



Generalized Kloosterman Sheaves

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Dedicated to the memory of Professor Le Dung Trang

Received: 19 March 2026 / Revised: 2 July 2026 / Accepted: 2 August 2026

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Abstract

We determine in many (but not all) cases the geometric monodromy group G_{geom} of the one-parameter local system $\mathcal{G}(f, a, \chi)$ on $\mathbb{G}_m$ in characteristic $p > 0$, whose trace function (at $t \in L^\times$ for large enough finite extensions L of the field $\mathbb{F}_p$) is

$$t \in L^\times \mapsto - \sum_{x \in L^\times} \psi(f(x) + t/x^a) \chi(x),$$

with ψ a (nontrivial) additive character, $f(x)$ a polynomial of degree $A \geq 1$, $a \geq 1$, and χ a (possibly trivial) multiplicative character. These local systems are generalizations of Kloosterman sheaves.

Keywords Local systems · Hypergeometric sheaves · Monodromy groups

Mathematics Subject Classification (2020) 11T23 · 20C15 · 20D06 · 20C33 · 20G40

1 Introduction

We work in characteristic $p > 0$, and denote by $\overline{\mathbb{F}_p}$ an algebraic closure of $\mathbb{F}_p$. We also fix a prime $\ell \neq p$ to be able to speak of $\overline{\mathbb{Q}_\ell}$ -adic cohomology. We fix integers

$$A \geq 1, a \geq 1$$

about which we assume

$$p \nmid Aa.$$

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We fix a polynomial

$$f(x) \in k[x], \deg(f) = A, \quad k \text{ some finite subfield of } \overline{\mathbb{F}}_p.$$

We make the assumption that $f(x)$ is Artin-Schreier reduced: this means that in the expression $f(x) = \sum_i c_i x^i$, we have $c_i = 0$ if $p|i$. We define

$$\gcd_{\deg}(f) := \gcd(\{i | c_i \neq 0\}),$$

the greatest common divisor of the degrees of the monomials appearing in f . We suppose

$$\gcd(a, \gcd_{\deg}(f)) = 1.$$

We also fix a (possibly trivial) multiplicative character χ of $k^\times$, with the convention that for $\chi \neq 1$, we have $\chi(0) = 0$, but $1(0) = 1$. We denote by $\mathcal{G}(f, a, \chi)$ the lisse sheaf on $\mathbb{G}_m/k$ whose trace function at time $t \in L^\times$, for L/k a finite extension, is

$$t \mapsto - \sum_{x \in L^\times} \psi_L(f(x) + t/x^a) \chi_L(x).$$

As we show in Lemmas 2.7 and 2.8, they are generalizations of Kloosterman sheaves.

The main results concern the geometric monodromy groups G_{geom} of the various $\mathcal{G}(f, a, \chi)$. A key technical step in determining G_{geom} is to ensure that G_{geom} , in its given representation, satisfies condition (S+), cf. Proposition 4.3. Condition (S+), introduced in [17, Definition 1.2] and motivated by [1] and [10], means that a Zariski closed subgroup G of $\text{GL}_D(\mathbb{C})$ acts irreducibly on $V = \mathbb{C}^D$, preserves no (nontrivial) imprimitive decomposition, no tensor decomposition, and no tensor induced decomposition of V , and furthermore $\mathbf{Z}(G)$ is finite. The later sections make essential use of condition (S+).

In Section 5, we give instances for which G_{geom} is an “interesting” finite group (Weyl groups of E_8 , E_6 , the Mitchell group), or is the symmetric group. We also give criteria which force G_{geom} to be infinite (see Proposition 5.10 and the following results in that section, as well as Lemma 6.1).

In Section 6, we prove the main result of the paper, Theorem 6.4, which describes G_{geom} in the cases when we know, from Section 5, that G_{geom} is infinite.

The careful reader will notice that, even with condition (S+), we do not have a criterion, in the case when χ is the quadratic character χ_2 , to determine whether a given $\mathcal{G}(f, a, \chi_2)$ has finite or infinite monodromy (although we have good information about what G_{geom} can be in the infinite case, see Theorem 6.4, and some partial results for the finite case in Propositions 5.5 and 5.14). That same reader may wonder to what extent the requirement that $A^2 > A + a$, so essential in Proposition 4.3 to ensure condition (S+) whenever χ is nontrivial, can be relaxed.

We end this Introduction by recalling, almost verbatim, from the Introduction to [18], some basic algebro-geometric notions. A connected scheme X has (once chosen a base point $\bar{\eta}$) a profinite étale fundamental group $\pi_1(X, \bar{\eta})$, which up to inner automorphism is independent of the auxiliary choice of base point. Given a topological ring R , an R -local system $\mathcal{F}$ of rank n on X , also called a lisse R -sheaf of rank n on X , is just a continuous homomorphism $\rho_{\mathcal{F}} : \pi_1(X, \bar{\eta}) \rightarrow \text{GL}_n(R)$. When X is a connected scheme over a finite field k such that $X \otimes \bar{k}$ is connected, we refer to $\pi_1(X)$ as the arithmetic fundamental group of X , and we refer to $\pi_1(X \otimes \bar{k})$ as its geometric fundamental group. For brevity, we denote these groups $\pi_1^{\text{arith}}(X)$ and $\pi_1^{\text{geom}}(X)$. In this situation, for each finite extension field K/k , and each point $x \in X(K)$, the group $\pi_1^{\text{arith}}(X)$ contains a well-defined Frobenius conjugacy class $\text{Frob}_{x,K}$. [When X/k is of finite type, these Frobenius conjugacy classes are dense; this

is the Chebotarev density theorem.] Given an R -local system $\mathcal{F}$ on X , with corresponding representation $\rho_{\mathcal{F}}$, the *trace function* of $\mathcal{F}$ is the rule which attaches to each pair (K, x) with K/k a finite field extension and $x \in X(K)$ the *trace of Frobenius*, i.e.,

$$\text{Trace}(\text{Frob}_{x,K}|\mathcal{F}) := \text{Trace}(\rho_{\mathcal{F}}(\text{Frob}_{x,K})).$$

It is often useful to think of this trace function as providing, for each finite field extension K/k , the R -valued function on the set $X(K)$ given by

$$x \in X(K) \mapsto \text{Trace}(\text{Frob}_{x,K}|\mathcal{F}).$$

A local system $\mathcal{F}$ on X is said to be *geometrically irreducible*, respectively *arithmetically irreducible*, if it is irreducible as a representation of $\pi_1^{\text{geom}}(X)$, respectively of $\pi_1^{\text{arith}}(X)$. Similarly, $\mathcal{F}$ is said to be *geometrically semisimple*, respectively *arithmetically semisimple*, if it is completely reducible as a representation of $\pi_1^{\text{geom}}(X)$, respectively of $\pi_1^{\text{arith}}(X)$.

We are typically concerned with the case when X/k is $\mathbb{G}_m/\mathbb{F}_q$, with $\mathbb{F}_q$ a finite extension of $\mathbb{F}_p$, the ring R is the field $\overline{\mathbb{Q}_{\ell}}$ for some prime $\ell \neq p$, and $\mathcal{F}$ is a local system whose trace function is given by a simple (in the sense of “simple to remember”) explicit formula involving exponential sums. One particular interest is in local systems $\mathcal{F}$ for which the images under $\rho_{\mathcal{F}}$ of $\pi_1^{\text{arith}}(X)$ and $\pi_1^{\text{geom}}(X)$ are finite groups, which we call the *arithmetic monodromy group*, respectively, *geometric monodromy group*, of $\mathcal{F}$. When $\mathcal{F}$ is a local system on $\mathbb{G}_m/\overline{\mathbb{F}_p}$, we frequently work with its local monodromy groups at 0 and ∞ : its inertia subgroup $I(0)$ with its wild inertia subgroup $P(0)$, and its inertia subgroup $I(\infty)$ with its wild inertia subgroup $P(\infty)$. We refer to [12, Chapter 1] for basic information on the *slopes* (also called “*breaks*”) of representations of $I(\infty)$ and $I(0)$, and to [12, 4.3] for the notions of the Kummer sheaves $\mathcal{L}_{\chi}$ and the Artin-Schreier sheaf $\mathcal{L}_{\psi}$.

2 Basic Facts about $\mathcal{G}(f, a, \chi)$

The local system $\mathcal{G}(f, a, \chi)$ is lisse of rank $D = A + a$ on $\mathbb{G}_m$, and pure of weight one. We may also view it as having trace function

$$t \mapsto - \sum_{x \in L^{\times}} \psi_L(f(1/x) + tx^a) \overline{\chi}_L(x).$$

Viewed this way, it is the Fourier Transform

$$\text{FT}_{\psi}([a]_{\star}(\mathcal{L}_{\psi(f(1/x))} \otimes \mathcal{L}_{\overline{\chi}(x)})).$$

Lemma 2.1 *Suppose $p \nmid Aa$, f is Artin-Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = 1$. We have the following information about $\mathcal{G}(f, a, \chi)$.*

- (i) *The $I(\infty)$ -representation of $\mathcal{G}(f, a, \chi)$ is irreducible. It has rank $A + a$ and all slopes $A/(A + a)$.*
- (ii) *If $\chi \neq 1$, the $I(0)$ -representation of $\mathcal{G}(f, a, \chi)$ is the direct sum*

$$\oplus_{\rho: \rho^a = \chi} \mathcal{L}_{\rho} \oplus (\overline{\mathbb{Q}_{\ell}})^A.$$

In particular, if $a = 1$, then $I(0)$ acts through complex reflections (pseudoreflections) of finite order.

(iii) If $\chi = \mathbb{1}$, the $I(0)$ -representation of $\mathcal{G}(f, a, \mathbb{1})$ is the direct sum

$$\bigoplus_{\rho: \rho^a = \mathbb{1}, \rho \neq \mathbb{1}} \mathcal{L}_\rho \oplus J_2 \oplus (\overline{\mathbb{Q}_\ell})^{A-1},$$

where J_2 denotes a unipotent Jordan block of size 2. In particular, if $\chi = \mathbb{1}$ and $a = 1$, then $I(0)$ acts through transvections. In general, if γ_0 denotes a generator of $I(0)$ and $\chi = \mathbb{1}$, then γ_0^a acts via a transvection, and $\mathcal{G}(f, a, \mathbb{1})$ has infinite G_{geom} .

Proof This is a straightforward application of Laumon's results on the local monodromy of FT_ψ . The input sheaf to FT_ψ is lisse on $\mathbb{G}_m$ of rank a , with $I(0)$ -slopes A/a and $I(\infty)$ -representation $\bigoplus_{\rho: \rho^a = \overline{\chi}} \mathcal{L}_\rho$. Then the $I(\infty)$ -representation of $\mathcal{G}(f, a, \chi)$ is $\text{FTloc}(0, \infty)(\text{rank } a, \text{slopes } A/a)$, which has rank $A + a$ and all slopes $A/(A + a)$, cf. [13, 7.4.4(4)].

The only subtle point is that the input sheaf has an irreducible $I(0)$ -representation. Once we know this, then from the fact [13, 7.4.4 (4)] that $\text{FTloc}(0, \infty)$ is an equivalence of categories between $I(0)$ -representations and $I(\infty)$ -representations with all slopes < 1 we get that the $I(\infty)$ -representation of $\mathcal{G}(f, a, \chi)$ is irreducible. The $I(0)$ -representation of the input is

$$[a]_* (\mathcal{L}_{\psi(f(1/x))} \otimes \mathcal{L}_{\overline{\chi}(x)}).$$

This $[a]_*$ direct image is irreducible if and only if for $\zeta \in \mu_a := \mu_a(\overline{\mathbb{F}_p})$, we have an $I(0)$ isomorphism

$$\mathcal{L}_{\psi(f(1/x))} \otimes \mathcal{L}_{\overline{\chi}(x)} \cong \mathcal{L}_{\psi(f(1/\zeta x))} \otimes \mathcal{L}_{\overline{\chi}(\zeta x)}$$

only for $\zeta = 1$. The isomorphism class of $\mathcal{L}_{\overline{\chi}(x)}$ is invariant by multiplicative translation, so we have such an isomorphism if and only if $\mathcal{L}_{\psi(f(1/x) - f(1/\zeta x))}$ is trivial on $I(0)$. But the polynomial $f(1/x) - f(1/\zeta x)$ is Artin-Schreier reduced (as a polynomial in $1/x$). Because $\gcd(a, \gcd_{\deg}(f)) = 1$, the polynomial $f(1/x) - f(1/\zeta x)$ is nonzero for $\zeta \neq 1$ in μ_a , so has a pole at 0 of some prime to p degree. Hence $\mathcal{L}_{\psi(f(1/x) - f(1/\zeta x))}$ has $\text{Swan}_0 > 0$ for such ζ , so is certainly nontrivial on $I(0)$.

The $I(0)$ -representation V of $\mathcal{G} := \mathcal{G}(f, a, \chi)$ has

$$V/(V^{I(0)}) = \text{FTloc}(\infty, 0)(\bigoplus_{\rho: \rho^a = \overline{\chi}} \mathcal{L}_\rho) = \bigoplus_{\rho: \rho^a = \chi} \mathcal{L}_\rho,$$

cf. [13, 7.4.3.1]. □

Corollary 2.2 *The geometric determinant of the local system $\mathcal{G}(f, a, \chi)$ is $\mathcal{L}_\chi$ if a is odd, and is $\mathcal{L}_{\chi\chi_2}$, with χ_2 the quadratic character, if a is even.*

Proof At ∞ , all slopes are < 1 , and at 0 all slopes are 0. Thus the geometric determinant is tame at both 0, ∞ , so is determined by its restriction to the $I(0)$ -representation. For $\chi \neq \mathbb{1}$, this determinant is $\mathcal{L}_\Lambda$ for $\Lambda = \prod_{\rho: \rho^a = \chi} \rho$. Choose an a^{th} root ρ_0 of χ : then $\prod_{\rho: \rho^a = \chi} \rho = \prod_{\sigma: \sigma^a = \mathbb{1}} (\rho_0 \sigma) = \chi \prod_{\sigma: \sigma^a = \mathbb{1}} \sigma$. The second factor is $\mathbb{1}$ for a odd, and χ_2 for a even. For $\chi = \mathbb{1}$, the determinant is $\prod_{\sigma: \sigma^a = \mathbb{1}, \sigma \neq \mathbb{1}} \sigma$, which is $\mathbb{1}$ if a is odd, and χ_2 if a is even. □

Recall from [17, 4.2] the following result.

Lemma 2.3 *Let $\mathcal{G}$ be a geometrically semisimple (e.g., geometrically irreducible) local system on $\mathbb{G}_m$, all of whose $I(\infty)$ -slopes are < 1 . Let $\Gamma \leq G_{\text{geom}}$ be the Zariski closure of the subgroup generated by all G_{geom} -conjugates of the image of “the” inertia group $I(0)$. Then $\Gamma = G_{\text{geom}}$.*

Interchanging 0 and ∞ , we have the following result.

Lemma 2.4 *Let $\mathcal{G}$ be a geometrically semisimple (e.g., geometrically irreducible) local system on $\mathbb{G}_m$, all of whose $I(0)$ -slopes are < 1 . Let $\Gamma \leq G_{\text{geom}}$ be the Zariski closure of the subgroup generated by all G_{geom} -conjugates of the image of “the” inertia group $I(\infty)$. Then $\Gamma = G_{\text{geom}}$.*

Corollary 2.5 *Suppose $p \nmid Aa$, f is Artin-Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = 1$. Let G_{geom} be the geometric monodromy group of $\mathcal{G}(f, a, \chi)$. Denote by $\Gamma_0 \leq G_{\text{geom}}$ the Zariski closure of the subgroup generated by all G_{geom} -conjugates of the image of “the” inertia group $I(0)$. Denote by $\Gamma_\infty \leq G_{\text{geom}}$ the Zariski closure of the subgroup generated by all G_{geom} -conjugates of the image of “the” inertia group $I(\infty)$. Then $\Gamma_0 = G_{\text{geom}} = \Gamma_\infty$.*

Proof Immediate from Lemmas 2.3 and 2.4, given Lemma 2.1. $\square$

Corollary 2.6 *Suppose $a = 1$, $p \nmid A$, and f is Artin-Schreier reduced. Suppose $\chi \neq \mathbb{1}$ has order $d \geq 2$. Let G_{geom} be the geometric monodromy group of $\mathcal{G}(f, 1, \chi)$. Then G_{geom} is the Zariski closure of the subgroup generated by complex reflections of order dividing d . In particular, if G_{geom} is finite, then it is generated by complex reflections of order dividing d .*

Proof Immediate from Corollary 2.5, given Lemma 2.1, according to which $I(0)$ is generated by a complex reflection of order d . $\square$

The following two lemmas explain the sense in which the local systems $\mathcal{G}(f, a, \chi)$ are generalizations of Kloosterman sheaves.

Lemma 2.7 *Suppose $A = 1$ and $p \nmid a$. Then $\mathcal{G}(x, a, \chi)$ is a Kloosterman sheaf, indeed it is a multiplicative translation of the Kloosterman sheaf $\mathcal{Kl}(\mathbb{1}, \text{all } a^{\text{th}} \text{ roots of } \chi)$.*

Proof Indeed, by Lemma 2.1, $\mathcal{G}(x, a, \chi)$ is lisse on $\mathbb{G}_m$, tame at 0, and totally wild at ∞ of rank $a + 1$ with all slopes $1/(a + 1)$. By [12, 8.7.1], it is a multiplicative translate of a Kloosterman sheaf, whose $I(0)$ -representation is specified by Lemma 2.1. $\square$

Lemma 2.8 *Suppose $a = 1$ and $p \nmid A$. For any ρ with $\rho^A = \chi$, $\mathcal{G}(x^A, 1, \chi)$ is the Kummer $[A]^*$ pullback of a multiplicative translate of the Kloosterman sheaf $\mathcal{Kl}(\rho, \text{Char}(A))$.*

Proof The trace function of $\mathcal{G}(x^A, 1, \chi)$ is

$$t \mapsto - \sum_x \psi(x^A + t/x) \chi(x) =$$

(by $x \mapsto tx$)

$$= - \sum_x \psi(t^A x^A + 1/x) \chi(tx).$$

Choose a character ρ with $\rho^A = \chi$. Then this sum is

$$-\rho(t^A) \sum_x \psi(t^A x^A + 1/x) \chi(x).$$

This last expression makes obvious that $\mathcal{G}(x^A, 1, \chi)$ is $[A]^* \mathcal{H}$ for the local system $\mathcal{H}$ whose trace function is

$$t \mapsto -\rho(t) \sum_x \psi(tx^A + 1/x) \chi(x).$$

This $\mathcal{H}$ is not unique; the possible descents are $\mathcal{H} \otimes \mathcal{L}_\sigma$ for any character σ with $\sigma^A = \mathbb{1}$. Because $\mathcal{G}(x^A, 1, \chi)$ is tame at 0 and totally wild at ∞ of rank $A + 1$ with all slopes $A/(A + 1)$, any $[A]^*$ descent is tame at 0 and totally wild at ∞ with all slopes $1/(A + 1)$. By [12, 8.7.1], any such descent is a Kloosterman sheaf. If $\chi \neq \mathbb{1}$, its local monodromy at 0 is diagonalizable (because this is true after $[A]^*$ pullback), therefore any descent is, up to translation, geometrically isomorphic to $\mathcal{K}l(\rho, \text{Char}(A))$ for some ρ with $\rho^A = \chi$. If $\chi = \mathbb{1}$, then its local monodromy has a Jordan block of size two, and in this case as well $\mathcal{K}l(\rho, \text{Char}(A))$, for any $\rho \in \text{Char}(A)$, is such a descent. $\square$

Lemma 2.9 *Suppose $p \nmid Aa(A+a)$, f is Artin-Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = 1$. Let $\gamma \in I(\infty)$ be a generator of $I(\infty)/P(\infty)$. Then the action of γ on $\mathcal{G}(f, a, \chi)$ has simple spectrum (i.e., has $D = A + a$ distinct eigenvalues) and finite order, in fact of central order D , and the image of $I(\infty)$ is finite.*

Proof (Compare [20, proof of Lemma 3.4 and Theorem 3.7].) By the original form [26, Proposition on p. 515] of Grothendieck's local monodromy theorem, the action of γ (or indeed of any element of $I(\infty)$) on $\mathcal{G}(f, a, \chi)$ is quasi-unipotent. Because $p \nmid (A + a)$, our irreducible $I(\infty)$ -representation, call it $W(A, A + a)$, is the Kummer direct image $[A + a]_*(\mathcal{L}$ of rank 1), cf. [12, 1.14.2]. Then the Kummer pullback $[A + a]^*(W_{A, A+a})$ is the direct sum of the multiplicative translates of $\mathcal{L}$ by the $A + a$ elements of $\mu_{A+a}(\overline{\mathbb{F}}_p)$, and γ , acting by conjugation, cyclically permutes these $A + a$ summands. Thus $\gamma^{A+a}|_{W_{A, A+a}}$ acts as a scalar, call it $\lambda \in \overline{\mathbb{Q}}_\ell^\times$, and the eigenvalues of γ on $W_{A, A+a}$ are the $(A + a)^{\text{th}}$ roots of λ . In particular, $\gamma|_{W_{A, A+a}}$ has all distinct eigenvalues. The scalar λ is a root of unity, (being an eigenvalue of γ^{A+a}), and hence γ has finite order and central order D .

