GENERALIZED AIRY SHEAVES: AN INTRODUCTION

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Dedicated to our colleague Hoang Xuan Phu, on the occasion of his seventieth birthday

Abstract. We develop the theory of generalized Airy sheaves, the one-parameter local systems $\mathcal{F}(f,a,\chi)$ on $\mathbb{A}^1$ in characteristic $p>0$, whose trace function (at $t\in L$ for large enough finite extensions L of the field $\mathbb{F}_p$) is

$$t\in L\mapsto -\sum_{x\in L}\psi(f(x)+tx^a)\chi(x),$$

with ψ a (nontrivial) additive character, $f(x)$ a polynomial of degree $A>a\geq 1$ with $\gcd(A,a)=1$, $p\nmid Aa$, and χ a (possibly trivial) multiplicative character. These local systems are generalizations of the Airy sheaves $\mathcal{F}(f,1,\mathbb{1})$ studied by Such [Such]. The main goal is to provide tools for the later determination of their monodromy groups in cases where $f(x)$ is not simply x^A, the x^A case having been completely treated in [KT9, Theorems 10.2.4, 10.2.6] when $a=1$ and in [KT9, Theorems 10.3.13, 10.3.21] when $a>1$.

Contents

1. Introduction

In classical analysis, Airy functions are the Fourier transforms of exponentials $e^{f(x)}$ of polynomials, (originally for the polynomial $f(x):=x^3/3$) and Airy differential equations are the linear differential equations $f'(d/dt)y+ty=0$ they satisfy. These differential equations have an irregular singularity at ∞, and have quite interesting differential galois groups. In the seminal paper [Such] of Such, he introduces their ℓ-adic finite field analogues, the local systems whose trace functions are of the form

$$t\mapsto -\sum_x \psi(f(x)+tx).$$

The local systems we are concerned with here are generalizations of these Airy local systems in two ways. We allow the "t term" tx to be replaced by tx^a, and we allow an "outside factor" $\chi(x)$ in the sum. Here is a more detailed discussion.

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We work in characteristic $p > 0$, and denote by $\overline{\mathbb{F}_p}$ an algebraic closure of $\mathbb{F}_p$. We also fix a prime $\ell \neq p$ to be able to speak of $\overline{\mathbb{Q}_\ell}$-adic cohomology. We fix integers

$$A > a \geq 1$$

about which we assume

$$\gcd(A, a) = 1, p \nmid Aa.$$

We fix a polynomial

$$f(x) \in k[x], \deg(f) = A,\ k \text{ some finite subfield of } \overline{\mathbb{F}_p},$$

and a (possibly trivial) multiplicative character χ of $k^\times$, with the convention that for $\chi \neq \mathbb{1}$, we have $\chi(0) = 0$, but $\mathbb{1}(0) = 1$. We denote by $\mathcal{F}(f, a, \chi)$ the lisse sheaf on $\mathbb{A}^1/k$ whose trace function at time $t \in L$, for L/k a finite extension, is

$$t \mapsto -\sum_{x \in L} \psi_L(f(x) + tx^a)\chi_L(x).$$

The condition $p \nmid Aa$ is natural here. Indeed, a monomial αx^{pd} inside ψ is Artin–Schreier equivalent to $\alpha^{1/p}x^d$. So with no loss of generality, we could instead require that $f(x) + tx$ is Artin–Schreier reduced: every nonzero exponent which occurs is prime to p.

We give some basic results about these local systems $\mathcal{F}(f, a, \chi)$. In the special case when $f(x) = x^A$, these local systems were completely analyzed in [KT9, 10.2.4, 10.2.6, 10.3.13, 10.3.21]. But that analysis depended on the fact that for $f(x) = x^A$, $\mathcal{F}(f, a, \chi)$ was related to a hypergeometric sheaf. No such relation seems to exist in the case of a more general $f(x)$, hence the motivation for this paper. Another motivation is this: in his paper [Such, 11.1, 11.6, 11.7], Such shows that if $\mathcal{F}(f, 1, \mathbb{1})$ is primitive, then for $d := \deg(f) - 1$, either its $G_{\rm geom}$ is finite or $G^\circ_{\rm geom}$ is one of SL_d or Sp_d (for even d), with two extra possible cases if $p = 2$. We would like to determine this $G_{\rm geom}$ in the cases when it is finite.

Recall, see e.g. [KT9, §1.1], that a local system $\mathcal{F}$ of rank D is said to satisfy ($\mathbf{S}-$) if, in the underlying representation $V = \mathbb{C}^D$, the geometric monodromy $G = G_{\mathrm{geom},\mathcal{F}}$ is irreducible, primitive, and tensor indecomposable. If in addition the G-module V is not tensor induced, and $\mathbf{Z}(G)$ is finite, then $\mathcal{F}$ is said to satisfy ($\mathbf{S}+$). Condition ($\mathbf{S}+$), which in the case of classical groups corresponds to Aschbacher's class $\mathcal{S}$ of maximal subgroups [Asch], has proved to be instrumental in the determination of geometric monodromy groups of ℓ-adic local systems, see e.g. [KRLT4], [KT5], and [KT9].

The main result of the paper, Theorem 4.10, is that, under fairly general hypotheses, the local systems $\mathcal{F}(f, a, \chi)$ satisfy ($\mathbf{S}+$), a first ingredient towards the eventual determination of their monodromy groups.

2. Basic facts

Lemma 2.1. *Let $\mathcal{F}$ be a local system on $\mathbb{A}^1/\overline{\mathbb{F}_p}$. Let $\Gamma \subset G_{\mathrm{geom},\mathcal{F}}$ be the Zariski closure of the subgroup generated by the images of all conjugates in $\pi_1(\mathbb{A}^1/\overline{\mathbb{F}_p})$ of "the" wild inertia group $P(\infty)$. Then $\Gamma = G_{\mathrm{geom},\mathcal{F}}$.*

Proof. Consider the composite homomorphism

$$\pi_1(\mathbb{A}^1/\overline{\mathbb{F}_p}) \to G_{\mathrm{geom},\mathcal{F}} \to G_{\mathrm{geom},\mathcal{F}}/\Gamma.$$

The image of $P(\infty)$ dies in $G_{\mathrm{geom},\mathcal{F}}/\Gamma$, so this homomorphism factors through $\pi_1^{\text{tame at }\infty}(\mathbb{A}^1/\overline{\mathbb{F}_p})$, which is the trivial group. But the composite homomorphism has Zariski dense image in $G_{\mathrm{geom},\mathcal{F}}/\Gamma$, so this last group is trivial. □

Proposition 2.2. *Suppose $p \nmid Aa$ and $\gcd(A, a) = 1$. Then we have have the following results.*

(i) *For* $\chi = \mathbb{1}$, $\mathcal{F}(f, a, \mathbb{1})$ *is lisse of rank* $A - 1$ *on* $\mathbb{A}^1$, *and geometrically irreducible. Its* $I(\infty)$*-representation is the direct sum*

$$[a]_\star(\mathbb{1})/\mathbb{1} \;\oplus W_{A,a}$$

with $W_{A,a}$ *a totally wild* $I(\infty)$*-representation of rank* $A - a$ *with all slopes* $A/(A - a)$.

(ii) *For* $\chi \neq \mathbb{1}$, $\mathcal{F}(f, a, \chi)$ *is lisse of rank* A *on* $\mathbb{A}^1$, *and geometrically irreducible. Its* $I(\infty)$*-representation is the direct sum*

$$[a]_\star(\overline{\chi}) \;\oplus W_{A,a},$$

with $W_{A,a}$ *a totally wild* $I(\infty)$*-representation of rank* $A - a$ *with all slopes* $A/(A - a)$.

Proof. In both cases, we view $\mathcal{F}(f, a, \chi)$ as a Fourier transform. To see this, write its trace function as

$$-\sum_x \psi(f(x) + tx^a)\chi(x) = -\sum_u \psi(tu) \sum_{x:x^a=u} \psi(f(x))\chi(x),$$

Thus $\mathcal{F}(f, a, \chi)$ is

$$\mathrm{FT}_\psi([a]_\star(\mathcal{L}_{\psi(f(x))} \otimes \mathcal{L}_{\chi(x)})).$$

The input has ∞-slope $A/a > 1$, so its FT is lisse on A^1, cf. [Ka-ESDE, §7.8, Class (1)]. To show that $\mathcal{F}(f, a, \chi)$ is geometrically irreducible, it suffices to show that the input is geometrically irreducible. By Frobenius reciprocity, with

$$\mathcal{L} := \mathcal{L}_{\psi(f(x))} \otimes \mathcal{L}_{\overline{\chi}(x)},$$

we have

$$< [a]_\star\mathcal{L}, [a]_\star\mathcal{L} > = < \mathcal{L}, [a]^\star[a]_\star\mathcal{L} > = [\mathcal{L}, \oplus_{\zeta\in\mu_a}\mathrm{MultTrans}_\zeta(\mathcal{L}) > .$$

