CONDITION (S+) IN RANKS 4, 8, AND 9

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ABSTRACT. Condition (**S**+), introduced in [KT5], plays a key role in the study of Kloosterman and hypergeometric l-adic local systems in positive characteristic p. Prior results of [KT5], [KT8] establish (**S**+) for primitive Kloosterman and hypergeometric sheaves, except possibly in ranks 4, 8, and 9. In this paper we study (**S**+) in these remaining ranks, and completely determine when (**S**+) does or does not hold.

1. Introduction

We work over an algebraically closed field $\mathbb{C}$ of characteristic zero, which we will take to be $\overline{\mathbb{Q}_\ell}$ for a suitable prime ℓ. Given a nonzero finite-dimensional $\mathbb{C}$-vector space V, a group Γ and a representation $\Phi : \Gamma \to \mathrm{GL}(V)$, we say that the pair (Γ, V) satisfies condition (**S**+) if each of the following five conditions is satisfied.

(i) The Γ-module V is irreducible.
(ii) The Γ-module V is primitive.
(iii) The Γ-module V is tensor indecomposable.
(iv) The Γ-module V is not tensor induced.
(v) The determinant $\det(\Gamma|V)$ is finite.

One knows [KT5, Lemma 1.6] that (Γ, V) satisfies condition (**S**+) if and only if for G the Zariski closure of $\Phi(\Gamma)$ in $\mathrm{GL}(V)$, the pair (G, V) satisfies condition (**S**+). Condition (**S**+) is a slightly strengthening of condition (**S**) introduced in [GT2], and roughly speaking, corresponds to Aschbacher's class $\mathcal{S}$ of maximal subgroups of classical groups [Asch].

The importance of condition (**S**+) is this, cf. [KT5, Lemma 1.1].

Lemma 1.1. *Suppose $G \leq \mathrm{GL}(V)$ is a Zariski closed subgroup, $\dim(V) > 1$, and (G, V) satisfies condition (**S**+). Then we have three possibiliites:*

(a) *The identity component G° is a simple algebraic group, and $V|_{G^\circ}$ is irreducible.*
(b) *G is finite, and almost quasisimple, i.e. there is a finite non-abelian simple group S such that $S \lhd G/\mathbf{Z}(G) < \mathrm{Aut}(S)$.*
(c) *G is finite and it is an "extraspecial normalizer" (in characteristic r), that is, $\dim(V) = r^n$ for a prime r, and G contains a normal r-subgroup $R = \mathbf{Z}(R)E$, where E is an extraspecial r-group E of order r^{1+2n} acting irreducibly on V, and either $R = E$ or $\mathbf{Z}(R) \cong C_4$.*

The application to hypergeometric sheaves is this. In a given characteristic p, we are given a prime $\ell \neq p$ and a (geometrically irreducible) $\overline{\mathbb{Q}_\ell}$-hypergeometric sheaf $\mathcal{H}$ of type (D, m) with $D > m \geq 0$ on $\mathbb{G}_m/\overline{\mathbb{F}_p}$, definable over some finite extension $k/\mathbb{F}_p$. We view $\mathcal{H}$ as a representation $\pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \to \mathrm{GL}_D(\overline{\mathbb{Q}_\ell})$. If this pair satisfies condition (**S**+), we say that $\mathcal{H}$ satisfies condition (**S**+).

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In previous papers [KT5] and [KT8], we showed that all primitive $\mathcal{H}$ of rank $D > 1$ satisfy condition ($\mathbf{S}+$) **with the possible exception** of ranks $4, 8, 9$. This paper gives the complete analysis of these exceptional cases. The inverse problem of which of the pairs (G, V) satisfying ($\mathbf{S}+$) can actually occur as geometric monodromy groups of ℓ-adic hypergeometric sheaves is the subject of several recent papers, see e.g. [KRLT3], [KRLT4], [KT1], [KT2], [KT3], [KT4], [KT5], [KT6], [KT7], [KT8], [Lee].

As defined above, the notion of ($\mathbf{S}+$) for a geometrically irreducible hypergeometric sheaf $\mathcal{H}$ requires, in addition to being tensor indecomposable and not tensor induced, being primitive. By [KT5, 2.3], Kloosterman sheaves of any rank are tensor indecomposable; their being primitive or not is irrelevant. By [KRLT3, 10.4]), hypergeometric sheaves of any type (D, m) with $D > m > 0$ and $D \neq 4$ are tensor indecomposable; their being primitive or not is irrelevant. [For $D = 4$, tensor indecomposability is more complicated, and can fail.] Whether or not a given hypergeometric sheaf is primitive can be visibly determined by its shape, see [KRLT3, Proposition 1.2].

The main result of the paper is summarized in the table below, in which we consider **only primitive** hypergeometric sheaves of a given type (D, m) in a given characteristic p. We specify for each listed type and characteristic whether all are ($\mathbf{S}+$), or whether there exist some which, despite being primitive, are not ($\mathbf{S}+$).

type (D, m)	all are ($\mathbf{S}+$) in characteristic	some are not ($\mathbf{S}+$) in characteristic
$(4, 0)$	$p = 2$	$p > 2$
$(4, 1)$	$p > 2$	$p = 2$
$(4, 2)$	$p = 2$	$p > 2$
$(4, 3)$	$p > 2$	$p = 2$
$(8, 0)$	$p > 2$	$p = 2$
$(8, 1)$	all p	
$(8, 2)$	$p = 2, 3$	$p \geq 5$
$(8, 3)$	all p	
$(8, 4)$	all p	
$(8, 5)$	all p	
$(8, 6)$	all p	
$(8, 7)$	all p	
$(9, 0)$	all p	
$(9, 1)$	$p \neq 3$	$p = 3$
$(9, 2)$	all p	
$(9, 3)$	$p = 2, 3$	$p \geq 5$
$(9, 4)$	all p	
$(9, 5)$	all p	
$(9, 6)$	all p	
$(9, 7)$	all p	
$(9, 8)$	all p	

Table 1. ($\mathbf{S}+$) for primitive hypergeometric sheaves in ranks $4, 8, 9$

2. Review of known results in rank 4

Lemma 2.1. *In characteristic $p = 2$, any primitive Kloosterman sheaf $\mathcal{K}l$ of rank 4 has* ($\mathbf{S}+$).