In general, given a finite dimensional, continuous $\overline{\mathbb{Q}}_\ell$ -representation ρ of D_∞ , the set

$$\Gamma := \{\alpha \in I_\infty \text{ such that } \rho(\alpha) \text{ is unipotent}\}$$

is an open subgroup of I_∞ , cf. [12, 7.0.5]. Whenever ρ is irreducible as D_∞ -representation (which is automatic if $\rho|_{I_\infty}$ is irreducible), the monodromy filtration, cf. [13, 7.0.6] can have only a single nonzero gr_i^W . Therefore if ρ is irreducible as D_∞ -representation, any element $\alpha \in \Gamma$ must have $\rho(\alpha) = \text{Id}$, which is to say that $\Gamma = \text{Ker}(\rho|_{I_\infty})$ is an open subgroup of I_∞ , i.e., I_∞ has finite image. $\square$

Lemma 2.10 *Let $\mathcal{F}$ be a Lie-irreducible local system of rank $D \geq 2$ on $\mathcal{G}_m/\overline{\mathbb{F}}_p$ which is Lie self-dual, i.e., the given representation $V := \mathcal{F}_{\overline{\eta}}$ of $G := G_{\text{geom}, \mathcal{F}}$ remains irreducible on the subgroup G° , and V_{G° is self-dual as a representation of G° . Suppose that all three of the following conditions are satisfied.*

- (i) $\mathcal{F}$ is tame at 0.
- (ii) For any generator γ of $I(0)/P(0)$, 1 is the only repeated eigenvalue of γ on V .
- (iii) All ∞ -slopes of $\mathcal{F}$ are < 1 .

Then $\mathcal{F}$ is self-dual.

Proof The dual $\mathcal{F}^\vee$ of $\mathcal{F}$ is itself Lie irreducible, and $\mathcal{F}$ and $\mathcal{F}^\vee$ become isomorphic on an open normal subgroup of finite index in $\pi_1^{\text{geom}}(\mathcal{G}_m)$. Therefore there exists a lisse, rank one $\mathcal{L}$ on $\mathcal{G}_m$ and an isomorphism

$$\mathcal{F}^\vee \cong \mathcal{F} \otimes \mathcal{L}.$$

Both $\mathcal{F}^\vee$ and $\mathcal{F}$ have all $I(\infty)$ -slopes < 1 , hence $\mathcal{L}$ is tame at ∞ . Both $\mathcal{F}$ and its dual are tame at 0, hence $\mathcal{L}$ is tame at 0. The action of an element $\gamma \in I(0)$ which generates $I(0)/P(0)$,

acting on either $\mathcal{F}$ or on $\mathcal{F}^\vee$, has 1 as its only repeated eigenvalue. Therefore $\mathcal{L}$ is trivial at 0. Being tame at both 0 and ∞ and trivial at 0, $\mathcal{L}$ must be trivial. Therefore $\mathcal{F}$ is self-dual. $\square$

Corollary 2.11 *Suppose $p \nmid Aa$, f is Artin-Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = 1$. Suppose $(A, a) \neq (1, 1)$. If $\mathcal{G}(f, a, \chi)$ is Lie self-dual, then it is self-dual.*

Proof Suppose first that either $A \geq 2$ or that $\chi = \mathbb{1}$. In view of Lemma 2.1, $\mathcal{G}(f, a, \chi)$ satisfies all the hypotheses of Lemma 2.10, and we are done. Suppose now that $A = 1$ and $\chi \neq \mathbb{1}$. Choose a character ρ with $\rho^a = \chi$. Then the $I(0)$ -representation of $\mathcal{G}(f, a, \chi)$ is $(\rho \text{Char}(a), \mathbb{1})$, and that of $\mathcal{G}(f, a, \chi)^\vee$ is $(\bar{\rho} \text{Char}(a), \mathbb{1})$. In the isomorphism $\mathcal{G}(f, a, \chi)^\vee \cong \mathcal{G}(f, a, \chi) \otimes \mathcal{L}$, write $\mathcal{L} = \mathcal{L}_\Lambda$. If $\Lambda = \mathbb{1}$, we are done. If not, comparing $I(0)$ -representations gives

$$(\bar{\rho} \text{Char}(a), \mathbb{1}) = (\rho \text{Char}(a), \mathbb{1}) \otimes \Lambda = (\Lambda \rho \text{Char}(a), \Lambda).$$

As $\Lambda \neq \mathbb{1}$, Λ lies in $\bar{\rho} \text{Char}(a)$. Then $(\Lambda \rho \text{Char}(a), \Lambda) = (\text{Char}(a), \Lambda)$. Thus

$$(\bar{\rho} \text{Char}(a), \mathbb{1}) = (\text{Char}(a), \Lambda).$$

Taking a^{th} powers, we get

$$(\bar{\chi}^{[a]}, \mathbb{1}^{[1]}) = (\mathbb{1}^{[a]}, \bar{\chi}^{[1]}).$$

Therefore we have $a = 1$, so we are in the excluded case $(A, a) = (1, 1)$. [Indeed, in this excluded case, by Lemma 2.7 we are dealing with the Kloosterman sheaf $\mathcal{Kl}(\mathbb{1}, \chi)$, whose dual is $\mathcal{Kl}(\mathbb{1}, \bar{\chi})$, which is indeed the $\bar{\chi}$ twist of $\mathcal{Kl}(\mathbb{1}, \chi)$. But unless χ is either $\mathbb{1}$ or χ_2 , $\mathcal{Kl}(\mathbb{1}, \chi)$ is not self-dual.] $\square$

Lemma 2.12 *Suppose $p \nmid Aa$, f is Artin-Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = 1$. Then the local system $\mathcal{G}(f, a, \chi)$ is geometrically self-dual if and only if Aa is odd, $f(x)$ is an odd polynomial, and $\chi^2 = \mathbb{1}$. Moreover, the autoduality is symplectic if $\chi = \mathbb{1}$ and is orthogonal if $\chi = \chi_2$.*

Proof The “moreover” statement is standard, cf. [13, pages 242–243].

The oddness conditions, and χ^2 trivial, imply autoduality. For $p = 2$, over even degree extensions $k/\mathbb{F}_2$ (coefficients of f), after the constant field twist by $1/\sqrt{\#k}$, the traces are real (in fact in $\mathbb{Q}$). And when p is odd, after the constant field twist by $1/\text{Gauss}(\psi, \chi_2)$ and over even degree extensions of $\mathbb{F}_p$ (coefficients of f), the traces are real. In general, an arithmetically irreducible local system which is pure of weight zero (for all embeddings of $\overline{\mathbb{Q}_\ell}$ into $\mathbb{C}$) is self-dual if and only if all its Frobenius traces are real: this is because the trace function of its dual has the complex conjugate trace function, and by Chebotarev’s density theorem, an arithmetically irreducible local system is determined by its trace function. Hence the specified constant-field twist of $\mathcal{G}(f, a, \chi)$ is arithmetically self-dual, so a fortiori $\mathcal{G}(f, a, \chi)$ is geometrically self-dual, as constant-field twists are geometrically trivial.

To prove the converse, we argue as follows. Since $\mathcal{G} := \mathcal{G}(f, a, \chi)$ is geometrically irreducible, it is self-dual if and only if $H_c^2(\mathcal{G}_m/\overline{\mathbb{F}_p}, \mathcal{G} \otimes \mathcal{G})$ is nonzero (and in fact has dimension 1). We compute this dimension as the limsup, over extensions k of $\mathbb{F}_p$ (coefficients of f, g), of the sums

$$\frac{1}{\#k(\#k - 1)} \sum_{t \in k^\times, x, y \in k^\times} \psi_k(f(1/x) + f(1/y) + t(x^a + y^a)) \chi_k(xy).$$

The $t = 0$ “missing” summand is

$$\frac{1}{(\#k(\#k - 1))} \sum_{x, y \in k^\times} \psi_k(f(1/x) + f(1/y)) \chi_k(xy) = \frac{1}{\#k(\#k - 1)} \left(\sum_{x \in k^\times} \psi_k(f(1/x)) \chi_k(x) \right)^2,$$

which is $O(1/(\#k - 1))$, because the sum being squared is, by the Weil bound, of absolute value $\leq (A)\sqrt{\#k}$.

So the limsup doesn’t change if we add this term. Then we have the limsup of

$$\frac{1}{(\#k - 1)} \sum_{x, y \in k^\times, x^a + y^a = 0} \psi_k(f(1/x) + f(1/y)) \chi_k(xy).$$

This is then the limsup of the sum of the a sums, one for each ζ with $\zeta^a = -1$,

$$S_\zeta := \frac{1}{(\#k - 1)} \sum_{x \in k^\times} \psi_k(f(1/x) + f(1/\zeta x)) \chi_k(\zeta x^2).$$

Recall that $f(x)$ is Artin-Schreier reduced and $\gcd(a, \gcd_{\deg}(f)) = 1$. Then the sum $f(x) + f(x/\zeta)$ is Artin-Schreier reduced. Unless this sum vanishes, the S_ζ summand is $O(1/\sqrt{\#k})$, again by the Weil estimate. And even if this sum does vanish, then the S_ζ still vanishes unless χ^2 is the trivial character. Thus in all cases, we must have χ^2 trivial if we are to have self-duality.

Suppose now that χ^2 is trivial and $\mathcal{G}$ is self-dual. Then for at least one ζ with $\zeta^a = -1$, $f(x) + f(x/\zeta) = 0$. In the case $p = 2$, f is an odd polynomial (because Artin-Schreier reduced), so there is nothing to prove.

Thus it remains to treat the case when p is odd. Suppose first that a is even. Then we claim that for any ζ with $\zeta^a = -1$, $f(x) + f(x/\zeta) \neq 0$. To see this, write $f(x) = \sum_n a_n x^n$, and define $\mathcal{E}_f := \{n | a_n \neq 0\}$, the set of exponents which occur in f . By hypothesis, we have

$$\gcd(a, \text{all } n \in \mathcal{E}_f) = 1.$$

We rewrite this as

$$\gcd(a, \text{all } n - a \text{ with } n \in \mathcal{E}_f) = 1.$$

If $f(x) + f(x/\zeta) = 0$, then $a_n(1 + 1/\zeta^n) = 0$ for all $n \in \mathcal{E}_f$, i.e., $\zeta^n = -1$ for all $n \in \mathcal{E}_f$, i.e., $\zeta^n = \zeta^a$ for all $n \in \mathcal{E}_f$, and finally $\zeta^{n-a} = 1$ for all $n \in \mathcal{E}_f$. Define

$$D := \gcd(\text{all } n - a \text{ with } n \in \mathcal{E}_f).$$

Then $\zeta^D = 1$. But $\gcd(a, D) = 1$, so there exist integers u, v with $au + Dv = 1$. Then $\zeta = (\zeta^a)^u (\zeta^D)^v = (-1)^u$. Thus ζ is ± 1 , neither of which has $\zeta^a = -1$ if a is even.

Suppose next that a is odd. Then the above argument shows that ζ is ± 1 . But of these two choices, only $\zeta = -1$ has $\zeta^a = -1$. For this $\zeta = -1$, we have $f(x) + f(-x) = 0$, which means precisely that f is an odd polynomial. $\square$

3 Elements with Special Spectra

Lemma 3.1 *Suppose $G \leq \text{GL}(V) = \text{GL}_D(\mathbb{C})$ contains a transvection t . Then G cannot fix any nontrivial tensor decomposition of V .*

Proof Assume the contrary: G fixes a tensor decomposition $V = V_1 \otimes V_2$ where $\dim V_1, \dim V_2 \geq 2$. Suppose $t = X \otimes Y$ with $X \in \text{GL}(V_1)$ and $Y \in \text{GL}(V_2)$. Then

for any eigenvalue γ of X , and for any eigenvalue δ of Y , the product $\gamma\delta = 1$. So we may scale X and Y so that each has 1 as an eigenvalue. Then $\gamma\delta = 1$ forces both X, Y to be unipotent. Write each as a sum of unipotent Jordan blocks, say

$$X = \oplus_i J(n_i), \quad Y = \oplus_j J(m_j).$$

If both X and Y have a Jordan block of size > 1 , then t has a Jordan block $J(n_i) \otimes J(m_j)$ with both $n_i, m_j \geq 2$. But the Jordan decomposition of $J(n_i) \otimes J(m_j)$ is $J(n_i + m_j - 1) \oplus$ lower terms, giving t a Jordan block of size ≥ 3 . So at least one of X, Y is the identity, say $X = \text{id}_{V_1}$. Then Y cannot be the identity. But any nontrivial Jordan block of Y is repeated $\dim(V_1) \geq 2$ times in t , also impossible. $\square$

Lemma 3.2 *Suppose $g \in \text{GL}(V) = \text{GL}_D(\mathbb{C})$ is any element whose spectrum on V , counting multiplicities, is of the form*

$$\{1, 1, \dots, 1, \alpha_1, \dots, \alpha_a\}$$

with $a \geq 1$ and $\alpha_1, \dots, \alpha_a$ pairwise distinct and different from 1. Then g cannot fix any nontrivial tensor decomposition of V , if at least one of the following two conditions holds.

- (i) $a < D - \sqrt{D}$.
- (ii) $a \leq D - 2$, $\alpha_1, \dots, \alpha_a$ are all the a^{th} roots of some complex number $\epsilon \neq 1$, and furthermore $(D, \epsilon) \neq (4, -1)$.

Proof Suppose we can write $g = X \otimes Y$ with $X \in \text{GL}_m(\mathbb{C})$ and $Y \in \text{GL}_n(\mathbb{C})$, where $m, n \geq 2$. If for instance the matrix Y is scalar, then every eigenvalue of g repeats at least m times, contrary to the assumption that α_1 is a simple eigenvalue. So we can write

$$\text{Spec}(X) = \{\gamma_1, \dots, \gamma_m\}$$

(counting multiplicities) with $\gamma_1 \neq \gamma_2$. Suppose δ is a multiple eigenvalue of Y . Then $\delta\gamma_1$ and $\delta\gamma_2$ are distinct multiple eigenvalues, whereas 1 is the only possible multiple eigenvalue of g , a contradiction. Thus Y has simple spectrum

$$\{\delta_1, \dots, \delta_n\},$$

and similarly X has simple spectrum. Now, for a fixed i , the set $\{\gamma_i \delta_j \mid 1 \leq j \leq n\}$ contains 1 at most once. Thus the multiplicity of 1 in $\text{Spec}(g)$ is at most m , and by symmetry of X and Y , at most n as well. Hence

$$D - a \leq \min(m, n) \leq \sqrt{D},$$

i.e. $a \geq D - \sqrt{D}$. This proves (i).

For (ii), because g has an eigenvalue 1 (in fact with multiplicity $D - a \geq 1$), we may assume that

$$\gamma_1 \delta_1 = 1. \quad (3.2.1)$$

As mentioned above, X and Y have simple spectra, so

$$\text{If } \gamma_i \delta_j = 1 \text{ for some } i, j \text{ with } 1 \in \{i, j\}, \text{ then } i = j = 1. \quad (3.2.2)$$

Moreover, when $2 \leq i \leq m$ and $2 \leq j \leq n$, $\gamma_i \delta_1 \neq 1$ and $\gamma_1 \delta_j \neq 1$ are eigenvalues of g , so $(\gamma_i \delta_1)^a = \epsilon = (\gamma_1 \delta_j)^a$, and hence

$$\gamma_2^a = \gamma_3^a = \dots = \gamma_m^a, \quad \delta_2^a = \delta_3^a = \dots = \delta_n^a. \quad (3.2.3)$$

By assumption $D - a \geq 2$, so in addition to (3.2.1) we can find $\gamma_s \delta_t = 1$ with $(s, t) \neq (1, 1)$. Together with (3.2.2), we may renumber so that $s = t = 2$, and thus

$$\gamma_2 \delta_2 = 1.$$

Repeating the above arguments, we now obtain that

$$\gamma_1^a = \gamma_3^a = \dots = \gamma_m^a, \quad \delta_1^a = \delta_3^a = \dots = \delta_n^a. \quad (3.2.4)$$

Now, $\epsilon = (\gamma_1 \delta_2)^a = (\gamma_1 / \gamma_2)^a$ and $\epsilon = (\gamma_2 \delta_1)^a = (\gamma_2 / \gamma_1)^a$, so $\epsilon^2 = 1$. In particular, as $\epsilon \neq 1$, $m, n \geq 2$, and $(D, \epsilon) \neq (4, -1)$, we must have $D > 4$. So we may assume that

$$m \geq 3.$$

It then follows from (3.2.3) and (3.2.4) that

$$\gamma_2^a = \gamma_3^a = \gamma_1^a,$$

and hence

$$1 = (\gamma_1 \delta_1)^a = (\gamma_2 \delta_1)^a = \epsilon,$$

a contradiction. $\square$

Remark 3.3 The bound in Lemma 3.2(i) is optimal. For instance, taking $(D, a) = (9, 6)$ and choosing the set $\{1, \zeta = \alpha_1, \dots, \alpha_6\}$ to be the set of 7th roots of unity, we see that g is conjugate in $\text{GL}(V)$ to $\text{diag}(1, \zeta, \zeta^3) \otimes \text{diag}(1, \zeta^6, \zeta^4)$. Here is the general setting of this example.

We take $N \geq 2$, $(D, a) := (N^2, N^2 - N)$, ζ a primitive $(N^2 - N + 1)$ th root of unity, $S = \{a_1, \dots, a_N\}$ a collection of integers mod $(N^2 - N + 1)$, and ask that

$$\text{diag}(\{\zeta^a\}_{a \in S}) \otimes \text{diag}(\{\zeta^{-a}\}_{a \in S})$$

have eigenvalues $\{1^{N-1}, \text{all elements of } \mu_{N^2-N+1}\}$. This happens in the following cases.

$$N = 2, \quad S = \{0, 1\}$$

$$N = 3, \quad S = \{0, 1, 3\}$$

$$N = 4, \quad S = \{0, 1, 3, 9\}$$

$$N = 5, \quad S = \{0, 1, 4, 14, 16\}$$

$$N = 6, \quad S = \{0, 1, 3, 8, 12, 18\}$$

$$N = 8, \quad S = \{0, 1, 3, 13, 32, 36, 43, 52\}.$$

$$N = 9, \quad S = \{0, 1, 3, 7, 15, 31, 36, 54, 63\}.$$

There are no solutions for $N = 7$; we do not understand why.

For the reader's convenience, we recall the following well-known fact:

Lemma 3.4 *Let $G \leq S_n$ be a transitive subgroup with $n \geq 2$. Then G contains a derangement, i.e. a fixed-point-free permutation.*

Proof Let ρ denote the permutation character of G acting on $\{1, 2, \dots, n\}$. Since G is transitive, $[\rho, 1_G]_G = 1$, i.e. $\sum_{g \in G} \rho(g) = |G|$. As $\rho(1) = n \geq 2$, we have

$$\sum_{1 \neq g \in G} \rho(g) = |G| - n < |G| - 1.$$

It follows that $\rho(h) < 1$ for some $h \in G$. Since $\rho(h) \in \mathbb{Z}_{\geq 0}$, we conclude that $\rho(h) = 0$, i.e. h is a derangement. $\square$

Lemma 3.5 *Let $G \leq \mathrm{GL}(V) = \mathrm{GL}_D(\mathbb{C})$ be a finite subgroup which does not fix any nontrivial tensor decomposition of V . Suppose G fixes a tensor-induced decomposition of V . Then the following statements hold.*

- (i) *If G is cyclic, then $|\mathrm{Trace}(g)| \leq \sqrt{D}$ for any generator g of G .*
- (ii) *G contains an element h such that $|\mathrm{Trace}(h)| \leq \sqrt{D}$.*

Proof By assumption, G acts transitively on the n tensor factors $U_1, \dots, U_n$ of a tensor-induced decomposition

$$V = U_1 \otimes U_2 \otimes \dots \otimes U_n.$$

Let $d := \dim U_i$.