So we must show that for $\zeta \in \mu_a, \zeta \neq 1$, $\mathcal{L}_{\psi(f(\zeta x))} \otimes \mathcal{L}_{\overline{\chi}(\zeta x)}$ is not isomorphic to $\mathcal{L}_{\psi(f(x))} \otimes \mathcal{L}_{\overline{\chi}(x)}$, or equivalently (remembering that $\mathcal{L}_\chi$ is geometrically translation invariant) that $\mathcal{L}_{\psi(f(\zeta x) - f(x))}$ is not geometrically trivial. But this is clear, because $f(\zeta x) - f(x) = (\zeta^A - 1)x^A$ + lower terms, and $(\zeta^A - 1) \neq 0$ because $\zeta \neq 1$ in μ_a and $\gcd(A, a) = 1$. Thus $\mathcal{L}_{\psi(f(\zeta x) - f(x))}$ has $\mathsf{Swan}_\infty = A$, so is not geometrically trivial. The shape of the $I(\infty)$-representation of $\mathcal{F}(f, a, \chi)$ is a straightforward application of Laumon's results [Lau] on the local monodromy of FT_ψ. ☐

3. Tensor decomposability and tensor induction

Lemma 3.1. *Suppose* $p \nmid Aa$ *and* $\gcd(A, a) = 1$. *Then the* $I(\infty)$*-representation of the local system* $\mathcal{F}(f, a, \chi)$ *on* $\mathbb{A}^1/\overline{\mathbb{F}_p}$ *is tensor indecomposable, and hence* $\mathcal{F}(f, a, \chi)$ *is itself tensor indecomposable, under each of the following conditions.*

(i) $\chi \neq \mathbb{1}$.

(ii) $\chi = \mathbb{1}$, $a > 1$, *and* $(A, a) \neq (5, 3)$ *if* $p \neq 2$.

Proof. Indeed, under these hypotheses, the $I(\infty)$-representation is tensor indecomposable, by [KRLT3, Corollary 10.4].

Consider first the case of $\chi \neq \mathbb{1}$. Then the rank is A and the tame rank is a. We need either $A \neq 4$, or $A = 4$, p odd and $a \neq 2$ (automatic by gcd) or $A = 4$, $p = 2$, and $a = 1$ (not allowed because $p \nmid A$).

Now consider the case $\chi = \mathbb{1}, a > 1$. Then the rank is $A - 1$ and the tame rank is $a - 1$. Then we need either $A \neq 5$, or $A = 5$, p odd and $a \neq 3$, or $A = 5$, $p = 2$, and $a \neq 2$. This last possible bad case $(A, a) = (5, 2)$ with $p = 2$ is ruled out by $p \nmid a$. ☐

Lemma 3.2. *Suppose* $p \nmid Aa$ *and* $\gcd(A, a) = 1$. *Suppose further that* $\chi = \mathbb{1}$, $a = 1$, *and* $p^2 \nmid (A - 1)$. *Then the* $I(\infty)$*-representation* $W_{A,1}$ *of the local system* $\mathcal{F}(f, 1, \mathbb{1})$ *on* $\mathbb{A}^1/\overline{\mathbb{F}_p}$ *is tensor indecomposable, and hence* $\mathcal{F}(f, 1, \mathbb{1})$ *is itself tensor indecomposable.*

Proof. The argument is inspired by [Such, first paragraph of the proof of Prop. 11.1]. Suppose $W_{A,1}$ is tensor decomposable, say with factors of dimensions d, e, each ≥ 2 with $de = A-1$. To fix ideas, suppose $p \nmid d$. Then by [KT5, Prop. 2.2], we can write $W_{A,1} \cong \mathcal{A} \otimes \mathcal{B}$ and (because $p \nmid d$) each of $\mathcal{A}, \mathcal{B}$ has all slopes $\leq A/(A-1) = 1 + 1/(A-1)$. Each of $\mathcal{A}, \mathcal{B}$ must be $I(\infty)$-irreducible (because their tensor product is), so each has a single slope, call it λ_d, λ_e. By the integrality of Swan conductors, each of $d\lambda_d$ and $e\lambda_e$ is an integer. But $d\lambda_d \leq d(1 + 1/(A-1)) = d + d/(A-1) < d+1$, so $d\lambda_d \leq d$, and hence $\mathcal{A}$ has all slopes ≤ 1. Similarly, $\mathcal{B}$ has all slopes ≤ 1. But then $\mathcal{A} \otimes \mathcal{B}$ has all slopes ≤ 1, contradiction. ☐

Remark 3.3. For $q = p^f$ with $f \geq 2$, the Pink–Sawin local system $\mathcal{F}(x^{q+1}, 1, \mathbb{1})$ has $G_{\rm geom}$ equal to the extraspecial group p_+^{1+2f} for p odd, and to 2_-^{1+2f} for $p = 2$, cf. [KT9, Theorem 7.3.8]. In both cases, $G_{\rm geom}$ is the image of $I(\infty)$. Its q-dimensional representation is the irreducible representation of dimension $q = p^f$ specified by the choice of ψ. One knows that for a given nontrivial additive character ψ of $\mathbb{F}_p$, this representation is the f-fold tensor product of the p-dimensional representations of the extraspecial groups p_+^{1+2} for p odd, and to the tensor product of the 2-dimensional representation of 2_-^{1+2} with the $(f-1)$-fold tensor product of the extraspecial groups 2_+^{1+2}. Thus $\mathcal{F}(x^{q+1}, 1, \mathbb{1})$ is tensor decomposable both for $I(\infty)$ and for $G_{\rm geom}$.

Lemma 3.4. *Let ρ be a finite dimensional, continuous $\overline{\mathbb{Q}_\ell}$-representation of the decomposition group $D(\infty)$. If ρ is irreducible on $D(\infty)$, then $\rho(I(\infty))$ is a finite group.*

Proof. In general, given a finite dimensional, continuous $\overline{\mathbb{Q}_\ell}$-representation ρ of $D(\infty)$, one knows (Grothendieck's local monodromy theorem) that the set

$$\Gamma := \{\alpha \in I_\infty \text{ such that } \rho(\alpha) \text{ is unipotent}\}$$

is an open subgroup of I_∞, cf. [Ka-GKM, 7.0.5]. Whenever ρ is irreducible as $D(\infty)$-representation (which is automatic if $\rho|_{I_\infty}$ is irreducible), the monodromy filtration, cf. [Ka-ESDE, 7.0.6] can have only a single nonzero gr_i^W. Therefore if ρ is irreducible as $D(\infty)$-representation, any element $\alpha \in \Gamma$ must have $\rho(\alpha) = \mathrm{Id}$, which is to say that $\Gamma = \mathrm{Ker}(\rho|I_\infty)$ is an open subgroup of I_∞, i.e., I_∞ has finite image. ☐

Corollary 3.5. *Suppose V is a finite dimensional, continuous $\overline{\mathbb{Q}_\ell}$-representation ρ of $D(\infty)$ which is a direct sum of irreducible $D(\infty)$-representations V_i. Then $\rho(I(\infty))$ is a finite group.*

Corollary 3.6. *Suppose that $p \nmid Aa$ and $\gcd(A, a) = 1$. In the $D(\infty)$-representation of $\mathcal{F}(f, a, \chi)$, $I(\infty)$ has finite image.*

Proof. If $a = 1$ and $\chi = \mathbb{1}$, the $D(\infty)$-representation is the irreducible $W_{A,1}$. If either $a > 1$ or $\chi \neq \mathbb{1}$, the the $D(\infty)$-representation is direct sum of the irreducible $W_{A,a}$ with (after a possible extension of finite ground field) the direct sum of one-dimensional tame pieces. ☐

Recall [KT9, Definition 1.1.5(b)] that an element in $\mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ is said to have *almost simple spectrum*, or to be an `asp`-*element*, if it is diagonalizable and has at least $n-1$ distinct eigenvalues.

Theorem 3.7. *Suppose that $p \nmid Aa$ and $\gcd(A, a) = 1$. Then in the $I(\infty)$-representation of $\mathcal{F}(f, a, \chi)$, the image of any element $\delta \in I(\infty)$ has finite order. Assume in addition that $p \nmid (A-a)$, and consider the action of an element $\gamma \in I(\infty)$ which modulo $P(\infty)$ is a topological generator of $I(\infty)/P(\infty)$. Then we have the following results.*

(i) *Suppose $\chi = \mathbb{1}$ and $a = 1$. Then γ has simple spectrum, i.e., it has $A-1$ distinct eigenvalues.*
(ii) *Suppose that either $\chi = \mathbb{1}$ and $a \geq 2$, or that $\chi \neq \mathbb{1}$. Then γ is an* `asp`*-element.*

Proof. The first statement is just Corollary 3.6.

Assume $p \nmid (A-a)$. Then the wild part $W_{A,a}$, which is irreducible of rank $A-a$, is the Kummer direct image $[A-a]_\star \mathcal{L}$ of a rank one $\mathcal{L}$ with $\mathsf{Swan}_\infty(\mathcal{L}) = A$, cf. [Ka-GKM, 1.14]. Then the Kummer pullback

$[A-a]^\star(W_{A,a})$ is the direct sum of the multiplicative translates of $\mathcal{L}$ by the $A-a$ elements of $\mu_{A-a}(\overline{\mathbb{F}_p})$, and γ, acting by conjugation, cyclically permutes these $A-a$ summands. Thus $\gamma^{A-a}|_{W_{A,a}}$ acts as a scalar, call it $\lambda \in \overline{\mathbb{Q}_\ell}^\times$, and the eigenvalues of γ on $W_{A,a}$ are the $(A-a)^{\mathrm{th}}$ roots of λ. In particular, $\gamma|_{W_{A,a}}$ has all distinct eigenvalues. By (i), the scalar λ is a root of unity, (being an eigenvalue of γ^{A-a}), and hence γ has finite order, first on $W_{A,a}$ and then (trivially) on Tame, on which it has order dividing $a \times \mathrm{order}(\chi)$.

