $I(\infty)$ representation of $\mathcal{G}(f, g, a, \chi)$ is totally wild of rank $A + a = 1 + a$, with all slopes $A/(A + a) = 1/(a + 1)$. By Pink's argument [Ka-MG, Lemma 11], if this $I(\infty)$ representation is induced, it is Kummer induced of some prime to p degree $D > 1$. As this D divides the rank $1 + a$, we see that $D|n_0$, and hence $D < p$, Thus $\mathcal{G}(f, g, a, \chi) = \pi_\star\mathcal{H}$ for some lisse $\mathcal{H}$ on a finite etale connected

$$\pi : U \to \mathbb{G}_m$$

of degree D.

On the complete nonsingular model X of U, there is a unique point x_∞ lying over ∞, simply because $\mathcal{G}_{I(\infty)}$ is irreducible.

Now consider the unique point $x_0 \in X$ over 0 for which $\pi_{x_0\star}\mathcal{H}$ contains $W_{B,a-B}$ as $I(0)$ representation. The degree d_0 of π_{x_0} is $\leq D$, hence is $< p$, hence is prime to p. Thus we have

$$\pi_{x_0\star}(\mathcal{H}_{I(x_0)}) = W_{B,a-B} \oplus (\overline{\mathbb{Q}_\ell})^n, \text{ for some } n \geq 0.$$

Because $d_0 := \deg(\pi_{x_0})$ is prime to p, we have

$$\mathsf{Swan}_{x_0}(\mathcal{H}_{I(x_0)}) = \mathsf{Swan}_0(W_{B,a-B} \oplus (\overline{\mathbb{Q}_\ell})^n) = B.$$

At any other point x_i lying over 0, the degree $d_i := \deg(\pi_{x_i})$ is again $\leq D < p$, hence is prime to p. At each such point, $\pi_{x_i}(\mathcal{H}_{I(x_0)})$ is a trivial $I(0)$ representation. This first implies that $\mathcal{H}_{I(x_i)}$ is tame, and then that both $d_i = 1$ (otherwise the Kummer direct image of any $\mathcal{L}_\chi$ by $[d_i]$ will not be entirely trivial) and that $\mathcal{H}_{I(x_i)}$ is just the direct sum $\overline{\mathbb{Q}_\ell}^{\mathrm{rank}(\mathcal{H})}$.

Now we give the EP argument. On the one hand, we have

$$\mathsf{EP}(U, \mathcal{H}) = \mathsf{EP}(\mathbb{G}_m, \mathcal{G}) = -\mathsf{Swan}_0(\mathcal{G}) - \mathsf{Swan}_\infty(\mathcal{G}) = -B - A,$$

while we also have

$$\mathsf{EP}(U, \mathcal{H}) = \mathsf{EP}(U)\mathrm{rank}(\mathcal{H}) - \sum_{x_i \text{ over } 0} \mathsf{Swan}_{x_i}(\mathcal{H}) - \mathsf{Swan}_{x_\infty}(\mathcal{H}) =$$

$$= \mathsf{EP}(U)\mathrm{rank}(\mathcal{H}) - B - A.$$

Comparing the two expressions for $\mathsf{EP}(U, \mathcal{H})$, we find $EP(U)\mathrm{rank}(\mathcal{H}) = 0$, and hence $EP(U) = 0$. But

$$EP(U) = 2 - 2g_X - \#\{x_i \text{ over } 0\} - 1,$$

hence $2g_X = 1 - \#\{x_i \text{ over } 0\}$. Hence $g_X = 0$ and there is precisely one point over 0, as well as precisely one point over ∞. Thus in suitable coordinates U is $\mathbb{G}_m$, $x_\infty = \infty$, $x_0 = 0$, and π is the D'th power map. At $x_0 = 0$, $\mathcal{H}$ cannot be totally wild (otherwise $[D]_\star(\mathcal{H})$ would be totally wild at 0), so must contain some $\mathcal{L}_\chi$. Then $[D]_\star(\mathcal{H})$ contains $[D]_\star(\mathcal{L}_\chi)$, which cannot be $I(0)$-trivial unless $D = 1$ (and $\chi = \mathbb{1}$). Thus $D = 1$, contradiction. ☐

Corollary 2.13. *Suppose that $p \nmid ABa$, f and g are both Artin-Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = \gcd(a, \gcd_{\deg}(g)) = 1$. Suppose that $A = 1$ and that $A + a = n_0 p^e$ with $0 \leq e \leq 1$ and $1 \leq n_0 < p$. If $e = 0$, suppose further that $A + a$ is not a power. Then $\mathcal{G}(f, g, a, \chi)$ satisfies $(\mathbf{S}+)$.*

Proof. Indeed, the $I(\infty)$-representation is tensor indecomposable, cf. Lemma 3.4 later on. Furthermore, if $e = 1$, then $D = A + a$ cannot be a power. Thus in all cases, $\mathcal{G}(f, g, a, \chi)$ cannot be tensor induced. ☐

3. Elements with special spectra and tensor induction

Let $V = \mathbb{C}^d$. We will say an element $g \in \mathrm{GL}(V)$ has *quasi-simple spectrum*, and write g is a qsp-*element*, if g is diagonalizable, and has at most one repeated eigenvalue but at least two distinct eigenvalues.