In the case of (i), g is tensor indecomposable, so g must induce an n -cycle on $\{U_1, \dots, U_n\}$. Since g has finite order, the formula for tensor induction [8] then shows that

$$|\mathrm{Trace}(\gamma)| \leq d \leq \sqrt{D}$$

as $D = d^n \geq d^2$.

To prove (ii), we apply Lemma 3.4 to get an element h which acts fixed-point-freely on $\{U_1, \dots, U_n\}$. Conjugating h suitably, we may assume that the h -orbits on this set are

$$\{U_1, \dots, U_{k_1}\}, \{U_{k_1+1}, \dots, U_{k_1+k_2}\}, \dots, \{U_{k_1+\dots+k_{s-1}+1}, \dots, U_{k_1+\dots+k_s}\},$$

where $k_i \geq 2$ for $1 \leq i \leq s$ and $n = \sum_{i=1}^s k_i$. Again, since g has finite order and $k_j \geq 2$, the formula for tensor induction [8] shows that the absolute value of the trace of h on

$$V_j := U_{k_1+\dots+k_{j-1}+1} \otimes \dots \otimes U_{k_1+\dots+k_j}$$

is at most

$$d \leq \sqrt{\dim V_j}.$$

As $V = V_1 \otimes \dots \otimes V_s$, we conclude that $|\mathrm{Trace}(h)| \leq \sqrt{D}$. □

4 Condition (S+) for $\mathcal{G}(f, a, \chi)$

Proposition 4.1 *The following statements hold for the local system $\mathcal{G}(f, a, \chi)$ of rank $D = A + a$ if $(D, \chi) \neq (4, \chi_2)$.*

- (i) *If $\chi = \mathbb{1}$, then the $I(0)$ -representation of $\mathcal{G}(f, a, \chi)$ (and hence $\mathcal{G}(f, a, \chi)$ itself) is tensor indecomposable and not tensor induced.*
- (ii) *Suppose $A \geq 2$ and $\chi \neq \mathbb{1}$. Then the $I(0)$ -representation of $\mathcal{G}(f, a, \chi)$ is tensor indecomposable (and hence $\mathcal{G}(f, a, \chi)$ is itself tensor indecomposable).*
- (iii) *Suppose $\chi \neq \mathbb{1}$ and $A^2 > D$. Then the $I(0)$ -representation of $\mathcal{G}(f, a, \chi)$ (and hence $\mathcal{G}(f, a, \chi)$ itself) is not tensor induced (and by (ii) is tensor indecomposable).*

Proof By Lemma 2.1, a generator $\gamma = \gamma_0$ of $I(0)$ has spectrum consisting of 1 with multiplicity $A + \delta_{\chi, \mathbb{1}}$, and all $a - \delta_{\chi, \mathbb{1}}$ non-identity a^{th} roots of χ (identified with an element of the same order in $\mathbb{C}^\times$) – note these latter eigenvalues occur with multiplicity one each. Furthermore, γ has finite order if $\chi \neq \mathbb{1}$. Clearly we may assume

$$D \geq 4. \tag{4.1.1}$$

(i) By Lemma 2.1, γ^a is a transvection. Hence tensor indecomposability follows from Lemma 3.1. If the $I(0)$ -representation is n -tensor induced for some $n \geq 2$, then the transvection $\gamma^{an!}$ is tensor decomposable, contradicting Lemma 3.1.

(ii) follows from Lemma 3.2(ii) applied to the element γ .

(iii) Recalling (4.1.1), we have $A > 2$. So γ is tensor indecomposable by (ii), and has finite order. Suppose that the underlying representation V for $I(0)$ is n -tensor induced for some $n \geq 2$: $V = U^{\otimes n}$ with $\dim(U) = d \geq 2$. By Lemma 3.5(i) we have

$$|\text{Trace}(\gamma)| \leq d \leq \sqrt{D}.$$

Now, if $a \geq 2$, then $\text{Trace}(\gamma) = A$, contrary to $A^2 > D$. Suppose $a = 1$. Then

$$(A - 1)^2 \leq |\text{Trace}(\gamma)|^2 \leq d^2 \leq D = A + 1,$$

showing $A \leq 3$, and hence $A = 3$, $D = 4$, and $|\text{Trace}(\gamma)| = 2$. The latter can happen only when χ is of order 2, which violates the assumption $(D, \chi) \neq (4, \chi_2)$. $\square$

Proposition 4.2 *Suppose $p \nmid Aa(A+a)$, f is Artin-Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = 1$. Then $\mathcal{G}(f, a, \chi)$ is primitive on $\mathcal{G}_m/\overline{\mathbb{F}}_p$, so long as either $A + a \geq 3$ or $A = a = 1$ and $\chi \neq \chi_2$ (the quadratic character).*

Proof We argue by contradiction. Suppose $\mathcal{G}(f, a, \chi) = \pi_* \mathcal{H}$ for some finite etale $\pi : U \rightarrow \mathcal{G}_m$ of degree $d > 1$ and some local system $\mathcal{H}$ on U . Then $d \times \text{rank}(\mathcal{H}) = \text{rank}(\mathcal{G}(f, a, \chi)) = A + a$ is prime to p . Then U is geometrically connected, otherwise $\pi_* \mathcal{H}$ is not irreducible. Denote by X the complete nonsingular model of U , and denote by $\pi : X \rightarrow \mathbb{P}^1$ the finite flat map on the complete curves. Let $C := \pi^{-1}(0)$ and $D = \pi^{-1}(\infty)$, of cardinalities c, d respectively.

For each point $x \in D$, denote by

$$\pi_x : \text{Spec}((K_X, x)^\wedge) \rightarrow \text{Spec}((K_{\mathbb{P}^1, \infty})^\wedge)$$

the induced map of the spec's of completed function fields. Then for $\mathcal{G} := \mathcal{G}(f, a, \chi)$, we have

$$\mathcal{G}|_{I(\infty)} = \bigoplus_{x \in D} \pi_{x*}(\mathcal{H}_{I(x)}).$$

But $\mathcal{G}|_{I(\infty)}$ is irreducible, hence there is precisely one point in D , call it ∞_{up} , and

$$\mathcal{G}|_{I(\infty)} = \pi_{\infty_{up}*}(\mathcal{H}_{I(\infty_{up})}).$$

Because ∞_{up} is the unique point lying over ∞ , the degree of $\pi_{\infty_{up}}$ is precisely $d := \deg(\pi)$, which is a divisor of $A + a$. Therefore $\deg(\pi_{\infty_{up}})$ is prime to p , hence is tame. By [14, 1.6.4.1], it follows that

$$\text{Swan}_{\infty_{up}}(\mathcal{H}) = \text{Swan}_{\infty}(\mathcal{G}).$$

Now we turn to a global argument. On the one hand, for $\mathcal{G} := \mathcal{G}(f, a, \chi)$ we have

$$\text{EP}(U, \mathcal{H}) = \text{EP}(\mathcal{G}_m, \mathcal{G}) = -\text{Swan}_0(\mathcal{G}) - \text{Swan}_{\infty}(\mathcal{G}) = -0 - A.$$

But

$$\begin{aligned} \text{EP}(U, \mathcal{H}) &= \text{EP}(U) \text{rank}(\mathcal{H}) - \sum_{x \in C} \text{Swan}_x(\mathcal{H}) - \text{Swan}_{\infty_{up}}(\mathcal{H}) = \\ &= (\text{EP}(X) - c - 1) \text{rank}(\mathcal{H}) - \sum_{x \in C} \text{Swan}_x(\mathcal{H}) - A. \end{aligned}$$

Subtracting these two expressions for $\text{EP}(U, \mathcal{H})$, we find

$$(\text{EP}(X) - c - 1) \text{rank}(\mathcal{H}) - \sum_{x \in C} \text{Swan}_x(\mathcal{H}) = 0,$$

which we rewrite as

$$(\mathrm{EP}(X) - c - 1)\mathrm{rank}(\mathcal{H}) = \sum_{x \in C} \mathrm{Swan}_x(\mathcal{H}).$$

The Swan conductors are non-negative integers, and $c \geq 1$. If X has genus $g > 0$, then $(\mathrm{EP}(X) - c - 1) < 0$, not possible. If $X = \mathbb{P}^1$ and $c \geq 2$, then also $(\mathrm{EP}(X) - c - 1) < 0$, not possible. Thus we must have $X = \mathbb{P}^1$, and (after a change of coordinates) U is $\mathcal{G}_m$, with 0 the unique point over 0, and ∞ the unique point over ∞ . Then π is the Kummer map $x \mapsto x^d$ for some divisor d of $A + a$.

Now we look at the local monodromies of $\mathcal{G}$ and $\mathcal{H}$ at 0. $\mathcal{H}$ as $I(0)$ -representation is tame (again by [14, 1.6.4.1]), and is a direct sum of various $\mathcal{L}_{\rho_i} \otimes J_{n_i}$, where J_n denotes the unipotent Jordan block of size n . The $I(0)$ -representation of its Kummer direct image $[d]_*(\mathcal{H})$ is the direct sum of the various

$$\bigoplus_i \bigoplus_{\lambda_{i,j}^d = \rho_i} \mathcal{L}_{\Lambda_{i,j}} \otimes J_{n_i}.$$

Suppose first that $\mathcal{G} := \mathcal{G}(f, a, \chi)$ has $\chi = \mathbb{1}$. Then the local monodromy of $\mathcal{G}$ at 0 is the direct sum of a diagonalizable part of rank $A + a - 2$ with a single J_2 . So in the above decomposition, there is a single i with $n_i > 1$; it has $n_i = 2$ but occurs just once. On the other hand, it occurs d times, hence $d = 1$, contradicting the imprimitivity of $\mathcal{G}$.

Suppose next that $\mathcal{G} := \mathcal{G}(f, a, \chi)$ has $\chi \neq \mathbb{1}$. Then all $n_i = 1$, the local monodromy of $\mathcal{H}$ at 0 is the direct sum of various $\mathcal{L}_{\rho_i}$, and the local monodromy of $\mathcal{G}$ at 0 is the direct sum

$$\bigoplus_i \bigoplus_{\lambda_{i,j}^d = \rho_i} \mathcal{L}_{\Lambda_{i,j}}.$$

Consequently, for any character σ of order dividing d , the two sets of characters with multiplicity $\{\Lambda_{ij}\}$ and $\{\sigma \Lambda_{ij}\}$ coincide. Each $\mathbb{1}$ occurs A times in the local monodromy at 0 of $\mathcal{G}$. Therefore each σ of order d occurs A times. But the only nontrivial characters in the local monodromy at 0 of $\mathcal{G}$ are the a^{th} roots of χ , and each of these occurs once. Therefore we must have $A = 1$. Thus $A + a = a + 1$, and hence $d|(a + 1)$. Therefore $\gcd(d, a) = 1$, and among the a^{th} roots of χ are all the characters σ of order d . But σ^a is then χ , but σ^a has order d (because $\gcd(d, a) = 1$). Therefore χ has order d . For a given ρ with $\rho^a = \chi$, either $\rho\sigma = \mathbb{1}$, or $\rho\sigma$ is another a^{th} root of χ . In the second case $\sigma^a = \mathbb{1}$. The second case is impossible, as σ has order d and $\gcd(d, a) = 1$. Therefore every ρ with $\rho^a = \chi$ has $\rho = 1/\sigma$ for every σ of order d . If $d > 1$, there is such a σ , and hence there is only one ρ . This in turn forces $a = 1$. Thus our situation is $A = a = 1$, $d = 2$, the local monodromy at 0 of $\mathcal{G}$ is $\rho + \mathbb{1}$, $\mathcal{H}$ has rank one with $\{\mathbb{1}, \rho\}$ being the two square roots of some Λ . Thus $\Lambda = \mathbb{1}$, the local monodromy at 0 of $\mathcal{G}$ is $\chi_2 + \mathbb{1}$, and our local system $\mathcal{G}$ is $\mathcal{G}(x, 1, \chi_2)$, the Kloosterman sheaf $\mathcal{Kl}(\psi; \mathbb{1}, \chi_2) = [2]_* \mathcal{L}_\psi$. $\square$

Putting together Propositions 4.1 and 4.2, we get

Proposition 4.3 *Suppose $p \nmid Aa(A+a)$, f is Artin-Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = 1$. If $\chi = \chi_2$, suppose also $D := A + a \neq 4$. Then $\mathcal{G}(f, a, \chi)$ satisfies (S+) if either $\chi = \mathbb{1}$ or $A^2 > D := A + a$.*

In the case when $\chi = \mathbb{1}$, we can drop the condition that $p \nmid (A + a)$ in Proposition 4.3.

Proposition 4.4 *Suppose $p \nmid Aa$, f is Artin-Schreier reduced, $\gcd(a, \gcd_{\deg}(f)) = 1$, and $\chi = \mathbb{1}$. Then $\mathcal{G}(f, a, \mathbb{1})$ satisfies (S+).*

Proof In view of Proposition 4.1(i) (which does not require $p \nmid (A + a)$), it remains to prove that $\mathcal{G} := \mathcal{G}(f, a, \mathbb{1})$ is primitive. To see this, we argue by contradiction. Let $k/\mathbb{F}_p$ be finite extension with $f \in k[x]$. Suppose that there exists a finite étale $\pi : U \rightarrow \mathcal{G}_m$ over $\mathbb{F}_p$ and a local system $\mathcal{H}$ on U with $\pi_* \mathcal{H} \cong \mathcal{G}$. Then $\mathcal{H}$ is irreducible (because $\pi_* \mathcal{H}$ is). By Frobenius reciprocity, $\mathcal{H}$ occurs in $\pi^* \mathcal{G}$ with multiplicity 1. The action of $\text{Gal}(\overline{\mathbb{F}_p}/k)$ permutes the geometrically irreducible constituents of $\pi^* \mathcal{G}$ which occur with multiplicity 1, so after a finite extension L/k , the local system $\mathcal{H}$ upstairs exists as a local system $\mathcal{H}_0$ on U_0/L (for some descent U_0 of U), and the morphism π descends to a map $\pi_0 : U_0 \rightarrow \mathcal{G}_m$ over L . Then $\pi_{0*} \mathcal{H}_0$ is geometrically isomorphic to $\mathcal{G}$ over L . As both are geometrically irreducible, their “ratio” is a linear character of $\pi_1^{\text{arith}}(\mathcal{G}_m)/\pi_1^{\text{geom}}(\mathcal{G}_m)$. Thus at the expense of twisting $\mathcal{H}_0$ by some $\alpha^{\deg}$, we have an arithmetic isomorphism $\pi_{0*} \mathcal{H}_0 \cong \mathcal{G}$ on $\mathcal{G}_m/L$.

Once $\mathcal{G}$ is arithmetically induced, from some open subgroup $\gamma \subset \pi_1^{\text{arith}}(\mathcal{G}_m)$, its trace function is supported in the union, inside $\pi_1^{\text{arith}}(\mathcal{G}_m)$, of all conjugates of γ . But in the world of profinite groups, such a union of conjugates is an open (and closed) subset of $\pi_1^{\text{arith}}(\mathcal{G}_m)$ whose Haar measure is < 1 (for the Haar measure giving $\pi_1^{\text{arith}}(\mathcal{G}_m)$ total mass 1). By Chebotarev, it follows that a positive proportion of Frobenius traces of $\mathcal{G}$ **vanish**.

On the other hand, for each finite extension $\mathbb{F}_q/L$ and each $t \in \mathbb{F}_q^\times$, the trace of $\text{Frob}_{t, \mathbb{F}_q} | \mathcal{G}$ is

$$-(\text{the sum of } q - 1 \text{ elements of } \mu_p(\mathbb{Z}[\zeta_p])).$$

Each p^{th} root of unity is 1 mod $\mathcal{P}$, so this trace is 1 mod $\mathcal{P}$ in $\mathbb{Z}[\zeta_p]$, hence is nonzero. Thus all Frobenius traces of $\mathcal{G}$ are nonzero, the desired contradiction. $\square$

5 Determination of Monodromy Groups: Finiteness

In this section we address the question when G_{geom} can be finite, and then determine G_{geom} in the finite cases. First we consider the cases where $\text{o}(\chi) \leq 2$.

Proposition 5.1 *Suppose $p \nmid Aa(A+a)$, f is Artin-Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = 1$. Suppose further $a = 1$ and $\chi \neq \chi_2$. If $A + a > 4$, then $G_{\text{geom}, \mathcal{G}(f, 1, \chi)}$ is infinite.*

Proof If $\chi = \mathbb{1}$, then G_{geom} contains transvections, so is certainly infinite. If $\chi^2 \neq \mathbb{1}$, then χ has order ≥ 3 , and by Lemmas 2.1 and 2.3, G_{geom} is generated by pseudoreflections of order ≥ 3 . Such a group cannot be finite by Mitchell’s theorem [24, Theorem 1]: no finite primitive group in more than four variables contains pseudoreflections of order greater than 2. $\square$

Proposition 5.2 *Suppose $p \nmid Aa(A+a)$, f is Artin-Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = 1$.*

- (i) *Suppose that $a = 1$ and $\chi = \chi_2$. If $A + a > 8$ and $G := G_{\text{geom}, \mathcal{G}(f, 1, \chi_2)}$ is finite, then G is the symmetric group S_{A+a+1} in its deleted permutation representation.*
- (ii) *If $G := G_{\text{geom}, \mathcal{G}(f, a, \chi)}$ is finite, and it is the symmetric group S_{A+a+1} in its deleted permutation representation, then $A + a + 1$ is a p -power.*

Proof (i) In dimension $D = A + a > 8$, the only finite primitive groups generated by reflections are S_{D+1} in its deleted permutation representation, cf. [24, Theorem 3] and [15, 1.4.4]. By Lemmas 2.1, 2.3, and 4.2, G is a finite primitive group generated by reflections, and hence $G = S_{D+1}$ in its deleted permutation representation.

(ii) Assume $G = S_{D+1}$ in its deleted permutation representation. Then, if φ denotes the character of G in the representation underlying $\mathcal{G}(f, a, \chi)$, then $\pi := \varphi + 1_G$ is the permutation character corresponding to the action of G on $D + 1$ letters.

Let $J = QC$ denote the image of $I(\infty)$ in G and let Q denote the image of $P(\infty)$ in G . By Lemma 2.1(i), $\varphi|_J$ is irreducible. In particular, $[\pi|_J, 1_J]_J = 1$, which means that J acts transitively on $D + 1$ letters, and so $\pi|_J = \text{Ind}_{J_1}^J(1_{J_1})$ where J_1 denotes the stabilizer of a fixed letter say c . Now, the irreducibility of $\varphi|_J$ implies that

$$2 = [\pi|_J, \pi|_J]_J = [\pi|_{J_1}, 1_{J_1}]_{J_1},$$

and hence J acts doubly transitively on $D + 1$ letters.

Suppose that $Q = \mathbf{O}_p(J) = 1$. Then $J = C$ is abelian (in fact cyclic). For any other letter b , there is some $g \in C$ such that $g(c) = b$, and the stabilizer of b is $gJ_1g^{-1} = J_1$. Thus J_1 fixes every letter, which contradicts the double transitivity of J .