If $\chi = \mathbb{1}$ and $a = 1$, then $W_{A,1}$ is the entire $I(\infty)$-representation of $\mathcal{F}(A,1,\mathbb{1})$. This proves (i).

If either $a \geq 2$ or if $\chi \neq \mathbb{1}$, then the tame part Tame of the $I(\infty)$-representation of $\mathcal{F}(A,1,\mathbb{1})$ consists either of all nontrivial characters of order dividing a (the case when $\chi = \mathbb{1}$), or it consists of all a^{th} roots of $\overline{\chi}$. In either case, $\gamma|_{\mathsf{Tame}}$ has all distinct eigenvalues and has finite order dividing $a \times \mathrm{order}(\chi)$. So the action of γ on the entire $I(\infty)$-representation of $\mathcal{F}(A,a,\chi)$ has all eigenvalues simple with the possible exception of those which occur in both Tame and in $W_{A,a}$, which each occur twice.

Now consider an eigenvalue α which occurs in both Tame and in $W_{A,a}$. On the one hand, $\alpha^{A-a} = \lambda$. On the other hand, $\alpha^a = \overline{\chi}(\gamma)$. There is at most one such eigenvalue: indeed, if β were a second, then α/β is both an $(A-a)^{\mathrm{th}}$ root of unity and an a^{th} root of unity. But $\gcd(A,a) = \gcd(A-a,a) = 1$, hence $\beta = \alpha$. □

Proposition 3.8. *Let $\mathcal{F}$ be a local system on $\mathbb{A}^1/\overline{\mathbb{F}_p}$ such that the underlying representation V is tensor indecomposable over $I(\infty)$. Assume furthermore that any $\gamma \in I(\infty)$ which is a topological generator of $I(\infty)$ modulo $P(\infty)$ acts as an* asp*-element of finite order on V. Suppose $G_{\mathrm{geom},\mathcal{F}}$ preserves an n-tensor induced decomposition $V = V_1 \otimes V_2 \otimes \ldots \otimes V_n$ with $d := \dim(V_i) > 1$ and $n > 1$. Then the following statements hold for the permutation π induced by the action of γ on the n tensor factors $V_1, \ldots, V_n$.*

(i) *The permutation π is nontrivial for at least one representative in the $P(\infty)$-coset of γ.*
(ii) *Either π is trivial, or π is a 2-cycle, or $d = 2$ and π is a 3-cycle or a disjoint product of a 2-cycle and a 3-cycle.*

Proof. For (i), suppose that all elements in the $P(\infty)$-coset of γ act trivially on the set of n tensor factors. Then every element in $P(\infty)$ also acts trivially on this set. This implies by Lemma 2.1 that $G_{\mathrm{geom},\mathcal{F}}$ acts trivially on the set $\{V_1, \ldots, V_n\}$; in particular, $I(\infty)$ acts trivially. But this also means that the $I(\infty)$-representation V is tensor decomposable, a contradiction.

For (ii), we may assume that $\pi \neq \mathrm{id}$. Now we can apply [KT9, Proposition 5.2.3]. □

Next we will prove some auxiliary results on permutation groups.

Lemma 3.9. *Let p be a prime, and let $J = P \rtimes C$ be a transitive subgroup of S_n with $n > 1$ such that P is a nontrivial normal p-subgroup and $C = \langle \gamma \rangle$ a cyclic p'-group. Suppose that every element in the coset γP is either trivial or a 3-cycle. Then one of the following statements holds.*

(i) *$p = n = |J| = |P| = 3$.*
(ii) *$n = 4$, $p = 2$, $P = C_2^2$, and $J = \mathsf{A}_4$.*

Proof. Let ρ denote the corresponding permutation character of J. Then the transitivity of J means that

$$\sum_{x \in J} \rho(x) = |J|. \tag{3.9.1}$$

First consider the case $J = P$. Then every nontrivial element x in P is a 3-cycle, whence $p = 3$, and $\rho(x) = n - 3$. It then follows from (3.9.1) that

$$|P| = \sum_{x \in P} \rho(x) = n + (|P| - 1)(n-3) = 3 + |P|(n-3),$$

i.e. $|P|(n-4) = -3$. As $|P| \geq 3$, we conclude that $n = 3$ and hence $|P| = 3$, as stated in (i).

Next suppose that $J > P$, i.e. $\gamma \notin P$. Then every element of γP, including γ, is a 3-cycle. In particular, $p \neq 3$, $n \geq 3$, and $|J| = 3|P|$. Also, $\rho(x) = n-3$ for all $x \in \gamma P$. The same argument applied to γ^{-1} shows that $\rho(x) = n-3$ for all $x \in \gamma^{-1}P$. On the other hand, $\sum_{y \in P} \rho(y)$ is $|P|$ times the number of P-orbits on $\{1, 2, \ldots, n\}$, and so it is at least $|P|$. It then follows from (3.9.1) that

$$3|P| = |J| = \sum_{x \in J} \rho(x) = \sum_{x \in \gamma P} \rho(x) + \sum_{x \in \gamma^{-1}P} \rho(x) + \sum_{x \in P} \rho(x) \geq (2n-5)|P|,$$

whence $2n \leq 8$ and $n \leq 4$. Now if $n = 3$, then $J = \mathsf{S}_3$ (as $P \neq 1$), but then $P = C_2$ cannot be normal in J. So $n = 4$, and $p = 2$ as $|P$ divides $|\mathsf{S}_4|/3 = 8$. Since the subgroup P of a Sylow 2-subgroup of S_4, which is dihedral of order 8, is normalized by a 3-cycle, we conclude that $P \cong C_2^2$ and so $J = \mathsf{A}_4$, as stated in (ii). ☐

Lemma 3.10. *Let $J = P \rtimes C$ be a transitive subgroup of S_n with $n > 1$ such that P is a nontrivial normal 2-subgroup and $C = \langle \gamma \rangle$ a cyclic group of odd order. Suppose that every element in the coset γP is either trivial, a 2-cycle, a 3-cycle, or a disjoint product of a 2-cycle and a 3-cycle. Then one of the following statements holds.*

(i) *$n = |J| = |P| = 2$.*
(ii) *$n = 4$, $P = C_2^2$, and $J = \mathsf{A}_4$.*

Proof. Again let ρ denote the corresponding permutation character of J, so that (3.9.1) holds.

First consider the case $J = P$. Then every nontrivial element $x \in P$, being a 2-element, must be a 2-cycle, and so $\rho(x) = n-2$. It then follows from (3.9.1) that

$$|P| = \sum_{x \in P} \rho(x) = n + (|P|-1)(n-2) = 2 + |P|(n-2),$$

i.e. $|P|(n-3) = -2$. As $|P| \geq 2$, we conclude that $n = 2$ and hence $|P| = 3$, as stated in (i).

From now on we may assume $J > P$. As $J/P \cong C = \langle \gamma \rangle$ is cyclic of odd order and $\gamma^6 = 1$ by assumption, we must have that $|J| = 3|P|$ and so $n \geq 3$. Now $\rho(x) \geq n-5$ for all $x \in \gamma P$. The same argument applied to γ^{-1} shows that $\rho(x) \geq n-5$ for all $x \in \gamma^{-1}P$. As in the proof of Lemma 3.9, we have $\sum_{y \in P} \rho(y) \geq |P|$. It then follows from (3.9.1) that

$$3|P| = |J| = \sum_{x \in J} \rho(x) = \sum_{x \in \gamma P} \rho(x) + \sum_{x \in \gamma^{-1}P} \rho(x) + \sum_{x \in P} \rho(x) \geq (2n-9)|P|, \tag{3.10.1}$$

whence $2n \leq 12$ and $n \leq 6$. Now if $n = 3$, then $J = \mathsf{S}_3$ (as $P \neq 1$), but then $P = C_2$ cannot be normal in J. If $n = 4$, then γ is a 3-cycle, and the subgroup P of a Sylow 2-subgroup of S_4, which is dihedral of order 8, is normalized by the 3-cycle γ, whence $P \cong C_2^2$ and so $J = \mathsf{A}_4$, as stated in (ii).

The case $n = 5$ is impossible since n divides $|J|$, 3 times a 2-power.