Proposition 3.1. *Let $V = V_1 \otimes \ldots \otimes V_n$ be a tensor product of $n \geq 2$ $\mathbb{C}$-vector spaces each of dimension $d \geq 2$. Suppose $g \in \big(\mathrm{GL}(V_1) \otimes \ldots \otimes \mathrm{GL}(V_n)\big) \rtimes \mathsf{S}_n$ induces a nontrivial permutation π on the set of n tensor factors V_i and that g has simple or quasi-simple spectrum, and finite order on V. Then the following statements hold.*

(i) *Suppose $d \geq 3$. Then π is either an n-cycle or a 2-cycle.*
(ii) *If $d = 2$, then π is either an n-cycle, a 2-cycle, a 3-cycle, or a disjoint product of a 2-cycle and a 3-cycle.*

Proof. Write $\pi = \sigma_1\sigma_2 \ldots \sigma_l$ as a product of disjoint cycles of non-increasing lengths

(3.1.1) $$k_1 \geq k_2 \geq \ldots \geq k_l \geq 1.$$

If $l = 1$, then π is an n-cycle, and we are done. Hence we will assume $l \geq 2$, and so $\dim(V) = d^n \geq 4$.

First we note that if $g = X \otimes Y$ is tensor decomposable, then both X and Y have simple spectra. Indeed, if X, say of size $s \times s$ with $s > 1$ has only a single eigenvalue, then each of the eigenvalues of g repeats $\geq s$ times, contrary to the assumption that g has a simple eigenvalue. Hence X has at least two distinct eigenvalues $\alpha_1 \neq \alpha_2$. Now if Y admits a multiple eigenvalue $\beta_1 = \beta_2$, then $\alpha_1\beta_1$ and $\alpha_2\beta_1$ are two distinct multiple eigenvalues of g, again a contradiction. Hence Y has a simple spectrum, and similarly does X.

Suitably conjugating g in $\mathrm{GL}(V)$, we may assume that

$$\pi = (1, 2, \ldots, k_1)(k_1 + 1, k_1 + 2, \ldots, k_1 + k_2) \ldots \bigl(\sum_{i=1}^{l-1} k_1 + 1, \sum_{i=1}^{l-1} k_i + 2, \ldots, n\bigr).$$

Now we can write $g = X \otimes Y$, where

$$X \in \mathrm{GL}(V_1 \otimes V_2 \otimes \ldots \otimes V_{k_1+\ldots+k_{l-1}})$$

permutes the $n - k_l$ tensor factors $V_1, \ldots, V_{n-k_l}$, inducing the permutation

$$(1, 2, \ldots, k_1)(k_1 + 1, k_1 + 2, \ldots, k_1 + k_2) \ldots \bigl(\sum_{i=1}^{l-2} k_1 + 1, \sum_{i=1}^{l-2} k_i + 2, \ldots, n - k_l\bigr),$$

and

$$Y \in \mathrm{GL}(V_{n-k_l+1} \otimes V_{n-k_l+2} \otimes \ldots \otimes V_n)$$

inducing the k_l tensor factors cyclically.

By the previous remark, both X and Y have simple spectra, and we may rescale X and Y so that both have finite order. Also, since π is nontrivial, we have $k_1 \geq 2$ and $k_l \leq k_1$ by (3.1.1). Assume first that $d \geq 3$. Then, applying [KT8, Proposition 5.2.3], we see that $k_1 = 2$, $k_2 = \ldots = k_{l-1} = 1$. Now if $k_l = 1$, then we arrive at (i). If $k_l = 2$, then we must have $l = 2$ by (3.1.1). In this case, the proof of [KT8, Lemma 5.2.2] shows that g has at least two distinct multiple eigenvalues on V, a contradiction.

Assume now that $d = 2$. Again applying [KT8, Proposition 5.2.3], we have that either

(a) $k_1 = 2$ and $k_2 = \ldots = k_{l-1} = 1$, or
(b) $k_1 = 3$ and $k_2 = \ldots = k_{l-1} = 1$, or
(c) $k_1 = 3$, $k_2 = 2$, and $k_3 = \ldots = k_{l-1} = 1$.

In the case of (a), we cannot have $(l, k_l) = (2, 2)$ again by [KT8, Lemma 5.2.2]. So $k_l = 1$, and we arrive at (ii).

Suppose we are in the case of (b). If $k_l = 1$ then we arrive at (ii). If $k_l = 2$, then $l = 2$ by (3.1.1), and (ii) holds again. If $k_l = 3$, then $l = 2$ by (3.1.1), and the proof of [KT8, Lemma 5.2.2] shows that g has at least two distinct multiple eigenvalues on V (namely $\gamma\delta$ and $\gamma\delta\zeta_3$ in its notation), a contradiction.

Finally, assume we are in the case of (c). If $k_l = 1$ then we arrive at (ii). If $k_l = 2$, then $l = 3$ by (3.1.1), and the proof of [KT8, Lemma 5.2.2] shows that g has at least two distinct multiple eigenvalues on V, again a contradiction. □

We rule out the case of n-cycle of Proposition 3.1 in a more special situation.

Proposition 3.2. *Let $r \geq 2$ be a prime and let $V = V_1 \otimes \ldots \otimes V_r$ be a tensor product of r $\mathbb{C}$-vector spaces each of dimension $d \geq 2$. Suppose $g \in \bigl(\mathrm{GL}(V_1) \otimes \ldots \otimes \mathrm{GL}(V_r)\bigr) \rtimes \mathsf{S}_r$ induces an r-cycle on the set of r tensor factors V_i. Assume in addition that g is conjugate to*

$$\mathrm{diag}\bigl(\underbrace{1, \ldots, 1}_{t \text{ times}}, \alpha, \alpha\zeta, \alpha\zeta^2, \ldots, \alpha\zeta^{w-1}\bigr),$$

where $t \geq 2$, $w \geq 1$, $\alpha \in \mathbb{C}^\times$ *is a root of unity,* $\zeta = \exp(2\pi i/w)$, *and* $(\alpha, w) \neq (1,1)$. *Then* $d = r = 2$, *and either* $(t, w, \alpha) = (3, 1, -1)$, *or* $t = w = 2$ *and* $\alpha = \pm 1$.