We have shown that $\mathbf{O}_p(J) \neq 1$, and hence J admits a nontrivial minimal normal subgroup Q_0 which is a p -group. By the general structure of doubly transitive permutation groups, see [4, Proposition 5.2], the socle of J must be Q_0 , which is then transitive on $D + 1$ letters by [4, Theorem 4.1]. This means that $D + 1 = |Q_0|$, a p -power. $\square$

Lemma 5.3 *Suppose p is odd, $f \geq 1$, $\chi = \chi_2$, and $q := p^f \geq 5$. Then $\mathcal{G}(x^{q-2}, 1, \chi_2)$ has $G_{\text{geom}} = S_q$.*

Proof By Lemma 2.8, applied with $\chi := \chi_2$, $\mathcal{G}(x^{q-2}, 1, \chi_2)$ is the $[A]^*$ pullback of a multiplicative translate of $\mathcal{Kl}(\chi_2, \text{Char}(A))$. By [17, the proof of Theorem 9.3(ii)], which applies Sawin's trick with $(n, k) = (q, 2)$, we find that $\mathcal{Kl}(\chi_2, \text{Char}(q - 2))$ has $G_{\text{geom}} = S_q$. $\square$

We will also need the following refinement of Mitchell's theorem [24] (which is [15, 1.4.4] in the case $D > 8$), given (but not explicitly stated!) in the proof of [19, 4.2.3].

Theorem 5.4 *Suppose G is a finite primitive subgroup of $\text{GL}_D(\mathbb{C})$ with $D \geq 5$ that contains a complex reflection of order d . Then $d = 2$, and $G = \mathbf{Z}(G)G_0$, where either*

- (a) G_0 is S_{D+1} in its deleted permutation representation, or
- (b) $5 \leq D \leq 8$ and $(D, G_0) = (5, 2 \times \text{SU}_4(2)), (6, W(E_6)), (6, 6_1 \cdot \text{PSU}_4(3) \cdot 2_2), (7, W(E_7)), \text{ or } (8, W(E_8))$.

Proof The fact that $d = 2$ follows from [24, Theorem 1], since G is primitive and $D \geq 5$. Let G_0 denote the (necessarily normal) subgroup of G that is generated by all reflections in G . Next, let $H < \text{GL}_D(\mathbb{C})$ denote the complex reflection group S_{D+1} (in its deleted permutation representation), or $2 \times \text{SU}_4(2)$, $W(E_6)$, $6_1 \cdot \text{PSU}_4(3) \cdot 2_2$, $W(E_7)$, $W(E_8)$, when $D = 5, 6, 6, 7$, or 8 , respectively. Correspondingly, let $S := A_{D+1}$, $\text{SU}_4(2)$, $\text{SU}_4(2)$, $\text{PSU}_4(3)$, $\text{Sp}_6(2)$, or $\Omega_8^+(2)$, so that S is the unique non-abelian composition factor of H . Then Mitchell's theorem implies that $G/\mathbf{Z}(G) \cong H/\mathbf{Z}(H)$. Note that $H/\mathbf{Z}(H) \cong S \cdot 2$, S , $S \cdot 2$, $S \cdot 2_2$, S , and $S \cdot 2$ in the above cases, and in each case, the induced projective representation of S on $\mathbb{C}^D$ is irreducible. Suppose S is not a composition factor of G_0 . Then S is not a composition factor of $G_0\mathbf{Z}(G)$. But $G_0\mathbf{Z}(G)/\mathbf{Z}(G)$ is a normal subgroup of $G/\mathbf{Z}(G) \cong H/\mathbf{Z}(H) \in \{S, S \cdot 2\}$. So we conclude that $G_0 \leq \mathbf{Z}(G)$, and hence G_0 cannot contain any reflection (as $D > 2$), a contradiction. Thus S is the unique non-abelian composition factor of G_0 , and G_0 is an irreducible subgroup of $\text{GL}_D(\mathbb{C})$ generated by reflections. Now we can apply the Shephard–Todd classification [27] to G_0 to see that $G_0 \cong H$; in particular, $G_0/\mathbf{Z}(G_0) \cong H/\mathbf{Z}(H) \cong G/\mathbf{Z}(G)$. Now, since $\mathbf{Z}(G) \cap G_0 \leq \mathbf{Z}(G_0)$, we have that $\mathbf{Z}(G)G_0/\mathbf{Z}(G) \cong G_0/(\mathbf{Z}(G) \cap G_0)$ has order divisible by $|G_0/\mathbf{Z}(G_0)| = |G/\mathbf{Z}(G)|$. Hence $G = \mathbf{Z}(G)G_0$, and the claim is proved. $\square$

If G is a finite primitive subgroup of $\mathrm{GL}_D(\mathbb{C})$ for any $D \geq 1$ that is generated by reflections, then G is one of the primitive groups in the Shephard–Todd list [27].

Proposition 5.5 *Suppose $p \nmid A \geq 4$, f is odd, and $\mathcal{G} = \mathcal{G}(f, 1, \chi_2)$ has finite $G = G_{\mathrm{geom}}$. Then G is a primitive complex reflection group, and $G_0 := G$ is as described in (a) or (b) of Theorem 5.4. Moreover, in the case of 5.4(a), $D + 1 = A + 2$ is a p -power.*

Proof By Corollary 2.6, G is generated by true reflections. Suppose G is imprimitive. Then by [27], $G = G(m, k, D)$ for some $m > 1$ and $k|m$. Now if m is not a 2-power, then m admits an odd divisor $m_0 \geq 3$, and G contains an element which is conjugate in $\mathrm{GL}_D(\mathbb{C})$ to $x := \mathrm{diag}(\zeta, \zeta, \zeta^{-2}, 1, 1, \dots, 1)$, where $\zeta \in \mathbb{C}$ is a primitive m_0^{th} root of unity. Similarly, if $4|m$, we have the element $x := \mathrm{diag}(i, i, i^{-2}, 1, 1, \dots, 1)$. But $G \leq \mathrm{O}_D(\mathbb{C})$ by Corollary 2.12, and no element in $\mathrm{O}_D(\mathbb{C})$ can be conjugate to x , a contradiction. Thus $m = 2$, and so G is a split extension of a 2-group by S_D . [In fact, G is the Weyl group of Sp_{2D} if $k = 1$, and of SO_{2D} if $k = 2$, but we will not use this more precise information.]

It follows that a Sylow p -subgroup of G embeds in S_D , realized as a permutational matrix subgroup of $\mathrm{GL}_D(\mathbb{C})$. In particular, it acts on the underlying space $V = \mathbb{C}^D$ as on the natural permutation module of S_D , and hence its character contains the trivial character. The same is true for the image Q of $P(\infty)$, i.e. the character of Q acting on V contains 1_Q . By Lemma 2.1(i), V is $I(\infty)$ -irreducible. As $P(\infty) \triangleleft I(\infty)$, the space of $P(\infty)$ -invariants in V is an $I(\infty)$ -stable subspace. Being nonzero, this subspace must, by irreducibility, be all of V . Thus $P(\infty)$ acts trivially on V , again a contradiction (otherwise V is an irreducible representation of the abelian group $I(\infty)/P(\infty)$ of dimension $D > 1$).

We have shown that G is primitive. It then follows from [27] that $G_0 = G$ satisfies either 5.4(a) or 5.4(b). Suppose we are in the case of 5.4(a). Arguing as in the proof of Proposition 5.2, we see that $D + 1$ is a p -power. $\square$

Theorem 5.6 *In characteristic $p = 5$, for any finite extension $k/\mathbb{F}_5$, and any $a_0, b_0 \in k^\times$, the local system $\mathcal{G}_{a_0, b_0} := \mathcal{G}(a_0x^7 + b_0x^3, 1, \chi_2)$ on $\mathcal{G}_m/k$ has $G_{\mathrm{geom}} = W(E_8)$, the Weyl group of type E_8 .*

Proof By Proposition 4.3, $\mathcal{G}(a_0x^7 + b_0x^3, 1, \chi_2)$ has (S^+) . Denote by $H := G_{\mathrm{geom}, \mathcal{G}_{a_0, b_0}}$. By [16, Theorem 5.1], the three parameter system $\mathcal{G}_{3\mathrm{par}}$ on $\mathcal{G}_m^3/\mathbb{F}_5$ whose trace function on finite extensions $L/\mathbb{F}_5$ is

$$(a, b, t) \in (L^\times)^3 \mapsto - \sum_{x \in L^\times} \psi_L(t/x + ax^7 + bx^3) \chi_{2,L}(x),$$

has $G_{\mathrm{geom}, \mathcal{G}_{3\mathrm{par}}} = W(E_8)$. As $\mathcal{G}_{a_0, b_0}$ is a pullback of $\mathcal{G}_{3\mathrm{par}}$, we have $H \leq W(E_8)$. On the other hand, H is a primitive group generated by reflections by Corollary 2.6. It then results from [27] that H is either S_9 in its deleted permutation representation or it is $W(E_8)$. The former case is ruled out by Proposition 5.5. $\square$

Theorem 5.7 *In characteristic $p = 3$, for any finite extension $k/\mathbb{F}_3$, and any $a_0, b_0 \in k^\times$, consider the local systems*

$$\mathcal{G}_{5, a_0, b_0} := \mathcal{G}(a_0x^7 + b_0x^5, 1, \chi_2)$$

and

$$\mathcal{G}_{1, a_0, b_0} := \mathcal{G}(a_0x^7 + b_0x, 1, \chi_2).$$

There are dense open sets U_1, U_5 in $\mathcal{G}_m^2/\overline{\mathbb{F}_3}$ which both contain the point $(1, 1)$ such that for $(a_0, b_0) \in (\mathcal{G}_m)^2(\overline{\mathbb{F}_3})$, we have

$$G_{\text{geom}, \mathcal{G}_{5, a_0, b_0}} = W(E_8) \text{ for } (a_0, b_0) \in U_5, \text{ and } = S_9 \text{ otherwise,}$$

$$G_{\text{geom}, \mathcal{G}_{1, a_0, b_0}} = W(E_8) \text{ for } (a_0, b_0) \in U_1, \text{ and } = S_9 \text{ otherwise.}$$

Proof Denote by H_{5, a_0, b_0} , respectively H_{1, a_0, b_0} , the G_{geom} of the named local system. Exactly as in the proof of Theorem 5.6, we use [16, Theorems 5.2 and 5.3] to show that both H_{5, a_0, b_0} and H_{1, a_0, b_0} are subgroups of $W(E_8)$. Each of these H is a primitive group generated by reflections. It then results from [27] that H is either S_9 in its deleted permutation representation or it is $W(E_8)$.

Exactly as in the proof of [16, Theorem 5.2], a Magma calculation over $\mathbb{F}_{3^4}$ shows that each of $H_{5, 1, 1}$ and $H_{1, 1, 1}$ has traces both 2 and -2 , so neither of these groups can be S_9 . So each must be $W(E_8)$. The conclusion now follows from the specialization result [16, Corollary 3.2].

For the reader's convenience, in what follows we make explicit the Magma calculation over $\mathbb{F}_{3^4}$ of the traces of each of $H_{5, 1, 1}$ and $H_{1, 1, 1}$. We give Magma definitions of ψ and of χ_2 as follows.

```
psi:=function(p,n,x);
K<ζ>:=CyclotomicField(p);
k:=GF(p^n);
x:=k!x;
x:=Trace(x);
x:=Integers()!x;
return ζx;
end function;
```

and

```
chiquad:=function(p,n,x);
K<ζ>:=CyclotomicField(p);
k:=GF(p^n);
x:=k!x;
x:=Norm(x);
x:=Integers()!x;
return JacobiSymbol(x,p);
end function;
```

Here we must look at the traces of the normalized (to have weight zero) sums

$$-\sum_{x \in \mathbb{F}_{3^4}^\times} \psi(x^7 + x^D + t/x) \chi_2(x)/3^2$$

as t runs over $\mathbb{F}_{3^4}^\times$, for $D = 1$, and for $D = 5$.

For each D , the list of all 80 of these traces, in the form $\text{trace}^{\text{multiplicity}}$ is produced by the Magma command

```
{* & + [-psi(3, 4, x7 + xD + t/x) * chiquad(3, 4, x) : x in GF(34)|x ne 0]/32 :
t in GF(34)|t ne 0*};
```

□

Theorem 5.8 *In characteristic $p = 3$, for any finite extension $k/\mathbb{F}_3$, and any $a_0, b_0 \in k^\times$, consider the local system*

$$\mathcal{G} := \mathcal{G}(a_0x^5 + b_0x, 1, \chi_2).$$

Then $G_{\text{geom}, \mathcal{G}} = W(E_6)$.

Proof From [16, Theorem 6.1], $G_{\text{geom}, \mathcal{G}}$ is finite by specializing. Now we can apply Proposition 5.5 and use self-duality of $\mathcal{G}$ to conclude that $G_{\text{geom}, \mathcal{G}} = W(E_6)$. $\square$

Theorem 5.9 *In characteristic $p = 3$, for any $(a_0, b_0) \in \overline{\mathbb{F}_3}^\times \times \overline{\mathbb{F}_3}$, consider the local system*

$$\mathcal{G}_{(a_0, b_0)} := \mathcal{G}(a_0x^5 + b_0x^2, 1, \chi_2).$$

Denote by $\text{Mit} = 6_1 \cdot \text{PSU}_4(3) \cdot 2_2$ the Mitchell group [24], viewed as a subgroup of GL_6 , and define

$$H(a_0, b_0) := G_{\text{geom}, \mathcal{G}_{(a_0, b_0)}}.$$

Then we have the following results

- (i) *For any $a_0 \in \overline{\mathbb{F}_3}^\times$, there is a dense open set $U_{a_0} \subseteq \mathbb{G}_m/\overline{\mathbb{F}_3}$ such that for $b_0 \in U_{a_0}$, we have $H(a_0, b_0) = \text{Mit}$, and for $b_0 \neq 0$ outside U_{a_0} we have $H(a_0, b_0) = G(3, 3, 6)$, the Shephard–Todd imprimitive group [27].*
- (ii) *There is a dense open set $U \subseteq \mathbb{G}_m^2/\overline{\mathbb{F}_3}$ such that for $(a_0, b_0) \in U$, we have $H(a_0, b_0) = \text{Mit}$, and for $(a_0, b_0) \in \mathbb{G}_m^2 \setminus U$ we have $H(a_0, b_0) = G(3, 3, 6)$.*

Proof On $\mathbb{G}_m \times \mathbb{A}^1 \times \mathbb{G}_m/\mathbb{F}_3$, with parameters (r, s, t) , we have the local system $\mathcal{G}_{\text{univ}}$ whose trace function at L -valued points, for $L/\mathbb{F}_3$ a finite extension, is

$$(r, s, t) \mapsto - \sum_{x \in L^\times} \psi_L(rx^5 + sx^2 + t/x) \chi_{2, L}(x).$$

By [16, Theorem 6.2] and the discussion at the beginning of [16, §5], $\mathcal{G}_{\text{univ}}$ has $G_{\text{geom}} = \text{Mit}$.

We first prove (i). Fix a choice of $a_0 \in \overline{\mathbb{F}_3}^\times$, and denote by $\mathcal{G}_{a_0}$ the $r = a_0$ pullback of $\mathcal{G}_{\text{univ}}$ to $\mathbb{A}^1 \times \mathbb{G}_m$. Its G_{geom} is a subgroup $\text{Mit}(a_0) \leq \text{Mit}$. Now view $\mathcal{G}_{a_0}$ as a local system on $\mathbb{A}^1 \times \mathbb{G}_m$, and map $\mathbb{A}^1 \times \mathbb{G}_m$ to $\mathbb{A}^1$ by $(s, t) \mapsto s$. The restriction of $\mathcal{G}_{a_0}$ to the fibre over a geometric point s_0 is precisely the local system $\mathcal{G}_{(a_0, s_0)}$, with $G_{\text{geom}} = H(a_0, s_0)$. By [16, Theorem 3.1 and Cor. 3.2], there is a subgroup $H(a_0) \leq \text{Mit}(a_0)$ with the following property: There is a dense open set $U_{a_0} \subseteq \mathbb{A}^1$ such that for $s_0 \in U_{a_0}$, the group $H(a_0, s_0)$ is conjugate in $\text{Mit}(a_0)$ to the group $H(a_0)$, and for every $s' \in \mathbb{A}^1$, $H(a_0, s')$ is conjugate in $\text{Mit}(a_0)$ to a subgroup of $H(a_0)$.

Now we show $H(a_0) = \text{Mit}$. On the one hand, the group $H(a_0, s_0 = 0)$ is G_{geom} for the local system $\mathcal{G}(a_0x^5, 1, \chi_2)$. By Proposition 5.5, $H(a_0, s_0 = 0)$ is $W(E_6)$. Thus $W(E_6) \leq H(a_0)$. Now let $s' \neq 0$ and consider the group $H(a_0, s')$. It is a finite irreducible group generated by true reflections inside GL_6 , and it is not self-dual, by Lemma 2.12. If $H(a_0, s')$ is primitive (in its given six-dimensional representation), then $H(a_0, s') = \text{Mit}$, by Proposition 5.5. But $H(a_0, s') \leq H(a_0)$, so we are done in this case.

Suppose $H(a_0, s')$ is imprimitive. Then it is a Shephard–Todd group $G(m, k, 6) = D_{m, k} \rtimes S_6$ for some $m > 1$ and $k|m$, where

$$D_{m, k} := \left\{ \text{diag}(\zeta_m^{a_1}, \dots, \zeta_m^{a_6}) \mid a_i \in \mathbb{Z}, \sum_{i=1}^6 a_i \equiv 0 \pmod{k} \right\}$$

is abelian of order m^6/k , $\zeta_m \in \mathbb{C}$ a primitive m^{th} root of unity, and S_6 acts via naturally permuting the 6 coordinates. In particular, any element of $G(m, k, 6)$ has all of its entries, and hence its trace, in $\mathbb{Q}(\zeta_m)$. Note that a priori $\mathcal{G}(a_0, s')$ has all traces in $\mathbb{Q}(\zeta_3)$. Now if $3 \nmid m$, then $\mathcal{G}(a_0, s')$ has traces in $\mathbb{Q}(\zeta_3) \cap \mathbb{Q}(\zeta_m) = \mathbb{Q}$, in which case $\mathcal{G}(a_0 x^5 + s' x^2, 1, \chi_2)$ with $s' \neq 0$ is self-dual, which however contradicts Lemma 2.12. Thus $3|m$, in which case

$$\begin{aligned} D_{m,k} &\geq D_{m,m} = \{\text{diag}(\zeta_m^{a_1}, \dots, \zeta_m^{a_6}) \mid a_i \in \mathbb{Z}\} \cap \text{SL}_6(\mathbb{C}) \\ &\geq \{\text{diag}(\zeta_3^{a_1}, \dots, \zeta_3^{a_6}) \mid a_i \in \mathbb{Z}\} \cap \text{SL}_6(\mathbb{C}) = D_{3,3}, \end{aligned}$$

whence $G(m, k, 6) = D_{m,k} \rtimes S_6$ contains $G(3, 3, 6) = D_{3,3} \rtimes S_6$. We have shown that $H(a_0) \leq \text{Mit}$ contains both $W(E_6)$ and $G(3, 3, 6)$. Modulo scalars, $W(E_6)$ projects onto $\text{SO}_5(3)$ and $G(3, 3, 6)$ projects onto $C_3^5 \rtimes S_6$, which are two maximal subgroups of distinct indices 126 and 112 of $\text{PSU}_4(3) \cdot 2_2$ [3], which is Mit modulo scalars. It follows that $H(a_0) = \text{Mit}$ modulo scalars. Suppose that $H(a_0) \neq \text{Mit}$. Note that the subgroup Z of scalars in Mit is $Z = \mathbf{Z}(\text{Mit}) = \langle \zeta_6 \cdot I_6 \rangle \cong C_6$, and the subgroup of scalars in $G(3, 3, 6)$ is $\langle \zeta_3 \cdot I_6 \rangle \cong C_3$, of index 2 in Z . Hence, $H(a_0)$ cannot contain Z , but it already contains $Z \cap G(3, 3, 6)$ which has index 2 in Z . So $H(a_0)$ has index 2 in Mit. The only subgroup of index 2 in Mit is the perfect group $6_1 \cdot \text{PSU}_4(3)$ [3] which however does not contain $W(E_6)$, a contradiction. Thus $H(a_0) = \text{Mit}$ in this case as well.