In the remaining case $n = 6$, we must have equalities throughout (3.10.1), and so $\sum_{x \in P} \rho(x) = |P|$, which means that P is transitive on 6 letters. But this is impossible since P is a 2-group. ☐

Theorem 3.11. *Let $\mathcal{F}$ be a local system on $\mathbb{A}^1/\overline{\mathbb{F}_p}$ such that the underlying representation V is tensor indecomposable over $I(\infty)$. Assume furthermore that any element $\gamma \in I(\infty)$ which is a topological generator of $I(\infty)$ modulo $P(\infty)$ acts as an* asp*-element of finite order on V. Choose such a γ that has p'-order on V* (cf. [Ka-GKM, p. 21] for the existence of such choices). *Suppose $G_{\mathrm{geom},\mathcal{F}}$ preserves an n-tensor induced decomposition $V = V_1 \otimes V_2 \otimes \ldots \otimes V_n$ with $d := \dim(V_i) > 1$ and $n > 1$. Then one of the following statements holds, where π is the permutation induced by γ while acting on $\{V_1, \ldots, V_n\}$.*

(i) *$p = n = 3$ and $d = 2$. Moreover, π is trivial.*
(ii) *$p = n = 2$. Moreover, π is trivial.*
(iii) *$p = 2$, $d = 2$, and $n = 4$. Moreover, π is a 3-cycle.*

Proof. Let $\Phi : G_{\mathrm{geom},\mathcal{F}} \to \mathsf{S}_n$ be induced by the action of $G_{\mathrm{geom},\mathcal{F}}$ on $\{V_1, \ldots, V_n\}$. Then $J := \Phi(I(\infty))$ is a transitive subgroup of S_n since V is tensor indecomposable. Moreover, $J = P \rtimes C$, where $P := \Phi(P(\infty))$ is a finite p-group and $C = \langle \pi \rangle$ is a cyclic p'-group. Now we can Proposition 3.8 to J. Note that $P \neq 1$, as otherwise $\Phi(G_{\mathrm{geom},\mathcal{F}}) = 1$ by Lemma 2.1, and so J cannot be transitive.

First we consider the case $p > 2$. Then all elements in P are of odd order, hence contained in A_n. By Lemma 2.1, the image of Φ is also contained in A_n. Thus, for any $\gamma' \in \gamma P(\infty)$, $\Phi(\gamma')$ is an even permutation, so either trivial or a 3-cycle by Proposition 3.8. It then follows from Lemma 3.9 that $p = n = |J| = |P| = 3$. By Proposition 3.8(i), we can choose $\gamma' \in \gamma P(\infty)$ such that $\Phi(\gamma') \neq 1$, hence a 3-cycle. This implies by Proposition 3.8(ii) that $d = 2$. Now, as $p = |J| = 3$ and π is of p'-order, we conclude that π is trivial, as stated in (i).

Next assume that $p = 2$. Then we can apply Lemma 3.10 to see that either $n = 2 = |J| = |P|$, or $n = 4$ and $J = \mathsf{A}_4$. In the former case, the $2'$-element π must be trivial, and we arrive at (ii). In the latter case, $\pi \neq 1$ as $J > P$, hence it is a 3-cycle as is every element in $\mathsf{A}_4 \smallsetminus C_2^2$. This implies by Proposition 3.8(ii) that $d = 2$, and we arrive at (iii). □

Proposition 3.12. *Under the notation and assumption of Theorem 3.7, suppose the action C of the element γ in the underlying representation V of $\mathcal{F}(f, a, \chi)$ can be written as the tensor product $C' \otimes C''$ of two matrices of sizes $m, n \geq 2$ with $mn > 4$. Then at least one of the following cases occurs.*

(i) $a = A - 1$.
(ii) $a = 1$.
(iii) $a = 2$ *and* $\chi = \mathbb{1}$.

Proof. (a) Assume for instance that $n \geq 3$. List the eigenvalues of C' and C'', counting multiplicities, as $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_n$. If $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, then $\alpha_1\beta_1$ is an eigenvalue of C with multiplicity ≥ 4. If If $\alpha_1 = \alpha_2$ and $\beta_1 \neq \beta_2$, then each of $\alpha_1\beta_1$ and $\alpha_1\beta_2$ is an eigenvalue of C with multiplicity ≥ 2. Each of these cases is impossible since C is asp. So C' and C'' both have simple spectrum, and we can list eigenvalues of C, counting multiplicities, as $\alpha_i\beta_j$, $1 \leq i \leq m$, $1 \leq j \leq n$. As C is invertible, $\alpha_i, \beta_j \neq 0$.

According to the proof of Theorem 3.7, there are $\lambda, \mu \in \mathbb{C}^\times$ such that we can represent the set S of distinct eigenvalues of C as $X \cup Y$, where

$$X := \{\delta \in S \mid \delta^{A-a} = \lambda\},\ Y := \{\delta \in S \mid \delta^a = \mu\},$$

and $|X \cap Y| \leq 1$. More concretely, X is the spectrum of γ on the wild part, and Y is the spectrum of γ on the tame part for $P(\infty)$.

(b) Consider the three pairwise distinct elements $\alpha_1\beta_j$, $1 \leq j \leq 3$, of S. Then we may assume that (at least) two of them belong to X, or to Y. For definiteness (but ignoring the asymmetry of $A-a$ and a with respect to the actual description of X and Y in the proof of Theorem 3.7), assume that $\alpha_1\beta_1, \alpha_1\beta_2 \in Y$. Then $(\alpha_1\beta_1)^a = \mu = (\alpha_1\beta_2)^a$, and so

$$(\beta_1/\beta_2)^a = 1. \tag{3.12.1}$$

Now, for any $1 \leq i \leq m$ we have

$$(\alpha_i\beta_1)^a = (\alpha_i\beta_2)^a. \tag{3.12.2}$$

Suppose both $\alpha_i\beta_1$ and $\alpha_i\beta_2$ belong to X, then

$$(\alpha_i\beta_1)^{A-a} = (\alpha_i\beta_2)^{A-a}. \tag{3.12.3}$$

Since $\gcd(A - a, a) = 1$, (3.12.2) and (3.12.3) imply that $\alpha_i\beta_1 = \alpha_i\beta_2$, and so $\beta_1 = \beta_2$, a contradiction. So we may assume that $\alpha_i\beta_1 \in Y$, i.e. $(\alpha_i\beta_1)^a = \mu$. By (3.12.2), we also have $(\alpha_i\beta_2)^a = \mu$, and hence $\alpha_i\beta_2 \in Y$.

(c) We have therefore shown that (3.12.1) implies that $\alpha_i\beta_1, \alpha_i\beta_2 \in Y$ for all $1 \leq i \leq m$. Now consider any $1 \leq i < j \leq m$. Since $(\alpha_i\beta_1)^a = \mu = (\alpha_j\beta_1)^a$, it follows that

$$(\alpha_i/\alpha_j)^a = 1. \tag{3.12.4}$$

The arguments in (b) then show that (3.12.4) implies that $\alpha_i\beta_k, \alpha_j\beta_k \in Y$ for all $1 \leq k \leq n$. Thus $S = Y$, whence $X \subseteq Y$ and so $|X| = |X \cap Y| \leq 1$.

(d) We have shown that $\min(|X|, |Y|) = |X \cap Y| \leq 1$. Recalling the description of X and Y in the proof of Theorem 3.7, we see that one of (i)–(iii) must occur. □

Theorem 3.13. *Consider the sheaf* $\mathcal{F} = \mathcal{F}(f, a, \chi)$ *with* $p \nmid A(A-a)a$ *and* $\gcd(A, a) = 1$. *If* $\chi = \mathbb{1}$ *and* $a > 1$, *assume in addition that* $(A, a) \neq (5, 3)$. *Suppose that the underlying representation* V *of* $G_{\mathrm{geom},\mathcal{F}}$ *is* n*-tensor induced for some* $n > 1$: $V = V_1 \otimes V_2 \otimes \ldots \otimes V_n$ *with* $d := \dim(V_i) > 1$. *Then* $p = n = 3$, $\chi \neq \mathbb{1}$, *and* $(A, a) = (8, 1), (8, 7)$.

Proof. Note that the condition $p \nmid Aa(A-a)$ implies that $p \neq 2$. By Lemma 3.1, the $I(\infty)$-representation V is tensor indecomposable. Hence by Theorem 3.7, we all the assumptions of Theorem 3.11 hold for $\mathcal{F}$. Applying Theorem 3.11, we see that $p = n = 3$, $\mathrm{rank}(\mathcal{F}) = 8$, and moreover γ fixes a nontrivial tensor decomposition of V. Applying Proposition 3.12, we see that $p = n = 3$ and $A - \delta_{\chi,\mathbb{1}} = \mathrm{rank}(\mathcal{F}) = 8$. As $p \nmid A$, $\chi \neq \mathbb{1}$, and so $A = 8$ and $a = 1, 7$. □

4. PRIMITIVITY

Theorem 4.1. *Suppose* $p \nmid Aa$ *and* $\gcd(A, a) = 1$. *For* $\chi \neq \mathbb{1}$, *the local system* $\mathcal{F}(f, a, \chi)$ *on* $\mathbb{A}^1/\overline{\mathbb{F}_p}$ *is primitive.*

Proof. In the case $a = 1$, this is [KRLT1, Lemma 1.1(ii)]. For the general case, we work over $\overline{\mathbb{F}_p}$. The rank is A and $\mathsf{Swan}_\infty = A$, hence the Euler–Poincaré characteristic $\mathsf{EP}(\mathbb{A}^1, \mathcal{F}(f, a, \chi))$ is 0. The hypothesis that $\gcd(A, a) = 1$ insures that $\mathcal{F}(f, a, \chi)$ is geometrically irreducible. Suppose that $\mathcal{F}(f, a, \chi)$ is induced. Then there exists a finite etale map

$$\pi : U \to A^1$$

of degree > 1 with U geometrically connected, and a geometrically irreducible local system $\mathcal{G}$ on U such that $\pi_\star\mathcal{G} \cong \mathcal{F}(f, a, \chi)$. Then

$$\mathsf{EP}(U, \mathcal{G}) = \mathsf{EP}(\mathbb{A}^1, \mathcal{F}(f, a, \chi)) = 0.$$

Write U as the complement in a projective, smooth, geometrically connected curve X of $d \geq 1$ $\overline{\mathbb{F}_p}$-valued points $x_1, \ldots, x_d$, with X of genus g. Then

$$0 = \mathsf{EP}(U, \mathcal{G}) = (2 - 2g - d)\mathrm{rank}(\mathcal{G}) - \sum_{i=1}^{d} \mathsf{Swan}_{x_i}(\mathcal{G}).$$

First observe that $g < 1$, otherwise the sum is strictly negative. If $g = 0$, then $d \leq 2$, otherwise the sum is strictly negative.