Proof. (a) The assumptions imply that g is a qsp-element of finite order, with 1 being the only multiple eigenvalue. Again conjugating g suitably in $\mathrm{GL}(V)$, we may assume that

$$g : V_1 \mapsto V_2 \mapsto \ldots \mapsto V_r \mapsto V_1.$$

In particular, g^r induces a semisimple element of $\mathrm{GL}(V_1)$, and thus we can find a basis $(e^1_1, \ldots, e^1_d)$ of V_1 in which g^r acts as $\mathrm{diag}(x_1, x_2, \ldots, x_d)$ for some roots of unity $x_i \in \mathbb{C}^\times$. Defining $e^i_j = g^{i-1}(e^1_j)$ for $2 \leq i \leq r$ and $1 \leq j \leq d$, we see that $(e^i_1, \ldots, e^i_d)$ is a basis of V_i. Now arguing as in the proof of [KT8, Proposition 5.2.1], we see that the spectrum of g can be written (counting multiplicities) as

(3.2.1) $$\mathrm{Spec}\,(g) = \{\underbrace{1, \ldots, 1}_{t \text{ times}}\} \sqcup Z = X \sqcup Y,$$

where $X = \{x_1, x_2, \ldots, x_d\}$, and Y consists of $(d^r - d)/r$ r-tuples, each being all the r^{th} roots of some $x_{i_1} x_{i_2} \ldots x_{i_r}$ with $1 \leq i_1, i_2, \ldots, i_r \leq d$ being not all the same, and

$$Z := \{\alpha, \alpha\zeta, \alpha\zeta^2, \ldots, \alpha\zeta^{w-1}\}.$$

In particular, Y is stable under the multiplication by the subgroup μ_r of $\mathbb{C}^\times$.

Suppose $d = r = 2$. Then (3.2.1) shows that

$$x_1 + x_2 = \mathrm{Trace}(g) = t + \alpha \sum_{i=1}^{w} \zeta_w^i.$$

If $w = 2$, then we get $t = 2$ and then $2 = x_1 + x_2$, which implies $x_1 = x_2 = 1$ for the roots of unity x_1, x_2. In this case, $X = \{1, 1\}$ and $Y = \{1, -1\}$, and so $\alpha = \pm 1$. If $w = 1$, then we get $t = 3$ and then $3 = x_1 + x_2 - \alpha$ which implies $x_1 = x_2 = 1 = -\alpha$ for the roots of unity x_1, x_2, α.

(b) We will now assume $(d, r) \neq (2, 2)$, so that

(3.2.2) $$(d^r - d)/r \geq 2.$$

In particular, Y contains at least two μ_r-cosets of $\beta, \gamma \in \mathbb{C}^\times$ (counting multiplicities).

First we show that

(3.2.3) $$r | w.$$

For suppose that $r \nmid w$. Then $|\{\beta, \beta\zeta_r\} \cap Z| \leq 1$, and similarly $|\{\gamma, \gamma\zeta_r\} \cap Z| \leq 1$. It follows from (3.2.1) that at least one of $\beta, \beta\zeta_r$ is 1, which means that the μ_r-coset of β is μ_r. Similarly, the μ_r-coset of γ is μ_r. Thus Y contains ζ_r twice, and hence $\zeta_r \neq 1$ is a multiple eigenvalue of g, a contradiction.

In particular, the elements in Z sum up to zero, and so

(3.2.4) $$\mathrm{Trace}(g) = t.$$

Next we show that the multi-set Y contains 1 at most once. Indeed, if Y contains 1 at least twice, then since Y is μ_r-stable, Y contains ζ_r at least twice, again a contradiction.

(c) Suppose that X contains 1 at least twice. Then, without loss we may assume $x_1 = x_2 = 1$. Now if $x_j \neq 1$ for some $j > 2$, then Y contains δ at least twice for

$$\delta^r = x_1^{r-1} x_j = x_2^{r-1} x_j \neq 1,$$

and thus $\delta \neq 1$ is a multiple eigenvalue of g, a contradiction. It follows that

$$x_1 = x_2 = \ldots = x_d = 1,$$

which means that g^r acts trivially on V_1 and hence $g^r = \mathrm{id}_V$. The formula for tensor induction [GI] and (3.2.4) then show that

$$t = \mathrm{Trace}(g) = d.$$

Note that, in this case, Y consists of $(d^r - d)/r$ copies of μ_r and X contains 1 exactly d times. So the multiplicity of 1 as an eigenvalue of g is

$$d + (d^r - d)/r.$$

But this multiplicity is at most $t + 1 = d + 1$, so we arrive at $(d^r - d)/r \leq 1$, contrary to (3.2.2).

We have therefore shown that X contains 1 at most once. But in (c) we showed that Y also contains 1 at most once. On the other hand, the multiplicity of 1 in $\mathrm{Spec}\,(g)$ is at least $t \geq 2$. So we conclude that $t = 2$, and each of X and Y contains 1 exactly once. In such a case, the μ_r-invariance of Y implies that $\zeta_r \in Y$. Since $\zeta_r \neq 1$, ζ_r belongs to the set Z which is μ_w-invariant. By (3.2.3), Z is also μ_r-invariant, and hence $1 = (\zeta_r)(\zeta_r)^{-1}$ belongs to Z. But then the multiplicity of 1 in $\mathrm{Spec}\,(g)$ becomes 3, a contradiction. □

Next we will prove an auxiliary result on finite permutation groups.