We have shown that $H(a_0) = \text{Mit}$, and that, for any $s' \neq 0$, $H(a_0, s')$ equals $H(a_0)$ if it is primitive; if it is imprimitive, then $H(a_0, s') = G(m, k, 6) \geq G(3, 3, 6)$ for some $3|m$ and $k|m$. Suppose that $H(a_0, s') > G(3, 3, 6)$ in the latter case. Since $G(m, k, 6)Z$ is still imprimitive, $G(m, k, 6)Z < \text{Mit}$. On the other hand, we have mentioned above that $G(3, 3, 6)Z$ is a maximal subgroup of Mit. It follows that $G(m, k, 6)Z = G(3, 3, 6)Z$. As $Z \cong C_6$ and $Z \cap G(3, 3, 6) \cong C_3$, we must have that $H(a_0, s') = G(m, k, 6) = C_2 \times G(3, 3, 6)$ which however is none of the Shephard–Todd groups. Hence any imprimitive $H(a_0, s')$ is $G(3, 3, 6)$, and this completes the proof of (i).

The proof of (ii) now goes by looking at the map of $\mathbb{G}_m \times \mathbb{A}^1 \times \mathbb{G}_m/\mathbb{F}_3$ to $\mathbb{G}_m \times \mathbb{A}^1$, $(r, s, t) \mapsto (r, s)$. By (i), there exists some (a_0, b_0) with $H(a_0, b_0)$ the full Mitchell group, and we conclude as above by [16, Theorem 3.1 and Cor. 3.2]. $\square$

Next we address the cases where $\text{o}(\chi) > 2$.

Proposition 5.10 *Suppose $p \nmid Aa(A+a)$, f is Artin-Schreier reduced, $\gcd(a, \gcd_{\deg}(f)) = 1$, and $(A, a, \chi) \neq (1, 1, \chi_2)$. Then $G_{\text{geom}, \mathcal{G}(f, a, \chi)}$ is infinite in each of the following situations.*

- (i) χ has order ≥ 6 .
- (ii) χ has order 4 or 5, and $A \neq a$.
- (iii) χ has order 3, and $A + a > \min(4a, 4A)$.

Proof Suppose $G = G_{\text{geom}, \mathcal{G}(f, a, \chi)}$ is finite. Then by Proposition 4.2 it is a finite primitive group, generated by diagonalizable elements g whose eigenvalues are of the form

$$(\text{all the } a^{\text{th}} \text{ roots of } \chi, A \text{ repeats of } 1).$$

The a^{th} power h of such an element is thus

$$(a \text{ repeats of } \chi, A \text{ repeats of } 1),$$

which is a quadratic element of drop a . If χ has order ≥ 6 , then by Blichfeldt's 60-degree theorem [2], this element is a scalar, nonsense. If χ has order 4, respectively 5, then $A = a$ by [30, Theorem 1], respectively by [14, Propositions AZ.1, AZ.2].

If χ has order 3, then G_{geom} contains the element

$$\text{diag}(a \text{ repeats of } \zeta_3, A \text{ repeats of } 1),$$

and the finite primitive group $\mu_3 G_{\text{geom}}$ contains the element

$$\text{diag}(a \text{ repeats of } 1, A \text{ repeats of } \zeta_3^2).$$

These elements have drops a and A respectively, hence $A + a \leq 4a$ and $A + a \leq 4A$, cf. [30, p. 606]. $\square$

We will now improve on the cases (ii) and (iii) of Proposition 5.10, under the condition $A^2 > D$.

Proposition 5.11 *Suppose $p \nmid Aa(A+a)$, f is Artin-Schreier reduced, $\gcd(a, \gcd_{\deg}(f)) = 1$, $\mathfrak{o}(\chi) = 5$, and $A^2 > D = A + a$. Then $G_{\text{geom}, \mathcal{G}(f, a, \chi)}$ is infinite.*

Proof (a) Assuming more generally that $3 \leq e := \mathfrak{o}(\chi) \leq 5$, suppose $G = G_{\text{geom}, \mathcal{G}(f, a, \chi)}$ is finite. Then by Proposition 4.3 it is a finite subgroup of $\text{GL}_D = \text{GL}(V)$ satisfying $(\mathbf{S}^+)$; moreover G contains an ss-element γ of central order D , as well as the element g in the proof of Proposition 5.10 such that $g^a = h$ is a quadratic element of order e . According to [17, Lemma 1.1], either $D = r^n$ for a prime r and G is an r -extraspecial normalizer, or G is almost quasisimple.

(b) Here we assume that G is an r -extraspecial normalizer. Then G contains a normal r -subgroup $R = \mathbf{Z}(R)E$, where E is extraspecial of order r^{1+2n} acting irreducibly on $V = \mathbb{C}^{r^n}$, and $G \leq \mathbf{Z}(\text{GL}(V))R \cdot \text{Sp}_{2n}(r)$, see e.g. part (i) of the proof of [17, Theorem 7.4]. We will show that either $r = 2$ and $n \leq 5$, or $r = e = 3$ and $n \leq 3$.

Suppose first that $r = 2$. Note that any unipotent Jordan block of size k over $\mathbb{F}_2$ has order $2^{\lceil \log_2 k \rceil}$. By considering the Jordan canonical form of any 2-element $x \in \text{Sp}_{2n}(2)$ in its action on $\mathbb{F}_2^{2n}$, we see that $\mathfrak{o}(x) \leq 2^{\lceil \log_2 2n \rceil}$. Since $\exp(R/\mathbf{Z}(R)) = \exp(E/\mathbf{Z}(E)) = r$, it follows that $\bar{\mathfrak{o}}(\gamma) \leq 2^{1+\lceil \log_2 2n \rceil}$. But we also know that $\bar{\mathfrak{o}}(\gamma) = D = 2^n$. It follows that $n \leq 2 + \lceil \log_2 n \rceil$, whence $n \leq 5$.

Assume now that $r \neq 2$; in particular D is odd and hence the two eigenvalues of h have distinct multiplicities. If $e = 4$ or 5 , this contradicts Proposition 5.10. Assume $e = 3$. Then any unipotent Jordan block of size k over $\mathbb{F}_r$ has order $r^{\lceil \log_r k \rceil}$. By considering the Jordan canonical form of any r -element $x \in \text{Sp}_{2n}(r)$ in its action on $E/\mathbf{Z}(E) \cong \mathbb{F}_r^{2n}$, we see that $\mathfrak{o}(x) \leq r^{\lceil \log_r 2n \rceil}$. Since $\exp(R/\mathbf{Z}(R)) = \exp(E/\mathbf{Z}(E)) = r$, it follows that $\bar{\mathfrak{o}}(\gamma) \leq r^{1+\lceil \log_r 2n \rceil}$. But we also know that $\bar{\mathfrak{o}}(\gamma) = D = r^n$. It follows that $n \leq 1 + \lceil \log_r 2n \rceil$, whence $n \leq 2$, or $n = 3$ and $r = 3, 5$. Assume in addition that $r \geq 5$. Then the quadratic element h of central order 3 projects onto a semisimple element $\bar{h}$ of order 3 of $\text{Sp}_{2n}(r)$, whence it has a fixed point subspace of dimension $\leq 2n - 2$ on $\mathbb{F}_r^{2n}$. It follows from [9, Lemma 2.4] that

$$|\text{Trace}(g)|^2 \leq r^{2n-2} = D^2/r^2 < D^2/4.$$

On the other hand

$$\begin{aligned} |\text{Trace}(g)|^2 &= (A + a\zeta_3)(A + a\zeta_3^2) \\ &= A^2 - aA + a^2 = (A + a)^2 - 3Aa \geq (A + a)^2/4 = D^2/4, \end{aligned}$$

a contradiction. Thus $D = 3^n$ with $n \leq 3$ when $e = 3$. The same computation shows that $A = a = D/2$ and $\text{CSp}_{2n}(2)(\bar{h}) = \text{Sp}_{2n-2}(2) \times C_3$ when $(r, e) = (2, 3)$.

Finally, we consider the case $r = e = 3$ and again consider the element $\bar{h}$ induced by the action of h on $R/\mathbf{Z}(R) \cong \mathbb{F}_3^{2n}$. If this action is trivial, then, as $\bar{o}(h) = 3$, h is a scalar multiple of an element in $R \setminus \mathbf{Z}(R)$, and so $\text{Trace}(h) = 0$, a contradiction. The above computation shows that $\bar{h}$ must have a $(2n - 1)$ -dimensional fixed point subspace on $\mathbb{F}_3^{2n}$. It follows from [9, Lemma 2.4] that $|\text{Trace}(h)|^2 = 3^{2n-1} = D^2/3$ or 0. Since $A^2 - aA + a^2 > 0$, we get

$$A^2 - aA + a^2 = |\text{Trace}(g)|^2 = D^2/3 = (A + a)^2/3,$$

and hence $\{A, a\} = \{2D/3, D/3\}$.

(c) Now we consider the case G is almost quasisimple, with $Z := \mathbf{Z}(G)$ and $L := G^{(\infty)}$. Then G/Z is almost simple, with a unique minimal normal subgroup $S = LZ/Z$, which is simple non-abelian. Let $H := \langle h^G \rangle \triangleleft G$. Since h is quadratic, h is not central in G , and so $H \not\leq Z$. Now, $1 \neq HZ/Z$ is a normal subgroup of G/Z , so it contains the minimal normal subgroup LZ/Z . So we get $HZ \geq LZ$, and so $L = [L, L] = [LZ, LZ] \leq [HZ, HZ] = [H, H] \leq H$. Hence $S = LZ/Z \triangleleft HZ/Z \leq G/Z \leq \text{Aut}(S)$, showing that HZ/Z is almost simple with socle S .

For $i \geq 1$, define $H_1 := H$ and $H_{i+1} = \langle h^{H_i} \rangle \triangleleft H_i$. The above arguments show by induction on i that $H_i Z \geq LZ$ and $H_i Z/Z$ is almost simple with socle S . But $G \geq H_i \geq H_{i+1} \geq L$, so by finiteness of $[G : LZ]$ we must have that $H_j = H_{j-1}$ for some $j \geq 2$. In this case,

$$H_j = \langle h^{H_{j-1}} \rangle = \langle h^{H_j} \rangle.$$

Since $L \triangleleft G$ and $L \not\leq Z$, L is irreducible on V by [10, Lemma 2.5]. As $H_j \geq L$, we conclude that $H_j < \text{GL}(V)$ is an irreducible subgroup generated by conjugates of the quadratic element h .

(d) Now we will specialize to the case $e = 5$. In the case of (b), we have $D = 2^n$ with $n \leq 5$. By [10, Lemma 2.5], $|\text{Trace}(\gamma)|^2$ is an integer. On the other hand, for a primitive fifth root of unity $\zeta = \exp(2\pi ic/5)$, $1 \leq c \leq 4$, we see that

$$|\text{Trace}(\gamma)|^2 = (A + a\zeta)(A + a\bar{\zeta}) = A^2 + a^2 + Aa(\zeta + \bar{\zeta}) = A^2 + a^2 + 2Aa \cos(2\pi c/5)$$

is not an integer, a contradiction.

In the case of (c), we have $D = 2$ by [31, Lemma (11.2)] applied to H_j , again a contradiction. $\square$

Proposition 5.12 Suppose $p \nmid Aa(A+a)$, f is Artin-Schreier reduced, $\gcd(a, \gcd_{\deg}(f)) = 1$, $\text{o}(\chi) = 3$, $A^2 > D = A + a$, and $D \neq 3$. Then $G_{\text{geom}, \mathcal{G}(f, a, \chi)}$ is infinite.

Proof Suppose $G = G_{\text{geom}, \mathcal{G}(f, a, \chi)}$ is finite. Then we repeat verbatim part (a) of the proof of Proposition 5.11 to see that either $D = r^n$ for a prime r and G is an r -extraspecial normalizer, or G is almost quasisimple. If necessary, we multiply h by ζ_3 or ζ_3^2 so that h has only two eigenvalues ζ_3 and ζ_3^2 , which implies that

$$h^2 + h + \text{Id}_V = 0. \quad (5.12.1)$$

(a) First we consider the case where G is an r -extraspecial normalizer. Then part (b) of the proof of Proposition 5.11 shows that $D = r^n$, and either $r = 2$ and $n \leq 5$, or $r = 3$ and $n \leq 3$.

Suppose that $r = 2$. As shown in part (b) of the proof of Proposition 5.11, $A = a = D/2 = 2^{n-1}$ and h , while acting on $R/\mathbf{Z}(R) \cong \mathbb{F}_2^{2n}$, induces an element $\bar{h}$ of order 3 in $\text{Sp}_{2n}(2)$ with $\text{C}_{\text{Sp}_{2n}(2)}(\bar{h}) = \text{Sp}_{2n-2}(2) \times C_3$. The assumption $A^2 > D$ implies that $D > 4$, and thus $n \geq 3$. Consider the element $\bar{g}$ induced by the action of g on $R/\mathbf{Z}(R)$, and recall

that $g^a = h$. If $\bar{g}$ does not act trivially on the fixed point subspace of $\bar{h}$ on $\mathbb{F}_2^{2n}$, then the $\bar{g}$ -fixed point subspace of $R/\mathbf{Z}(R)$ has dimension $\leq 2n - 3$, and so using [9, Lemma 2.4] we get $|\text{Trace}(g)| \leq 2^{(2n-3)/2} < 2^{n-1} = A$, a contradiction. Hence $\bar{g}$ acts trivially on the $(2n - 2)$ -dimensional $\bar{h}$ -fixed point subspace, which implies that $\bar{g} = \bar{h}$ (as $\text{Sp}_2(2) \cong S_3$ does not contain any element of order 6). It follows that $g^3 \in \mathbf{Z}(G)R$, and hence $\bar{o}(g)|6$. But this is a contradiction since $\bar{o}(g) = 3a = 3 \cdot 2^{n-1} \geq 12$.

Suppose that $r = 3$. As shown in part (b) of the proof of Proposition 5.11, $\{A, a\} = \{2D/3, D/3\}$ and h , while acting on $R/\mathbf{Z}(R) \cong \mathbb{F}_3^{2n}$, induces a transvection $\bar{h}$ of order 3 in $\text{Sp}_{2n}(3)$. The assumption $D \neq 3$ implies that $n \geq 2$. Consider the element $\bar{g}$ induced by the action of g on $R/\mathbf{Z}(R)$, and recall that $g^a = h$. Also let d denote the dimension of the $\bar{g}$ -fixed point subspace on $\mathbb{F}_3^{2n}$. Then $A^2 = |\text{Trace}(g)|^2 \in \{0, 3^d\}$ by [9, Lemma 2.4]. It follows that $A = D/3 = 3^{n-1}$ and $d = 2n - 2$. The assumption $A^2 > D$ now implies that $n = 3$, and so $a = 18$. Now $(\bar{g}^2)^9 = \bar{h}$ has order 3 in $\text{Sp}_6(3)$, so $\bar{g}^2 \in \text{Sp}_6(3)$ must have order 27 in $\text{Sp}_6(3)$, which is impossible by inspecting the character table of $\text{Sp}_6(3)$ in [3] or [7].

(b) From now on we may assume that G is almost quasisimple. Then, as shown in part (c) of the proof of Proposition 5.11, G contains a subnormal subgroup H_j such that $K := H_j = \langle h^{H_j} \rangle \geq L$, and K is irreducible on V . Since G is almost quasisimple, the generalized Fitting subgroup $F^*(K)$ of K is $\mathbf{Z}(K)L$, where $L := G^{(\infty)}$. Now using (5.12.1) and reducing the representation of K on V modulo 3 as in [30, §4] (so that one can apply results of [5], [6]), we arrive at the possibilities listed in [30, Theorem 3] for L , which we will now analyze case-by-case.

(c) $L = \text{Sp}_{2n}(3)$ or $\text{PSp}_{2n}(3)$, and $n \geq 3$. Since $\bar{o}(h) = 3$ and L has no outer automorphism of order 3, we must have that $K = \mathbf{Z}(K)L$. Then we can apply the results of [32, 33], recorded in [28, Theorem 3.1] to see that h is (the image in L of) a transvection. Now, applying [28, Theorem 3.2], we conclude that $V|_L$ is a Weil representation, of degree $D = (3^n + \kappa)/2$, where $\kappa = \pm 1$. In such a representation, $G = \mathbf{Z}(G)L$ since the outer automorphism of L does not fix the equivalence class of $V|_L$. Moreover, the two eigenvalues of h have multiplicities 3^{n-1} and $(3^{n-1} + \kappa)/2$, whence

$$\{A, a\} = \{3^{n-1}, (3^{n-1} + \kappa)/2\}.$$

Suppose $a = 3^{n-1}$. Recalling $h = g^a$, where g is the element in the proof of Proposition 5.10, we see that L contains an element with central order $3a = 3^n$, and hence $\text{Sp}_{2n}(3)$ contains a 3-element g' of order 3^n . But the size of any Jordan block of g' is at most $2n$, so $n \leq \lceil \log_3 2n \rceil$, a contradiction since $n \geq 3$.

Thus $a = (3^{n-1} + \kappa)/2$. Again using $h = g^a$, we obtain an element $g' \in \text{Sp}_{2n}(3)$, such that g'^a is a transvection and g' has trace $|\text{Trace}(g')| = A = 3^{n-1}$ on V . If $n = 3$, then using calculations with the character table of $\text{Sp}_6(3)$ in [7] one can check that $\text{Sp}_6(3)$ has no element with trace 9 in a Weil representation. So $n \geq 4$, and hence $3^{n-1} - 1$, respectively $3^{2n-2} - 1$, admits a primitive prime divisor $\ell \geq 5$ by [34], which then divides $3a = o(g')$. As $(g')^{3a/\ell}$ has prime order ℓ , the choice of ℓ implies that g' acting on $\mathbb{F}_3^{2n}$ has at least $n - 1$ pairwise distinct eigenvalues not equal to ± 1 (which constitute the orbit of an eigenvalue under the map $\lambda \mapsto \lambda^3$). Thus the fixed point subspaces of g' and zg' on $\mathbb{F}_3^{2n}$ both have dimension $\leq n + 1$, where z is the central involution of $\text{Sp}_{2n}(3)$. Hence, if ξ_n and η_n denote the irreducible characters of $\text{Sp}_{2n}(3)$ occurring in a reducible Weil representation (that contains V), then

$$|\xi_n(g') + \eta_n(g')| \leq 3^{(n+1)/2}, \quad |\xi_n(zg') + \eta_n(zg')| \leq 3^{(n+1)/2}$$

by [9, Lemma 2.4]. Using $\{\xi_n(zg'), \eta_n(zg')\} = (\pm 1)\{\xi_n(g'), -\eta_n(g')\}$, we deduce that

$$3^{n-1} = |\text{Trace}(g')| \leq (|\xi_n(zg') + \eta_n(zg')| + |\xi_n(g') + \eta_n(g')|)/2 \leq 3^{(n+1)/2},$$

a contradiction because $n > 3$.

