ON A QUESTION OF BROWNING AND HEATH-BROWN

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To Klaus Roth, with admiration and respect

1. Introduction, and Statement of the Main Result

Let k be a finite field, p its characteristic, and

$$\psi : (k, +) \to \mathbb{Z}[\zeta_p]^\times \subset \mathbb{C}^\times$$

a nontrivial additive character of k. We are given a polynomial f in $n \geqslant 1$ variables over k of degree $d \geqslant 1$ which is a "Deligne polynomial", *i.e.* its degree d is prime to p and its highest degree term, say f_d, is a homogeneous form of degree d in n variables which is nonzero, and whose vanishing, if $n \geqslant 2$, defines a smooth hypersurface in the projective space $\mathbb{P}^{n-1}$. For a Deligne polynomial f as above, one has Deligne's fundamental estimate [2, 8.4]

$$\left| \sum_{x \in k^n} \psi(f(x)) \right| \leqslant (d-1)^n (\#(k))^{n/2}.$$

Suppose now that g is an arbitrary polynomial over k in n variables, of degree $e < d$. Then for every $t \in k^\times$, the polynomial

$$tf(x) + g(x) + \frac{1}{t}$$

is still a Deligne polynomial of degree d in n variables over k. Browning and Heath-Brown [1] asked when one had cancellation in the sum

$$\sum_{t \in k^\times} \sum_{x \in k^n} \psi\left(tf(x) + g(x) + \frac{1}{t}\right),$$

whose absolute value is, in view of Deligne's result, trivially bounded by

$$(d-1)^n (\#(k))^{(n+2)/2}.$$

In their application [1], $d = 3$, and g is linear, and so the following theorem applies.

Theorem 1.1. *Let $d \geqslant 3$ be prime to p, f a Deligne polynomial of degree d in n variables over k, and g an arbitrary polynomial over k in n variables. Suppose that p is odd, and that g is of degree $e < d/2$. Then we have the estimate*

$$\left| \sum_{t \in k^\times} \sum_{x \in k^n} \psi\left(tf(x) + g(x) + \frac{1}{t}\right) \right| \leqslant 2(d-1)^n (\#(k))^{(n+1)/2}.$$

It will be useful, for certain inductive arguments, to say that a polynomial f in $n = 0$ variables over k, *i.e.* a constant in k, is a "Deligne polynomial", of any prime to p degree one likes, if that constant is nonzero. With this interpretation, the theorem above remains valid when $n = 0$; it becomes the usual estimate for classical Kloosterman sums, proven by Weil [14] and foreseen by Hasse [7] as a consequence of the Riemann hypothesis for curves over finite fields.

Our methods are cohomological, and allow us to treat some other sums "with the same shape", as well as "twists" by multiplicative characters χ, for example, sums of the form

$$\sum_{t\in k^\times}\sum_{x\in k^n}\chi(t)\,\psi\!\left(tf(x)+g(x)+\frac{1}{t}\right),$$

cf. Section 8. We also give estimates for certain sums of the form

$$\sum_{t\in k^\times}\sum_{x\in k^n}\psi(tf(x)+g(x)),$$

cf. Section 5.

2. Cohomological Reformulation of the Main Result

In this section, we continue to work over the finite field k. We fix $d \geqslant 2$ prime to p, f a Deligne polynomial of degree d in $n \geqslant 0$ variables over k, and g an arbitrary polynomial over k in n variables of degree $e < d$. We also choose a prime number $\ell \neq p$, and a group isomorphism $\mu_p(\mathbb{C}) \cong \mu_p(\overline{\mathbb{Q}}_\ell)$, by means of which we view our chosen additive character of k as now having values in $\overline{\mathbb{Q}}_\ell$,

$$\psi : (k,+) \to \mathbb{Z}[\zeta_p]^\times \subset \overline{\mathbb{Q}}_\ell^\times.$$

Henceforth, we work entirely with $\overline{\mathbb{Q}}_\ell$-sheaves.

On $\mathbb{A}^1/k$ we have the Artin–Schreier sheaf $\mathcal{L}_\psi$, and on the product space

$$\mathbb{A}^n \times \mathbb{G}_m/k,$$

with coordinates (x,t), we have the pulled back Artin–Schreier sheaf

$$\mathcal{L}_{\psi(tf(x)+g(x)+1/t)}.$$

The fundamental object of interest to us is the sheaf $\mathcal{F}$ on $\mathbb{G}_m/k$ we obtain by projecting $\mathbb{A}^n \times \mathbb{G}_m/k$ onto the second factor and taking compact cohomology of this Artin–Schreier sheaf along the fibres. More precisely, we take

$$\mathcal{F} := R^n(pr_2)_!\mathcal{L}_{\psi(tf(x)+g(x)+1/t)}.$$

When $n \geqslant 1$,

$$t \mapsto tf(x)+g(x)+\frac{1}{t}$$

is a one-parameter family of Deligne polynomials over G_m/k, so we know [13, 3.5.11] that $\mathcal{F}$ is lisse on G_m/k of rank $(d-1)^n$, and pure of weight n. Moreover, we know that for $i \neq n$, the sheaves $R^i(pr_2)_!\mathcal{L}_{\psi(tf(x)+g(x)+1/t)}$ all vanish.

In the case $n = 0$, f is a nonzero constant, g is an arbitrary constant, the projection pr_2 above is the identity, and the sheaf $\mathcal{F}$ on G_m/k, lisse of rank one and pure of weight zero, is $\mathcal{L}_{\psi(tf+g+1/t)}$.

The trace function of $\mathcal{F}$ is given as follows. For a finite extension field E/k, denote by ψ_E the additive character of E obtained by composition with the trace map $\mathrm{Trace}_{E/k}$. For any such E/k, and for any point $t \in E^\times = \mathbb{G}_m(E)$, we have

$$\mathrm{Trace}(\mathrm{Frob}_{E,t} \,|\mathcal{F}) = (-1)^n \sum_{x \in E^n} \psi_E\left(tf(x) + g(x) + \frac{1}{t}\right).$$

By the Lefschetz trace formula applied to $\mathcal{F}$, we have, for any finite extension E/k, the identity

$$\sum_{t \in E^\times} (-1)^n \sum_{x \in E^n} \psi_E\left(tf(x) + g(x) + \frac{1}{t}\right)$$
$$= \mathrm{Trace}(\mathrm{Frob}_E \,|H^2_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F})) - \mathrm{Trace}(\mathrm{Frob}_E \,|H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F})).$$

By Deligne's main theorem [4, 3.3.1], the fact that $\mathcal{F}$ is lisse and pure of weight n assures us that the cohomology group $H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F})$ is mixed of weight $\leqslant n+1$, *i.e.* each eigenvalue α of Frob_E is an algebraic number which is pure of some integer weight $w = w(\alpha) \leqslant n+1$ (all of its complex absolute values are $\mathrm{Card}(E)^{w/2}$). On the other hand, one knows that $H^2_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F})$ is pure of weight $n+2$.

Thus Theorem 1.1 follows from the following results on the cohomology of $\mathcal{F}$.

Theorem 2.1. *If p is odd and $\deg(g) := e < d/2$, then we have the following two results:*
(i) $H^2_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}) = 0$.
(ii) $\dim H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}) \leqslant 2(d-1)^n$.

Remark 2.2. Indeed, assertion (i) of the above theorem is equivalent to the statement that there exists a constant C such that for every finite extension E/k, we have the estimate

$$\left|\sum_{t \in E^\times} (-1)^n \sum_{x \in E^n} \psi_E\left(tf(x) + g(x) + \frac{1}{t}\right)\right| \leqslant C(\#(E))^{(n+1)/2}.$$

That assertion (i) implies such an estimate, with C taken to be the dimension of $H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F})$, is obvious from the known weights of the compact cohomology groups of $\mathcal{F}$. Conversely, if such an estimate holds, then certainly the $\limsup_E$ of the ratios

$$\left|\sum_{t \in E^\times} (-1)^n \sum_{x \in E^n} \psi_E\left(tf(x) + g(x) + \frac{1}{t}\right)\right| \Big/ (\#(E))^{(n+2)/2}$$

vanishes. But in view of the fact that $H^2_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F})$ is pure of weight $n+2$, while the H^1 is mixed of lower weight, this $\limsup_E$ is precisely the dimension of $H^2_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F})$, *cf.* [8, 2.2.2.1].