Lemma 3.3. *Let p be a prime, and let $J = P \rtimes C$ be a transitive subgroup of S_n with $n > 1$ such that P is a transitive normal p-subgroup and $C = \langle \gamma \rangle$ is a cyclic p'-group. Suppose that every element in the coset γP is either trivial, a 2-cycle, a 3-cycle, or a disjoint product of a 2-cycle and a 3-cycle. Then one of the following statements holds.*

(i) $p = n = |J| = |P| = 2$.
(ii) $p = n = |J| = |P| = 3$.
(iii) $p = n = 3$, $P = C_3$, *and* $J = \mathsf{S}_3$. *Furthermore,* γ *is a 2-cycle.*
(iv) $n = 4$, $p = 2$, $P = C_2^2$, *and* $J = \mathsf{A}_4$. *Furthermore,* γ *is a 3-cycle.*

Proof. Let ρ denote the corresponding permutation character of J. Then the transitivity of J means that

(3.3.1)
$$\sum_{x \in J} \rho(x) = |J|.$$

Also, since P is transitive, we have

(3.3.2)
$$\sum_{x \in P} \rho(x) = |P|, \text{ and } n = p^c \leq |P| \text{ for some } c \in \mathbb{Z}_{\geq 1}.$$

(a) First consider the case $J = P$. Then either $p = 2$ and every nontrivial element x in P is a 2-cycle, in which case $\rho(x) = n - 2$, or $p = 3$ and every nontrivial element x in P is a 3-cycle, in which case $\rho(x) = n - 3$. Using (3.3.2), in the former case we have

$$|P| = \sum_{x \in P} \rho(x) = n + (|P| - 1)(n - 2) = 2 + |P|(n - 2),$$

i.e. $|P|(n - 3) = -2$. As $|P| \geq 2$, we must have that $n = 2$ and hence $|P| = 2$, as stated in (i). In the latter case we have

$$|P| = \sum_{x \in P} \rho(x) = n + (|P| - 1)(n - 3) = 3 + |P|(n - 3),$$

i.e. $|P|(n - 4) = -3$. As $|P| \geq 3$, we conclude that $n = 3$ and hence $|P| = 3$, as stated in (ii).

(b) From now on we will assume that $J > P$, i.e. $\gamma \notin P$. By assumption, $\rho(x) \geq n-5$ for all $x \in \gamma P$; furthermore, $x^6 = 1$, so $J/P \hookrightarrow C_6$. It follows from (3.3.1) and (3.3.2) that

$$6|P| \geq |J| = \sum_{x\in J}\rho(x) = \sum_{x\in P}\rho(x) + \sum_{x\in\gamma P}\rho(x) \geq (n-4)|P|, \tag{3.3.3}$$

whence $p^c = n \leq 10$.

Assume in addition that $p \geq 5$. Then in fact we have $n = p \in \{5,7\}$ and hence $P \cong C_p$. Now

$$\mathbf{N}_{\mathsf{S}_p}(P) = P \rtimes \langle\sigma\rangle,$$

where σ is a $(p-1)$-cycle; in particular, any $1 \neq \sigma^i$ has a unique fixed point. As $P \lhd J$ and $\gamma \notin P$, we have $1 \neq \sigma^j \in \gamma P$ for some $j \in \mathbb{Z}$. Thus σ^i is either a 2-cycle, a 3-cycle, or a disjoint product of a 2-cycle and a 3-cycle, none of which can have exactly one fixed point.

Now we consider the case $p = 3$. As $\gamma \neq 1$ is a $3'$-element, it must be a 2-cycle, and thus $\gamma^2 = 1$. It follows that $J/P = C_2$, so instead of (3.3.3) we now have

$$2|P| = |J| = \sum_{x\in J}\rho(x) = \sum_{x\in P}\rho(x) + \sum_{x\in\gamma P}\rho(x) \geq (n-4)|P|.$$

Thus $3^c = n \leq 6$. It follows that $n = 3$, $P = C_3$ and $J = \mathsf{S}_3$ as $J > P$, and we arrive at (iii).

Finally, let $p = 2$. As $\gamma \neq 1$ is a $2'$-element, it must be a 3-cycle, and thus $\gamma^3 = 1$. It follows that $J/P = C_3$. Furthermore, any element $x \in J$ belongs to γP if and only if $x^{-1} \in \gamma^{-1}P$, and so we also have $\rho(y) \geq n-5$ for all $y \in \gamma^{-1}P$. So instead of (3.3.3) we now have

$$3|P| = |J| = \sum_{x\in J}\rho(x) = \sum_{x\in P}\rho(x) + \sum_{x\in\gamma P}\rho(x) + \sum_{x\in\gamma^{-1}P}\rho(x) \geq (2n-9)|P|.$$

Thus $2^c = n \leq 6$. The case $n = 2$ is impossible since $J > P \geq C_2$. So $n = 4$. Since the subgroup P of a Sylow 2-subgroup of S_4, which is dihedral of order 8, is normalized by the 3-cycle γ, we conclude that $P \cong C_2^2$ and so $J = \mathsf{A}_4$, as stated in (iv). □

Now we establish some basic lemmas about tensor indecomposability and lack of tensor induction for $I(\infty)$ of ℓ-adic local systems.