(d) $L = \mathrm{SU}_n(2)$ or $\mathrm{PSU}_n(2)$, and $n \geq 4$. Since $\bar{o}(h) = 3$ and L has no outer automorphism of order 3, we must have that $K = \mathbf{Z}(K)L$ and $\mathbf{C}_K(L) = \mathbf{Z}(K)$. Viewing h as an element of $\mathrm{GU}_n(2)$ and applying [6, Theorem B(a)], we see that h is a scalar multiple of a pseudoreflection. Now, applying [28, Theorem 5.2], we conclude that $V|_L$ is a Weil representation, of degree $D = (2^n + 2(-1)^n)/3 - (-1)^n\kappa$ where $\kappa \in \{0, 1\}$. In such a representation, the two eigenvalues of h have multiplicities

$$\{A, a\} = \{(2^{n-1} - (-1)^{n-1})/3, (2^{n-1} - (-1)^{n-1})/3 - (-1)^n\kappa\}.$$

Note that $(2^{n-1} - (-1)^{n-1})/3 - (-1)^n\kappa = 2(2^{n-2} - (-1)^{n-2})/3$ when $\kappa = 1$. Thus a is either $(2^{n-1} - (-1)^{n-1})/3$ or $2(2^{n-2} - (-1)^{n-2})/3$; in particular $a > 1$. Recalling $h = g^a$, we have an element $g \in G$, such that g^a is a scalar multiple of a pseudoreflection and g has trace $|\mathrm{Trace}(g)| = A \geq (2^{n-1} - 4)/3$ on V . In the case $n = 5$, in fact we have $|\mathrm{Trace}(g)| \geq 5$, which is impossible as one can check using character tables of $\mathrm{SU}_5(2)$ and $\mathrm{SU}_5(2) \cdot 2$ in [3] or [7]. In the case $n = 4$, we have $A \geq \sqrt{D} \geq \sqrt{5}$, so $|\mathrm{Trace}(g)| \geq 3$, which is impossible as one can check again using character tables of $\mathrm{SU}_4(2)$ and $\mathrm{SU}_4(2) \cdot 2$ in [7]. So $n \geq 6$. Then $(-2)^{n-1} - 1$, respectively $(-2)^{n-2} - 1$, admits a primitive prime divisor $\ell \geq 5$ by [34], which then divides $o(g)$. As $g_1 := g^{3 \cdot o(g)/\ell}$ has prime order $\ell > 2$ and $|\mathrm{Out}(L)| = 2$, we may assume that $g_1 \in \mathrm{GU}_n(2)$, and then the choice of ℓ implies that g_1 acting on $\mathbb{F}_4^n$ has at least $n - 2$ pairwise distinct eigenvalues not contained in μ_3 (which constitute the orbit of an eigenvalue under the map $\lambda \mapsto \lambda^{-2}$). The same applies to $g_2 := g^{o(g)/\ell}$ (as we have $g_1 = g_2^3$). Also, we still have $|\mathrm{Trace}(g_2)| = A$. Thus the fixed point subspaces of $z^i g_2$, $0 \leq i \leq 3$ and $\mathbb{F}_4^n$ all have dimension ≤ 2 , where $(z) = \mathbf{Z}(\mathrm{GU}_n(2))$. It follows from [28, Lemma 4.1] that

$$(2^{n-1} - 4)/3 \leq A = |\mathrm{Trace}(g_2)| \leq 2^2,$$

a contradiction because $n \geq 6$.

(e) $L = 2A_n$ and $n \geq 5$. Since $\bar{o}(h) = 3$ and L has no outer automorphism of order 3, we must have that $K = \mathbf{Z}(K)L$ and $\mathbf{C}_K(L) = \mathbf{Z}(K)$. Viewing $\bar{h} := h\mathbf{Z}(K)$ as an element of A_n and applying [6, Theorem B(b)], we see that $\bar{h}$ is a 3-cycle. Now, applying [29, Theorem 8.1], we conclude that $V|_L$ is a basic spin representation, of degree $D = 2^{\lfloor n/2 \rfloor - 1}$. In such a representation, the two eigenvalues of h have equal multiplicities $A = a = D/2$. The assumption $A^2 > D$ implies that $D > 4$, and hence $n \geq 8$. Again using $h = g^a$, we obtain an element $g' \in S_n$, such that g'^a is a 3-cycle and $|\mathrm{Trace}(g')| = D/2$. When $n = 8, 9$, one can check using character tables of $2A_8$ and $2A_9$ in [3] or [7] that no such element exists. When $n \geq 10$, we have that $\mathbf{C}_{S_n}(g'^a) \cong A_3 \times S_{n-3}$ contains the element g'^3 of order $a = 2^{\lfloor n/2 \rfloor - 2}$, a contradiction.

(f) In the remaining cases,

$$|\mathrm{Trace}(h)| = A = a = D/2 > 1, \text{ whence } |\mathrm{Trace}(g)| = D/2, \quad (5.12.2)$$

and L is $\mathrm{SU}_3(3)$, $6_1 \cdot \mathrm{PSU}_4(3)$, $2 \cdot J_2$, $2 \cdot G_2(4)$, $2 \cdot C_{O_1}$, $2 \cdot \Omega_8^+(2)$, $2 \cdot \mathrm{Sp}_6(2)$, $6 \cdot \mathrm{Suz}$. In the first two cases, using (5.12.2) and character tables of L as given in [7] we can check that $D = 6$. Then, checking the character tables of $\mathrm{SU}_3(3) \cdot 2$ and $6_1 \cdot \mathrm{PSU}_4(3) \cdot 2_2$ in [7], we see that there is no element g of central order $3a$ that satisfies (5.12.2), a contradiction. In the last six cases, $K = \mathbf{Z}(K)L$ by [6, Theorem A], and furthermore the conjugacy class of $\langle h \rangle$ in K is determined by [6, Theorem B], and we may assume that $h \in L$. Again using (5.12.2) and character tables of L as given in [7], we see that $D = 6$, respectively $D = 12$, $D = 24$, $D = 8$, $D = 8$, and $D = 12$. Then, checking the character tables of $2 \cdot J_2$, respectively $2 \cdot G_2(4)$ and $2 \cdot G_2(4) \cdot 2$, $2 \cdot C_{O_1}$, $2 \cdot \Omega_8^+(2)$ and $2 \cdot \Omega_8^+(2) \cdot 2$, $2 \cdot \mathrm{Sp}_6(2)$, and $6 \cdot \mathrm{Suz}$

in [7], we see that there is no element g of central order $3a$ that satisfies (5.12.2), again a contradiction. $\square$

Proposition 5.13 *Suppose $p \nmid Aa(A+a)$, f is Artin-Schreier reduced, $\gcd(a, \gcd_{\deg}(f)) = 1$, $\circ(\chi) = 4$, and $A^2 > D = A + a$. Then $G_{\text{geom}, \mathcal{G}(f, a, \chi)}$ is infinite.*

Proof Suppose $G = G_{\text{geom}, \mathcal{G}(f, a, \chi)}$ is finite. Then we repeat verbatim part (a) of the proof of Proposition 5.11 to see that either $D = r^n$ for a prime r and G is an r -extraspecial normalizer, or G is almost quasisimple. By Proposition 5.10 we have $A = a = D/2$; in particular, $2|D$ and $|\text{Trace}(h)|^2 = D^2/2$. The assumption $A^2 > D$ now implies that $D \geq 6$.

(a) First we consider the case G is almost quasisimple. Then, as shown in part (c) of the proof of Proposition 5.11, G contains a subnormal subgroup H_j such that $K := H_j = \langle h^{H_j} \rangle \geq L$, and K is irreducible on V . Since G is almost quasisimple, the generalized Fitting subgroup $F^*(K)$ of K is $\mathbf{Z}(K)L$, where $L := G^{(\infty)}$, and furthermore $F^*(G) = \mathbf{Z}(G)L$. Recall that G contains the element γ with simple spectrum on $V = \mathbb{C}^D$ and L is irreducible on V . Now the proof of [19, Theorem 3.1.6] shows that K is primitive and tensor indecomposable on V , unless K is imprimitive on V and, moreover, either $(L/\mathbf{Z}(L), D)$ is $(\text{PSL}_n(q), (q^n - 1)/(q - 1))$ with $2 \nmid q$ or $(M_{12}, 12)$, or $D = 6$ and $L/\mathbf{Z}(L) = A_5, A_6$ (recall that $2|D$). If we are not in these two exceptions, then we can apply the main result of [22] to conclude that K contains an irreducible normal 2-subgroup (and so is an extraspecial normalizer). But this is impossible since $F^*(K) = \mathbf{Z}(K)L$.

The cases of $D = 6$ are excluded since in this case G contains no element with trace of absolute value $3\sqrt{2}$, by inspecting character tables of the group G in question as given in [7].

The case of $(M_{12}, 12)$ is excluded since in this case G contains no element with trace of absolute value $6\sqrt{2}$, again using character tables of G as given in [7].

Assume that $(L/\mathbf{Z}(L), D) = (\text{PSL}_n(q), (q^n - 1)/(q - 1))$ with $2 \nmid q$. The proof of [19, Theorem 3.1.6] shows that the imprimitive action of K on $V = \mathbb{C}^D$ induces a 2-transitive permutation representation π of degree $D = (q^n - 1)/(q - 1) \geq 6$, where $H := \pi(K)$ has socle $\text{PSL}_n(q)$. Note that $\text{Ker}(\pi)$ acts on V diagonally, hence $\text{Ker}(\pi) \triangleleft K$ is abelian. But $F^*(K) = \mathbf{Z}(K)L$, so

$$\text{Ker}(\pi) \leq \mathbf{Z}(K) \leq \mathbf{Z}(G).$$

It follows from [4] that $\text{PSL}_n(q) \triangleleft H \leq \Gamma\text{L}_n(q)$. We may assume that H is contained in the image of the action of the group $\Gamma\text{L}(W)$ of all invertible semilinear transformations of $W = \langle e_1, \dots, e_n \rangle_{\mathbb{F}_q} \cong \mathbb{F}_q^n$ on the set $\mathbb{P}W$ of 1-dimensional $\mathbb{F}_q$ -subspaces of W . Since $|\text{Trace}(h)| \geq D/\sqrt{2}$, the set Ω of h -fixed points on $\mathbb{P}W$ has size at least $D/\sqrt{2}$. Now if Ω does not span W , then $|\Omega| \leq (q^{n-1} - 1)/(q - 1) = (D - 1)/q < D/\sqrt{2}$, a contradiction. So without loss of generality we may assume that $\langle e_i \rangle_{\mathbb{F}_q} \in \Omega$ for all $1 \leq i \leq n$. If $q = r^f$ for a prime r , then, changing h to a suitable generator of $\langle h \rangle$, we can find some $0 \leq j \leq f - 1$, subject to $j|f$ if $j \geq 1$, and some $a_i \in \mathbb{F}_q^\times$, $1 \leq i \leq n$, such that

$$h : \sum_{i=1}^n x_i e_i \mapsto \sum_{i=1}^n a_i x_i^{r^j} e_i. \quad (5.13.1)$$

Note that $\text{PSL}_n(q) \triangleleft K = \langle h^K \rangle$, so $\pi(h)$ cannot be trivial. Now if $j = 0$, then using (5.13.1) we can show that either

$$|\Omega| \leq (q^{n-1} - 1)/(q - 1) < D/q < D/\sqrt{2},$$

or $a_1 = a_2 = \dots = a_n$, i.e. $\pi(h) = 1$, both leading to a contradiction. So $1 \leq j \leq f/2$ and $j|f$. In this case, the line spanned by $\sum_{i=1}^n x_i e_i$ is h -fixed if and only if there exists some

$b \in \mathbb{F}_q^\times$ such that $a_i x_i^{r^j} = b x_i$ for all i . For a given i , the equation $a_i x_i^{r^j} = b x_i$ has at most r^j solutions in $\mathbb{F}_q$. It follows that

$$|\Omega| \leq \frac{(q-1)r^{nj}}{q-1} = r^{nj}.$$

Now if $n \geq 3$, then

$$|\Omega| \leq r^{nj} \leq r^{nf/2} = q^{n/2} \leq \frac{q^n}{q^{3/2}} \leq \frac{q^n}{q\sqrt{2}} = \frac{q^{n-1}}{\sqrt{2}} < D/\sqrt{2},$$

a contradiction. So $n = 2$. If $j \leq f/3$, then $f \geq 3$ and

$$|\Omega| \leq r^{2j} \leq r^{2f/3} = \frac{q}{r^{f/3}} \leq \frac{q}{2} < \frac{q+1}{\sqrt{2}} = D/\sqrt{2},$$

again a contradiction. We have shown that $j = f/2$, and hence

$$h^2 : \sum_{i=1}^n x_i e_i \mapsto \sum_{i=1}^n a_i^{\sqrt{q}+1} x_i e_i.$$

Again, the line spanned by $x_1 e_1 + x_2 e_2$ is h -fixed if and only if there exists some $b \in \mathbb{F}_q^\times$ such that $a_i x_i^{\sqrt{q}} = b x_i$ for all i . We may assume that $x_1 \neq 0$, whence

$$1 = x_1^{q-1} = (b/a_1)^{\sqrt{q}+1},$$

showing that there are at most $\sqrt{q} + 1$ choices for b . For any such b , there are at most $q - 1$ solutions $(x_1, x_2) \neq (0, 0)$. It follows that

$$|\Omega| \leq \frac{(q-1)(\sqrt{q}+1)}{q-1} = \sqrt{q} + 1 < (q+1)/\sqrt{2} = D/\sqrt{2},$$

a contradiction.

(b) Assume now that G is an r -extraspecial normalizer. Then $2|D = r^n$, so $r = 2$. But $D \geq 6$, so $n \geq 3$. Since $|\text{Trace}(h)|^2 = D^2/2 = 2^{2n-1}$, h , while acting on $R/\mathbf{Z}(R) \cong \mathbb{F}_2^{2n}$, induces an element $\bar{h}$ in $\text{Sp}_{2n}(2)$ with an $(2n-1)$ -dimensional fixed point subspace, so a transvection. Consider the element $\bar{g}$ induced by the action of g on $R/\mathbf{Z}(R)$, and recall that $g^a = h$. Thus $\bar{g}^{2^{n-1}} = \bar{h}$ is a transvection. It follows that $\bar{g} \in \text{Sp}_{2n}(2)$ has order 2^n . Also, as $a > 1$, $|\text{Trace}(g)|^2 = A^2 = 2^{2n-2}$, which implies by [9, Lemma 2.4] that $\bar{g}$ has a $(2n-2)$ -dimensional fixed point subspace on $\mathbb{F}_2^{2n}$. Thus the size of any Jordan block of the 2-element $\bar{g}$ is at most 3, and so its order is at most 4, which is a contradiction since $n \geq 3$. $\square$

We also offer a partial result for χ of order 2:

Proposition 5.14 *Suppose $p \nmid Aa(A+a)$, f is Artin-Schreier reduced, $\gcd(a, \gcd_{\deg}(f)) = 1$, $\circ(\chi) = 2$, and $A^2 > D = A + a$. Then one of the following conclusions holds for $G := G_{\text{geom}, \mathcal{G}(f, a, \chi)}$*

- (i) G is infinite.
- (ii) G is finite and $G^{(\infty)}$ is A_{D+1} acting on its deleted permutation module.
- (iii) $D \leq \min(80a, 160A)$.

Proof Suppose G is finite but (ii) does not hold. Then we repeat verbatim part (a) of the proof of Proposition 5.11 to see that either $D = r^n$ for a prime r and G is an r -extraspecial normalizer, or G is almost quasisimple, moreover G contains an ss-element γ of central order D , as well as the element g in the proof of Proposition 5.10. By Proposition 5.2, we may assume $a > 1$, and also

$$D > \min(80a, 160A). \quad (5.14.1)$$

Since $a > 1$, we have

$$\text{Trace}(g) = A, \text{Trace}(g^a) = A - a.$$

If G is an r -extraspecial normalizer, then [9, Lemma 2.4] implies that

$$|\text{Trace}(h)|/D \leq 1/\sqrt{r} \leq 1/\sqrt{2} < 2/3$$

for any $h \in G$. Suppose G is almost quasisimple. Since $D > 80$ and we are not in (ii), the results of [17, §6] imply that $G^{(\infty)}$ is not an alternating group. It follows from [11, §2] that

$$|\text{Trace}(h)|/D \leq 79/80$$

for any $h \in G$. Now

$$(D - a)/D = |\text{Trace}(g)|/D \leq 79/80,$$

implying $D \leq 80a$ and hence $160A < D \leq 80a$ by (5.14.1). Thus $a > 2A$, and

$$(D - 2A)/D = |\text{Trace}(g^a)|/D \leq 79/80,$$

implying $D \leq 160A$, a contradiction. $\square$

6 Determination of Monodromy Groups: Infinite Cases

Lemma 6.1 Suppose $p \nmid Aa$, f is Artin-Schreier reduced, $\gcd(a, \gcd_{\deg}(f)) = 1$, and $\chi = 1$. Then the following statements hold for $G = G_{\text{geom}, \mathcal{G}(f, a, 1)}$.

- (i) If $2 \nmid Aa$ and f is an odd polynomial, then $G = \text{Sp}_{A+a}$.
- (ii) If $2 \mid Aa$, or f is not an odd polynomial, then $G^\circ = \text{SL}_{A+a}$.

Proof For $\gamma \in I(0)$ which generates $I(0)/P(0)$, γ^a is a transvection, cf. Lemma 2.1. Therefore $G_{\text{geom}, \mathcal{G}(f, a, 1)}$ is infinite. Because $\mathcal{G}(f, a, 1)$ satisfies (S+) by Proposition 4.4, $G_{\text{geom}, \mathcal{G}(f, a, 1)}^\circ$ acts irreducibly, and is normalized by a transvection. It then follows from [13, Theorem 1.5] applied to the case of a pseudoreflexion of determinant 1 (i.e., a transvection) that $G^\circ = \text{SL}_{A+a}$ or Sp_{A+a} . Applying Lemma 2.12, we are done. $\square$

Lemma 6.2 Suppose $p \nmid Aa$, f is Artin-Schreier reduced, $\gcd(a, \gcd_{\deg}(f)) = 1$, $\chi = 1$, and $G_{\text{geom}, \mathcal{G}(f, a, 1)}^\circ = \text{SL}_{A+a}$. Then

$$G_{\text{geom}, \mathcal{G}(f, a, 1)} = \text{SL}_{A+a}$$

if a is odd, and

$$G_{\text{geom}, \mathcal{G}(f, a, 1)} = \{g \in \text{GL}_{A+a} \mid \det(g)^2 = 1\}$$

if a is even.

Proof The determinant $\mathcal{L} := \det(\mathcal{G}(f, a, \mathbb{1}))$ is tame at ∞ (because $\mathcal{G}(f, a, \mathbb{1})$ has all ∞ -slopes < 1) and is tame at 0 (because $\mathcal{G}(f, a, \mathbb{1})$ is tame at 0). If a is odd, the action of an element $\gamma \in I(0)$ which generates $I(0)/P(0)$ has determinant 1, while it has determinant -1 if a is even. $\square$

Proposition 6.3 Suppose $p \nmid Aa(A+a)$, f is Artin-Schreier reduced, $a = 1$, $\chi = \chi_2$, and $A^2 > A + 1 \neq 4$ (i.e., $A \geq 2$ but $A \neq 3$). If $G_{\text{geom}, \mathcal{G}(f, 1, \chi_2)}$ is infinite, then either $G_{\text{geom}, \mathcal{G}(f, 1, \chi_2)} = \mathbf{O}_{A+a}$, or $G_{\text{geom}, \mathcal{G}(f, 1, \chi_2)} = \{g \in \mathbf{GL}_{A+a} \mid \det(g)^2 = 1\}$. Moreover, the first case occurs if and only if f is an odd polynomial.

Proof If $G := G_{\text{geom}, \mathcal{G}(f, 1, \chi_2)}$ is infinite, then by Proposition 4.3, G° is a simple algebraic group acting irreducibly. It cannot be $\mathbf{SO}_4$, because $\mathbf{SO}_4$ is not simple. It is normalized by a reflection (the action of any $\gamma \in I(0)$ which generates $I(0)/P(0)$). So by [13, Theorem 1.5], applied to the case of a pseudoreflection of determinant -1 (i.e., a true reflection), G° is either $\mathbf{SO}_{A+1}$ or $\mathbf{SL}_{A+1}$.

Suppose that $G^\circ = \mathbf{SO}_{A+1}$. Then $\mathcal{G}$ is self-dual by Corollary 2.11. It then follows from Lemma 2.12 that f is an odd polynomial. Conversely, if f is an odd polynomial, then $G \leq \mathbf{O}_{A+1}$ by Lemma 2.12; and hence $G^\circ = \mathbf{SO}_{A+1}$ as shown above. As the determinant of $\mathcal{G}$ is $\mathcal{L}_{\chi_2}$ by Lemma 2.2, in this case we have $G = \mathbf{O}_{A+1}$. $\square$

Theorem 6.4 Suppose that $p \nmid aA(A+a)$, f is Artin-Schreier reduced, $\gcd(a, \gcd_{\deg}(f)) = 1$, $A^2 > D := A + a$. If χ has order 2, assume $G := G_{\text{geom}, \mathcal{G}(f, a, \chi)}$ is infinite. If χ has order 3, suppose $D \neq 3$. Then exactly one of the following statements holds for G .

- (i) $G^\circ = \mathbf{SL}_D$.
- (ii) $\chi = \mathbb{1}$, $2 \nmid Aa$, f is an odd polynomial, and $G = \mathbf{Sp}_D$.
- (iii) $\chi = \chi_2$, $2 \nmid Aa$, f is an odd polynomial, and $\mathbf{SO}_D = G^\circ \leq G \leq \mathbf{O}_D$.