Proof. By [KT5, proof of 2.3], all Kloosterman sheaves are tensor indecomposable for $I(\infty)$. Suppose $\mathcal{K}l$ is 2-tensor induced. Then the map of $I(\infty)$ to S_2 is trivial on $P(\infty)$ (because $w = 4 \geq 2$), while the group $I(\infty)/P(\infty)$ has odd pro-order. So $\mathcal{K}l$ is tensor decomposable for $I(\infty)$, contradiction. □

Lemma 2.2. *In odd characteristic p, there exist primitive Kloosterman sheaves $\mathcal{K}l$ of rank 4 which are tensor induced. More precisely, choose a character χ of order $r \geq 5$, $p \nmid r$. Then the (primitive) Kloosterman sheaf*

$$\mathcal{K}l_\psi(\chi, \overline{\chi}, \mathbb{1}, \chi_{\mathrm{quad}})$$

is 2-tensor induced.

Proof. This is an instance of [Ka-CC, 6.3]. □

Lemma 2.3. *In characteristic $p = 2$, there exist primitive hypergeometric sheaves $\mathcal{H}$ of type $(4, 1)$ which are tensor decomposable. More precisely, choose two odd primes r, s, a character χ of order r and a character ρ of order s. Then the (primitive) hypergeometric sheaf*

$$\mathcal{H}yp_\psi(\chi\rho, \chi\overline{\rho}, \overline{\chi}\rho, \overline{\chi\rho} : \mathbb{1})$$

is tensor decomposable.

Proof. This is an instance of [Ka-CC, 5.2]. □

Lemma 2.4. *In odd characteristic p, every hypergeometric sheaf of type $(4, 1)$ has $(\mathbf{S}+)$.*

Proof. By [KRLT3, 10.4], any such $\mathcal{H}$ is tensor decomposable on $I(\infty)$. If it were 2-tensor induced, the map of $I(\infty)$ to S_2 would be trivial on $P(\infty)$. The image of a generator of $I(\infty)/P(\infty)$ is a three cycle (if it were trivial, $\mathcal{H}$ would be tensor decomposed for $I(\infty)$). Then $[2]^\star\mathcal{H}$ would be tensor decomposable for $I(\infty)$. But $[2]^\star\mathsf{Wild}_{\mathcal{H}}$ is totally wild and $I(\infty)$-irreducible (all its slopes are $2/3$), contradicting [KRLT3, 10.4]. □

Lemma 2.5. *In characteristic $p = 2$, every hypergeometric sheaf $\mathcal{H}$ of type $(4, 2)$ has $(\mathbf{S}+)$.*

Proof. This is an instance of [KT5, 3.3]. □

Lemma 2.6. *In any odd characteristic p, there are primitive hypergeometric sheaves $\mathcal{H}$ of type $(4, 2)$ which are 2-tensor induced. More precisely, choose two odd primes r, s, $p \nmid rs$. Choose characters α of order r and β of order s. Then the (primitve) hypergeometric sheaf*

$$\mathcal{H}yp_\psi(\alpha, \beta, \text{ both square roots of } \alpha\beta : \mathbb{1}, \alpha\beta)$$

is 2-tensor induced.

Proof. This is an instance of [Ka-CC, 6.5]. □

Lemma 2.7. *In characteristic $p = 2$, there exist primitive hypergeometric sheaves $\mathcal{H}$ of type $(4, 3)$ which are not $(\mathbf{S}+)$.*

Proof. Consider the (primitive) hypergeometric sheaf $\mathcal{H}yp_\psi(\mathsf{Char}(5) \smallsetminus \{\mathbb{1}\}; \mathbb{1}, \mathbb{1}, \mathbb{1})$. By [Ka-ESDE, 8.8.1 and 8.8.2], it is orthogonally self-dual. Its G_{geom} is not finite, because its "downstairs" characters, all $\mathbb{1}$, are not pairwise distinct. By [KT8, 4.1.5], it follows that $G_{\mathrm{geom}}^\circ = \mathrm{SO}(4)$. Therefore $\mathcal{H}$ cannot be $(\mathbf{S}+)$, because its G_{geom}° is not a simple algebraic group. □

Lemma 2.8. *In characteristic $p = 3$, primitive hypergeometric sheaves $\mathcal{H}$ of type $(4, 3)$ satisfy $(\mathbf{S}+)$.*

Proof. Since $p = 3$ and $w = 1$, the image Q of $P(\infty)$ in $G := G_{\rm geom}$ of $\mathcal{H}$ is generated by a single element h which is a complex reflection of order 3. If G_0 denotes the normal closure of Q in G, then G/G_0 is cyclic of order coprime to 3 by [KT5, Theorem 4.7].

First we show that G_0 is irreducible in the underlying representation V. As $G_0 \lhd G$, G permutes the m isotypic components of $V|_{G_0}$. But G is assumed to be primitive, so $m = 1$. This means that if φ_0 is an irreducible constituent of the character of the representation $V|_{G_0}$, then φ_0 is G-invariant. But G/G_0 is cyclic, so φ_0 extends to an irreducible character θ of G. As φ lies above φ_0, by Gallagher's theorem [Is, (6.17)], $\varphi = \theta\lambda$ for some irreducible character λ of G/G_0. In this case, $\lambda(1) = 1$ and $\varphi|_{G_0} = \theta_{G_0} = \varphi_0$, which means G_0 is irreducible.

Suppose that G_0 is an irreducible, but imprimitive subgroup of $\mathrm{GL}(V)$ that is generated by complex reflections of order 3. Such a group, by [ST], has index r for some $r \in \mathbb{Z}_{\geq 1}$ in $C_m \wr \mathsf{S}_4$ for some $m \in \mathbb{Z}_{\geq 1}$ divisible by $3r$: $G_0 = A \rtimes \mathsf{S}_4$, where

$$A = \left\{ \mathrm{diag}(\epsilon^{a_1}, \ldots, \epsilon^{a_4}) \mid a_i \in \mathbb{Z},\ r | \sum_{i=1}^{4} a_i \right\},$$

$\epsilon \in \mathbb{C}^\times$ has order m, and S_4 consists of permutational 4×4-matrices. The group G_0 contains exactly 3^8 complex reflections of order 3, each conjugate in G_0 to $\mathrm{diag}(\epsilon^{m/3}, 1, 1, 1)$ or $\mathrm{diag}(\epsilon^{-m/3}, 1, 1, 1)$. All these elements are contained in the normal subgroup A of G_0, so they do not generate G_0, a contradiction.

The remaining possibility is that G_0 is irreducible and primitive. Then the classification theorem of [ST] implies that the primitive complex reflection group G_0 in dimension 4 must be $3 \times \mathrm{Sp}_4(3)$ in one of its 4-dimensional reflection representations, for which it is easy to verify $(\mathbf{S}+)$ for G_0 and hence for G as well. ☐

Lemma 2.9. *In characteristic $p \geq 5$, all primitive hypergeometric sheaves $\mathcal{H}$ of type $(4, 3)$ have* $(\mathbf{S}+)$.