Lemma 3.4. *Let $\mathcal{F}$ be an irreducible $I(\infty)$-representation of rank $D \geq 2$ all of whose slopes are N/D with $N \geq 1$ and $\gcd(N,D) = 1$. Suppose further that $p^2 \nmid D$. Then $\mathcal{F}$ is tensor indecomposable.*

Proof. By (the $I(\infty)$-version of) [KT5, 2.2], if $\mathcal{F}$ is tensor decomposable, we can write it as $\mathcal{A}\otimes\mathcal{B}$ where both $\mathcal{A},\mathcal{B}$ are $I(\infty)$-representations of dimensions ≥ 2. Because $p^2 \nmid D$, at least one of $\mathcal{A},\mathcal{B}$ has dimension prime to p, say $\mathcal{A}$ has dimension prime to p. By the argument proving [KT5, 2.2(ii)], we may do so in such a way that $\mathcal{A}$ has $G_{\mathrm{geom}} \leq \mathrm{SL}_{\dim(\mathcal{A})}$, and then infer that both $\mathcal{A},\mathcal{B}$ have all slopes $\leq N/D$. Each of $\mathcal{A},\mathcal{B}$ is irreducible (otherwise their tensor product is reducible). Let λ be the unique slope of $\mathcal{A}$ (unique because $\mathcal{A}$ is $I(\infty)$-irreducible). Then for $d := \dim(\mathcal{A})$, $d\lambda \in \mathbb{Z}$. This integrality shows that $\lambda < N/D$; indeed, if $\lambda = N/D$, then $dN/D \in \mathbb{Z}$ with $d < D$, impossible because $\gcd(N,D) = 1$. Similarly, $\mathcal{B}$ has unique slope $\mu < N/D$, and hence $\mathcal{A}\otimes\mathcal{B}$ has slopes $\leq \sup(\lambda,\mu) < N/D$, contradiction. □

Lemma 3.5. *Let $\mathcal{F}$ be an $I(\infty)$-representation of rank $D \geq 2$ all of whose slopes are N/D with $N \geq 1$ and $\gcd(N,D) = 1$. Then $\mathcal{F}$ is $I(\infty)$-irreducible.*

Proof. Indeed, any nonzero irreducible subrepresentation V has all slopes N/D, and the product $\dim(V) \times N/D \in \mathbb{Z}$, impossible if $\dim(V) < D$. □

Combining Lemmas 3.4 and 3.5, we get the following corollary.

Corollary 3.6. *Suppose* $\gcd(a, A) = 1$ *and* $p^2 \nmid D := A + a$. *Then* $\mathcal{G}(f, g, a, \chi)$ *is both* $I(\infty)$-*irreducible and* $I(\infty)$-*tensor indecomposable.*

Proof. Here the $I(\infty)$-slopes are $A/(A + a)$ with $\gcd(A, a + A) = 1$. $\square$

Lemma 3.7. *Let* $\mathcal{F}$ *be an irreducible* $I(\infty)$-*representation of rank* $D \geq 2$ *all of whose slopes are* N/D *with* $N \geq 1$ *and* $\gcd(N, D) = 1$. *Suppose further that* $p^2 \nmid D$. *Suppose that* $D = d^n$ *with* $n \geq 2$, $n < p$, *and* $\gcd(n, D) = 1$. *Then* $\mathcal{F}$ *is not* n-*tensor induced.*

Proof. If $\mathcal{F}$ were n-tensor induced, the map $I(\infty) \mapsto \mathsf{S}_n$, giving the action on the tensor factors, is trivial on $P(\infty)$, simply because $P(\infty)$ is a pro-p group while S_n for $n < p$ has order prime to p. So the image of $I(\infty)$, is a cyclic subgroup of S_n, generated by the image π of a chosen element $\gamma \in I(\infty)$ which generates $I(\infty)/P(\infty)$. We first claim that π is an n-cycle. For if not, write it as a product of disjoint cycles to see that γ preserves a tensor decomposition, and (hence) that every power of γ, times any element of $P(\infty)$, preserves this same tensor decomposition. Thus the entire group $I(\infty)$ preserves this tensor decomposition. By Lemma 3.4, this contradicts the tensor indecomposablity of $\mathcal{F}$. Once γ induces an n-cycle, γ (and then the entire group $I(\infty)$ preserves the tensor decomposition of the Kummer pullback $[n]^\star\mathcal{F}$. But this Kummer pullback $[n]^\star\mathcal{F}$ has rank D and all slopes nN/D, so is irreducible when $\gcd(n, D) = 1$ (because then $\gcd(nN, D) = 1$), hence is tensor indecomposable, the desired contradiction. $\square$

Lemma 3.8. *Suppose* $\mathcal{F}$ *is an* $I(\infty)$-*representation of the form* $T \oplus W$, *with* T *tame of rank* $t \geq 1$ *and with* W *irreducible of rank* $w \geq 1$ *with all slopes* m/w *with* $m \geq 1$ *and* $\gcd(m, w) = 1$. *Suppose further that* $t + w \neq 4$. *Suppose that* $D := t + w$, *the rank of* $\mathcal{F}$, *is a power* $D = d^n$ *with* $n \geq 2$, $n < p$, *and* $\gcd(n, w) = 1$. *Then* $\mathcal{F}$ *is not* n-*tensor induced.*

Proof. By [KRLT3, 10.4], $\mathcal{F}$ is tensor indecomposable. If it were n-tensor induced, then precisely as in the proof of Lemma 3.7, the image of γ must be an n-cycle. Then $[n]^\star\mathcal{F}$ is tensor decomposed. But $[n]^\star\mathcal{F} = [n]^\star T \oplus [n]^\star W$. Here $[n]^\star T$ is tame of the same rank t,and $[n]^\star W$ has rank w and all slopes nm/w. Then $[n]^\star W$ is irreducible by Lemma 3.5, and by [KRLT3, 10.4], $[n]^\star\mathcal{F}$ is tensor indecomposable, the desired contradiction. $\square$

Lemma 3.9. (Compare to [KT5, 3.2].) *Suppose* $A, a \geq 1$, *and* $\mathcal{F}$ *an* $I(\infty)$-*representation of rank* $D := A + a$ *all of whose slopes are* $A/(A + a)$. *Suppose that* $\mathcal{F}$ *is tensor indecomposable over* $I(\infty)$. *Suppose that* $\mathcal{F}$ *is* n-*tensor induced for some* $n \geq 2$. *Consider the map* $\phi : I(\infty) \to \mathsf{S}_n$ *giving the action on the tensor factors. If* $(n - 2)A < a$, *then* ϕ *is trivial on* $P(\infty)$, *and the image of* ϕ *is the cyclic group generated by an* n-*cycle. Moreover,* n *is prime to* p.