3. Further Reductions, via Dwork-Regularity

The theorem of the last section is geometric, *i.e.* it concerns what happens after we extend scalars from k to $\overline{k}$. This allows us to pass from k to a finite extension E/k and work instead with the sheaf $\mathcal{F}$ on $\mathbb{G}_m/E$. The advantage of doing this is that we can make a linear change of coordinates in $\mathbb{A}^n$ in which our Deligne polynomial f, and more precisely its leading form f_d, is particularly easy to work with.

Let us say that a homogeneous form F in $n \geqslant 1$ variables $X_1, \ldots, X_n$ over E is Dwork-regular with respect to the coordinates $X_1, \ldots, X_n$ if the subscheme of $\mathbb{P}^{n-1}$ defined by the vanishing of F and of all the $X_i \partial F/\partial X_i$ is empty, *cf.* [5, Introduction and last 2 pages], where this notion plays an important role. When F has degree d prime to p, it is the same to say that the $X_i \partial F/\partial X_i$ define the empty subscheme of $\mathbb{P}^{n-1}$, since we have the Euler relation $\mathrm{d}F = \sum_i X_i \partial F/\partial X_i$. It is essentially tautologous that F is Dwork-regular if and only if the following conditions hold: for every nonempty subset $S \subset \{1, \ldots, n\}$, the homogeneous form F_S obtained from F by setting to zero the variables X_i with $i \notin S$ is a Deligne polynomial in the variables X_s, $s \in S$; *i.e.* F_S is nonzero, and, if $\#S \geqslant 2$, its vanishing defines a smooth hypersurface in $\mathbb{P}^{\#S-1}$.

Let us recall from [5, last 2 pages] the following lemma, whose elementary proof we include for the convenience of the reader.

Lemma 3.1. *Let F be a nonzero homogeneous form in $n \geqslant 1$ variables $X_1, \ldots, X_n$ of some degree $d \geqslant 1$, over an algebraically closed field E. If $n \geqslant 2$, assume that the vanishing of F defines a smooth hypersurface in $\mathbb{P}^{n-1}$. Then there exist new coordinates $Y_1, \ldots, Y_n$, i.e.*

$$X_i = \sum_j a_{i,j} Y_j, \quad (a_{i,j}) \in GL(n, E),$$

such that F, written as a homogeneous form in the Y_i, is Dwork-regular with respect to $Y_1, \ldots, Y_n$.

Proof. If $n = 1$, F is already Dwork-regular. If $n \geqslant 2$, we argue as follows. The key fact is that given a finite list of smooth connected closed subschemes $Z_\nu \subset \mathbb{P}^{n-1}$, there exists an E-rational hyperplane, say $H \subset \mathbb{P}^{n-1}$, which is transverse to each Z_ν, *i.e.* if $\dim(Z_\nu) = 0$, then $Z_\nu \cap H = \emptyset$; if $\dim(Z_\nu) \geqslant 1$, then $Z_\nu \cap H$ is smooth of dimension one less; and if Z_ν is empty, there is no condition. We apply this first to find a hyperplane $H_1 \subset \mathbb{P}^{n-1}$ transverse to each member of our first list, which we define to be $\mathbb{P}^{n-1}$ and $\mathrm{Var}(F)$ (which is the hypersurface in $\mathbb{P}^{n-1}$ defined by the vanishing of F). We then apply it to find a hyperplane $H_2 \subset \mathbb{P}^{n-1}$ which is transverse to each member of our second list, which we define to consist of the first list together with the intersections of H_1 with each member of the first list. At the $(i+1)$-st stage, we find a hyperplane $H_{i+1} \subset \mathbb{P}^{n-1}$ which is transverse to each member of our $(i+1)$-st list, which we define to consist of the i-th list together with the intersections of H_i with each member of the i-th list.

In this way, we obtain a transverse system of hyperplanes $H_1, \ldots, H_n$ in $\mathbb{P}^{n-1}$. Taking Y_i to be a linear form whose vanishing defines H_i, we obtain the required coordinate system $Y_1, \ldots, Y_n$. □

Now let us return to the situation of the theorem. Thus f is a Deligne polynomial of degree d prime to p in $n \geqslant 1$ variables over the finite field k. After possibly extending scalars from k to a finite extension, we use the above lemma to reduce to the case where the leading form f_d in f is Dwork-regular with respect to the given coordinates $X_1, \ldots, X_n$. Let us say that the polynomial f itself is affine-Dwork-regular if the form F of degree d in $n+1$ variables $X_0, \ldots, X_n$ obtained from f by homogenization,

$$F(X_0, \ldots, X_n) := X_0^d f\left(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}\right),$$

is Dwork-regular with respect to $X_0, \dots, X_n$. Of course, if a Deligne polynomial of degree d prime to p in n variables is affine-Dwork-regular, then its leading form f_d is Dwork-regular, simply because f_d is $F|_{X_0=0}$.

Lemma 3.2. *Let f be a Deligne polynomial of degree d prime to p in $n \geqslant 1$ variables over the finite field k, whose leading form f_d in f is Dwork-regular with respect to the given coordinates $X_1, \dots, X_n$. Then for all but at most d^n elements $a \in \overline{k}$, the polynomial $f(x) + a$ is affine-Dwork-regular.*

Proof. By hypothesis, f_d is Dwork-regular with respect to $X_1, \dots, X_n$. As d is prime to p, this means simply that in $\mathbb{P}^{n-1}$, the n forms of degree d, $X_i \partial f_d / \partial X_i$ for $i = 1, \dots, n$, define the empty subscheme. As f_d is $F\|_{X_0=0}$, and the formation of $X_i \partial F / \partial X_i$ for $i = 1, \dots, n$ commutes with putting X_0 to zero, the subscheme $\mathcal{J} \subset \mathbb{P}^n$ defined by the vanishing of the n forms of degree d, $X_i \partial f_d / \partial X_i$, has empty intersection with the hyperplane $X_0 = 0$. Therefore by Bertini this subscheme must be finite over k. As it is defined by the vanishing of n forms, each of degree d, it has at most d^n $\overline{k}$-valued points, and each of these points has X_0 invertible. Now consider the function

$$\phi := \left(X_0 \frac{\partial F}{\partial X_0} \right) \Big/ X_0^d$$

on $\mathcal{J}$. We claim that so long as $a \in \overline{k}$ is not of the form $-\phi(j)/d$ for any $j \in \mathcal{J}(\overline{k})$, then $f + a$ is affine-Dwork-regular, *i.e.* the form $F + aX_0^d$ is Dwork-regular with respect to $X_0, \dots, X_n$. Because d is prime to p, this amounts to showing that the subscheme $\mathcal{K}$ of $\mathbb{P}^n$ defined by the vanishing of the $n+1$ forms

$$X_i \frac{\partial (F + aX_0^d)}{\partial X_i}, \quad i = 0, \dots, n,$$

is empty, so long as $a \in \overline{k}$ is not of the form $-\phi(j)/d$ for some $j \in \mathcal{J}(\overline{k})$. These forms are

$$X_0 \frac{\partial F}{\partial X_0} + daX_0^d$$

for $i = 0$, and $X_i \partial F / \partial X_i$ for $i = 1, \dots, n$. Thus a point of $\mathcal{K}$ is a point of $\mathcal{J}$ (from the vanishing of these last n forms) at which

$$\frac{\partial F}{\partial X_0} + daX_0^d$$

also vanishes. But as X_0 is invertible at every point of $\mathcal{J}$, it is the same to say that a point of $\mathcal{K}(\overline{k})$ is a point $j \in \mathcal{J}(\overline{k})$ at which $\phi(j) + da = 0$. For the allowed values of a, the function $\phi + da$ is invertible on $\mathcal{J}$, and hence the scheme $\mathcal{K}$ is empty, as required. □

Remark 3.3. The reader should be cautioned that if a Deligne polynomial f has its leading form f_d Dwork-regular, f itself need not be affine-Dwork-regular, even if the affine hypersurface $f = 0$ is smooth. The simplest example is

$$x_1^d + x_1 + \sum_{i=2}^{n} x_i^d,$$

for any degree $d \geqslant 2$ prime to p, and any $n \geqslant 1$. Indeed, f cannot be affine-Dwork-regular unless its constant term $f(0) \neq 0$; if $f(0) = 0$ then the homogenization $F(X_0, X_1, \dots, X_n)$

will vanish when we set X_i to 0 for $i = 1, \ldots, n$, rather than being a Deligne polynomial in the remaining variable X_0.