Proof (A) Using Lemma 6.2 we will assume throughout the proof that $\chi \neq \mathbb{1}$. By Propositions 5.10–5.13, G is infinite. By Proposition 4.3, G in its given D -dimensional representation $V = \mathbb{C}^D$ has $(\mathbf{S}^+)$. Thus G° is a simple algebraic group, acting irreducibly on V via its simple module $L(\lambda)$. By Lemma 2.9, G contains a simple spectrum element g_∞ . Hence we can apply [19, Theorem 3.3.4] to determine the simple G° -module $V = L(\lambda)$.

Note that $(A, a) \neq (1, 1)$ as we assume $A^2 > A + a$. Since $\mathbf{o}(\chi) \geq 2$, it follows from Lemma 2.12 and Corollary 2.11 that either

- (a) V is not self-dual over G° , or
- (b) the G° -module V is self-dual, of quadratic type, $\chi = \chi_2$, and then $G \leq \mathbf{O}(V)$.

In view of [19, Theorem 3.3.4], first we consider the case $g_\infty \notin \mathbf{Z}(G)G^\circ$. Then V is listed in [19, Theorem 3.3.4(B)(b)], and it is always self-dual as a G° -module, hence (b) holds.

If $V = L(\varpi_1)$ for G° of type D_n , then $D = 2n$, and we arrive at (iii).

If $V = L(\varpi_2)$ for $G^\circ = \mathbf{SL}_4/\mu_2$, then $D = 6$, and we can view G° as $\mathbf{SO}_6$ and again arrive at (iii).

Suppose $V = L(\varpi_3)$ with $D = 20$ for $G^\circ = \mathbf{SL}_6/\mu_3$. In this case, V is of symplectic type, cf. [23, Appendices A.2, A.3], contrary to (b).

The final case listed in [19, Theorem 3.3.4(B)] is $D = 8$, and $V = L(\varpi_1 + \varpi_2)$ for $G^\circ = \mathbf{SL}_3/\mu_3$. As V is self-dual, $\chi = \chi_2$ and $2 \nmid Aa$ by (b). Note that $a \neq 1$ by Proposition 6.3, and $A \geq 3$ by assumption. So we may assume $3 \leq a \leq 5$. Now some multiple h of $(g_\infty)^2$ belongs to G° and has spectrum consisting of $\beta(\mu_a \setminus \{1\})$ together with β repeated $A + 1 \geq 4$ times on V . We may assume $h = \text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ in $\mathbf{SL}_3$, so that the spectrum

of h on V consists of $(\varepsilon_i \varepsilon_j^{-1})^{\pm 1}$, $1 \leq i < j \leq 3$, and 1 repeated twice. Since β is the only repeated eigenvalue, we get $\beta = 1$. But β is repeated ≥ 4 times, so we may assume $\varepsilon_1 = \varepsilon_2$. In this case, $\varepsilon_1 \varepsilon_3^{-1} = \varepsilon_2 \varepsilon_3^{-1}$ is repeated, so the common value is $\beta = 1$, which shows that $\varepsilon_3 = \varepsilon_1$, and thus $h|_V = \text{Id}$, a contradiction.

(B) From now on, we may assume that $g_\infty \in \mathbf{Z}(G)G^\circ$. We first show that

$$G = \mathbf{Z}(G)G^\circ, \quad (6.4.1)$$

except possibly in the case G° is of type D_4 . Indeed, it suffices to prove the claim in the case $\mathbf{Z}(G)G^\circ$ has index 2 in G . By Lemma 2.9, the image of $I(\infty)$ in G is finite. The condition $p \nmid Aa(A+a)$ implies that $p > 2$, and hence the image Q of $P(\infty)$, a finite p -group, is contained in $\mathbf{Z}(G)G^\circ$. As g_∞ is chosen to generate together with Q the image of $I(\infty)$ and $g_\infty \in \mathbf{Z}(G)G^\circ$, we have $G = \mathbf{Z}(G)G^\circ$ by Corollary 2.5.

When (6.4.1) holds, it follows from Lemma 2.1(ii) that G° admits an element g whose spectrum on $V = L(\lambda)$ is

$$\{\alpha_1, \dots, \alpha_a, \underbrace{\beta, \dots, \beta}_A\},$$

where $\alpha_1, \dots, \alpha_a, \beta$ are pairwise distinct. In particular, $g \notin \mathbf{Z}(G)$. Our subsequent arguments will use the key observations that if we have two linearly independent eigenvectors of g with eigenvalues γ_1 and γ_2 , then $\gamma_1 = \gamma_2$ implies $\gamma_1 = \beta = \gamma_2$, whereas $\gamma_1 \neq \gamma_2$ implies $\alpha_i \in \{\gamma_1, \gamma_2\}$ for some $1 \leq i \leq a$. We will frequently embed g in a maximal torus T of G° .

(B1) First we consider the case $V = L(\varpi_k) = \wedge^k(\mathbb{C}^n)$ for G° the image of SL_n , and $1 \leq k \leq n-1$. By duality we may assume that $1 \leq k \leq n/2$, and we arrive at (i) if $k = 1$. If $k = 2 = n/2$, then we can view $L(\lambda)$ as $L(\varpi_1)$ for $\text{SO}_6 \cong \text{SL}_4/\mu_2$, and arrive at (iii). By [19, Theorem 3.3.4], we may assume that $g_\infty \in \mathbf{Z}(G)G^\circ$, $2 \leq k \leq n/2$ and $n \geq 5$. Hence we may assume that g is the image in SL_D of $\text{diag}(\varepsilon_1, \dots, \varepsilon_n)$, so that the eigenvalues of g on V are the products

$$\varepsilon_X := \prod_{i \in X} \varepsilon_i$$

for all the k -subsets X of $I := \{1, 2, \dots, n\}$.

Suppose first that $\varepsilon_1, \dots, \varepsilon_n$ are pairwise distinct. We claim that for each $1 \leq i \leq n$, there is at most one k -subset X containing i such that $\varepsilon_X = \beta$. Indeed, suppose for instance that

$$\varepsilon_{\{1\} \cup Y} = \beta = \varepsilon_{\{1\} \cup Z}$$

for two different $(k-1)$ -subsets $Y, Z \subseteq \{2, 3, \dots, n\}$. Since $|Y \cup Z| \leq 2k-2 \leq n-2$, we may assume that $2 \notin Y \cup Z$. As $\varepsilon_{\{1\} \cup Y} = \varepsilon_1 \varepsilon_Y \neq \varepsilon_2 \varepsilon_Y = \varepsilon_{\{2\} \cup Y}$, we must have that $\varepsilon_{\{2\} \cup Y} = \alpha_i$ for some $1 \leq i \leq a$. Similarly $\varepsilon_{\{2\} \cup Z} = \alpha_j$ for some $1 \leq j \leq a$. It follows that

$$\frac{\alpha_i}{\beta} = \frac{\varepsilon_{\{2\} \cup Y}}{\varepsilon_{\{1\} \cup Y}} = \frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_{\{2\} \cup Z}}{\varepsilon_{\{1\} \cup Z}} = \frac{\alpha_j}{\beta},$$

and thus $\alpha_i = \alpha_j$ occurs with multiplicity ≥ 2 , a contradiction.

The proved claim implies that the X 's with $\varepsilon_X = \beta$ are all disjoint k -subsets of I , whence $A \leq n/k$. On the other hand, if $k = 2$ then $D = \binom{n}{2} > (n/2)^2$, and if $3 \leq k \leq n/2$ then

$$D = \binom{n}{k} \geq \binom{n}{3} > (n/3)^2 \geq (n/k)^2.$$

Thus in all cases $A^2 < D$, contrary to our assumption.

We may therefore assume that $\varepsilon_1 = \varepsilon_2 =: \gamma$. Now for any $(k-1)$ -subset $S \subseteq \{3, 4, \dots, n\}$, $\varepsilon_{\{1\} \cup S} = \varepsilon_{\{2\} \cup S}$ is a repeated eigenvalue of g on V , so must be β , and hence $\varepsilon_S = \beta/\varepsilon_1$

independently from S . Taking $S = \{3\} \cup T$ and $\{4\} \cup T$ for a $(k-2)$ -subset T of $\{6, \dots, n\}$ (which is possible since $n-5 \geq 2k-4 \geq k-2$ when $k < n/2$ and $n-5 \geq 2k-5 \geq k-2$ when $k = n/2 \geq 3$), we deduce that $\varepsilon_3 = \varepsilon_4$, and so, more generally,

$$\varepsilon_3 = \varepsilon_4 = \dots = \varepsilon_n =: \delta.$$

Now for the same T , $\varepsilon_{\{1,3\} \cup T} = \varepsilon_{\{1,4\} \cup T}$ is repeated, so $\varepsilon_{\{1,3\} \cup T} = \beta$. Similarly, $\varepsilon_{\{3,4\} \cup T} = \varepsilon_{\{3,5\} \cup T}$ is repeated, so $\varepsilon_{\{3,4\} \cup T} = \beta$. It follows that $\gamma = \varepsilon_1 = \varepsilon_4 = \delta$, and hence $g \in \mathbf{Z}(G)$, a contradiction.

(B2) Next we consider the case $V = L(k\varpi_1)$ for G° the image of SL_n , or $V = L(k\varpi_{n-1})$, with $k \geq 2$ and $n \geq 2$. By duality we may assume $V = L(k\varpi_1) = S^k(\mathbb{C}^n)$. One can rule out this case following the proof of [19, Theorem 6.1.16] using Lemma 2.9 and the (highly technical) results in [19, §6.1]. Instead, we will give a more direct argument.

We may assume that g is the image in SL_D of $\mathrm{diag}(\varepsilon_1, \dots, \varepsilon_n)$, so that the eigenvalues of g on V are the monomials (in the variables $\varepsilon_1, \dots, \varepsilon_n$)

$$\varepsilon_X := \prod_{i \in X} \varepsilon_i$$

for all the k -multisets X of $I = \{1, 2, \dots, n\}$.

Suppose first that $\varepsilon_1, \dots, \varepsilon_n$ are pairwise distinct. We claim that for each $1 \leq i \leq n$, there is at most one k -multiset X containing i such that $\varepsilon_X = \beta$. Indeed, suppose for instance that

$$\varepsilon_{\{1\} \cup Y} = \beta = \varepsilon_{\{1\} \cup Z}$$

for two different $(k-1)$ -multisets $Y, Z \subseteq I$. As $\varepsilon_{\{1\} \cup Y} = \varepsilon_1 \varepsilon_Y \neq \varepsilon_2 \varepsilon_Y = \varepsilon_{\{2\} \cup Y}$, we must have that $\varepsilon_{\{2\} \cup Y} = \alpha_i$ for some $1 \leq i \leq a$. Similarly $\varepsilon_{\{2\} \cup Z} = \alpha_j$ for some $1 \leq j \leq a$. It follows that

$$\frac{\alpha_i}{\beta} = \frac{\varepsilon_{\{2\} \cup Y}}{\varepsilon_{\{1\} \cup Y}} = \frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_{\{2\} \cup Z}}{\varepsilon_{\{1\} \cup Z}} = \frac{\alpha_j}{\beta},$$

and thus $\alpha_i = \alpha_j$ occurs with multiplicity ≥ 2 , a contradiction.

The proved claim implies that the X 's with $\varepsilon_X = \beta$ are all disjoint k -multisets of I . Counting the distinct ε_i occurring in those X 's, we deduce that $A \leq n$. On the other hand, if $k \geq 3$ then

$$D = \binom{n+k-1}{k} \geq \binom{n+2}{3} \geq n^2$$

(as $n \geq 2$), contrary to the assumption that $A^2 > D$. It follows that $k = 2$. In this case, the distinctness of ε_i 's implies that there are at most $c \leq 2$ multisets $X = \{i, i\}$ with $\varepsilon_X = \beta$, say for $1 \leq i \leq c$. The remaining $A - c$ multisets Y with $\varepsilon_Y = \beta$ are disjoint 2-subsets of $\{c+1, \dots, n\}$, showing $A - c \leq (n - c)/2$, and thus $A \leq (n + c)/2 \leq (n + 2)/2$. Using $A^2 > D = n(n+1)/2$, we obtain $n \leq 3$. If $n = 3$, then $A \leq (n + 2)/2$ implies $A \leq 2$, contradicting $A^2 > D = 6$. If $n = 2$, then we can view $L(\lambda)$ as $L(\varpi_1)$ for $\mathrm{SL}_2/\mu_2 \cong \mathrm{SO}_3$ and arrive at (iii).

We may therefore assume that $\varepsilon_1 = \varepsilon_2$. Now for any $(k-1)$ -multiset S of I , $\varepsilon_{\{1\} \cup S} = \varepsilon_{\{2\} \cup S}$ is a repeated eigenvalue of g on V , so must be β , and hence $\varepsilon_S = \beta/\varepsilon_1$ independently from S . For any $i \neq j$, taking $S = \{i\} \cup T$ and $\{j\} \cup T$ for a $(k-2)$ -multiset T of I , we deduce that $\varepsilon_i = \varepsilon_j$. Thus

$$\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_n,$$

i.e. $g \in \mathbf{Z}(G)$, a contradiction.

(B3) Now we consider the case where G° is of type D_n with $n \geq 4$. If $V = L(\varpi_1)$, so $D = 2n$, then it is self-dual and (iii) holds. So let $V = \mathbb{C}^{2^{n-1}}$ be a half-spin representation of $G^\circ = \text{Spin}_{2n}$. If $n = 4$, then $V(\lambda)$ is quasi-equivalent to $L(\varpi_1)$. So we may assume $n \geq 5$ (in particular (6.4.1) holds), and embed g in a maximal torus T of G° . Choose an orthonormal basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$ and realize the set of all T -weights of V (written additively) as $\sum_{i=1}^n a_i e_i / 2$ with $a_i = \pm 1$ and $\prod_{i=1}^n a_i = (-1)^n$. Write any such weight as

$$\mu = \mu_X = -\sum_{i=1}^n e_i / 2 + \sum_{i \in X} e_i \quad (6.4.2)$$

with $X \subseteq I := \{1, 2, \dots, n\}$ of even size.

Suppose that $(e_1 + \kappa e_2)(g) \neq 1$ for both $\kappa = +1$ and $\kappa = -1$. Then, for each odd-size subset Y of $\{3, 4, \dots, n\}$, $\mu_{\{1\} \cup Y}(g) \neq \mu_{\{2\} \cup Y}(g)$ by (6.4.2), so at most one of these values can be β . Moreover, if there are at least three such Y such that one of $\mu_{\{1\} \cup Y}(g)$, $\mu_{\{2\} \cup Y}(g)$ is β (so the other is some α_i), then we may assume that

$$(\mu_{\{1\} \cup Y_1}(g), \mu_{\{2\} \cup Y_1}(g)) = (\alpha_i, \beta), \quad (\mu_{\{1\} \cup Y_2}(g), \mu_{\{2\} \cup Y_2}(g)) = (\alpha_j, \beta)$$

for two such Y_1, Y_2 . It follows that

$$\frac{\alpha_i}{\beta} = \frac{\mu_{\{1\} \cup Y_1}(g)}{\mu_{\{2\} \cup Y_1}(g)} = \frac{e_1(g)}{e_2(g)} = \frac{\mu_{\{1\} \cup Y_2}(g)}{\mu_{\{2\} \cup Y_2}(g)} = \frac{\alpha_j}{\beta},$$

showing $\alpha_i = \alpha_j$ has multiplicity at least 2 as an eigenvalue of g , a contradiction.

The same arguments applied to pairs of weights μ_Z and $\mu_{\{1,2\} \cup Z}$ for any even-size subset Z of $\{3, 4, \dots, n\}$ show that these weights can take value β at g at most twice. It follows that $A \leq 4$, a contradiction since $A^2 > D \geq 2^4$.

So, conjugating g by an element of the Weyl group that changes the sign of e_2 if necessary, we may assume that $e_1(g) = e_2(g) =: \gamma$. Then for any $3 \leq i \leq n$, $\mu_{\{1,i\}}(g) = \mu_{\{2,i\}}(g)$ is a repeated eigenvalue, so it must be β , and thus $e_3(g) = e_4(g) = \dots = e_n(g)$. It follows that $\mu_{\{3,4\}}(g) = \mu_{\{3,5\}}(g)$ is a repeated eigenvalue, so it must be equal to $\beta = \mu_{\{1,4\}}(g)$, and hence $e_3(g) = e_1(g) = \gamma$. Writing $\delta := \mu_\emptyset(g)$ we have $\beta = \mu_{\{1,3\}}(g) = \gamma^2 \delta$ by (6.4.2). Also, $\mu_{\{1,2,3,4\}}(g) = \mu_{\{1,2,3,5\}}(g) = \gamma^4 \delta$ is a repeated eigenvalue, whence $\gamma^4 \delta = \beta = \gamma^2 \delta$, yielding $\gamma^2 = 1$. Consequently, $(e_i + e_j)(g) = \gamma^2 = 1 = (e_i - e_j)(g)$ for all $1 \leq i < j \leq n$. Since $\{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}$ is the set of T -weights for the adjoint module of G° , we have shown that $g \in \mathbf{Z}(G)$, a contradiction.

(B4) Next we consider the case where G° is of type B_n with $n \geq 2$. If $V = L(\varpi_1)$, so $D = 2n + 1$, then it is self-dual and (iii) holds. Consider the case $V = \mathbb{C}^{2^n}$ is the spin representation of $G^\circ = \text{Spin}_{2n+1}$. We may assume $n \geq 3$, since V is symplectic for $G^\circ \cong \text{Sp}_4$ when $n = 2$. Choose an orthonormal basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$ and realize the set of all T -weights of V (written additively) as $\sum_{i=1}^n a_i e_i / 2$ with $a_i = \pm 1$. Write any such weight as

$$\mu = \mu_X = -\sum_{i=1}^n e_i / 2 + \sum_{i \in X} e_i \quad (6.4.3)$$

with $X \subseteq I := \{1, 2, \dots, n\}$. Since these weights come in pairs $\pm \mu$, β^{-1} is also a repeated eigenvalue of g , whence $\beta = \pm 1$. On the other hand, each α_i has multiplicity 1 and occurs as some $v(g)$, so $-v(g) \neq v(g)$, i.e. $\alpha_i \neq \pm 1$. Thus the α_i 's also come in pairs, showing $2|a$ and $2|A$. Write $\delta := \mu_\emptyset(g)$.

First suppose that $(e_i + \kappa e_j)(g) = 1$ for some $\kappa = \pm 1$ and some $i \neq j$. Conjugating g by an element of the Weyl group that changes the sign of e_2 if necessary, we may assume

that $e_1(g) = e_2(g) =: \gamma$. Then $\mu_{\{1\}}(g) = \mu_{\{2\}}(g)$ by (6.4.3), so $\beta = \mu_{\{1\}}(g) = \gamma\delta$. In this case, for any $3 \leq i \leq n$, $\mu_{\{1,i\}}(g) = \mu_{\{2,i\}}(g)$ is a repeated eigenvalue, so it must be β , and thus $e_3(g) = e_4(g) = \dots = e_n(g) = \beta/\gamma\delta = 1$. Now $\delta = \mu_{\emptyset}(g) = \mu_{\{3\}}(g)$, forcing $\delta = \beta = \gamma\delta$, and thus $\gamma = 1$. Since $\{\pm e_i, \pm e_i \pm e_j \mid 1 \leq i < j \leq n\}$ is the set of $\mathcal{T}$ -weights for the adjoint module of G° , we have shown that $g \in \mathbf{Z}(G)$, a contradiction.

We may now assume that, for any $i \neq j$, $(e_i + \kappa e_j)(g) \neq 1$ for both $\kappa = +1$ and $\kappa = -1$. Then, for each subset Y of $\{3, 4, \dots, n\}$, $\mu_{\{1\} \cup Y}(g) \neq \mu_{\{2\} \cup Y}(g)$ by (6.4.3), so at most one of these values can be β . Moreover, if there are at least three such Y such that one of $\mu_{\{1\} \cup Y}(g)$, $\mu_{\{2\} \cup Y}(g)$ is β (so the other is some α_i), then we may assume that

$$(\mu_{\{1\} \cup Y_1}(g), \mu_{\{2\} \cup Y_1}(g)) = (\alpha_i, \beta), \quad (\mu_{\{1\} \cup Y_2}(g), \mu_{\{2\} \cup Y_2}(g)) = (\alpha_j, \beta)$$

for two such Y_1, Y_2 . It follows that

$$\frac{\alpha_i}{\beta} = \frac{\mu_{\{1\} \cup Y_1}(g)}{\mu_{\{2\} \cup Y_1}(g)} = \frac{e_1(g)}{e_2(g)} = \frac{\mu_{\{1\} \cup Y_2}(g)}{\mu_{\{2\} \cup Y_2}(g)} = \frac{\alpha_j}{\beta},$$

showing $\alpha_i = \alpha_j$ has multiplicity at least 2 as an eigenvalue of g , a contradiction.