Proof. To show that ϕ is trivial on $P(\infty)$, view $\mathsf{S}_n \leq \mathrm{O}_{n-1}$ by the deleted permutation representation. It suffices to show that $\phi : I(\infty) \to \mathrm{O}_{n-1}$ has $\mathsf{Swan}_\infty < 1$. Note that

$$\mathsf{Swan}_\infty \leq (n - 1)(\text{the largest slope of } \mathcal{F}) = (n - 1)A/(A + a).$$

Thus $\mathsf{Swan}_\infty < 1$ is the condition

$$(n - 1)A < A + a, \text{ i.e., } (n - 2)A < a.$$

Let $\gamma \in I(\infty)$ be a generator of $I(\infty)/P(\infty)$. Then the image of ϕ is the cyclic subgroup of S_n generated by $\phi(\gamma)$. If $\phi(\gamma)$ were not an n-cycle, it (and every power of it, and hence every element of $I(\infty)$) would preserve some given tensor decomposition of $\mathcal{F}$, contradicting the tensor indecomposablility of $\mathcal{F}$ over $I(\infty)$. Because $I(\infty)/P(\infty)$ has (pro) order prime to p, its image under ϕ has order prime to p. $\square$

Remark 3.10. In the above Lemma 3.9, the condition $(n - 2)A < a$ is always satisfied for $n = 2$. For the extreme case $A = 1$, the condition is $n < a + 2$, which is satisfied by **all** $n \geq 2$. Indeed, if

$n \geq a+2 > a+1$, while $a+1 = D = d^n$, then $n > d^n$, which is false for all $n \geq 1$ and all $d \geq 2$: worst case being $d = 2$, for which $2^n \geq n+1$. If in this $A = 1$ case we also had both $p^2 \nmid (a+1)$ and $\gcd(n, 1+a) = 1$ whenever $1+a = d^n$, then we would rule out $\mathcal{F}$ being tensor induced. [But already for $n = 2$, we have a problem with a odd and $1+a$ is a square (i.e., when $a = 4k^2 - 1$, any $k \geq 1$.]

Theorem 3.11. *Suppose $p \nmid ABa$, f, g are both Artin-Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = 1$, $\gcd(a, \gcd_{\deg}(g)) = 1$. Suppose that $\mathcal{G} = \mathcal{G}(f, g, a, \chi)$ is n-tensor induced as a representation of $\pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p})$ for some $n \geq 2$. Assume in addition that $p \nmid w = a - B$ and $D = a + A > 4$. Then all the following conditions hold.*

(i) *Either $n = p = 3$ or $(D, n, p) = (16, 4, 2)$.*
(ii) *$\mathcal{G}$ is tensor decomposable over $I(\infty)$.*
(iii) *If in addition $\gcd(a, A) = 1$, then $p^n | D$.*

Proof. By assumption, $G = G_{\text{geom}}$ stabilizes a tensor induced decomposition $V = V_1 \otimes V_2 \otimes \ldots \otimes V_n$ of the underlying representation V, with $d := \dim(V_i)$ and $D = d^n$. Let $\pi : G \to \mathsf{S}_n$ denote the permutation representation of G while acting on the n tensor factors of V. By Proposition 2.8, $\mathcal{G}$ is tensor indecomposable over $I(0)$, hence $\pi(I(0))$ is a transitive subgroup of S_n. Furthermore, since $D > 4$, we must have $w > 1$ by Proposition 2.9(b).

Fix a p'-generator γ of $I(0)$ over $P(0)$, and write $\pi(I(0)) = J = P \rtimes C$, where $P = \pi(P(0))$ and $C = \langle \pi(\gamma) \rangle$. By Lemma 2.1, the condition $p \nmid w$ implies that the action of γ on V has spectrum

$$\operatorname{diag}\big(\underbrace{1, \ldots, 1}_{t \text{ times}}, \alpha, \alpha\zeta, \alpha\zeta^2, \ldots, \alpha\zeta^{w-1}\big), \tag{3.11.1}$$

where $t = A + B \geq 2$, $w = a - B \geq 1$, $\alpha \in \mathbb{C}^\times$ is a root of unity, $\zeta = \exp(2\pi i/w)$. As $w > 1$, $\gamma|_V$ is a qsp-element. In fact, this also holds for any element in the coset $\gamma P(0)$ for the same reason.