Proof. This is [KT8, 4.1.1]. ☐

3. PREVIOUSLY KNOWN CASES OF TENSOR INDUCTION IN RANKS 8 AND 9

Lemma 3.1. *In characteristic $p \geq 5$, there exist hypergeometric sheaves of type $(8, 2)$ which are 3-tensor induced. More precisely, the tensor induction*

$$[3]_{\otimes\star}\mathcal{K}l_\psi(\mathsf{Char}(3) \smallsetminus \mathsf{Char}(1))$$

is geometrically isomorphic to a multiplicative translate of

$$\mathcal{H}yp_\psi(\mathsf{Char}(9) \smallsetminus \mathsf{Char}(1); \mathsf{Char}(4) \smallsetminus \mathsf{Char}(2)).$$

Proof. This is the special case of [Ka-ESDE, 10.6.11] with its χ_1, χ_2 taken to be the the two characters of order 3. ☐

Lemma 3.2. *In characteristic $p \geq 5$, there exist hypergeometric sheaves of type $(9, 3)$ which are 2-tensor induced. More precisely, choose a prime $r \geq 7, r \neq p$, and a character χ of order r. Then the tensor induction*

$$[2]_{\otimes\star}\mathcal{K}l_\psi(\chi, \chi^2, \chi^{-3})$$

is geometrically isomorphic to a multiplicative translate of

$$\mathcal{H}yp_\psi(\chi, \chi^2, \chi^{-3}, \text{both square roots of each of } \chi^3, \chi^{-2}, \chi^{-1}; \mathsf{Char}(3)).$$

Proof. This is the special case of [Ka-ESDE, 10.6.9] with its χ_1, χ_2, χ_3 taken to be the the three characters χ, χ^2, χ^{-3}. ☐

Remark 3.3. In both Lemmas 3.1 and 3.2, the indicated examples of tensor induced sheaves can be checked to be primitive.

4. Kloosterman sheaves of rank 8 in characteristic $p = 2$

In Lemma 3.1, the "downstairs" characters are the two characters of order 4, which make no sense in characteristic $p = 2$. So we simply erase them.

Theorem 4.1. *In characteristic $p = 2$, the Kloosterman sheaf $\mathcal{K}l_\psi(\mathsf{Char}(9) \smallsetminus \mathsf{Char}(1))$ is 3-tensor induced. More precisely, the tensor induction*

$$[3]_{\otimes\star}\mathcal{K}l_\psi(\mathsf{Char}(3) \smallsetminus \mathsf{Char}(1))$$

is geometrically isomorphic to

$$\mathcal{K}l_\psi(\mathsf{Char}(9) \smallsetminus \mathsf{Char}(1)).$$

Proof. The argument is not conceptual, but rather by means of a Magma calculation. First we recall from [KRLT2, Lemma 1.2] some descent results.The sheaf $\mathcal{K}l_\psi(\mathsf{Char}(3) \smallsetminus \mathsf{Char}(1))$ has a descent to $\mathbb{G}_m/\mathbb{F}_4$, given by the pure of weight zero lisse sheaf $\mathcal{A}$ whose trace function is given as follows: for $k/\mathbb{F}_4$ a finite extension, and $t \in k^\times$,

$$\mathrm{Trace}(\mathsf{Frob}_{t,k}|\mathcal{A}) = (-1/2^{\deg(k/\mathbb{F}_4)}) \sum_{x \in k} \psi_k(x^3/t + x).$$

Let us denote

$$\mathcal{A}(t,k) := \mathrm{Trace}(\mathsf{Frob}_{t,k}|\mathcal{A}).$$

The sheaf $\mathcal{K}l_\psi(\mathsf{Char}(9) \smallsetminus \mathsf{Char}(1))$ has a descent to $\mathbb{G}_m/\mathbb{F}_4$, given by the pure of weight zero lisse sheaf $\mathcal{B}$ whose trace function is given as follows: for $k/\mathbb{F}_4$ a finite extension, and $t \in k^\times$,

$$\mathrm{Trace}(\mathsf{Frob}_{t,k}|\mathcal{B}) = (-1/2^{\deg(k/F_4)}) \sum_{x \in k} \psi_k(x^9/t + x).$$

Let us denote

$$\mathcal{B}(t,k) := \mathrm{Trace}(\mathsf{Frob}_{t,k}|\mathcal{B}).$$

[In both cases, we consider these descents to live on $\mathbb{G}_m/\mathbb{F}_4$ rather than on $\mathbb{G}_m/\mathbb{F}_2$ in order both to have integer traces and to be pure of weight zero.]

It suffices to show that the Kummer pullback $[3]^\star(\mathcal{B})$ and the triple tensor product

$$\mathcal{C} := \bigotimes_{\zeta \in \mu_3} [t \mapsto \zeta t]^\star(\mathcal{A})$$

are are geometrically isomorphic. Indeed, once we have this, we argue as follows. The tensor induction $[3]_{\otimes\star}\mathcal{A}$ is a descent through]3] of sC, , cf, [Ka-ESDE, 10.3.5]. Because $\mathcal{B}$ has all slopes 1/8, its pullback $[3]^\star(\mathcal{B})$ has all slopes 3/8, so is geometrically irreducible (indeed $I(\infty)$irreducible). Therefore $\mathcal{C}$ is geometrically irreducible. A fortiori, its descent $[3]_{\otimes\star}\mathcal{A}$ is geometrically irreducible. Thus both $\mathcal{B}$ and $[3]_{\otimes\star}\mathcal{A}$ are geometrically irreducible, and their $[3]^\star$ pullbacks are geometrically isomorphic. Therefore for some Kummer sheaf $\mathcal{L}_\rho$ with $\rho^3 = \mathbb{1}$, we have a geometric isomorphism of $[3]_{\otimes\star}\mathcal{A}$ with $\mathcal{L}_\rho \otimes \mathcal{B}$. By [Ka-ESDE, 10.6.9], the $I(0)$-representation of $[3]_{\otimes\star}\mathcal{A}$ is precisely $\mathsf{Char}(9) \smallsetminus \mathsf{Char}(1)$. Since $\mathcal{B}$ itself has $\mathsf{Char}(9) \smallsetminus \mathsf{Char}(1)$ as its $I(0)$-representation, then both $\mathcal{B}$ and $\mathcal{L}_\rho \otimes \mathcal{B}$ have this $I(0)$-representation, and hence $\rho = \mathbb{1}$.