Lemma 3.4. *Let f be a Deligne polynomial of degree d prime to p in $n \geqslant 1$ variables over the finite field k, which is affine-Dwork-regular. Then for any integer $r \geqslant 1$ prime to p, the polynomial $f(X^r) := f(X_1^r, \ldots, X_n^r)$ is a Deligne polynomial of degree dr prime to p in n variables over the finite field k, which is affine-Dwork-regular. Equivalently, if $F(X_0, \ldots, X_n)$ is a homogeneous form of degree d prime to p in $n+1$ variables over the finite field k which is Dwork-regular, then for any integer $r \geqslant 1$ prime to p, the form $F(X^r) := F(X_0^r, \ldots, X_n^r)$ is a form of degree dr prime to p in $n+1$ variables over the finite field k, which is Dwork-regular.*

Proof. We will prove the assertion in its homogeneous version. Let us put

$$F_i := X_i \frac{\partial F}{\partial X_i}, \quad G := F(X^r), \quad G_i := X_i \frac{\partial G}{\partial X_i}.$$

Then we have the identities

$$G_i(X) = rF_i(X^r).$$

As r is prime to p, if $x \in \mathbb{P}^n(\overline{k})$ is a common zero of the G_i, then x^r is a common zero of the F_i. But the F_i have no common zero in $\mathbb{P}^n(\overline{k})$. Thus the G_i have no common zero in $\mathbb{P}^n(\overline{k})$. □

In the case when f is a Deligne polynomial of degree d prime to p in n variables over the finite field k, which is affine-Dwork-regular, we have the following more precise "purity theorem" concerning the cohomology of the sheaf $\mathcal{F}$.

Theorem 3.5 (Purity theorem)**.** *Suppose that f is a Deligne polynomial of degree d prime to p in $n \geqslant 1$ variables over the finite field k, which is affine-Dwork-regular, and g is an arbitrary polynomial. If p is odd and* $\deg(g) := e < d/2$*, then we have the following three results:*

(i) $H_c^2(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}) = 0$.
(ii) $\dim H_c^1(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}) = 2(d-1)^n$.
(iii) $H_c^1(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F})$ *is pure of weight* $n+1$.

Once again, this theorem remains valid for $n = 0$ with the convention that a polynomial f in $n = 0$ variables over k, *i.e.* a constant in k, is a Deligne polynomial which is affine-Dwork-regular, of any prime to p degree one likes, if that constant is nonzero. It is simply the (cohomological underpinning of the) usual estimate for classical Kloosterman sums.

4. The Purity Theorem Implies Theorem 2.1

Let us put ourselves in the situation of Theorem 2.1. Thus f is a Deligne polynomial of degree d prime to p in n variables over the finite field k, g is an arbitrary polynomial of $\deg(g) := e < d/2$, and p is odd. We denote by $\mathcal{F}$ the lisse sheaf

$$\mathcal{F} := R^n(pr_2)_! \mathcal{L}_{\psi(tf(x)+g(x)+1/t)}$$

attached to this data.

At the expense of extending scalars from k to a finite extension, we may further assume that f_d is Dwork-regular, and that there exists an element $a \in k$ such that the polynomial $f+a$ is affine-Dwork-regular. The lisse sheaf $\mathcal{F}_a$ attached to the data $(f+a, g)$,

$$\mathcal{F}_a := R^n(pr_2)_!\mathcal{L}_{\psi(t(f(x)+a)+g(x)+1/t)}$$

is related to the original sheaf $\mathcal{F}$ by

$$\mathcal{F}_a = \mathcal{F} \otimes \mathcal{L}_{\psi(at)},$$

as is immediate from the projection formula.

Thus $\mathcal{F}_a = \mathcal{F} \otimes \mathcal{L}_{\psi(at)}$ is a lisse sheaf on $\mathbb{G}_m/k$, pure of weight n, whose H^1_c is pure of weight $n+1$. We now apply the following lemma.

Lemma 4.1. *Let k be a finite field, $a \in k$, and $\mathcal{F}$ a lisse sheaf on $\mathbb{G}_m/k$ which is pure of weight n. Suppose that $H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F} \otimes \mathcal{L}_{\psi(at)})$ is pure of weight $n+1$. Then we have the following results:*

- (i) *Viewing $\mathcal{F}$ as a representation of $\pi_1(\mathbb{G}_m)$, its restriction to the inertia group $I(0)$ at $0 \in \mathbb{A}^1$ has no nonzero invariants or coinvariants, i.e. $\mathcal{F}^{I(0)} = \mathcal{F}_{I(0)} = 0$.*
- (ii) *$H^2_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}) = 0$.*

Proof. Denote by $j : \mathbb{G}_m \subset \mathbb{P}^1$ the inclusion, and write $\mathcal{F}_a := \mathcal{F} \otimes \mathcal{L}_{\psi(at)}$. We have a short exact excision sequence of sheaves on $\mathbb{P}^1$,

$$0 \to j_!\mathcal{F}_a \to j_\star\mathcal{F}_a \to (\mathcal{F}_a^{I(0)})_{\text{pct at }0} \oplus (\mathcal{F}_a^{I(\infty)})_{\text{pct at }\infty} \to 0.$$

The long exact cohomology sequence gives an exact sequence

$$0 \to H^0(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}_a) \to \mathcal{F}_a^{I(0)} \oplus \mathcal{F}_a^{I(\infty)} \to H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}_a).$$

By assumption the last term is pure of weight $n+1$. But as $\mathcal{F}_a$ is pure of weight n, we know by [4, 1.8.1] that both $\mathcal{F}_a^{I(0)}$ and $\mathcal{F}_a^{I(\infty)}$ are mixed of weight at most n. So the last arrow must be the zero map, and so we find an isomorphism

$$H^0(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}_a) \cong \mathcal{F}_a^{I(0)} \oplus \mathcal{F}_a^{I(\infty)}.$$

But each of the two restriction maps

$$H^0(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}_a) \to \mathcal{F}_a^{I(0)}, \quad H^0(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}_a) \to \mathcal{F}_a^{I(\infty)}$$

is injective. Thus we have

$$\dim \mathcal{F}_a^{I(0)} + \dim \mathcal{F}_a^{I(\infty)} = \dim H^0 \leqslant \dim \mathcal{F}_a^{I(\infty)},$$

and hence $\mathcal{F}_a^{I(0)} = 0$. [Similarly, we find $\mathcal{F}_a^{I(\infty)} = 0$, but we will not use that fact here.]

The sheaf $\mathcal{L}_{\psi(at)}$ is lisse on $\mathbb{A}^1$, so trivial as a representation of $I(0)$, and hence $\mathcal{F}$ and $\mathcal{F}_a := \mathcal{F} \otimes \mathcal{L}_{\psi(at)}$ are isomorphic as representations of $I(0)$. So we have $\mathcal{F}^{I(0)} \cong \mathcal{F}_a^{I(0)} = 0$. For any ℓ-adic representation V of $I(0)$, here $\mathcal{F}$, one knows that $\dim V^{I(0)} = \dim V_{I(0)}$. This proves (i).