If $d = 1$, then U is $\mathbb{A}^1$. The map π is then given by a polynomial $\phi(x)$ whose derivative $d\phi/dx$ has no zeroes, so is a nonzero constant, say $\alpha \neq 0$. Then $\phi(x) - \alpha x$ has derivative zero, so is a p^{th} power, say $\phi = \alpha x + (h(x))^p$. We have $\deg(h) > 0$, otherwise ϕ has degree one. Thus ϕ has degree divisible by p. This in turn forces $\pi_\star\mathcal{G}$ to have rank divisible by p. But this rank is A, which is prime to p.

If $d = 2$, then U is $\mathbb{G}_m$, in which case

$$0 = \mathsf{EP}(\mathbb{G}_m, \mathcal{G}) = -\mathsf{Swan}_0(\mathcal{G}) - \mathsf{Swan}_\infty(\mathcal{G}).$$

Thus $\mathcal{G}$ is tame at both 0 and ∞, and is geometrically irreducible. Therefore $\mathcal{G}$ is a Kummer sheaf $\mathcal{L}_\rho$. Now consider the finite etale map $\pi : \mathbb{G}_m \to \mathbb{A}^1$. Denote by $\pi(0)$ and $\pi(\infty)$, the maps induced on formal completions upstairs, and by d_0 and $D(\infty)$ their degrees. We cannot have both d_0 and $D(\infty)$ prime to p, otherwise the covering of $\mathbb{A}^1$ by $\mathbb{G}_m$ is tame at ∞, but $\pi_1^{\mathrm{tame}}(\mathbb{A}_1)$ is trivial. They cannot both be divisible by p, otherwise π has degree divisible by p, not possible because $\pi_\star\mathcal{G}$ has rank A prime to p. After a possible inversion on $\mathbb{G}_m$, we may assume that $p \nmid d_0$ and that $D(\infty) = p^a d_1$ with $a \geq 1$ and d_1 prime to p.

Then the $I(\infty)$-representation of $\mathcal{F}(f, a, \chi)$ is the direct sum

$$\pi(0)_\star\rho \oplus \pi(\infty)_\star\rho.$$

A tame character Λ occurs in $\pi(0)_\star\rho$ if and only if $\Lambda^{d_0} = \rho$. And a tame character Λ occurs in $\pi(\infty)_\star\rho$ if and only if $\Lambda^{p^a d_1} = \rho$. The ratios of the various d_0 roots of ρ give all characters of order dividing d_0, and the ratio of all $p^a d_1$ roots of ρ give all characters of order dividing d_1. But the tame characters in the $I(\infty)$-representation of $\mathcal{F}(f, a, \chi)$ are all the a roots of $\overline{\chi}$, and their ratios give all characters of order dividing a. So looking at ratios, we see that $d_0|a$ and $d_1|a$.

On the other hand, $d_0 + d_1$ is the dimension of the tame part, so $d_0 + d_1 = a$, while $d_0 + p^a d_1 = A$. But $\gcd(A, a) = 1$. Therefore we must have $\gcd(d_0, d_1) = 1$ (because this gcd divides both a and A). Thus $a = d_0 + d_1$ is the sum of two divisors of a. Neither d_0 nor d_1 can be a, so each is at most $a/2$; if their sum is a, each must be $a/2$. But $\gcd(d_0, d_1) = 1$, hence we have $d_0 = d_1 = 1$, and $a = 2$. In this case, we have $p \neq 2$ because $p \nmid a$, and $A = d_0 + p^a d_1 = 1 + p^a$ is therefore even, contradicting $\gcd(A, a) = 1$. ☐

Theorem 4.2. *Suppose that* $\chi = \mathbb{1}$, $\gcd(A, a) = 1$ *and* $p \nmid Aa$. *Then* $\mathcal{F}(f, a, \mathbb{1})$ *is primitive in each of the following cases.*

(i) $a = 1$ *and* $p \nmid (A - 1)$.
(ii) $a \geq 2$ *and* $(a - 1) \nmid (A - 1)$.
(iii) $a \geq 2$, $(a-1)|(A-1)$, *and the ratio* $(A-1)/(a-1)$ *is not* $\equiv 1 \pmod{p}$, *or equivalently,* $(a-1)|(A-1)$ *but* $(A - a)/(a - 1)$ *is prime to* p.

Proof. For (i), see [KRLT1, Lemma 1.1(i)]. In general, in this $\chi = \mathbb{1}$ case, $\mathcal{F}(f, a, \mathbb{1})$ is geometrically irreducible, its rank is $A - 1$ and its Swan_∞ is A, hence $\mathsf{EP}(\mathbb{A}^1, \mathcal{F}(f, a, \chi)) = -1$. Just as in the proof of the previous theorem, we look for $\pi : U \to A^1$ finite etale of degree > 1 and a geometrically irreducible local system $\mathcal{G}$ on U with $\pi_\star\mathcal{G} \cong \mathcal{F}(f, a, \mathbb{1})$, Then $\mathsf{EP}(U, \mathcal{G}) = -1$. Write U as the complement in a projective, smooth, geometrically connected curve X of $d \geq 1$ $\overline{\mathbb{F}_p}$-valued points $x_1, \ldots x_d$, with X of genus g. Then

$$-1 = \mathsf{EP}(U, \mathcal{G}) = (2 - 2g - d)\mathrm{rank}(\mathcal{G}) - \sum_{i=1}^{d} \mathsf{Swan}_{x_i}(\mathcal{G}).$$

We must have $g \leq 1$, otherwise the sum is too negative.

If $g = 1$, then U must be $E \smallsetminus \{\infty\}$, the complement of a single point in a elliptic curve, and $\mathcal{G}$ is lisse of rank one and tame at ∞. The finite etale map $\pi : E \smallsetminus \{\infty\} \to \mathbb{A}^1$ has degree equal to the order of pole of π at ∞. This degree cannot be 1 (otherwise E would be $\mathbb{P}^1$), and it cannot be prime to p, otherwise the covering of $\mathbb{A}^1$ is tame at ∞, contradicting the vanishing of $\pi_1^{\mathrm{tame}}(\mathbb{A}^1)$. Thus the covering has degree divisible by p. Write the degree as dp^e with $p \nmid d$ and $e \geq 1$. Because $\mathcal{G}$ is lisse of rank 1 and tame at ∞, its $I(\infty)$-representation is a single tame character χ. Then the tame part of the $I(\infty)$ of $\pi_\star\mathcal{G}$ is the direct sum of the d dp^e roots of χ. Thus $d = a - 1$. On the other hand, the ratio argument shows that d divides a. Thus $a - 1$ divides a, possible only for $a = 2$. Hence $d = 1$. In this $a = 2$ case, p must be odd, as $p \nmid a$. Because $d = 1$, the wild part of $I(\infty)$ of $\pi_\star\mathcal{G}$ thus has rank $p^e - 1$. Thus $\pi_\star\mathcal{G}$ has rank p^e, thus $A = p^e - 1$. But $p^e - 1$ is even (because p is odd), contradicting the hypothesis that $\gcd(A, a) = 1$. So this elliptic curve case does not occur.

If $g = 0$, we cannot have $U = \mathbb{A}^1$, because as we have seen in the proof of the previous result, any finite etale map of $\mathbb{A}^1$ to itself of degree > 1 has degree a multiple of p, but $\pi_\star\mathcal{G}$ has rank $A - 1$, which by hypothesis is prime to p. In fact, this case really occurs, at least in the case $a = 1$ of Airy sheaves, as the following lemma shows.