We now prove that $[3]^\star(\mathcal{B})$ and $\mathcal{C}$ are geometrically isomorphic. Because $[3]^\star(\mathcal{B})$ is geometrically irreducible and of the same rank (8) as $\mathcal{C}$, it suffices to show there is a nonzero hom (as local systems on $\mathbb{G}_m/\overline{\mathbb{F}_4}$ from $[3]^\star(\mathcal{B})$ to $\mathcal{C}$; any such map is an isomorphism. Up to scalars there is at most one isomorphism, as the "ratio" of two is an automorphism of $[3]^\star(\mathcal{B})$; as $[3]^\star(\mathcal{B})$ is geometrcally irreducible, its only endomorphisms are scalars. Thus the hom group

$$H^2_c(\mathbb{G}_m/\overline{\mathbb{F}_4}, [3]^\star(\mathcal{B})^\vee \otimes \mathcal{C})$$

either vanishes, or has dimenion one. As $\mathcal{B}$ is self-dual [Ka-ESDE, 8.8.1], this hom group is also

$$H^2_c(\mathbb{G}_m/\overline{\mathbb{F}_4}, [3]^\star(\mathcal{B}) \otimes \mathcal{C}).$$

We next calculate the Euler-Poincaré characteristic of $[3]^\star(\mathcal{B}) \otimes \mathcal{C}$. The first factor $[3]^\star(\mathcal{B})$ has all slopes $3/8$. The second factor has all slopes $\le 1/2$, simply because $\mathcal{A}$ and each of its multiplicative translates has all slopes $1/2$. Therefore $[3]^\star(\mathcal{B}) \otimes \mathcal{C}$ has all slopes $\le 1/2$, and rank 64. Thus $\mathsf{Swan}_\infty([3]^\star(\mathcal{B}) \otimes \mathcal{C}) \le 32$ (and $[3]^\star(\mathcal{B}) \otimes \mathcal{C}$) is tame at 0). For any lisse sheaf $\mathcal{F}$ on $\mathbb{G}_m$ which is tame at 0, the Euler-Poincaré formula gives

$$h^1_c(\mathbb{G}_m/\overline{\mathbb{F}_4}, \mathcal{F}) - h^2_c(\mathbb{G}_m/\overline{\mathbb{F}_4}, \mathcal{F}) = \mathsf{Swan}_\infty(\mathcal{F}),$$

So either

$$h^2_c(\mathbb{G}_m/\overline{\mathbb{F}_4}, [3]^\star(\mathcal{B}) \otimes \mathcal{C}) = 1 \text{ and } h^1_c(\mathbb{G}_m/\overline{\mathbb{F}_4}, [3]^\star(\mathcal{B}) \otimes \mathcal{C}) \le 33,$$

or

$$h^2_c(\mathbb{G}_m/\overline{\mathbb{F}_4}, [3]^\star(\mathcal{B}) \otimes \mathcal{C}) = 0 \text{ and } h^1_c(\mathbb{G}_m/\overline{\mathbb{F}_4}, [3]^\star(\mathcal{B}) \otimes \mathcal{C}) \le 32.$$

We next calculate the Euler-Poincaré characteristic of $[3]^\star(\mathcal{B}) \otimes [3]^\star(\mathcal{B})^\vee$. Here all slopes are $\le 3/8$, so $\mathsf{Swan}_\infty([3]^\star(\mathcal{B}) \otimes [3]^\star(\mathcal{B})^\vee) \le 64 \times (3/8) = 24$. Here the $h^2_c = 1$, and for any finite extension $\mathbb{F}_q/\mathbb{F}_4$, the eigenvalue of $\mathsf{Frob}_{\mathbb{F}_q}$ on this H^2_c is q. Thus $h^1_c \le 25$. By Deligne's fundamental result [De, 3.3.1], the H^1_c is mixed of weight ≤ 1. So by the Lefschetz trace formula, for any finite extension $\mathbb{F}_q/\mathbb{F}_4$, the estimate

$$\left| q - \sum_{t \in \mathbb{F}_q^\times} (\mathrm{Trace}(\mathsf{Frob}_{t,\mathbb{F}_q} | [3]^\star(\mathcal{B})))^2 \right| \le 25\sqrt{q}.$$

Suppose now that $[3]^\star(\mathcal{B})$ and $\mathcal{C}$ are not geometrically isomorphic. We obtain a contradiction as follows. The H^2_c of $[3]^\star(\mathcal{B}) \otimes \mathcal{C}$ vanishes, and for any finite extension $\mathbb{F}_q/\mathbb{F}_4$ we have the estimate

$$\left| \sum_{t \in \mathbb{F}_q^\times} \mathrm{Trace}(\mathsf{Frob}_{t,\mathbb{F}_q} | [3]^\star(\mathcal{B})) \mathrm{Trace}(\mathsf{Frob}_{t,\mathbb{F}_q} | \mathcal{C}) \right| \le 32\sqrt{q}.$$

A Magma calculation shows that $[3]^\star(\mathcal{B})$ and $\mathcal{C}$ have the same traces at all points of $\mathbb{G}_m(\mathbb{F}_{4^6})$. Thus the sum

$$\sum_{t \in \mathbb{F}_{4^6}^\times} \mathrm{Trace}(\mathsf{Frob}_{t,\mathbb{F}_{4^6}} | [3]^\star(\mathcal{B})) \mathrm{Trace}(\mathsf{Frob}_{t,\mathbb{F}_q} | \mathcal{C})$$

ie equal to the sum

$$\sum_{t \in \mathbb{F}_{4^6}^\times} \mathrm{Trace}(\mathsf{Frob}_{t,\mathbb{F}_{4^6}} | [3]^\star(\mathcal{B}))^2.$$

This first sum has absolute value $\le 32 * 2^6 = 2048$, while the second sum is within $25 \times 2^6 = 1600$ of $q = 4^6 = 4048$. So the first sum is at most 2048, while the second sum is at least 2448, the desired contradiction. $\square$

Remark 4.2. In any characteristic $p \ne 3$, the Kloosterman sheaf $\mathcal{K}l_\psi(\mathsf{Char}(9) \smallsetminus \mathsf{Char}(1))$ is primitive.

5. Hypergeometric sheaves of type $(9,1)$ in characteristic $p=3$

In Lemma 3.2, the "downstairs" characters are $\mathsf{Char}(3)$, of which only $\mathbb{1}$ makes sense in characteristic $p=3$. So we erase the others.