Since $J \leq \mathsf{S}_n$ is transitive and $P \lhd J$, P acts on the set $\{V_1, V_2, \ldots, V_n\}$ with $e \geq 1$ orbits $\Omega_1, \ldots, \Omega_e$, all of length n/e and permuted cyclically by $\pi(\gamma)$. Suppose that $e > 1$. Letting U_j be the tensor product of the V_i in Ω_j for $1 \leq j \leq e$, we see that γ permutes the e tensor factors of the decomposition $V = U_1 \otimes U_2 \otimes \ldots \otimes U_e$ cyclically, say

$$U_1 \mapsto U_2 \mapsto U_3 \mapsto \ldots \mapsto U_e \mapsto U_1.$$

Choosing a prime divisor r of e, we see that γ permutes the r sets

$$\Delta_j := \{U_i \mid 1 \leq i \leq r, i \equiv j \pmod r\},\ 1 \leq j \leq r$$

cyclically. Letting W_j be the tensor product of the U_i in Δ_j, $1 \leq j \leq r$, we now have that the element γ with spectrum (3.11.1) permutes the r tensor factors of the decomposition $V = W_1 \otimes W_2 \otimes \ldots \otimes W_r$ cyclically. But this is impossible by Proposition 3.2 and the assumption that $D > 4$.

Thus $P \leq \mathsf{S}_n$ is a transitive subgroup; in particular, $n = p^c$ for some $c \geq 1$. By Proposition 3.2 applied to any $\gamma' \in \gamma P(0)$, $\pi(\gamma')$ cannot be an n-cycle (because $D > 4$). Applying Proposition 3.1 to γ', we see that any element in the coset $\pi(\gamma)P$ is either trivial, a 2-cycle, a 3-cycle, or a disjoint product of a 2-cycle and a 3-cycle. Hence we can apply Lemma 3.3 to J.

In the cases of 3.3(i) or 3.3(ii), $\pi(\gamma)P = P$, and so some element $\gamma_1 \in \gamma P(0)$ has $\pi(\gamma_1)$ being an n-cycle, a contradiction. Thus we are in the case of 3.3(iii), whence $n = p = 3$, or of 3.3(iv), whence $(n, p) = (4, 2)$. Observe that in the latter case $D = 16$. Indeed, in this case $\pi(\gamma)$ is a 3-cycle, so we may write the action of γ as $X \otimes Y$, where $X \in \mathrm{GL}(V_1 \otimes V_2 \otimes V_3)$ permutes V_1, V_2, V_3 cyclically, and $Y \in \mathrm{GL}(V_4)$. By the proof of Proposition 3.1, X has simple spectrum. Applying Proposition 3.1 to X, we see that $d = 2$ and hence $D = 16$. Thus we have proved (i).

In these two remaining cases, we now show that

$$a > (n-2)A. \tag{3.11.2}$$

Assume we are in the case of 3.3(iii), so that $\pi(\gamma)$ is a 2-cycle and $D = d^3$. Since $\gamma|_V$ has finite order and flips, say, V_1 and V_2, the formula for tensor induction [GI] shows that

$$|\mathrm{Trace}(\gamma|_V)| \leq d^2 \leq D/2.$$

Now, (3.11.1) implies that $\mathrm{Trace}(\gamma|_V) = t$, whence $t + w = D \geq 2t$, and so

$$a - B = w \geq t = A + B,$$

implying (3.11.2).

Next suppose we are in the case of 3.3(iv), so that $\pi(\gamma)$ is a 3-cycle and $D = d^4 = 16$. Since $\gamma|_V$ has finite order and permutes, say, V_1, V_2, and V_3, cyclically, the formula for tensor induction [GI] shows that

$$|\mathrm{Trace}(\gamma|_V)| \leq d^2 = D/4.$$

Now, (3.11.1) implies that $\mathrm{Trace}(\gamma|_V) = t$, whence $t + w = D \geq 4t$, and so

$$a - B = w \geq 3t = 3A + 3B,$$

implying (3.11.2).

Thus we have proved (3.11.2). Now, if $\mathcal{G}$ is tensor indecomposable over $I(\infty)$, then the equality $n = p$ contradicts Lemma 3.9. So $\mathcal{G}$ is tensor decomposable over $I(\infty)$, proving (ii).

Assume in addition that $\gcd(a, A) = 1$. Then the tensor decomposability over $I(\infty)$ of $\mathcal{G}$ implies by Lemma 3.4 that $p|D$. But $D = d^n$, so $p^n|D$, establishing (iii). ☐

Corollary 3.12. *Suppose $p \nmid ABa$, f, g are both Artin-Schreier reduced, and* $\gcd(a, \gcd_{\deg}(f)) = 1$, $\gcd(a, \gcd_{\deg}(g)) = 1$. *Suppose in addition that $p \nmid w = a - B$ and $D = a + A > 4$. If $\mathcal{G}(f, g, a, \chi)$ is primitive (e.g., by Theorem 2.10), then it satisfies* $(\mathbf{S}+)$ *if either $p \geq 5$ or if $p^2 \nmid D$.*

Remark 3.13. In cases when $p|w$, there are other ways to prove $(\mathbf{S}+)$. We can sometimes apply Theorems 2.11 or 2.12 to prove primitivity, and we can sometimes apply Propositions 2.8 and 2.9 to prove the absence of tensor induction.

References

[Asch] Aschbacher, M., Maximal subgroups of classical groups, On the maximal subgroups of the finite classical groups, *Invent. Math.* **76** (1984), 469–514.

[BH] Beukers, F.; Heckman, G., Monodromy for the hypergeometric function ${}_nF_{n-1}$, *Invent. Math.* **95** (1989), 325–354.

[Atlas] Conway, J. H., Curtis, R. T., Norton, S. P., Parker, R. A. and Wilson, R. A., Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. Oxford University Press, Eynsham, 1985.

[Cam] Cameron, P. J., Finite permutation groups and finite simple groups, *Bull. Lond. Math. Soc.* **13** (1981), 1–22.

[De] Deligne, P., La conjecture de Weil II. *Publ. Math. IHES* **52** (1981), 313–428.