Once we have proven (i), (ii) follows. Indeed, the H^2_c is the Tate-twisted group of coinvariants under $\pi_1^{\text{geom}}(\mathbb{G}_m)$,

$$H^2_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F})) \cong \mathcal{F}_{\pi_1^{\text{geom}}(\mathbb{G}_m)}(-1).$$

But this last group is a quotient of $\mathcal{F}_{I(0)}(-1)$, hence vanishes. ☐

To conclude the deduction of Theorem 2.1 from the Purity theorem, we argue as follows. We know that for all but a set S of at most d^n possible values of $a \in \overline{k}$, $f + a$ is affine-Dwork-regular. So we look at the sheaf on $\mathbb{G}_m \times \mathbb{A}^1$, with coordinates (t, a), which is $\mathcal{F} \otimes \mathcal{L}_{\psi(at)}$, and we take its H^1_c along the fibres of the second projection. Thus we consider the constructible sheaf $\mathcal{G}$ on $\mathbb{A}^1$ given by

$$\mathcal{G} := R^1(pr_2)_!(\mathcal{F} \otimes \mathcal{L}_{\psi(at)}).$$

Its stalk at a point $a \in \overline{k}$ is the group $H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F} \otimes \mathcal{L}_{\psi(at)})$. By the Purity theorem, this stalk has dimension $2(d-1)^n$, for all but at most d^n values of a. Thus the generic rank of $\mathcal{G}$ is $2(d-1)^n$. The sheaf $\mathcal{G}$ on $\mathbb{A}^1$ has no nonzero punctual sections, because it is a sheaf of perverse origin, *cf.* [12]. In particular, its stalk at any point has dimension at most the generic rank of $\mathcal{G}$. As the stalk at 0 is $H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F})$, we find the required inequality

$$\dim H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}) \leqslant 2(d-1)^n.$$

5. A Generalization: the (A, B) Purity Theorem

For later application, we first formulate a mild generalization of the Purity theorem. Here f is a Deligne polynomial of degree d prime to p in n variables over the finite field k, and g is an arbitrary polynomial of degree $\deg(g) := e < d$. We are also given two integers $A \geqslant 1$, $B \geqslant 1$, and a one-variable polynomial $P_B(t)$ over k of degree $\deg(P_B) \leqslant B$. We consider the one-parameter family of Deligne polynomials

$$t \mapsto t^A f(x) + g(x) + P_B\left(\frac{1}{t}\right)$$

over $\mathbb{G}_m$. On $\mathbb{A}^n \times \mathbb{G}_m$ we have the lisse sheaf $\mathcal{L}_{\psi(t^A f(x)+g(x)+P_B(1/t))}$, and we form the sheaf

$$\mathcal{F} := R^n(pr_2)_!\mathcal{L}_{\psi(t^A f(x)+g(x)+P_B(1/t))}$$

on $\mathbb{G}_m$. Again, $\mathcal{F}$ is lisse of rank $(d-1)^n$, and pure of weight n.

Theorem 5.1 ((A, B) purity theorem)**.** *In the above situation, suppose in addition that f is affine-Dwork-regular, that p is prime to $AB(A+B)$, that $\deg(P_B)$ is exactly B, and that $\deg(g) := e < (B/(A+B))d$. Then we have the following three results:*

(i) $H^2_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}) = 0$.
(ii) $\dim H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}) = (A+B)(d-1)^n$.
(iii) $H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F})$ *is pure of weight* $n+1$.

In the case $(A, B) = (1, 1)$, this theorem gives back the Purity Theorem 3.5. It perhaps clarifies the significance of the hypothesis that p be odd in that theorem.

Once again, this theorem remains valid for $n = 0$ with the convention that a polynomial f in $n = 0$ variables over k, *i.e.* a constant in k, is a Deligne polynomial which is affine-Dwork-regular, of any prime to p degree one likes, if that constant is nonzero. Now it becomes the (cohomological underpinning of the) usual estimate for additive character sums with one-variable Laurent polynomials. Exactly as in the previous section, we can do a specialization argument to get information about what happens in either of two mild degenerations.

Corollary 5.2. *If in the (A, B) Purity Theorem 5.1 we drop only the hypothesis that the Deligne polynomial f be affine-Dwork-regular, or we drop only the hypothesis that* $\deg(P_B)$ *be exactly B, then we have the following results:*

(i) *In the first case, the inertial invariants of $\mathcal{F}$ at zero vanish, i.e. $\mathcal{F}^{I(0)} = 0$. In the second case, the inertial invariants of $\mathcal{F}$ at ∞ vanish, i.e. $\mathcal{F}^{I(\infty)} = 0$.*

(ii) $H^2_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}) = 0$.

(iii) $\dim H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}) \leqslant (A + B)(d - 1)^n$.

(iv) *We have the estimate*

$$\left|\sum_{t\in k^\times}\sum_{x\in k^n}\psi\left(t^A f(x) + g(x) + P_B\left(\frac{1}{t}\right)\right)\right| \leqslant (A + B)(d - 1)^n(\#(k))^{(n+1)/2}.$$

Proof. Suppose first that we drop only the hypothesis that f be affine-Dwork-regular. The question being geometric, we may make a finite extension of scalars and find a coordinate system in which f_d is Dwork-regular. Then for most $a \in \overline{k}$, $f + a$ is affine-Dwork-regular, and we apply the (A, B) purity theorem to the sheaf $\mathcal{F}_a := \mathcal{F} \otimes \mathcal{L}_{\psi(at^A)}$. Exactly as in Lemma 4.1 above, the purity implies the vanishing of $\mathcal{F}_a^{I(0)}$. Again we have $\mathcal{F}_a^{I(0)} \cong \mathcal{F}^{I(0)}$, so we get the vanishing of $H^2_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}) = 0$. Assertion (iii) follows by the "sheaf of perverse origin" argument. Assertion (iv) simply spells out the diophantine consequence of parts (ii) and (iii).

Suppose now that we drop only the hypothesis that $\deg(P_B)$ be exactly B. Then for most $b \in \overline{k}$, the polynomial $P_B + bt^B$ has exact degree B, and we apply the (A, B) purity theorem to the sheaf $\mathcal{F}_b := \mathcal{F} \otimes \mathcal{L}_{\psi(b/t^B)}$. Now we exploit the fact that the purity implies the vanishing of $\mathcal{F}_b^{I(\infty)}$. We have $\mathcal{F}_b^{I(\infty)} \cong \mathcal{F}^{I(\infty)}$, and we conclude as above. □

Let us make explicit two special cases of this corollary, when we drop only the hypothesis that P_B has exact degree B, and where we take $A = 1$.

Corollary 5.3. *Suppose that f is an affine-Dwork-regular polynomial of degree d prime to p in n variables over the finite field k, and g is an arbitrary polynomial of lower degree, i.e.* $\deg(g) := e < d$. *Suppose that p is odd. Denote by δ the least strictly positive integer such that both $e < (\delta/(1 + \delta))d$ and such that p does not divide $\delta(1 + \delta)$ – thus δ can always be taken to be d, unless $p \mid d + 1$, in which case δ can be taken to be $d + 2$. Then we have the following estimates:*

(i) *We have the estimate*

$$\left|\sum_{t\in k^\times}\sum_{x\in k^n}\psi\left(tf(x) + g(x) + \frac{1}{t}\right)\right| \leqslant (1 + \delta)(d - 1)^n(\#(k))^{(n+1)/2}.$$

(ii) *We have the estimate*

$$\left|\sum_{t\in k^\times}\sum_{x\in k^n}\psi(tf(x) + g(x))\right| \leqslant (1 + \delta)(d - 1)^n(\#(k))^{(n+1)/2}.$$

(iii) *We have the estimate*

$$\left|\sum_{t\in k^\times}\sum_{x\in k^n}\psi(tf(x))\right| \leqslant 2(d - 1)^n(\#(k))^{(n+1)/2}.$$

Proof. The first two assertions are instances of the case $(A, B) = (1, \delta)$ of the above corollary, with P_B taken successively to be t and 0. The third assertion is the special case $(A, B) = (1, 1)$, with g and P_B both taken to be 0. □

Remark 5.4. Let us see how sharp the third estimate of this corollary is. Denote by $V_f \subset \mathbb{A}^n$ the affine hypersurface defined by $f = 0$. If we add to the sum in question the terms with $t = 0$, we get, writing $q := \#k$, $q\#V_f(k)$. Thus we have

$$\sum_{t \in k^\times} \sum_{x \in k^n} \psi(tf(x)) = q\#V_f(k) - q^n.$$

So the third assertion of the corollary amounts to the estimate

$$|\#V_f(k) - q^{n-1}| \leqslant 2(d-1)^n q^{(n-1)/2}.$$

Denote by $F = F(X_0, \ldots, X_n)$ the homogenization of f. Then V_f is the complement of the projective hypersurface $Z_{f_d} \subset \mathbb{P}^{n-1}$ defined by $f_d = 0$ in the projective hypersurface $Z_F \subset \mathbb{P}^n$ defined by $F = 0$. Thus

$$\#V_f(k) = \#Z_F(k) - \#Z_{f_d}(k).$$

Because f is affine-Dwork-regular, both Z_F and Z_{f_d} are smooth. From the excision sequence and the known cohomological structure of smooth projective hypersurfaces, one sees that the compact cohomology groups $H^i_c(V_f \otimes \overline{k}, \overline{\mathbb{Q}}_\ell)$ vanish except for $i = n-1$ and for $i = 2n-2$, and that for $n \geqslant 2$ we have

$$\dim H^{n-1}_c(V_f \otimes \overline{k}, \overline{\mathbb{Q}}_\ell) = (d-1)^n,$$

and

$$H^{2n-2}_c(V_f \otimes \overline{k}, \overline{\mathbb{Q}}_\ell) \cong \overline{\mathbb{Q}}_\ell(-(n-1)).$$

By Deligne's fundamental estimate [4, 3.3.1], the group H^{n-1}_c is mixed of weight $\leqslant n-1$, so for $n \geqslant 2$ we find

$$|\#V_f(k) - q^{n-1}| \leqslant (d-1)^n q^{(n-1)/2}.$$

For $n = 1$, this is trivially true as well. Thus the estimate in part (iii) of the above corollary is overly conservative, by a factor of 2.