Theorem 5.1. *In characteristic $p=3$, pick an prime $r \ge 7$, and fix a character χ of order r. Then either the hypergeometric sheaf*

$$\mathcal{H}yp_\psi(\chi,\chi^2,\chi^{-3},\text{both square roots of each of } \chi^3,\chi^{-2},\chi^{-1};\mathbb{1})$$

or the hypergeometric sheaf

$$\mathcal{H}yp_\psi(\chi,\chi^2,\chi^{-3},\text{both square roots of each of } \chi^3,\chi^{-2},\chi^{-1};\chi_{\mathrm{quad}})$$

with χ_{quad} the quadratic chacter, is 2-tensor induced. More precisely, a multiplicative translate of one of them is the tensor induction

$$[2]_{\otimes\star}\mathcal{K}l_\psi(\chi,\chi^2,\chi^{-3}).$$

Proof. All Kloosterman sheaves $\mathcal{K}l_\psi(\rho_1,\rho_2,\rho_3)$ with $\rho_1\rho_2\rho_3=\mathbb{1}$ have isomorphic $I(\infty)$-representations, cf. [Ka-ESDE, 8.6.4], call it Wild_3. Because $p=3$, Wild_3 is $P(\infty)$-irreducible. The dual of $\mathcal{K}l_\psi(\rho_1,\rho_2,\rho_3)$ is $\mathcal{K}l_{\overline{\psi}}(\overline{\rho_1},\overline{\rho_2},\overline{\rho_3})$, which is in turn geometrically isomorphic to $[t\mapsto -t]^\star\mathcal{K}l_\psi(\overline{\rho_1},\overline{\rho_2},\overline{\rho_3})$. Looking at the $I(\infty)$-representations, we find an isomorphism

$$\mathsf{Wild}_3^\vee \cong [t\mapsto -t]^\star\mathsf{Wild}_3.$$

Let us denote

$$\mathcal{A} := \mathcal{K}l_\psi(\chi,\chi^2,\chi^{-3}), \mathcal{C} := \mathcal{K}l_\psi(\chi,\chi^2,\chi^{-3})\otimes[t\mapsto -t]^\star\mathcal{K}l_\psi(\chi,\chi^2,\chi^{-3}),$$

and

$$\mathcal{B} := \mathcal{H}yp_\psi(\chi,\chi^2,\chi^{-3},\text{both square roots of each of } \chi^3,\chi^{-2},\chi^{-1};\mathbb{1}).$$

By [Ka-ESDE, 10.6.5(2(1)], $[2]_{\otimes\star}\mathcal{A}$ and $\mathcal{B}$ have the same $I(0)$-representations as each other.

What about their $I(\infty)$-representations? By [Ka-ESDE, 10.3.5], the $I(\infty)$-representation of $\mathcal{C}=[2]^\star[2]_{\otimes\star}(\mathcal{A})$ is

$$\mathsf{Wild}_3\otimes[t\mapsto -t]\mathsf{Wild}_3 \cong \mathsf{Wild}_3\otimes\mathsf{Wild}_3^\vee = \mathbb{1}\oplus\mathrm{End}^0(\mathsf{Wild}_3).$$

Because Wild_3 is $P(\infty)$-irreducible, the space of $P(\infty)$-invariants in $\mathsf{Wild}_3\otimes\mathsf{Wild}_3^\vee$ is one-dimensional. Thus $\mathrm{End}^0(\mathsf{Wild}_3)$ is totally wild. The slopes of $\mathrm{End}^0(\mathsf{Wild}_3)$ are $\le 1/3$, and its rank is 8. By the integrality of Swan conductors, we have $\mathsf{Swan}_\infty(\mathrm{End}^0(\mathsf{Wild}_3))\le 2$. Recalling that

$$\mathsf{Swan}_\infty([2]^\star[2]_{\otimes\star}(\mathcal{A})) = \mathsf{Swan}_\infty(\mathbb{1}\oplus\mathrm{End}^0(\mathsf{Wild}_3)) = \mathsf{Swan}_\infty(\mathrm{End}^0(\mathsf{Wild}_3)),$$

Thus the $I(\infty)$-representation of $[2]^\star[2]_{\otimes\star}(\mathcal{A})$ is the direct sum of a totally wild part of rank 8, with $\mathbb{1}$. Therefore the $I(\infty)$-representation of $[2]_{\otimes\star}(\mathcal{A})$ is the direct sum of a totally wild part of rank 8, and either $\mathbb{1}$ or χ_{quad}. Thus $\mathsf{Swan}_\infty([2]_{\otimes\star}(\mathcal{A})\ge 1$, while $\mathsf{Swan}_\infty([2]^\star[2]_{\otimes\star}(\mathcal{A}))\le 2$. Therefore

$$\mathsf{Swan}_\infty([2]_{\otimes\star}(\mathcal{A})=1,\ \mathsf{Swan}_\infty([2]^\star[2]_{\otimes\star}(\mathcal{A}))=2.$$

Thus the semisimplification of $[2]_{\otimes\star}(\mathcal{A})$ is either the direct sum of a Kloosterman sheaf of rank 8 with one of $\mathbb{1},\chi_{\mathrm{quad}}$, or it is a multiplicative translate of one of the asserted hypergeometrics. As neither $\mathbb{1}$ nor χ_{quad} is among the characters occurring in the $I(0)$-representation of $[2]_{\otimes\star}(\mathcal{A})$, it must be the latter. ∎

Remark 5.2. In Theorem 5.1, each of the specified local systems of type $(9,1)$ can be checked to be primitive.

6. The case $p = 2$

Theorem 6.1. *In characteristic $p = 2$, no primitive, geometrically irreducible hypergeometric sheaf $\mathcal{H}$ of type $(8, m)$ with $8 > m > 0$ is tensor induced. In the case $(8, m)$ with $6 \geq m > 0$, primitivity is not needed.*

Proof. Consider first the case $(8, 7)$. If G_{geom} is infinite, we are done by [KT8, 4.1.5]. Suppose G_{geom} is finite and primitive. Since $p = 2$ and $w = 1$, the image Q of $P(\infty)$ in G_{geom} is generated by a single element h which is a (true) reflection; let G_0 denote the normal closure of Q in G_{geom}. Then G_{geom}/G_0 is cyclic of odd order by [KT5, Theorem 4.7]. Moreover, as shown in the proof of [KT8, Theorem 4.2.3], G_0 is either S_9 in its deleted permutation representation, or it is the Weyl group $W(E_8)$ in its reflection representation. In both of these cases, one know that $(\mathbf{S}+)$ holds. Indeed, the quasisimple subgroup $G_0^{(\infty)}$ (which is either A_9 or $2 \cdot \Omega_8^+(2)$) acts irreducibly in the representation in question, but has no proper subgroup of index ≤ 8 and no nontrivial irreducible projective representation of degree < 8, see [Atlas], and hence $(\mathbf{S}+)$ already holds for $G_0^{(\infty)}$.

Consider next the case of an $\mathcal{H}$ of type $(8, m)$ with $6 \geq m > 0$, and the map of G_{geom} to S_3 arising if $\mathcal{H}$ is 3-tensor induced. The image of $P(\infty)$ is either trivial or it is a 2-group inside S_3.