Lemma 4.3. *Let $\mathcal{F}$ be an Airy sheaf (i.e., $\mathcal{F} = \mathrm{FT}_\psi(\mathcal{L})$ for a lisse, rank one $\mathcal{L}$ on $\mathbb{A}^1$ with $\mathsf{Swan}_\infty(\mathcal{L}) \geq 2$, cf.* [Such]*). Denote by $\mathcal{P} : \mathbb{A}^1 \to \mathbb{A}^1$ the Artin–Schreier map $x \mapsto x - x^p$. Then $\mathcal{P}_\star\mathcal{F}$ is an Airy sheaf: more precisely, its Fourier transform is given by*

$$\mathsf{Frob}_p^\star \mathrm{FT}_\psi(\mathcal{P}_\star\mathcal{F}) \cong (\mathcal{P})^\star(\mathrm{FT}_{\overline{\psi}}(\mathcal{F})).$$

Equivalently, for $\mathcal{F} := \mathrm{FT}_\psi(\mathcal{L})$, Fourier inversion gives

$$\mathsf{Frob}_p^\star \mathrm{FT}_\psi(\mathcal{P}_\star(\mathrm{FT}_\psi(\mathcal{L})) \cong (\mathcal{P})^\star(\mathcal{L}).$$

Applying $\mathrm{FT}_{\overline{\psi}}$ to both sides, and using the fact that $\mathrm{FT}_{\overline{\psi}} \circ \mathsf{Frob}_p^\star = \mathsf{Frob}_p^\star \circ \mathrm{FT}_{\overline{\psi}}$, we get the equivalent

$$\mathsf{Frob}_p^\star(\mathcal{P}_\star(\mathrm{FT}_\psi(\mathcal{L})) \cong \mathrm{FT}_{\overline{\psi}}((\mathcal{P})^\star(\mathcal{L})).$$

Proof. It suffices to verify the first identity. We will show it on trace functions, and leave to the reader the formal expression of the proof. We begin with $\mathrm{FT}_\psi(\mathcal{P}_\star\mathcal{F})$, whose trace function is

$$y \mapsto -\sum_x \psi(xy) \sum_{z: z - z^p = x} \mathcal{F}(z) = -\sum_z \psi((z - z^p)y)\mathcal{F}(z).$$

So the trace function of its $\mathsf{Frob}_p^\star$ is

$$y \mapsto -\sum_z \psi((z - z^p)y^p)\mathcal{F}(z).$$

The key point is that

$$\psi((z - z^p)y^p) = \psi(zy^p - (zy)^p) = \psi(zy^p - zy) = \overline{\psi}(z(y - y^p)) = \overline{\psi}(z\mathcal{P}(y)).$$

and hence the trace of $\mathsf{Frob}_p^\star \mathrm{FT}_\psi(\mathcal{P}_\star\mathcal{F})$ is

$$y \mapsto -\sum_z \overline{\psi}(z\mathcal{P}(y))\mathcal{F}(z),$$

which is precisely the trace function of $(\mathcal{P})^\star(\mathrm{FT}_{\overline{\psi}}(\mathcal{F}))$. ☐

Remark 4.4. Even in simple cases, this Lemma 4.3 leads to striking identities. For example, take p odd, and consider $\mathcal{L}_{\psi(x^2)}$, whose FT_ψ is $\mathcal{L}_{\psi(x^2/4)}$. Then

$$\mathsf{Frob}_p^\star(\mathcal{P}_\star)(\mathcal{L}_{\psi(x^2/4)}) \cong \mathrm{FT}_{\overline{\psi}}(\mathcal{L}_{\psi((x-x^p)^2)}) = \mathrm{FT}_{\overline{\psi}}(\mathcal{L}_{\psi(x^2-2x^{p+1}+x^{2p})}) = \mathrm{FT}_{\overline{\psi}}(\mathcal{L}_{\psi(2x^2-2x^{p+1})}).$$

In particular, $\mathcal{F}(x^2 - 2x^{p+1} + x^{2p}, 1, \mathbb{1})$ is Artin–Schreier induced. More generally, for any polynomial $f(x)$ of prime to p degree $d \geq 2$, $\mathcal{F}(f(x - x^p), 1, \mathbb{1})$ is Artin–Schreier induced.

We now return to the proof of Theorem 4.2.

We now claim that if $a > 1$, then no $\mathcal{F}(f, a, \mathbb{1})$ can be induced through a finite etale map $\pi : \mathbb{A}^1 \to \mathbb{A}^1$. Suppose $\mathcal{F}(f, a, \mathbb{1})$ were $\pi_\star\mathcal{G}$ for some lisse $\mathcal{G}$ on $\mathbb{A}^1$. Because $\pi_\star\mathcal{G}$ is geometrically irreducible, $\mathcal{G}$ must be geometrically irreducible. It cannot be tame at ∞, otherwise (because $\pi_1^{\mathrm{tame}}(\mathbb{A}^1)$ is trivial) it would be the the constant sheaf $\overline{\mathbb{Q}_\ell}$, but the induction of the constant sheaf always contains the constant sheaf, whereas $\pi_\star\mathcal{G}$ is geometrically irreducible. Consider the $I(\infty)$-representation of $\mathcal{G}$. It is of the form $T \oplus W$ with a (possibly zero) tame part T and an nonzero wild part W. The induction of the wild part $\pi_\star W$ has no tame part, because W contains no tame characters, in particular it contains no $\pi^\star(\mathcal{L}_\chi)$. On the other hand, the tame part T must vanish, because for each tame character ρ in T, its induction $\pi_\star\mathcal{L}_\rho$ has a tame part consisting of all the $\deg(\pi)^{\mathrm{th}}$ roots of ρ. Write $\deg(\pi) = dp^e$ with $e \geq 1$ and $p \nmid d$. Then $\pi_\star\mathcal{L}_\rho$ has rank dp^e, with a tame part of rank d, so a nonzero wild part. But the wild part of $\mathcal{G} = \mathcal{F}(f, a, \mathbb{1})$ is irreducible (because $\gcd(A, a) = 1$), so we cannot have such a second wild part. Thus $\mathcal{G}$ is totally wild at ∞, and, as noted above, its induction is totally wild. Therefore $\mathcal{F}(f,a,\mathbb{1})$ is totally wild at ∞, which happens precisely for $a = 1$.

If $g = 0$, we might have $U = \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ with $\mathcal{G}$ lisse of rank one and everywhere tame. The finite etale map

$$\pi : \mathbb{P}^1 \smallsetminus \{0, 1, \infty\} \to \mathbb{A}^1$$

has poles of orders D_0, D_1, D_∞ at the three upstairs points. Write these pole orders as

$$d_0 p^{e_0}, d_1 p^{e_1}, d_\infty p^{e_\infty}$$

with the d_i prime to p, and the $e_i \geq 0$. Then the tame part of $\pi_\star\mathcal{G}$ is the direct sum of the d_0^{th} roots of $\mathcal{G}|I(0)$, the d_1^{th} roots of $\mathcal{G}|I(1)$, and the d_∞ roots of $\mathcal{G}|I(\infty)$. Therefore the dimension of the tame part of $\mathcal{F}_{f,a,\mathbb{1}}$ (which is $\mathsf{Char}_{\text{nontriv}}(a)$) is given by

$$a - 1 = d_0 + d_1 + d_\infty.$$

The ratios argument shows that each d_1 divides a.

Suppose first that a is odd. Then each d_i is odd, and hence their sum is odd. But $a - 1$ is even if a is odd, so this a odd case cannot arise.

Suppose next that a is even. On the one hand, we cannot have all d_i prime to p, otherwise the covering is tame at ∞. Suppose to fix ideas that D_0 is divisible by p, i.e. that $e_0 \geq 1$.

The tame part of the $I(\infty)$-representation coming from $\pi(0)_\star(\mathcal{G}|I(0))$ has rank d_0, so the wild part has dimension $D_0 - d_0 = d_0(p^{e_0} - 1)$.

Since the wild part of the $I(\infty)$-representation of $\mathcal{F}(f,a,\mathbb{1})$ is a single irreducible, there can be only one point whose D_i is divisible by p. Thus the wild part of $\mathcal{F}(f,a,\mathbb{1})$ is the wild part from the upstairs point 0, so has rank $d_0(p^{e_0} - 1)$. Because a is even, p must be odd. Then the wild part has dimension divisible by $p - 1$, hence is even. Recall that the wild part has dimension $A - a$. But $\gcd(A, a) = 1$ and a even forces A to be odd, hence forces $A - a$ to be odd, again a contradiction. So this case cannot occur.

Finally, we could have $U = \mathbb{G}_m$, with $\mathcal{G}$ a geometrically irreducible local system with $\mathsf{Swan}_0(\mathcal{G}) + \mathsf{Swan}_\infty(\mathcal{G}) = 1$, which is to say that $\mathcal{G}$ is a hypergeometric sheaf. The finite etale map $\pi : \mathbb{G}_m \to \mathbb{A}^1$ has poles of orders D_0, D_∞ at $0, \infty$ respectively. As above, write $d_0 = d_0 p^a$, $D_\infty = d_\infty p^b$ with $a, b \geq 0$ and d_0, d_∞ prime to p. We cannot have both D_0 and D_∞ prime to p, otherwise the covering of $\mathbb{A}^1$ is tame at ∞. We cannot have both D_0, D_∞ divisible by p, because then the inductions from both 0 and ∞ have nonzero wild parts, impossible because $\mathcal{F}(f,a,\mathbb{1})$ has its wild part irreducible.

Interchanging $0, \infty$, we may assume that D_0 is prime to p, i.e. $D_0 = d_0$, but $D_\infty = d_\infty p^b$ with $b \geq 1$.