Remark 5.5. It may also be worth pointing out that the second and third estimates both become false if we allow f to be a Deligne polynomial whose leading form f_d is Dwork-regular, but such that f is not affine-Dwork-regular. To clarify the situation, consider once again the lisse of rank $(d-1)^n$, pure of weight n sheaf $\mathcal{F}$ on $\mathbb{G}_m$ attached to the one parameter family of Deligne polynomials over $\mathbb{G}_m$,

$$t \mapsto tf(x) + g(x).$$

The second and third estimates fail precisely when $H^2_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}) \neq 0$.

To quantify this failure, recall that two finite-dimensional $\overline{\mathbb{Q}}_\ell$-representations of $\mathrm{Gal}(\overline{k}/k)$, say V and W, have isomorphic semisimplifications (as representations), written $V^{ss} \cong W^{ss}$, if and only if $\det(1 - T\gamma|V) = \det(1 - T\gamma|W)$ for every element $\gamma \in \mathrm{Gal}(\overline{k}/k)$. As $\mathrm{Gal}(\overline{k}/k)$ is pro-cyclic, generated by the geometric Frobenius element Frob_k, we have

$$V^{ss} \cong W^{ss} \iff \det(1 - T\,\mathrm{Frob}_k\,|V) = \det(1 - T\,\mathrm{Frob}_k\,|W).$$

Consider first the third estimate. If we take $f = f_d$, then V_f is the affine cone over the projective smooth hypersurface $X_f \subset \mathbb{P}^{n-1}$ defined by $f = f_d$. Thus, again writing $q := \#k$,

$$\begin{aligned}\#V_f(k) &= 1 + (q-1)\#X_f(k) \\ &= 1 + (q-1)(\#\mathbb{P}^{n-2}(k) + (-1)^{n-2}\,\mathrm{Trace}(\mathrm{Frob}_k \mid \mathrm{Prim}^{n-2}(X_f))) \\ &= q^{n-1} + (-1)^{n-2}(q-1)\,\mathrm{Trace}(\mathrm{Frob}_k \mid \mathrm{Prim}^{n-2}(X_f)).\end{aligned}$$

Here $\mathrm{Prim}^{n-2}(X_f)$, the primitive part of $H^{n-2}(X_f \otimes \overline{k}, \overline{\mathbb{Q}}_\ell)$, is pure of weight $n-2$ and of strictly positive dimension

$$\frac{d-1}{d}((d-1)^{n-1} - (-1)^{n-1}).$$

So the "error term" $\#V_f(k) - q^{n-1}$ is $O((\#(k))^{n/2})$, but it is not $O((\#(k))^{(n-1)/2})$. Indeed, this identity, applied over all finite extensions of k (combined with the fact that $H^2_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F})$ is pure of weight $n+2$, while $H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F})$ is mixed of weight $\leqslant n+1$, which allows us to "separate terms"), shows that

$$H^2_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F})^{ss} \cong \mathrm{Prim}^{n-2}(X_f)^{ss}(-2)$$

and

$$H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F})^{ss} \cong \mathrm{Prim}^{n-2}(X_f)^{ss}(-1).$$

Consider now the second estimate. Suppose $n \geqslant 4$. Given $d \geqslant 3$ prime to p, take

$$f = f_d = x_{n-1}^d - x_n^d + \sum_{i=1}^{n-2} x_i^d, \quad g = x_{n-1} - x_n.$$

Denote by $X_f \subset \mathbb{P}^{n-1}$ the projective smooth hypersurface defined by $f = 0$, and by $H \subset \mathbb{P}^{n-1}$ the hyperplane defined by $g = 0$. Because g is a nontrivial linear form, $\sum_{x \in k^n} \psi(g(x)) = 0$, so we have

$$\sum_{t \in k^\times} \sum_{x \in k^n} \psi(tf(x) + g(x)) = \sum_{t \in k} \sum_{x \in k^n} \psi(tf(x) + g(x)) = q \sum_{x \in V_f(k)} \psi(g(x)).$$

Now for each point $x \in X_f(k)$, choose a representative $X \in V_f(k)$ lying over it. Then the points, other than the origin, in $V_f(k)$ are precisely the $k^\times$-multiples of the chosen points X, so our sum is

$$\begin{aligned}q\left(1 + \sum_{x \in X_f(k)} \sum_{\lambda \in k^\times} \psi(\lambda g(X))\right) &= q\left(1 - \#X_f(k) + \sum_{x \in X_f(k)} \sum_{\lambda \in k} \psi(\lambda g(X))\right) \\ &= q(1 - \#X_f(k) + q\#(X_f \cap H)(k)).\end{aligned}$$

But the intersection $X_f \cap H$ is the hypersurface in $\mathbb{P}^{n-2}$, with homogeneous coordinates $(x_1, \ldots, x_{n-1})$, defined by the equation $\sum_{i=1}^{n-2} x_i^d = 0$, an equation which does not involve the last variable x_{n-1}. So if we denote by $Z \subset \mathbb{P}^{n-3}$, with homogeneous coordinates $(x_1, \ldots, x_{n-1})$, the smooth hypersurface defined by this same equation, then we have

$$\#(X_f \cap H)(k) = 1 + q\#Z(k).$$

So our sum is

$$q(1 - \#X_f(k) + q(1 + q\#Z(k))) = q(1 + q) - q\#X_f(k) + q^3\#Z(k).$$

But

$$\#X_f(k) = \#\mathbb{P}^{n-2}(k) + (-1)^{n-2}\,\mathrm{Trace}(\mathrm{Frob}_k \,|\, \mathrm{Prim}^{n-2}(X_f))$$

and

$$\#Z(k) = \#\mathbb{P}^{n-4}(k) + (-1)^{n-4}\,\mathrm{Trace}(\mathrm{Frob}_k \,|\, \mathrm{Prim}^{n-4}(Z)),$$

and so our sum is $(-1)^n$ times

$$q^3\,\mathrm{Trace}(\mathrm{Frob}_k \,|\, \mathrm{Prim}^{n-4}(Z)) - q\,\mathrm{Trace}(\mathrm{Frob}_k \,|\, \mathrm{Prim}^{n-2}(X_f)),$$

which, because of the first term, is $O((\#(k))^{(n+2)/2})$ but is not $O((\#(k))^{(n+1)/2})$. Once again, we have the more precise result that

$$H^2_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F})^{ss} \cong \mathrm{Prim}^{n-4}(Z)^{ss}(-3)$$

and

$$H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F})^{ss} \cong \mathrm{Prim}^{n-2}(X_f)^{ss}(-1).$$

Remark 5.6. As Browning and Heath-Brown [1] have noticed, there is something that can be salvaged of the last corollary if we allow f to be "merely" a Deligne polynomial, but now impose some transversality conditions on the interaction of g and f. To state the result, let us define, given n, d, e strictly positive integers, the Bombieri constant

$$C(n, 1, d, e) := (4\max\{d + 1, e\} + 5)^{n+1},$$

cf. [11, page 877 and Theorem 4].