Suppose first that the image of $P(\infty)$ is nontrivial. Then up to conjugation it is the cyclic group generated by the transposition $(1, 2)$. But the image of $I(\infty)$ normalizes the image of $P(\infty)$. Therefore the image of $I(\infty)$ is again the cyclic group generated by $(1, 2)$. In this case, $\mathcal{H}$ is tensor decomposable as an $I(\infty)$-representation, a contradiction by [KRLT3, 10.4].

Suppose next that the image of $P(\infty)$ is trivial. In this case, the map to S_3 factors through the group $I(\infty)/P(\infty)$, a pro-cyclic group of odd pro-order. So either the image of $I(\infty)$ is trivial, or is the cyclic group generated by a 3-cycle. If the image is trivial, then $\mathcal{H}$ is tensor decomposable as an $I(\infty)$-representation, contradiction. If the image is nontrivial, then the Kummer pullback $[3]^\star\mathcal{H}$ is tensor decomposable. If $w := 8 - m$, the dimension of the wild part $\mathsf{Wild}_{\mathcal{H}}$ of $\mathcal{H}$, is prime to 3, then $[3]^\star\mathsf{Wild}_{\mathcal{H}}$ is still $I(\infty)$-irreducible and totally wild (all slopes $3/w$), and again a contradiction by [KRLT3, 10.4].

This $3 \nmid m$ consideration leaves only the cases when $\mathcal{H}$ has type $(8, 5)$ or $(8, 2)$.

Let us treat first the case of $(8, 2)$. Here the wild part $\mathsf{Wild}_{\mathcal{H}}$ has rank 6, so is the Kummer direct image $[3]_\star\mathsf{Wild}_2$ of a totally wild $I(\infty)$-representatiomn of rank 2 with both slopes $1/2$. Then $[3]^\star\mathsf{Wild}_{\mathcal{H}}$ is

$$[3]^\star\mathsf{Wild}_{\mathcal{H}} = [3]^\star[3]_\star\mathsf{Wild}_2 \cong \bigoplus_{\zeta \in \mu_3} [t \mapsto \zeta t]^\star\mathsf{Wild}_2.$$

At this point, we invoke the following lemma.

Lemma 6.2. *Let p be a prime, q a (possibly trivial) power p^e of p for some $e \geq 0$. Let Wild_q be an irreducible $I(\infty)$-representation of dimension q with $\mathsf{Swan}_\infty(\mathsf{Wild}_q) = 1$. Then Wild_q is $P(\infty)$-irreducible, and for any $\lambda \neq 1$ in $\overline{\mathbb{F}_p}^\times$, Wild_q is not $P(\infty)$-isomorphic to $[t \mapsto \lambda t]^\star\mathsf{Wild}_q$.*

Proof. In the case $q = 1$, Wild_1 is of the form $\mathcal{L}_\rho \otimes \mathcal{L}_{\psi(ax)}$ for some Kummer sheaf $\mathcal{L}_\rho$ and some $a \in \overline{\mathbb{F}_p}^\times$. So in this case the assertion amounts to the observation that

$$\mathcal{L}_{\psi(ax)} \otimes \mathcal{L}^\vee_{\psi(\lambda ax)} \cong \mathcal{L}_{\psi(a(1-\lambda)x)}$$

is nontrivial on $P(\infty)$.

Suppose now that $q > 1$. By [Ka-GKM, 8.6.3(2)], for any $\lambda \neq 1$ in $\overline{\mathbb{F}_p}^\times$, we have

$$\det(\mathsf{Wild}_q) = \det([t \mapsto \lambda t]^\star\mathsf{Wild}_q).$$

That Wild_q is $P(\infty)$-irreducible is [Ka-GKM, 1.14.2]. We now argue by contradiction. Suppose that for some $\lambda \neq 1$ in $\overline{\mathbb{F}_p}^\times$, we have a $P(\infty)$-isomorphism $\mathsf{Wild}_q \cong [t \mapsto \lambda t]^\star \mathsf{Wild}_q$. Because $P(\infty) \lhd I(\infty)$, it follows that for some Kummer sheaf $\mathcal{L}_\rho$, we have an $I(\infty)$-isomorphism

$$\mathcal{L}_\rho \otimes \mathsf{Wild}_q \cong [t \mapsto \lambda t]^\star \mathsf{Wild}_q.$$

Comparing determinants, we see that $\det(\mathcal{L}_\rho \otimes \mathsf{Wild}_q) = \det(\mathsf{Wild}_q)$. But

$$\det(\mathcal{L}_\rho \otimes \mathsf{Wild}_q) = \mathcal{L}_{\rho^q} \otimes \det(\mathsf{Wild}_q).$$

Therefore $\det(\mathsf{Wild}_q) = \mathcal{L}_{\rho^q} \otimes \det(\mathsf{Wild}_q)$, and hence $\rho^q = \mathbb{1}$. Being in characteristic p, this forces $\rho = \mathbb{1}$. Thus we find an $I(\infty)$-isomorphism $\mathsf{Wild}_q \cong [t \mapsto \lambda t]^\star \mathsf{Wild}_q$, contradicting [Ka-ESDE, 8.6.3(1)]. □

We now return to $\mathcal{H}$ of type $(8,2)$ in characteristic $p = 2$. We argue by contradiction. If $\mathcal{H}$ is 3-tensor induced, then $[3]^\star\mathcal{H}$ is tensor decomposable, and hence [KRLT3, 10.1, 10.4] linearly tensor decomposable, as $I(\infty)$-representation. Then $[3]^\star\mathcal{H}$ is linearly tensor decomposable as $P(\infty)$-representation. This representation is

$$2 \cdot \mathbb{1} + \bigoplus_{\zeta \in \mu_3} [t \mapsto \zeta t]^\star \mathsf{Wild}_2.$$

The key point is that we have three pairwise nonisomorphic irreducible $P(\infty)$-representations of dimension 2, along with a two dimensional trivial representation.

Suppose that there exist two dimensional representations $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of $P(\infty)$ such that

$$\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C} \cong 2 \cdot \mathbb{1} + \bigoplus_{\zeta \in \mu_3} [t \mapsto \zeta t]^\star \mathsf{Wild}_2.$$

It cannot be the case that each of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ is the direct sum of two linear characters, for then their tensor product is the sum of eight linear characters. So at least one of them, say $\mathcal{A}$, is $P(\infty)$-irreducible. Write $\mathcal{D} := \mathcal{B} \otimes \mathcal{C}$. Then $\mathcal{A} \otimes \mathcal{D}$ has a two dimensional space of $P(\infty)$-invariants. In other words, $\mathcal{A}^\vee$ occurs with multiplicity 2 in $\mathcal{D}$. But $\mathcal{D}$ has rank 4, while $\mathcal{A}$ has rank 2, so we must have $\mathcal{D} = 2\mathcal{A}^\vee$. But then $\mathcal{A} \otimes \mathcal{D} = 2\mathrm{End}(\mathcal{A})$ has all multiplicities even. This is a contradiction, since Wild_2 occurs with multiplicity one.