Suppose first that the hypergeometric $\mathcal{G}$ has type (n, m) with $n < m$, i.e. that $\mathcal{G}$ is tame at ∞ and has a nonzero wild part at 0, of rank $m - n$ and all slopes $1/(m - n)$. Then already the wild contribution from 0 is the Kummer direct image $[d_0]_\star(\mathcal{G}(0)^{\text{wild}})$, which has $\mathsf{Swan} = 1$ and all slopes $1/((m-n)d_0)$. But this slope is ≤ 1, whereas all slopes of $\mathcal{F}(f,a,\mathbb{1})$ are $A/(A - a) > 1$.

So it remains to treat the case when $\mathcal{G}$ is a hypergeometric of type (n, m) with $n > m$. Then $\mathcal{G}|I(\infty)$ is the direct sum $T_m + W_{n-m}$ of a tame part of rank m and a wild part of rank $n - m$. As $\pi(\infty)$ has degree $d_\infty p^a$, each tame character, if any, has a direct image whose tame part has rank d_∞, and hence whose wild part has nonzero rank $D_\infty - d_\infty = (p^b - 1)d_\infty$. The direct image of W_{n-m} is completely wild (simply because no tame character χ has χ^{D_∞} occurring in W_{n-m}), so this last direct image is totally wild of rank $d_\infty p^b(n - m)$. But $\mathcal{F}_{f,a,\mathbb{1}}$ has a wild part which is irreducible. Therefore there are no tame characters in $\mathcal{G}|I(\infty)$, i.e., $\mathcal{G}$ is a Kloosterman sheaf $\mathcal{K}l$ of rank n, and the wild part of $\mathcal{F}_(f,a,\mathbb{1})$ has rank $d_\infty p^b n$. But this rank is $A - a$, which we assumed prime to p.

We can be more precise in this Kloosterman case. Recall that the degree of $\pi(0)$ is the prime to p integer d_0. For n the rank of the Kloosterman sheaf, with characters $\chi_1, \ldots, \chi_n$ occurring at 0, the tame characters in $\pi_\star\mathcal{K}l$ are all the d_0^{th} roots of the χ_i. The ratio argument then shows that $d_0|a$. On the other hand, the entire tame part of $\pi_\star\mathcal{K}l$ is precisely these characters, while the tame part of $\mathcal{F}(f,a,\mathbb{1})$

has dimension $a-1$. Thus $d_0 n = a-1$. But also $d_0 | a$, so d_0 divides both a and $a-1$. Thus $d_0 = 1$. Hence $n = a-1$ and our $\mathcal{K}l$ is the one whose upstairs characters are $\mathsf{Char}_{\text{nontriv}}(a)$. Thus π has degree $1 + p^a d_\infty$, and so $\mathcal{F}(f, a, \mathbb{1})$ has rank $(a-1)(1+p^a d_\infty)$, with a tame part of dimension $a-1$ and a wild part of dimension $(a-1)(p^a d_\infty)$. So this case can only possibly occur if $a \ge 2$ (as $a-1$ is the rank of $\mathcal{K}l$), if $(a-1)|(A-1)$ and if the ratio $(A-1)/(a-1)$ is $\equiv 1 \pmod{p}$. This last condition is a clumsy way of saying that $(a-1)|(A-1)$ and $p|(A-a)/(a-1)$. It can very well fail in cases when $p|(A-a)$, for example $a = q+1$, $A = 2q+1$ for q a power of p. □

Remark 4.5. As mentioned in Remark 3.3, for $q = p^f$ with $f \ge 1$, the Pink–Sawin local system $\mathcal{F}(x^{q+1}, 1, \mathbb{1})$ has G_{geom} equal to the extraspecial group p_+^{1+2f} for p odd, and to 2_-^{1+2f} for $p = 2$, cf. [KT9, Theorem 7.3.8]. It is well known, see e.g. [Is, Corollary (6.14)], that any finite nilpotent group G is an M-group, that is, every complex irreducible representation of G is induced from a one-dimensional representation of a subgroup of G. Thus G_{geom} is not primitive.

Now take $p = 2$ and $q = 8$. Then the Pink–Sawin local system $\mathcal{F}(x^{q+1}, 1, \mathbb{1})$ is the $[9]^\star$ Kummer pullback of the Kloosterman sheaf $\mathcal{K}l_\psi(\mathsf{Char}_{\text{ntriv}}(9))$, and even the latter local system is also known to be 3-tensor induced, see [KT8, Theorem 4.1].

Theorem 4.6. *Suppose that* $p \nmid Aa$ *and that* $\gcd(A, a) = 1$. *Then* $\mathcal{F}(f, a, \chi)$ *satisfies* $(\mathbf{S}-)$ *under each of the following conditions.*

(i) $\chi \ne \mathbb{1}$.
(ii) $\chi = \mathbb{1}$, $a = 1$, *and* $p \nmid (A-1)$.
(iii) $\chi = \mathbb{1}$, $a \ge 2$, *and* $(a-1) \nmid (A-1)$.
(iv) $\chi = \mathbb{1}$, $a \ge 2$, $(A, a) \ne (5, 3)$ *if* p *is odd,* $(a-1)|(A-1)$ *but* $(A-a)/(a-1)$ *is prime to* p.

Proof. Immediate from Lemmas 3.1 and 3.2 and Theorems 4.1 and 4.2. □

Proposition 4.7. *Keep the assumptions of Theorem 4.6, and suppose we are in one of the cases* (i)–(iv), *so that* $(\mathbf{S}-)$ *holds. Then in fact the local system* $\mathcal{F} = \mathcal{F}(f, a, \chi)$ *in characteristic* p *satisfies* $(\mathbf{S}+)$ *if at least one of the following conditions holds.*

(a) *The rank of* $\mathcal{F}$, *namely* $D = A - \delta_{\chi,\mathbb{1}}$, *is not a* p^{th} *power* c^p *of an integer* $c > 1$.
(b) *For the ratio* $\tau := (a - \delta_{\chi,\mathbb{1}})/D$, *we have* $\tau > 3/4$ *when* $p = 2$, $\tau > 1/(p-1)$ *when* $p = 3, 5$, *and* $\tau \ge 1/(p-1)$ *if* $p \ge 7$.

Proof. We need to show that $\mathcal{F}$ is not tensor induced. Assume the contrary: $\mathcal{F}(f, a, \chi)$ is m-tensor induced for some $m > 1$. Consider the homomorphism $\pi : G \to \mathsf{S}_m$ induced by the action of $G := G_{\text{geom},\mathcal{F}}$ on the set Ω of m tensor factors. By Lemmas 3.1 and 3.2 and our assumptions, $\mathcal{F}$ is $I(\infty)$-tensor indecomposable. Hence $\pi(I(\infty))$ is a transitive subgroup of S_m, which contains $\pi(P(\infty))$ as a normal p-subgroup. It follows that all the orbits of $\pi(P(\infty))$ on Ω have the same length, say l, and l is a power of p which divides m. If $l = 1$, then $P(\infty)$ acts trivially on Ω, in which case G also acts trivially on Ω by Lemma 2.1, and thus $\mathcal{F}$ is tensor decomposable, a contradiction. Thus $p|l$, and so $p|m$, which is impossible in the case of (a).

Suppose (b) holds. We continue to assume $\mathcal{F}(f, a, \chi)$ is m-tensor induced for some $m > 1$. As in the previous paragraph, $\mathcal{F}$ being tensor indecomposable implies that P:= the image of $P(\infty)$ in G acts nontrivially on the m factors. Thus for $Q := P \cap \mathrm{Ker}(\pi)$, we have $Q < P$. Let φ denote the character of G acting on the underlying representation, so that $\varphi(1) = D$. In particular, $|\varphi(x)| \le D$ for all $x \in Q$, and $|Q| \le |P|/p$. We also have that $D = d^m$ if d is the common dimension of the tensor factors. Now if $x \in P \smallsetminus Q$, then the formula for tensor induction [GI] implies that $|\varphi(x)| \le D/d^{p-1}$. Now for the dimension $a - \delta_{\chi,\mathbb{1}}$ of the tame part we have

$$\tau = \frac{a - \delta_{\chi,\mathbb{1}}}{D} = \frac{[\varphi|_P, 1_P]}{D} = \frac{\sum_{x \in P} \varphi(x)}{|P|D} \leq \frac{|Q| + (|P| - |Q|)/d^{p-1}}{|P|}$$
$$= \frac{1}{|P|}\Big(|Q|(1 - \frac{1}{d^{p-1}}) + \frac{|P|}{d^{p-1}}\Big) \leq \frac{1}{p} + (1 - \frac{1}{p})\frac{1}{d^{p-1}}.$$

This however violates the assumption on τ in (b), since $d \geq 2$. ☐

Remark 4.8. By Mihailescu's proof [Mih] of the Catalan conjecture, the only case where both A and $A-1$ are proper powers is $A = 9$. This means that in Theorem 4.6, if $A \neq 9$, then at least one of $\mathcal{F}(f, a, \chi)$ with $\chi \neq \mathbb{1}$, or $\mathcal{F}(f, a, \mathbb{1})$ with $a \geq 2$, will satisfy ($\mathbf{S}+$).