[GI] Gluck, D., and Isaacs, I. M., Tensor induction of generalized characters and permutation characters, *Illinois J. Math.* **27** (1983), 514–518.

[GT2] Guralnick, R. M. and Tiep, P. H., Symmetric powers and a conjecture of Kollár and Larsen, *Invent. Math.* **174** (2008), 505–554.

[GT3] Guralnick, R. M. and Tiep, P. H., A problem of Kollár and Larsen on finite linear groups and crepant resolutions, *J. Europ. Math. Soc.* **14** (2012), 605–657.

[Is] Isaacs, I. M., Character Theory of Finite Groups, AMS-Chelsea, Providence, 2006.

[Ka-CC] Katz, N.. From Clausen to Carlitz: low-dimensional spin groups and identities among character sums, *Mosc. Math. J.* **9** (2009), 57–89.

[Ka-ESDE] Katz, N., Exponential sums and differential equations. Annals of Mathematics Studies, 124. Princeton Univ. Press, Princeton, NJ, 1990. xii+430 pp.

[Ka-GKM] Katz, N., Gauss sums, Kloosterman sums, and monodromy groups, Annals of Mathematics Studies, **116**. Princeton Univ. Press, Princeton, NJ, 1988. ix+246 pp.

[Ka-LAMM] Katz, N., Larsen's alternative, moments, and the monodromy of Lefschetz pencils. Contributions to automorphic forms, geometry, and number theory, 521–560, Johns Hopkins Univ. Press, Baltimore, MD, 2004.

[Ka-MG] Katz, N., On the monodromy groups attached to certain families of exponential sums, *Duke Math. J.* **54** (1987), no. 1, 41–56.

[Ka-MMP] Katz, N., Moments, Monodromy, and Perversity. Annals of Mathematics Studies, **159**. Princeton University Press, Princeton, NJ, 2005. viii+475 pp.

[Ka-TLFM] Katz, N., Twisted L-functions and monodromy. Annals of Mathematics Studies, 150. Princeton University Press, Princeton, NJ, 2002.

[KRLT2] Katz, N., Rojas-León, A., and Tiep, P.H., Rigid local systems with monodromy group the Conway group Co_2, *Int. J. Number Theory* **16** (2020), 341–360.

[KRLT3] Katz, N., Rojas-León, A., and Tiep, P.H., A rigid local system with monodromy group the big Conway group $2.\mathsf{Co}_1$ and two others with monodromy group the Suzuki group $6.\mathsf{Suz}$, *Trans. Amer. Math. Soc.* **373** (2020), 2007–2044.

[KRLT4] Katz, N., Rojas-León, A., and Tiep, P.H., Rigid local systems and sporadic simple groups, *Mem. Amer. Math. Soc.* (to appear).

[KRLT5] Katz, N., Rojas-León, A., and Tiep, P.H.,On Airy sheaves of Laurent type (submitted).

[KT1] Katz, N., with an Appendix by Tiep, P.H., Rigid local systems on $\mathbb{A}^1$ with finite monodromy, *Mathematika* **64** (2018), 785–846.

[KT2] Katz, N., and Tiep, P.H., Rigid local systems and finite symplectic groups, *Finite Fields Appl.* **59** (2019), 134–174.

[KT3] Katz, N., and Tiep, P.H., Local systems and finite unitary and symplectic groups, *Advances in Math.* **358** (2019), 106859, 37 pp.

[KT4] Katz, N., and Tiep, P.H., Rigid local systems and finite general linear groups, *Math. Z.* **298** (2021), no. 3-4, 1293-1321.

[KT5] Katz, N., and Tiep, P.H., Monodromy groups of Kloosterman and hypergeometric sheaves, *Geom. Funct. Analysis* **31** (2021), 562–662.

[KT6] Katz, N., and Tiep, P.H., Exponential sums and total Weil representations of finite symplectic and unitary groups, *Proc. Lond. Math. Soc.* **122** (2021), 745–807.

[KT7] Katz, N., and Tiep, P.H., Hypergeometric sheaves and finite symplectic and unitary groups, *Cambridge J. Math.* **9** (2021), 577–691.

[KT8] Katz, N., and Tiep, P.H., Exponential sums, hypergeometric sheaves, and monodromy groups, (submitted).

[Lee] Lee, T. Y., Hypergeometric sheaves and finite general linear groups, preprint.

[Mih] Mihailescu, P., Primary cyclotomic units and a proof of Catalan's conjecture, *J. Reine Angew. Math.* **572** (2004), 167–195.

[Mitchell] Mitchell, H., Determination of all primitive collineation groups in more than four variables which contain homologies, *Amer. J. Math.* **36** (1914), 1–12.

[Se-Ta] Serre, J.-P., and Tate, J., Good reduction of abelian varieties,Annals of Mathematics , Nov., 1968, Second Series, Vol. 88, No. 3 (Nov., 1968), pp. 492-517.

[ST] Shephard, G. C., and Todd, J. A., Finite unitary reflection groups, *Can. J. Math.* **6** (1954), 274–304.

[Such] Šuch, O., Monodromy of Airy and Kloosterman sheaves, *Duke Math. J.* **103** (2000), 397–444.

[Wales] Wales, D., Quasiprimitive linear groups with quadratic elements, *J. Algebra* **245** (2001), 584–606.

[Zal] Zalesskii, A. E., Linear groups, *Russian Math. Surveys* **36** (1981), 63–128.

Department of Mathematics, Princeton University, Princeton, NJ 08544
E-mail address: nmk@math.princeton.edu

Department of Mathematics, Rutgers University, Piscataway, NJ 08854
E-mail address: tiep@math.rutgers.edu