We now turn to the case of an $\mathcal{H}$ of type $(8,5)$. Here the $P(\infty)$-representation of $[3]^\star\mathcal{H}$ is

$$5 \cdot \mathbb{1} + \alpha + \beta + \gamma,$$

with α, β, γ being three distinct nontrivial linear characters of $P(\infty)$. Suppose this is $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}$. In any of the factors is $P(\infty)$-irreducible,say $\mathcal{A}$, then exactly as in the $(8,2)$ case the dimension of the space of $P(\infty)$-invariants is the multiplicity of $\mathcal{A}^\vee$ in $\mathcal{B} \otimes \mathcal{C}$. But this multiplicity is at most 2 (rather than 5). So each of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ is the sum of two linear characters, say

$$(A + B)(S + T)(X + Y).$$

Among the 8 linear characters we get by multiplying out, precisely 5 are trivial. Write $\mathcal{D} := (S + T)(X + Y)$. If $A \otimes \mathcal{D}$ contains 4 trivial characters, then $\mathcal{D}$ is $4A^\vee$, and all multiplicites are multiples of 4, a contradiction. If $A \otimes \mathcal{D}$ contains just one trivial character, then $B \otimes \mathcal{D}$ contains 4 trivial characters, again a contradiction. At the expense of interchanging A and B, we may assume that

$A \otimes \mathcal{D}$ contains 3 trivial characters, $B \otimes \mathcal{D}$ contains 2 trivial characters.

Thus among the four characters of $\mathcal{D}$, namely SX, SY, TX, TY, precisely 3 are $A^\vee$, and precisely 2 are $B^\vee$. At the expense of interchanging S and T, and of interchanging X and Y, we may assume that each of SX, SY, TX is $A^\vee$. Then $SX = SY$ and hence $X = Y$. But then $\mathcal{D}$ has even multiplicities, hence also $\mathcal{A}\otimes\mathcal{D}$ has even multiplicities, a contradiction. This completes the $(8,5)$ case, and, with it, the proof of Theorem 6.1. ☐

Theorem 6.3. *In characteristic $p=2$, no geometrically irreducible hypergeometric sheaf $\mathcal{H}$ of type $(9,m)$ with $9>m\geq 0$ is tensor induced.*

Proof. The case of Kloosterman sheaves of rank 9 is done in [KT5, 3.4]. The case $(9,8)$ is done by combining [KT8, 4.2.3] and [KT8, 4.1.5]. For $\mathcal{H}$ of type $(9,m)$ with $7\geq m>0$, we argue as follows. In these cases, the dimension $w:=9-m$ of the wild part is ≥ 2. So if $\mathcal{H}$ were 2-tensor induced, the resulting map of $I(\infty)$ to S_2 would be trivial on $P(\infty)$, and the image of a generator of $I(\infty)/P(\infty)$ would be a transposition, cf. [KT5, 3.2(ii)]. But $I(\infty)/P(\infty)$ has pro-order prime to $p=2$. So $\mathcal{H}$ is tensor decomposable, contradicting [KRLT3, 10.1, 10.4]. ☐

7. The case $p=3$

Theorem 7.1. *In characteristic $p=3$, no geometrically irreducible hypergeometric sheaf $\mathcal{H}$ of type $(8,m)$ with $8>m\geq 0$, is tensor induced.*

Proof. The case of Kloosterman sheaves of rank 8 is done in [KT5, 3.4]. The case of type $(8,7)$ is done in [KT8, 4.4.1]. Suppose now that $\mathcal{H}$ has type $(8,m)$ with $6\geq m>0$. In these cases, $\mathcal{H}$ is tensor indecomposable by [KT5, Lemma 2.4]. By way of contradiction, assume $\mathcal{H}$ is 3-tensor induced.

Consider the case $m<6$, so that the dimension $w:=8-m$ of the wild part is ≥ 3. Then the resulting map of $I(\infty)$ to S_3 would be trivial on $P(\infty)$, and the image of a generator of $I(\infty)/P(\infty)$ would be a 3-cycle, cf. [KT5, 3.2(ii)]. But $I(\infty)/P(\infty)$ has pro-order prime to $p=3$. So $\mathcal{H}$ is tensor decomposable, a contradiction.

Suppose finally that $m=6$. Then the wild part has dimension 2, and so the image Q of $P(\infty)$ in $G_{\rm geom}$ is generated by an element h of order 3 which acts in the underlying representation V as $\mathrm{diag}(\zeta_3,\overline{\zeta_3},1,1,1,1,1,1)$. In particular, h has trace 5. Suppose h permutes the 3 tensor factors V_1,V_2,V_3 of V nontrivially. Without any loss of generality, we may assume that

$$h: V_1\to V_2\to V_3\to V_1.$$

Then the arguments in the proof of [GT3, Lemma 2.25] show that $\mathrm{Trace}(h)=2$. More precisely, if (e^1_1,e^1_2) is a basis of V_1, then $(e^1_i\otimes e^2_j\otimes e^3_k\mid 1\leq i,j,k\leq 2)$ is a basis of V, where

$$h: e^1_i\mapsto e^2_i\mapsto e^3_i\mapsto e^1_i$$

for $i=1,2$. Now observe that h permutes the indicated 8 basis vectors of V, fixing exactly two of them: $e^1_1\otimes e^2_1\otimes e^3_1$ and $e^1_2\otimes e^2_2\otimes e^3_2$. Hence $\mathrm{Trace}(h)=2$.

Since the element h has trace 5 on V, we conclude that h acts trivially on $\{V_1,V_2,V_3\}$. Thus Q acts trivially on $\{V_1,V_2,V_3\}$. This closed condition also holds for every conjugate of Q in $G_{\rm geom}$. Hence it holds for the Zariski closure G_0 of the normal closure of Q in $G_{\rm geom}$. In other words, G_0 acts trivially on $\{V_1,V_2,V_3\}$. On the other hand, G/G_0 is a finite cyclic group of order coprime to 3 by [KT5, Theorem 4.7]. It follows that G cannot permute V_1,V_2,V_3 transitively, again a contradiction. ☐

Theorem 7.2. *In characteristic $p=3$, no hypergeometric sheaf $\mathcal{H}$ of type $(9,m)$ with $9>m\geq 0$ and $m\neq 1$ is tensor induced.*

Proof. The case of $(9,8)$ is done in [KT8, 4.1.1]. It remains to treat the types $(9,m)$ with $7\geq m>0$. In these cases, the dimension $w:=9-m$ of the wild part is ≥ 2. So if $\mathcal{H}$ were 2-tensor induced, the resulting map of $I(\infty)$ to S_2 would be trivial on $P(\infty)$, and the image of a generator of $I(\infty)/P(\infty)$

would be a transposition, cf. [KT5, 3.2(ii)]. Therefore $[2]^\star\mathcal{H}$ would be tensor decomposed, and hence linearly tensor decomposed.