Theorem 5.7. *Given integers $n \geqslant 1$, $d \geqslant 1$, $e \geqslant 1$, define*

$$B(n, d, e) := C(n + 1, 1, 1, e) + C(n, 1, d, e).$$

Suppose that f is a polynomial of degree d prime to p in n variables over the finite field k, with leading form f_d, and that g is a polynomial of degree e prime to p, with leading form g_e. Denote by X_{f_d} and X_{g_e} the projective hypersurfaces in $\mathbb{P}^{n-1}$ defined by the vanishing of f_d and of g_e respectively. We make the following assumptions:

(a) *$X_{f_d} \subset \mathbb{P}^{n-1}$ is a smooth hypersurface, i.e. f is a Deligne polynomial.*
(b) *$X_{g_e} \subset \mathbb{P}^{n-1}$ is a hypersurface which is either smooth, or has at worst isolated singularities, i.e. $\dim \mathrm{Sing}(X_{g_e}) \leqslant 0$.*
(c) *The scheme-theoretic intersection $X_{f_d} \cap X_{g_e} \subset \mathbb{P}^{n-1}$ is smooth of codimension 2.*

Then we have the following estimates:

(i) *We have*

$$\left|\sum_{t\in k^\times}\sum_{x\in k^n}\psi(tf(x) + g(x))\right| \leqslant B(n, d, e)(\#(k))^{(n+1)/2}.$$

(ii) *If $e < d$ and p is odd, we can replace $B(n, d, e)$ by the constant $(1 + \delta)(d - 1)^n$, where δ is the least strictly positive integer such that both $e < (\delta/(1 + \delta))d$ and such that p does not divide $\delta(1 + \delta)$.*

(iii) *If g is a Deligne polynomial and e is arbitrary, we can replace $B(n,d,e)$ by the constant*

$$(e-1)^n + \left|\textit{coeff of } L^{n-1} \textit{ in } \frac{d(1+L)^n}{(1+dL)(1+eL)}\right|.$$

Proof. Denote by $F = F_d(X_0,\dots,X_n)$ the degree d homogenization of f, by $G = G_e(X_0,\dots,X_n)$ the degree e homogenization of g. Denote by $X_F \subset \mathbb{P}^n$ the hypersurface defined by $F = 0$, by $H \subset \mathbb{P}^n$ the hypersurface defined by $G = 0$, and by $L \subset \mathbb{P}^n$ the hyperplane defined by X_0. And denote by $V_f \subset \mathbb{A}^n$ the affine hypersurface defined by $f = 0$.

We first show that the sum in question is $O((\#(k))^{(n+1)/2})$. To do this, we first "complete" the sum by adding the $t = 0$ terms:

$$\begin{aligned}\sum_{t\in k^\times}\sum_{x\in k^n}\psi(tf(x)+g(x)) &= -\sum_{x\in k^n}\psi(g(x)) + \sum_{t\in k}\sum_{x\in k^n}\psi(tf(x)+g(x))\\ &= -\sum_{x\in k^n}\psi(g(x)) + (\#k)\sum_{x\in V_f(k)}\psi(g(x)).\end{aligned}$$

The first term is (minus) the sum over $\mathbb{P}^n[1/X_0](k)$ of $\psi(G/X_0^e)$. Here $\mathbb{P}^n \cap L$ is of course smooth, being $\mathbb{P}^{n-1}$, and $\mathbb{P}^n \cap L \cap H$ is the hypersurface X_{g_e} in that $\mathbb{P}^{n-1}$. By assumption, X_{g_e} has at worst isolated singularities, so [11, Theorem 4], with $\epsilon = -1$ and $\delta \leqslant 0$, shows that the absolute value of this sum is $\leqslant C(n+1,1,1,e)(\#(k))^{(n+1)/2}$.

The second term is ($\#k$ times) the sum over $X_F[1/X_0](k)$ of $\psi(G/X_0^e)$. Here $X_F \cap L$ is of smooth, being X_{f_d}, and $X_F \cap L \cap G$ is smooth, being $X_{f_d} \cap X_{g_e} \subset \mathbb{P}^{n-1}$. So once again [11, Theorem 4], with $\epsilon = -1$ and $\delta = -1$, shows that the absolute value of this sum is $\leqslant C(n,1,d,e)(\#(k))^{(n+1)/2}$.

It remains to get the asserted constants either if both $e < d$ and p is odd, or if g is a Deligne polynomial. Suppose first that $e < d$ and p is odd. The question being geometric, we may make a finite extension of scalars, and find new coordinates in which f_d is Dwork-regular. Then for all but at most d^n values of $a \in \overline{k}$, $f + a$ is affine-Dwork-regular. On the product space $\mathbb{G}_m \times \mathbb{A}^1$, with coordinates (t, a), we have the lisse sheaf $(pr_1^\star \mathcal{F}) \otimes \mathcal{L}_{\psi(at)}$. So the sheaf

$$\mathcal{G} := Rpr_{2!}((pr_1^\star\mathcal{F}) \otimes \mathcal{L}_{\psi(at)})$$

on $\mathbb{A}^1$ is a sheaf of perverse origin, and hence its stalk $\mathcal{G}_0$ at $a = 0$, which is the group $H_c^1(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F})$, has rank at most equal to the generic rank of $\mathcal{G}$. This generic rank is attained at all but finitely many points, so certainly at some point a where $f + a$ is affine-Dwork-regular. Now apply Corollary 5.2(iii) to $f + a$ and to g, with $A = 1$ and $B = \delta$.

Now suppose that g is a Deligne polynomial of arbitrary degree e. Then the cohomology groups $H_c^i(\mathbb{A}^n \otimes \overline{k}, \mathcal{L}_{\psi(g)})$ vanish for $i \neq n$, and $H_c^n(\mathbb{A}^n \otimes \overline{k}, \mathcal{L}_{\psi(g)})$ has dimension $(e-1)^n$. What about the groups $H_c^i(V_f \otimes \overline{k}, \mathcal{L}_{\psi(g)})$? A close reading of the proof of [11, Theorem 4] shows that, under our hypotheses, these groups vanish for $i \geqslant n$. Because V_f is a hypersurface, and so certainly locally a complete intersection, the lisse sheaf $\mathcal{L}_{\psi(g)}$, placed in degree $1-n$, *i.e.* $\mathcal{L}_{\psi(g)}[n-1]$, is a perverse sheaf on V_f. Since V_f is affine, its compact cohomology groups with perverse coefficients vanish in strictly negative degree.

Thus we have

$$H^i_c(V_f \otimes \overline{k}, \mathcal{L}_{\psi(g)}[n-1]) = H^{n-1+i}_c(V_f \otimes \overline{k}, \mathcal{L}_{\psi(g)}) = 0, \quad i < 0.$$

So the groups $H^i_c(V_f \otimes \overline{k}, \mathcal{L}_{\psi(g)})$ vanish for $i \neq n-1$. Thus our sum is given by

$$\sum_{t \in k^\times} \sum_{x \in k^n} \psi(tf(x) + g(x)) = -(-1)^n \operatorname{Trace}(\mathrm{Frob}_k \,|H^n_c(\mathbb{A}^n \otimes \overline{k}, \mathcal{L}_{\psi(g)}))$$
$$+ (-1)^{n-1}(\#k) \operatorname{Trace}(\mathrm{Frob}\,|H^{n-1}_c(V_f \otimes \overline{k}, \mathcal{L}_{\psi(g)})).$$

It remains to bound the dimension of $H^{n-1}_c(V_f \otimes \overline{k}, \mathcal{L}_{\psi(g)})$. Again, the question is geometric, so we may make a finite extension of scalars and find coordinates such that f_d is Dwork-regular. Once again, $f + a$ is affine-Dwork-regular for all but at most d^n values of $a \in \overline{k}$. Now we consider the smooth hypersurface $V \subset \mathbb{A}^n \times \mathbb{A}^1$, with coordinates (x, a), of equation $f(x)+a = 0$. We have the lisse sheaf $pr_1^\star \mathcal{L}_{\psi(g)}$ on V, and the morphism $pr_2 : V \to \mathbb{A}^1$, which is a complete intersection morphism of relative dimension $n-1$. So the sheaf

$$\mathcal{G} := R^{n-1}pr_{2!}(pr_1^\star \mathcal{L}_{\psi(g)})$$

on $\mathbb{A}^1$ is a sheaf of perverse origin. Its stalk $\mathcal{G}_0$ at $a = 0$, which is the group

$$H^{n-1}_c(V_f \otimes \overline{k}, \mathcal{L}_{\psi(g)}),$$

has rank at most equal to the generic rank of $\mathcal{G}$. This generic rank is attained at all but finitely many points, so certainly at some point a where $f + a$ is affine-Dwork-regular. Pick such a point a. Then $\mathcal{G}_a$ is the group $H^{n-1}_c(V_{f+a} \otimes \overline{k}, \mathcal{L}_{\psi(g)})$. By the "nonsingular" case [8, 5.1.1, applied to $X \subset \mathbb{P}^n$ there defined by $F + aX_0^d = 0$ and the function G/X_0^e], we know that this group is pure of weight $n-1$, and its dimension is given by

$$(-1)^{n-1} \dim H^{n-1}_c(V_{f+a} \otimes \overline{k}, \mathcal{L}_{\psi(g)}) = \text{coeff of } L^n \text{ in } \frac{dL(1+L)^{n+1}}{(1+dL)(1+L)(1+eL)}. \quad \square$$