Remark 4.9. If D is of the form $\frac{N^3-1}{N-1}$ for some integer $N \geq 2$, $N \neq 18$, then D is not a perfect power. This is (part of) a theorem of Ljunggren and Nagell, cf. [Bu-Mi, Theorem NL]. Suppose now that q is a power of p, and $D = q^2 - q + 1 = \frac{(q-1)^3-1}{(q-1)-1}$. So long as $(p, q) \neq (19, 19)$, $\chi \neq \mathbb{1}$, and f has degree this D, the local system $\mathcal{F}(f, a, \chi)$ on $\mathbb{A}^1/\overline{\mathbb{F}_p}$ satisfies condition ($\mathbf{S}+$). In particular, we note that for $p = 3$ and each $q = 3^{2n+1}$, the Ree(q) candidate local system on $\mathbb{A}^1/\mathbb{F}_3$ proposed in [Ka-ESRS] geometrically satisfies condition ($\mathbf{S}+$): its f has degree $q^2 - q + 1$, and its χ is χ_2. We might also add that if $p = q = 19$ and $D = 19^2 - 19 + 1 = 7^3$, and $\chi \neq \mathbb{1}$, then even in this case we cannot be tensor induced. Indeed, being on $\mathbb{A}^1/\overline{\mathbb{F}_p}$, the tensor induction homomorphism $\pi_1 \to \mathsf{S}_3$ giving the action on the tensor factors must be trivial, simply because S_3 has order prime to 19. Hence we would be tensor decomposed, contradicting Lemma 3.1

Combining the results in this section with Theorem 3.13, under the additional assumption that $p \nmid (A - a)$, we have the following ($\mathbf{S}+$) result, which is the main result of the paper. On one hand, its hypotheses rule out $p = 2$ entirely. On the other hand, as shown in Remarks 3.3, 4.5, the condition $p \nmid (A - a)$ is necessary in various instances.

Theorem 4.10. *Let* $p \nmid A(A - a)a$, $\gcd(A, a) = 1$, *and* $(A, a) \neq (5, 3)$. *Then* $\mathcal{F}(f, a, \chi)$ *satisfies* ($\mathbf{S}+$) *in each of the following cases.*

(i) $p \geq 5$.
(ii) $p = 3$, *and* $(A, a) \neq (8, 1)$, $(8, 7)$ *if* $\chi \neq \mathbb{1}$.

Proof. We first check that Theorem 4.6 applies to show each such $\mathcal{F}(f, a, \chi)$ has ($\mathbf{S}-$). We have $(A, a) \neq (5, 3)$ in odd characteristic by hypothesis. We have $p \nmid A(A - a)a$ and $\gcd(A, a) = 1$ by hypothesis. If $\chi \neq \mathbb{1}$ we have ($\mathbf{S}-$). If $\chi = \mathbb{1}$ and $a = 1$, we have ($\mathbf{S}-$) because $p \nmid (A - 1)$ in this $a = 1$ case. If $\chi = \mathbb{1}$, $a \geq 2$, and $(a - 1) \nmid (A - 1)$, or equivalently $(a - 1) \nmid (A - a)$, then we have ($\mathbf{S}-$). If $(a - 1)|(A - a)$, then $p \nmid ((A - a)/(a - 1))$ (simply because $p \nmid (A - a)$), so we have ($\mathbf{S}-$). Now apply Theorem 3.13. ☐

5. Remarks on the case $p|(A - a)$, and a question

Although the condition $p \nmid (A - a)$ was essential to the proof of Theorem 4.10, there are nonethless many situations in which $p|(A - a)$ but in which we know a great deal. We now give two examples, each based on [KT9, 11.2.3 (i) and (iii)] together with the specialization theorem [KT10, Proposition 11.3]. See also [KT10, Remark 11.4].

Here is a simple example. Take $q = p^f$ an arbitrary power of p, and two odd integers $N > M \geq 1$ with $\gcd(N, M) = 1$. Take

$$A := (q^N + 1)/(q + 1), a := (q^M + 1)/(q + 1).$$

Then $p|(A-a)$, and one knows [KT9, 10.2.6(ii), 10.3 13(ii)] that $\mathcal{F}(x^A, a, \mathbb{1})$ has G_{geom} equal to the image of $\mathrm{SU}_N(q)$ in its $A-1$ dimensional Weil representation. Now choose a set of $k \geq 1$ odd integers $N > m_1 > \ldots > m_k \geq 1$, none of which is equal to M, and write

$$A_i := (q^{m_i}+1)/(q+1).$$

Given a polynomial

$$f = \sum_x c_i x^{A_i},$$

consider the local system

$$\mathcal{F}(x^A + f(x), a, \mathbb{1}),$$

and denote by $G_{\mathrm{geom},f}$ its G_{geom} (so that $G_{\mathrm{geom},f=0}$ is the known group, namely the image of $\mathrm{SU}_N(q)$ in its $A-1$ dimensional Weil representation). Then for every such f, we have $G_{\mathrm{geom},f} \leq G_{\mathrm{geom},f=0}$, and for f in an open dense set of the $\mathbb{A}^k$ of allowed f, we have $G_{\mathrm{geom},f} = G_{\mathrm{geom},f=0}$.

For another simple example, take $q = p^f$ an arbitrary power of an odd prime p, and two integers $N > M \geq 1$ with $\gcd(N, M) = 1$. Take

$$A = (q^N+1)/2, a := (q^M+1)/2.$$

Then $p|(A-a)$, and one knows [KT9, 10.3.13(i)] that $\mathcal{F}(x^A, a, \mathbb{1})$ has G_{geom} equal to the image of $\mathrm{Sp}_{2N}(q)$ in an $(A-1)$-dimensional Weil representation. Now choose a set of $k \geq 1$ integers $N > m_1 > \ldots > m_k \geq 1$, none of which is equal to M, and write

$$A_i := (q^{m_i}+1)/2.$$

Given a polynomial

$$f = \sum_x c_i x^{A_i},$$

consider the local system

$$\mathcal{F}(x^A + f(x), a, \mathbb{1}),$$

and denote by $G_{\mathrm{geom},f}$ its G_{geom} (so that $G_{\mathrm{geom},f=0}$ is the known group, namely the image of $\mathrm{Sp}_{2N}(q)$ in an $(A-1)$-dimensional Weil representation). Then for every such f, we have $G_{\mathrm{geom},f} \leq G_{\mathrm{geom},f=0}$, and for f in an open dense set of the $\mathbb{A}^k$ of allowed f, we have $G_{\mathrm{geom},f} = G_{\mathrm{geom},f=0}$.

Along these lines, we can make use of the results [KT9, 11.2.3] on multi-parameter local systems on $\mathbb{A}^k$, $k \geq 2$, with (known) finite G_{geom} to produce explicit examples of situations in which $\mathcal{F}(f, a, \chi)$ has finite G_{geom}. Take one of the "universal" local systems

$$(t_1, \ldots, t_k) \in L^k \mapsto -\sum_x \psi_L\big(x^A + t_1 x^{B_1} + \ldots + t_k x^{B_k}\big)\chi(x)$$

listed there, and specialize all but one of the variable coefficients t_i, say for $i \neq i_0$, to arbitrary constants c_i. Then with f taken to be

$$f := x^A + \sum_{i \neq i_0} c_i x^{B_i}$$

the local system $\mathcal{F}(f, B_{i_0}, \chi)$ has finite G_{geom} (simply because it is a pullback to $\mathbb{A}^1$ of a "universal" local system on $\mathbb{A}^k$ which had finite G_{geom}).

Are there other ways than these to produce instances of $\mathcal{F}(f, a, \chi)$ having finite G_{geom}? Here is an equivalent formulation of this question.

Question 5.1. Let k be a finite subfield of $\overline{\mathbb{F}_p}$, $n \in \mathbb{Z}_{\geq 2}$, and let $f(x) = \sum_{i=1}^{n} c_i x^{A_i} \in k[x]$, where $A = A_1 > A_2 > \ldots > A_n \geq 1$, $\prod_{i=1}^{n} c_i \neq 0$, and $p \nmid \prod_{i=1}^{n} A_i$. Suppose for some integer $1 \leq a < A$ coprime to p, $a \notin \{A_1, \ldots, A_n\}$, $\gcd(a, A) = 1$, and some (possibly trivial) multiplicative character χ of $k^\times$, the local system $\mathcal{F}(f, a, \chi)$ has finite geometric monodromy group. Is it true that the multi-parameter system $\mathcal{G}(A_1, A_2, \ldots, A_n, a, \chi)$ on $\mathbb{A}^n$, with trace function

$$(s, t_2, \ldots, t_n) \in L^n \mapsto -\sum_{x \in L} \psi_L\big(x^A + \sum_{i=2}^{n} t_i x^{A_i} + s x^a\big)\chi_L(x),$$

for variable finite extensions L/k, has finite geometric monodromy group?

A weaker, but still completely open, version of this question is this:

Question 5.2. Given an $\mathcal{F}(f, a, \chi)$ with finite geometric monodromy group and $\deg(f) = A > a$, $\gcd(A, a) = 1$, does the one-parameter local system $\mathcal{F}(x^A, a, \chi)$,

$$s \mapsto -\sum_{x} \psi(x^A + s x^a)\chi(x),$$

have finite geometric monodromy group?

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