If w is odd, the $[2]^\star\mathsf{Wild}_\mathcal{H}$ is totally wild and $I(\infty)$-irreducible, contradicting [KRLT3, 10.1, 10.4].

It remains to treat the types $(9,7),(9,5),(9,3)$. The case $(9,3)$ is done in [KT5, 3.6].

For $(9,5)$ and $(9,7)$, we argue as we did in the $p=2$ treatment of the case of $(8,5)$. Consider first an $\mathcal{H}$ of type $(9,5)$. The $P(\infty)$-representation of $[4]^\star\mathcal{H}$ is

$$5\cdot\mathbb{1}+\alpha+\beta+\gamma+\delta,$$

with $\alpha,\beta,\gamma,\delta$ being four distinct nontrivial linear characters of $P(\infty)$. Suppose this is $\mathcal{A}\otimes\mathcal{B}$. We cannot have $\mathcal{A}$ an irreducible $P(\infty)$-representation, otherwise the dimension of the invariants in $\mathcal{A}\otimes\mathcal{B}$ is the multiplicity of $\mathcal{A}^\vee$ in $\mathcal{B}$, which is at most 1 (rather than 5).

As irreducible representations of $P(\infty)$ are either linear or of dimension $\geq p=3$, each of $\mathcal{A},\mathcal{B}$ is the sum of three linear characters, say

$$(A+B+C)(X+Y+Z).$$

Then of the nine characters we get by multiplying out, precisely 5 are trivial. We cannot have $A(X+Y+Z)=3\mathbb{1}$, otherwise each of X,Y,Z is $A^\vee$ and all multiplicties would be divisible by 3. Similarly for $B(X+Y+Z)$ and $C(X+Y+Z)$. At the expense of reordering A,B,C, we may assume that each of $A(X+Y+Z)$ and $B(X+Y+Z)$ contains precisely two trivial characters, and $C(X+Y+Z)$ contains precisely one trivial character. At the expense of reordering X,Y,Z, we may assume that $X=Y=A^\vee$. Precisely two of X,Y,Z are $B^\vee$, so at least one of X,Y is equal to B^vee. Therefore $A^\vee=B^\vee$, i.e., $A=B$. Then

$$\mathcal{A}\otimes\mathcal{B}=(2A+C)(2A^\vee+Z)=4\cdot\mathbb{1}+2AZ+2CA^\vee+CZ.$$

But $C(X+Y+Z)=C(2A^\vee+Z)$ contains $\mathbb{1}$ precisely once, so we must have $CZ=\mathbb{1}$. Then $\mathcal{A}\otimes\mathcal{B}$ is $5\cdot\mathbb{1}+2AZ+2CA^\vee$, contradicting the fact that each of $\alpha,\beta,\gamma,\delta$ occurs with multiplicity one. Thus $[4]^\star\mathcal{H}$, and a fortiori $[4]^\star\mathcal{H}$ is tensor indecomposable for $P(\infty)$,

In the case of an $\mathcal{H}$ of type $(9,7)$, the $P(\infty)$-representation of $[2]^\star\mathcal{H}$ is

$$5\cdot\mathbb{1}+\alpha+\beta,$$

with α,β two distinct nontrivial linear characters of $P(\infty)$. Exactly as in the $(9,5)$ case, each of $\mathcal{A},\mathcal{B}$ is the sum of three linear characters, say

$$(A+B+C)(X+Y+Z).$$

None of $A(X+Y+Z)B(X+Y+Z)C(X+Y+Z)$ can be $3\cdot\mathbb{1}$. So each contains at most two trivial characters, giving at most 6 trivial characters (rather than 7). Thus $[2]^\star\mathcal{H}$ is tensor indecomposable for $P(\infty)$, ☐

8. The case $p\geq 5$

Theorem 8.1. *In character $p\geq 5$, no geometrically irreducible hypergeometric sheaf of type $(8,m)$ with $8>m\geq 0$, $m\neq 2$, is tensor induced.*

Proof. The case of Kloosterman sheaves of rank 8 is done in [KT5, 1.7]. If $\mathcal{H}$ of type $(8,m)$ with $8>m>0$ is tensor induced, the map of $I(\infty)$ to S_3 must be trivial on the p-group $P(\infty)$ (because $p\geq 5$), and the image of a generator of $I(\infty)/P(\infty)$ must be a three cycle (if it were trivial, $\mathcal{H}$ would be tensor decomposable for $I(\infty)$, contradicting [KRLT3, 10.4]). Then $[3]^\star\mathcal{H}$ is tensor decomposable, hence linearly tensor decomposable, for $I(\infty)$, and a fortiori for $P(\infty)$. If the dimension $w=8-m$ of the wild part is prime to 3, then $[3]^\star\mathsf{Wild}_\mathcal{H}$ is totally wild and $I(\infty)$-irreducible, contradicting [KRLT3, 10.4]. It remains to treat the case $(8,5)$. Here we repeat verbatim the $p=2$ discussion of the $(8,5)$ case. ☐

Theorem 8.2. *In character $p \geq 5$, no geometrically irreducible hypergeometric sheaf of type $(9, m)$ with $9 > m \geq 0$, $m \neq 3$, is tensor induced.*

Proof. The case of Kloosterman sheaves of rank 9 is done in [KT5, 1.7]. If $\mathcal{H}$ of type $(9, m)$ with $9 > m > 0$ is tensor induced, the map of $I(\infty)$ to S_2 must be trivial on the p-group $P(\infty)$ (because p is odd), and the image of a generator of $I(\infty)/P(\infty)$ must be a transposition (if it were trivial, $\mathcal{H}$ would be tensor decomposable for $I(\infty)$, contradicting [KRLT3, 10.4]). Then $[2]^\star\mathcal{H}$ is tensor decomposable, hence linearly tensor decomposable, for $I(\infty)$, and a fortiori for $P(\infty)$. If the dimension $w = 9 - m$ of the wild part is odd, then $[2]^\star\mathsf{Wild}_{\mathcal{H}}$ is totally wild and $I(\infty)$-irreducible, contradicting [KRLT3, 10.4].

Thus it remains to treat the cases $(9, 7), (9, 5), (9, 1)$. The case $(9, 1)$ is done by [KT5, 1.9]. The cases of $(9, 7)$ and $(9, 5)$ are done exactly as they were in the $p = 3$ case. ☐

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