6. Proof of the (A, B) Purity Theorem when $A + B = d$

Let us recall the situation. We are given

(a) an affine-Dwork-regular Deligne polynomial f of degree d prime to p in $n \geqslant 1$ variables over the finite field k,
(b) a partition $d = A + B$ with $A \geqslant 1$, $B \geqslant 1$, and both A, B prime to p,
(c) a polynomial $g(x)$ in n variables over the finite field k, of degree $\deg(g) := e < B = (B/(A+B))d$,
(d) a one-variable polynomial $P_B(t)$ over k of degree B without constant term, so that $P_B(0) = 0$.

We consider the one-parameter family of Deligne polynomials

$$t \mapsto t^A f(x) + g(x) + P_B\left(\frac{1}{t}\right)$$

over $\mathbb{G}_m$. On $\mathbb{A}^n \times \mathbb{G}_m$ we have the lisse sheaf $\mathcal{L}_{\psi(t^A f(x)+g(x)+P_B(1/t))}$, and we form the sheaf

$$\mathcal{F} := R^n(pr_2)_! \mathcal{L}_{\psi(t^A f(x)+g(x)+P_B(1/t))}$$

on $\mathbb{G}_m$. Again, $\mathcal{F}$ is lisse of rank $(d-1)^n$, and pure of weight n. Because $\mathcal{F}$ is the only one of the $R^i(pr_2)_!\mathcal{L}_{\psi(t^A f(x)+g(x)+P_B(1/t))}$ which is nonzero, the Leray spectral sequence degenerates at E_2, and gives isomorphisms

$$H_c^2(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}) \cong H_c^{n+2}((\mathbb{A}^n \times \mathbb{G}_m) \otimes \overline{k}, \mathcal{L}_{\psi(t^A f(x)+g(x)+P_B(1/t))})$$

and

$$H_c^1(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}) \cong H_c^{n+1}((\mathbb{A}^n \times \mathbb{G}_m) \otimes \overline{k}, \mathcal{L}_{\psi(t^A f(x)+g(x)+P_B(1/t))}).$$

What we must prove, then, is that the groups

$$H_c^i((\mathbb{A}^n \times \mathbb{G}_m) \otimes \overline{k}, \mathcal{L}_{\psi(t^A f(x)+g(x)+P_B(1/t))})$$

vanish for $i \neq n+1$, and that the H_c^{n+1} is pure of weight $n+1$, and has dimension $(A+B)(d-1)^n = d(d-1)^n$.

We apply the automorphism $t \mapsto 1/t$, $x \mapsto x$ to our situation. Then we must prove that the groups

$$H_c^i((\mathbb{A}^n \times \mathbb{G}_m) \otimes \overline{k}, \mathcal{L}_{\psi(t^{-A} f(x)+g(x)+P_B(t))})$$

vanish for $i \neq n+1$, and that the H_c^{n+1} is pure of weight $n+1$, and has dimension $(A+B)(d-1)^n = d(d-1)^n$.

This statement is a special case of a result proven in [8, 5.4.1] and amplified in [13, 4.1.12, but note that here the earlier condition 4.1.3 is assumed to remain in force]. Let us state that result in the case when the ambient space X there is $\mathbb{P}^{n+1}$ (here the case $n=0$ is perfectly fine), with coordinates $(T, X_0, X_1, \dots, X_n)$, the integer r there is 2, the integers d_1, d_2 there are both 1, and the prime to p integers b_1, b_2 of [8, 5.4.1], denoted by e_1, e_2 in [13, 4.1.12], are A, B. We take for Z_1 and Z_2 there the transverse hyperplanes of equation $T=0$ and $X_0=0$ respectively. We are given a degree d hypersurface of equation $H(T, X_0, X_1, \dots, X_n) = 0$. We assume the following:

(a) The intersection $(H=0) \cap (T=0)$ is smooth of codimension 2 in $\mathbb{P}^{n+1}$.
(b) The intersection $(H=0) \cap (X_0=0)$ is smooth of codimension 2 in $\mathbb{P}^{n+1}$.
(c) The intersection $(H=0) \cap (X_0=0) \cap (T=0)$ is smooth of codimension 3 in $\mathbb{P}^{n+1}$.

On the the open set $V := \mathbb{P}^{n+1}[1/TX_0] \subset \mathbb{P}^{n+1}$ where both T and X_0 are invertible, we have the function $H/T^A X_0^B$, and we form the Artin–Schreier sheaf $\mathcal{L}_{\psi(H/T^A X_0^B)}$. The theorem asserts that:

(i) The groups $H_c^i(V \otimes \overline{k}, \mathcal{L}_{\psi(H/T^A X_0^B)})$ vanish for $i \neq n+1$.
(ii) The remaining group H_c^{n+1} is pure of weight $n+1$.
(iii) Denote by L the class of a hyperplane, and by $c(\mathbb{P}^{n+1}) = (1+L)^{n+2}$ the total Chern class of $\mathbb{P}^{n+1}$. The dimension of the group H_c^{n+1} is given by the formula

$$\begin{aligned}(-1)^{n+1} \dim H_c^{n+1} &= \chi(V \otimes \overline{k}, \mathcal{L}_{\psi(H/T^A X_0^B)}) = \int_{\mathbb{P}^{n+1}} \frac{c(\mathbb{P}^{n+1})}{(1+dL)(1+L)^2} \\ &= \text{coeff of } L^{n+1} \text{ in } \frac{(1+L)^n}{1+dL} = \sum_{i=0}^{n} \binom{n}{i} (-d)^{n+1-i} \\ &= (-d)(1-d)^n = (-1)^{n+1} d(d-1)^n.\end{aligned}$$

In order to apply this result, let us denote by

$$F(X) := F(X_0, \ldots, X_n) := X_0^d f\left(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}\right)$$

the degree d homogenization of f, by

$$G_e(X) := G_e(X_0, \ldots, X_n) := X_0^e g\left(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}\right)$$

the degree e homogenization of g, and by

$$P_d(T, X_0) := T^A X_0^B P_B\left(\frac{T}{X_0}\right)$$

the degree d homogenization of $t^A P_B(t)$.

We then apply the cited result with

$$H(T, X_0, X_1, \ldots, X_n) := F(X) + T^A X_0^{B-e} G_e(X) + P_d(T, X_0).$$

The space $\mathbb{P}^{n+1}[1/TX_0]$ is just the product $\mathbb{A}^n \times \mathbb{G}_m$, with coordinates $x_i := X_i/X_0$, $i = 1, \ldots, n$, and $t = T/X_0$. The function $H/T^A X_0^B$ on this space is just the function

$$t^{-A} f(x) + g(x) + P_B(t).$$

The intersection $(H = 0) \cap (T = 0)$ is the hypersurface $F = 0$ in $\mathbb{P}^n$, which is smooth because f is affine-Dwork-regular. The intersection $(H = 0) \cap (X_0 = 0)$ is the smooth hypersurface in the $\mathbb{P}^n$ with homogeneous coordinates $(X_1, \ldots, X_n, T)$ of equation

$$f_d(X_1, \ldots, X_n) + (\text{leading coeff of } P_B(t))T^d = 0,$$

which is again smooth because d is prime to p, and because f_d defines a smooth hypersurface in $\mathbb{P}^{n-1}$. [It is the degree restriction on g which ensures that the term $T^A X_0^{B-e} G_e(X)$ vanishes when we set either T or X_0 to 0.] Finally, the intersection $(H = 0) \cap (X_0 = 0) \cap (T = 0)$ is the smooth hypersurface in $\mathbb{P}^{n-1}$ defined by f_d. This concludes the proof of the case $A + B = d$ of the (A, B) purity theorem.