The same arguments applied to pairs of weights μ_Z and $\mu_{\{1,2\} \cup Z}$ for any subset Z of $\{3, 4, \dots, n\}$ show that these weights can take value β at g at most twice. It follows that $A \leq 4$. Since $A^2 > D \geq 2^n$, $2|A$, and $n \geq 3$, we must have that $A = 4$ and $n = 3$. Hence we may assume that $\mu_{\{1,3\}}(g) = \beta = \mu_X$ for X or $I \setminus X$ being one of $\emptyset$ and $\{3\}$. Now $1 \neq e_1(g)e_3(g) = \beta\delta^{-1}$, i.e. $\beta \neq \delta = \mu_{\emptyset}(g)$. So $\mu_{\{3\}}(g) = \beta = \mu_{\{2\}}(g)$, which implies that $e_3(g) = e_2(g)$, again a contradiction.

(B5) Here we consider the case where $V = L(\varpi_1) = \mathbb{C}^{27}$ for $G^\circ = E_6$. It is convenient to use the following realization of the weight system of type E_6 given in [25, Table 1]. Namely, in the Euclidean space $\mathbb{R}^6$ one can find seven vectors $f, e_1, \dots, e_6$ with $(e_i, e_j) = \delta_{i,j} - 1/6$, $(f, e_i) = 0$, $(f, f) = 1/2$, and $\sum_{i=1}^6 e_i = 0$, so that the set of 27 $\mathcal{T}$ -weights of V is

$$\{e_i \pm f, e_{ij} := -e_i - e_j \mid 1 \leq i < j \leq 6\}.$$

Suppose first that $f(g)^2 = 1$. Then, for each $1 \leq i \leq 6$, $(e_i + f)(g) = (e_i - f)(g)$ is a repeated eigenvalue for g , so it must be β . It follows that $e_i(g) = f(g)\beta$ for all i . But then $(-e_i - e_j)(g) = \beta^{-2}$ is also a repeated eigenvalue for all $i \neq j$, whence $\beta = \beta^{-2}$ and g acts on V as $\beta \cdot \text{Id}$, a contradiction.

We have shown that $f(g)^2 \neq 1$. Then, for each $1 \leq i \leq 6$, $(e_i + f)(g) \neq (e_i - f)(g)$, so at most one of these values can be β . Moreover, if there are at least three such i such that one of $(e_i \pm f)(g)$ is β (so the other is some α_i), then we may assume that

$$((e_1 + f)(g), (e_1 - f)(g)) = (\alpha_i, \beta), \quad ((e_2 + f)(g), (e_2 - f)(g)) = (\alpha_j, \beta).$$

It follows that

$$\frac{\alpha_i}{\beta} = \frac{(e_1 + f)(g)}{(e_1 - f)(g)} = f(g)^2 = \frac{(e_2 + f)(g)}{(e_2 - f)(g)} = \frac{\alpha_j}{\beta},$$

showing $\alpha_i = \alpha_j$ has multiplicity at least 2 as an eigenvalue of g , a contradiction.

Thus at most 2 among the weights $e_i \pm f$, $1 \leq i \leq 6$, can take value β at g . Since β has multiplicity A with $A^2 > D = 27$, at least 4 among the weights $-e_i - e_j$ (with $1 \leq i < j \leq 6$) take value β at g . Each such weight corresponds to a pair $\{i, j\}$, so at least two among those pairs must intersect. So we may assume that $(-e_1 - e_6)(g) = \beta = (-e_2 - e_6)(g)$, and hence $e_1(g) = e_2(g) =: \gamma$. In this case, for each $3 \leq j \leq 6$, $(-e_1 - e_j)(g) = (-e_2 - e_j)(g)$ is repeated, whence $(-e_1 - e_j)(g) = \beta$ and thus $e_3(g) = e_4(g) = e_5(g) = e_6(g) =: \delta$ and $\beta = 1/\gamma\delta$. Also, $1/\delta^2 = (-e_3 - e_4)(g) = (-e_3 - e_5)(g)$ is repeated, so $1/\delta^2 = \beta = 1/\gamma\delta$,

implying $\gamma = \delta$. Thus the 15 eigenvalues $(-e_i - e_j)(g)$ are all equal to $\beta = 1/\gamma^2$. Next, the 6 eigenvalues $(e_i + f)(g)$ are the same, so they must be β . Similarly, the 6 eigenvalues $(e_i - f)(g)$ are the same, so they must also be equal to β . Thus g acts on V as $\beta \cdot \text{Id}$, a contradiction.

(B6) Consider the case where $V = L(\varpi_1) = \mathbb{C}^7$ for $G^\circ = G_2$. We use the following realization of the weight system of type G_2 given in [25, Table 1]. In the Euclidean space $\mathbb{R}^2$ one can find vectors $f, e_1, \dots, e_3$ with $(e_i, e_j) = \delta_{i,j} - 1/3$ and $\sum_{i=1}^3 e_i = 0$, so that the set of 7 T -weights of V is $\{0, \pm e_i \mid 1 \leq i \leq 3\}$. Since this set is closed under the multiplication by -1 , β^{-1} is also an eigenvalue of g with multiplicity $A \geq 3$ (as $A^2 > D = 7$), and so $\beta = \pm 1$. Using $A \geq 3$, we may assume $\kappa e_1(g) = \beta$ for some $\kappa = \pm 1$, which then implies that both $\pm e_1(g)$ are equal to β . Now if $\kappa' e_2(g) = \beta$ for some $\kappa' = \pm 1$, then again both $\pm e_2(g)$ are equal to β . In this case, $e_3(g) = (-e_1 - e_2)(g) = \beta^{-2} = 1$, and hence $(-e_3)(g) = 1$. Thus 1 is a repeated eigenvalue, so $\beta = 1$, and $g|_V = \text{Id}$, a contradiction. So we have $\pm e_i(g) \neq \beta$ for $i = 2, 3$. But $A \geq 3$, hence $\beta = 1$ and $e_1(g) = 1$. It follows that $e_3(g) = (-e_1 - e_2)(g) = (-e_2)(g)$ is a repeated eigenvalue, which is however not equal to β , again a contradiction.

(B7) Now we consider the case where G° is of type C_n with $n \geq 3$. If $V = L(\varpi_1)$, then it is self-dual of symplectic type, a contradiction.

The remaining case of $V = L(\varpi_3) = \mathbb{C}^{14}$ for $G^\circ = \text{Sp}_6$ is also ruled out because V is symplectic [23, Appendices A.2, A.3]. For a potential application, we give another argument without using the type of autoduality. Embed g in a maximal torus T of G° . Then the set of T -weights on V is $\{\pm e_i, \pm e_1 \pm e_2 \pm e_3 \mid 1 \leq i \leq 3\}$, where (e_1, e_2, e_3) is an orthonormal basis of $\mathbb{R}^3$. Since this set is closed under the multiplication by -1 , β^{-1} is also an eigenvalue of g with multiplicity $A \geq 4$ (as $A^2 > D = 14$, and so $\beta = \pm 1$).

Suppose first that $2e_i(g) \neq 1$ for all i . Then $\pm e_i(g) \neq \beta$ for all i . But $A \geq 4$, so without loss we may assume $\pm(e_1 + e_2 + e_3)(g) = \beta = (e_1 + e_2 - e_3)(g)$. But in this case $2e_3(g) = 1$, a contradiction. Hence we may assume that $e_1(g) = (-e_1)(g)$, which then implies that both $\pm e_1(g)$ are equal to β . But then, for any choice of $\kappa_2, \kappa_3 = \pm 1$, $(e_1 + \kappa_2 e_2 + \kappa_3 e_3)(g) = (-e_1 + \kappa_2 e_2 + \kappa_3 e_3)(g)$, which implies that they are both equal to β . We now have $(e_1 + e_2 + e_3)(g) = \beta = (e_1 - e_2 + e_3)(g)$, and hence $2e_2(g) = 1$. Thus $e_2(g) = (-e_2)(g)$, implying that the common value is β . Similarly $e_3(g) = (-e_3)(g) = \beta$. Thus $g|_V = \beta \cdot \text{Id}$, a contradiction.

(B8) Finally, the case where $V = L(\varpi_1) = \mathbb{C}^{56}$ for $G^\circ = E_7$ is also ruled out because V is symplectic [23]. Again, we give another argument without using the type of autoduality. Embed g in a maximal torus T of G° . Consider a 3-dimensional $\mathbb{F}_2$ -vector space U with basis (a, b, c) , and an orthogonal basis $(e_u \mid u \in U)$ of the Euclidean space $\mathbb{R}^8$ with $(e_u, e_u) = 1/2$. Then the root system of type E_8 can be realized as

$$\{\pm 2e_u, \pm e_x \pm e_y \pm e_z \pm e_t \mid x, y, z, t \in U, x + y + z + t = 0, x, y, z, t \text{ pairwise distinct}\}.$$

Imposing the condition $u \neq 0$ on the roots, one obtains the root system of type E_7 , and the set of 56 T -weights of V is given by

$$\{\pm e_x \pm e_y \pm e_z \mid x, y, z \in U, x + y + z = 0, x, y, z \text{ pairwise distinct}\}. \quad (6.4.4)$$

For brevity, we will label elements of U by $\{\bar{0}, \bar{1}, \dots, \bar{7}\}$ as follows:

$$0 \mapsto \bar{0}, a \mapsto \bar{1}, b \mapsto \bar{2}, a + b \mapsto \bar{3}, c \mapsto \bar{4}, a + c \mapsto \bar{5}, b + c \mapsto \bar{6}, a + b + c \mapsto \bar{7}.$$

Then the seven (unordered) triples (x, y, z) occurring in (6.4.4) are

$$(\bar{1}, \bar{2}, \bar{3}), (\bar{1}, \bar{4}, \bar{5}), (\bar{1}, \bar{6}, \bar{7}), (\bar{2}, \bar{4}, \bar{6}), (\bar{2}, \bar{5}, \bar{7}), (\bar{3}, \bar{5}, \bar{6}), (\bar{3}, \bar{4}, \bar{7}). \quad (6.4.5)$$

First suppose that $2e_i(g) \neq 1$ for all $1 \leq i \leq 7$. Then, for each triple (x, y, z) in (6.4.5) and for each choice of $\kappa_y, \kappa_z = \pm 1$, $(-e_x + \kappa_y e_y + \kappa_z e_z)(g) \neq (e_x + \kappa_y e_y + \kappa_z e_z)(g)$, so at most one of these values can be β . Moreover, if

$$(\kappa_x e_x + \kappa_y e_y + \kappa_z e_z)(g) = \beta = (\kappa_x e_x + \kappa'_u e_u + \kappa'_v e_v)(g)$$

for two different such 6-tuples $(x, y, z, \kappa_x, \kappa_y, \kappa_z)$ and $(x, u, v, \kappa'_u, \kappa'_v)$, then we have

$$(-\kappa_x e_x + \kappa_y e_y + \kappa_z e_z)(g) = \alpha_i, \quad (-\kappa_x e_x + \kappa'_u e_u + \kappa'_v e_v)(g) = \alpha_j.$$

It follows that

$$\frac{\alpha_i}{\beta} = \frac{(-\kappa_x e_x + \kappa_y e_y + \kappa_z e_z)(g)}{(\kappa_x e_x + \kappa_y e_y + \kappa_z e_z)(g)} = \frac{(-2\kappa_x e_x)(g)}{(\kappa_x e_x + \kappa_y e_y + \kappa_z e_z)(g)} = \frac{(-\kappa_x e_x + \kappa'_u e_u + \kappa'_v e_v)(g)}{(\kappa_x e_x + \kappa'_u e_u + \kappa'_v e_v)(g)} = \frac{\alpha_j}{\beta},$$

showing $\alpha_i = \alpha_j$ has multiplicity at least 2 as an eigenvalue of g , a contradiction.

Since $A^2 > 56$, i.e. $A > 7$, we may assume that β occurs at least twice among the 8 values $(\pm e_1 + \pm e_2 + \pm e_3)(g)$. Changing the sign of e_x if necessary, we may assume that

$$(e_1 + e_2 + e_3)(g) = \beta. \quad (6.4.6)$$

The preceding observation implies that the other occurrence of β among $(\pm e_1 + \pm e_2 + \pm e_3)(g)$ can only be attained as

$$(-e_1 - e_2 - e_3)(g) = \beta. \quad (6.4.7)$$

Since $A > 2$, we can find a triple $\{x, y, z\} \neq \{\bar{1}, \bar{2}, \bar{3}\}$ and some $\kappa_x, \kappa_y, \kappa_z = \pm 1$ such that

$$(\kappa_x e_x + \kappa_y e_y + \kappa_z e_z)(g) = \beta.$$

Inspecting (6.4.5), we see that $\{x, y, z\}$ intersects $\{\bar{1}, \bar{2}, \bar{3}\}$ at exactly one point, which, by the symmetry of $\bar{1}, \bar{2}, \bar{3}$, may be assumed to be $x = \bar{1}$. Applying the same observation, we see that $\kappa_x \neq 1$ by (6.4.6), and $\kappa_x \neq -1$ by (6.4.7), a contradiction.

Hence we may assume that $2e_{\bar{1}}(g) = 1$. Now for each choice of $\kappa_2, \kappa_3 = \pm 1$,

$$(e_{\bar{1}} + \kappa_2 e_2 + \kappa_3 e_3)(g) = (-e_{\bar{1}} + \kappa_2 e_2 + \kappa_3 e_3)(g),$$

so must be equal to β . But then $(e_{\bar{1}} + e_2 + e_3)(g) = \beta = (e_{\bar{1}} - e_2 + e_3)(g)$, and hence $2e_2(g) = 1$. Repeating this argument, we deduce that $2e_x(g) = 1$ for all x and moreover all $(\pm e_x \pm e_y \pm e_z)(g)$ are equal to β for all triples (x, y, z) occurring in (6.4.5). Thus g acts on V as $\beta \cdot \text{Id}$, a contradiction. $\square$

Acknowledgements The second author gratefully acknowledges the support of the NSF (grant DMS-2200850) and the Joshua Barlaz Chair in Mathematics. The authors are grateful to the referee for a careful reading and for helpful comments that improved the exposition of the paper.

References

1. Aschbacher, M.: On the maximal subgroups of the finite classical groups. *Invent. Math.* **76**, 469–514 (1984)
2. Blichfeldt, H.: *Finite Collineation Groups*. Univ. of Chicago Press (1917)
3. Conway, J.H., Curtis, R.T., Norton, S.P., Parker, R.A., Wilson, R.A.: *Atlas of Finite Groups. Maximal Subgroups and Ordinary Characters for Simple Groups*. Oxford University Press, Eynsham (1985)
4. Cameron, P.J.: Finite permutation groups and finite simple groups. *Bull. Lond. Math. Soc.* **13**, 1–22 (1981)
5. Chermak, A.: Quadratic pairs without components. *J. Algebra* **258**, 442–476 (2002)
6. Chermak, A.: Quadratic pairs. *J. Algebra* **277**, 36–72 (2004)
7. The GAP group: GAP - groups, algorithms, and programming, Version 4.4 (2004). <http://www.gap-system.org>

8. Gluck, D., Isaacs, I.M.: Tensor induction of generalized characters and permutation characters. *Illinois J. Math.* **27**, 514–518 (1983)
9. Guralnick, R.M., Tiep, P.H.: Cross characteristic representations of even characteristic symplectic groups. *Trans. Amer. Math. Soc.* **356**, 4969–5023 (2004)
10. Guralnick, R.M., Tiep, P.H.: Symmetric powers and a conjecture of Kollár and Larsen. *Invent. Math.* **174**, 505–554 (2008)
11. Guralnick, R.M., Tiep, P.H.: A problem of Kollár and Larsen on finite linear groups and crepant resolutions. *J. Europ. Math. Soc.* **14**, 605–657 (2012)
12. Katz, N.: Gauss Sums, Kloosterman Sums, and Monodromy Groups. *Annals of Mathematics Studies* 116. Princeton Univ. Press, Princeton, NJ, ix+246 pp. (1988)
13. Katz, N.: Exponential Sums and Differential Equations. *Annals of Mathematics Studies* 124. Princeton Univ. Press, Princeton, NJ, xii+430 pp. (1990)
14. Katz, N.: Twisted L-functions and Monodromy. *Annals of Mathematics Studies* 150. Princeton University Press, Princeton, NJ (2002)
15. Katz, N.: Larsen's alternative, moments, and the monodromy of Lefschetz pencils. In: *Contributions to automorphic forms, geometry, and number theory*, pp. 521–560. Johns Hopkins Univ. Press, Baltimore, MD (2004)
16. Katz, N., Rojas-León, A., Tiep, P.H.: On some Airy sheaves of Laurent type. *Eur. J. Math.* **10**(4), Paper No. 65, 34 pp. (2024)
17. Katz, N., Tiep, P.H.: Monodromy groups of Kloosterman and hypergeometric sheaves. *Geom. Funct. Analysis* **31**, 562–662 (2021)
18. Katz, N., Tiep, P.H.: Exponential sums and total Weil representations of finite symplectic and unitary groups. *Proc. Lond. Math. Soc.* **122**, 745–807 (2021)
19. Katz, N., Tiep, P.H.: Exponential Sums, Hypergeometric Sheaves, and Monodromy Groups. *Annals Math. Studies* 220. Princeton University Press, Princeton, NJ (2025)
20. Katz, N., Tiep, P.H.: Generalized Airy sheaves: an introduction. *Vietnam J. Math.* (to appear)
21. Kleidman, P.B., Liebeck, M.W.: *The Subgroup Structure of the Finite Classical Groups*. London Math. Soc. Lecture Note Ser. no. 129, Cambridge University Press (1990)
22. Korljukov, A.V.: Finite linear groups generated by quadratic elements of order 4. *Vestsi Akad. Navuk BSSR, Ser. Fiz.-Mat. Navuk* **4**, 38–40 (1986)
23. Lübeck, F.: Small degree representations of finite Chevalley groups in defining characteristic. *LMS J. Comput. Math.* **4**, 135–169 (2001)
24. Mitchell, H.: Determination of all primitive collineation groups in more than four variables which contain homologies. *Amer. J. Math.* **36**, 1–12 (1914)
25. Onishchik, A.L., Vinberg, E.B.: *Lie Groups and Algebraic Groups*. Springer-Verlag (1990)
26. Serre, J.-P., Tate, J.: Good reduction of abelian varieties. *Ann. of Math.* **88**, 492–517 (1968)
27. Shephard, G.C., Todd, J.A.: Finite unitary reflection groups. *Can. J. Math.* **6**, 274–304 (1954)
28. Tiep, P.H., Zalesskii, A.E.: Some characterizations of the Weil representations of the symplectic and unitary groups. *J. Algebra* **192**, 130–165 (1997)
29. Wales, D.: Some projective representations of S_n . *J. Algebra* **61**, 37–57 (1979)
30. Wales, D.: Quasiprimitive linear groups with quadratic elements. *J. Algebra* **245**, 584–606 (2001)
31. Zalesskii, A.E.: Linear groups. *Russian Math. Surveys* **36**, 63–128 (1981)
32. Zalesskii, A.E.: The normalizer of the extraspecial linear group. *Vestsi Acad. Navuk Belorussian SSR, Ser. Fiz.-Mat. Navuk* **6**, 11–16 (1985). (in Russian)
33. Zalesskii, A.E.: Spectra of elements of order p in complex representations of finite Chevalley groups of characteristic p . *Vestsi Acad. Navuk Belorussian SSR, Ser. Fiz.-Mat. Navuk* **6**, 20–25 (1986). (in Russian)
34. Zsigmondy, K.: Zur Theorie der Potenzreste. *Monatsh. Math. Phys.* **3**, 265–284 (1892)

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