Remark 6.1. The case $n = 0$ of the underlying result we have cited gives the case $n = 0$ of the (A, B) purity theorem.

7. Proof of the (A, B) Purity Theorem in the General Case

We begin by recalling two general principles. Let X and Y be two separated k-schemes of finite type.

Suppose we are given a morphism $\pi : X \to Y$ which is finite and flat of some rank $n \geqslant 1$. Then for any constructible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on Y, $\mathcal{F}$ is a direct summand of $\pi_\star \pi^\star \mathcal{F}$, a retraction being furnished by $(1/n)Tr_\pi$, *cf.* [16, Exposé XVII, 6.2.3, (Var 4)]. For $\mathcal{G}$ any constructible $\overline{\mathbb{Q}}_\ell$-sheaf on X, we have

$$H^i_c(X \otimes \overline{k}, \mathcal{G}) \cong H^i_c(Y \otimes \overline{k}, \pi_\star \mathcal{G}).$$

Taking $\mathcal{G} := \pi^\star \mathcal{F}$, we get

$$H^i_c(X \otimes \overline{k}, \pi^\star \mathcal{F}) \cong H^i_c(Y \otimes \overline{k}, \pi_\star \pi^\star \mathcal{F}).$$

Thus $H^i_c(Y \otimes \overline{k}, \mathcal{F})$ is a direct summand of $H^i_c(X \otimes \overline{k}, \pi^\star \mathcal{F})$.

Suppose now that G is a finite group of order n prime to $p := \mathrm{char}(k)$, and that $\pi : X \to Y$ is a finite étale G-covering. Then for any constructible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on Y, its compact Euler characteristic

$$\chi_c(Y \otimes \overline{k}, \mathcal{F}) := \sum_i (-1)^i \dim H^i_c(Y \otimes \overline{k}, \mathcal{F})$$

obeys the Riemann–Hurwitz formula for this covering:

$$\chi_c(Y \otimes \overline{k}, \mathcal{F}) = \frac{1}{\#G} \chi_c(X \otimes \overline{k}, \pi^\star \mathcal{F}),$$

cf. [8, 5.5.2, Corollary 1].

We now are in a position to prove the (A, B) purity theorem. Let us recall the situation: We are given two integers $A \geqslant 1$, $B \geqslant 1$, such that p is prime to $AB(A+B)$. We are also given an integer $d \geqslant 1$ prime to p and an affine-Dwork-regular Deligne polynomial f of degree d in $n \geqslant 1$ variables over the finite field k, and an arbitrary polynomial g of degree $\deg(g) := e < (B/(A+B))d$. Finally, we are given a one-variable polynomial $P_B(t)$ over k of degree B. We consider the one-parameter family of Deligne polynomials

$$t \mapsto t^A f(x) + g(x) + P_B\left(\frac{1}{t}\right)$$

over $\mathbb{G}_m$. On $\mathbb{A}^n \times \mathbb{G}_m$ we have the lisse sheaf $\mathcal{L}_{\psi(t^A f(x) + g(x) + P_B(1/t))}$, and we form the sheaf

$$\mathcal{F} := R^n(pr_2)_! \mathcal{L}_{\psi(t^A f(x)+g(x)+P_B(1/t))}$$

on $\mathbb{G}_m$. Again, $\mathcal{F}$ is lisse of rank $(d-1)^n$, and pure of weight n. Inverting t, what we must prove is that the groups

$$H^i_c((\mathbb{A}^n \times \mathbb{G}_m) \otimes \overline{k}, \mathcal{L}_{\psi(t^{-A} f(x)+g(x)+P_B(t))})$$

vanish for $i \neq n+1$, and that the H^{n+1}_c is pure of weight $n+1$, and has dimension $(A+B)(d-1)^n$.

We first consider the finite flat morphism

$$\pi : \mathbb{A}^n \times \mathbb{G}_m \to \mathbb{A}^n \times \mathbb{G}_m, \quad (x_1, \ldots, x_n, t) \mapsto (x_1^{A+B}, \ldots, x_n^{A+B}, t^d),$$

of rank $d(A+B)^n$. Visibly, we have

$$\pi^\star \mathcal{L}_{\psi(t^{-A} f(x)+g(x)+P_B(t))} = \mathcal{L}_{\psi(t^{-dA} f(x^{A+B})+g(x^{A+B})+P_B(t^d))}.$$

But the data

$$(Ad, Bd, (A+B)d, f(x^{A+B}), g(x^{A+B}), P_B(t^d))$$

is input for the case $Ad + Bd = (A+B)d$ of the theorem proven in the last section. So we know that the the groups

$$H^i_c((\mathbb{A}^n \times \mathbb{G}_m) \otimes \overline{k}, \mathcal{L}_{\psi(t^{-dA} f(x^{A+B})+g(x^{A+B})+P_B(t^d))})$$

vanish for $i \neq n+1$, and that the group H^{n+1}_c is pure of weight $n+1$, and has dimension $(Ad+Bd)(d(A+B)-1)^n$. Therefore the "downstairs" groups

$$H^i_c((\mathbb{A}^n \times \mathbb{G}_m) \otimes \overline{k}, \mathcal{L}_{\psi(t^{-A} f(x)+g(x)+P_B(t))}),$$

which are subgroups of these "upstairs" groups, themselves vanish for $i \neq n+1$, and the group H^{n+1}_c is pure of weight $n+1$.

It remains only to compute the compact Euler characteristic. The question is geometric, so we may assume k contains all the roots of unity of order dividing $(A+B)d$. We decompose $\mathbb{A}^n \times \mathbb{G}_m$ set-theoretically as a finite disjoint union of locally closed subschemes Z_S, indexed by subsets $S \subseteq \{1, \ldots, n\}$, as follows. We define

$$Z_S := \{(x_1, \ldots, x_n, t) : x_i \neq 0 \text{ if } i \in S,\ x_i = 0 \text{ otherwise}\}.$$

The merit of this stratification is that under the map π, we have $\pi^{-1}Z_S = Z_S$ for each S, and π makes Z_S into a finite étale Galois covering of itself, with Galois group $\mu_{A+B}^S \times \mu_d$. Let us also introduce the closed subschemes $W_S \subset \mathbb{A}^n \times \mathbb{G}_m$ defined by

$$W_S := \{(x_1, \ldots, x_n, t) : x_i = 0 \text{ if } i \notin S\}.$$

Thus $W_S \cong \mathbb{A}^S \times \mathbb{G}_m$. Because f is affine-Dwork-regular, if we take the data

$$(A, B, d, f, g, P_B)$$

and set the variables x_i with $i \notin S$ to zero, we get data on $W_S \cong \mathbb{A}^S \times \mathbb{G}_m$ which is input for the (A, B) purity theorem. Similarly, if we take the data

$$(Ad, Bd, (A+B)d, f(x^{A+B}), g(x^{A+B}), P_B(t^d))$$

and set the variables x_i with $i \notin S$ to zero, we get data on $W_S \cong \mathbb{A}^S \times \mathbb{G}_m$ which is input for the (Ad, Bd) purity theorem in the proven case $Ad + Bd = (A+B)d$.

For ease of notation in the combinatorics to follow, let us define

$$\begin{aligned}
\chi(W_S, \text{down}) &:= \chi_c(W_S \otimes \overline{k}, \mathcal{L}_{\psi(t^{-A}f(x)+g(x)+P_B(t))}|W_S),\\
\chi(Z_S, \text{down}) &:= \chi_c(W_S \otimes \overline{k}, \mathcal{L}_{\psi(t^{-A}f(x)+g(x)+P_B(t))}|Z_S),\\
\chi(W_S, \text{up}) &:= \chi_c(W_S \otimes \overline{k}, \mathcal{L}_{\psi(t^{-dA}f(x^{A+B})+g(x^{A+B})+P_B(t^d))}|W_S),\\
\chi(Z_S, \text{up}) &:= \chi_c(W_S \otimes \overline{k}, \mathcal{L}_{\psi(t^{-dA}f(x^{A+B})+g(x^{A+B})+P_B(t^d))}|Z_S).
\end{aligned}$$

What we must show is that $\chi(W_S, \text{down}) = (-A-B)(1-d)^S$, for S the entire set $\{1, \ldots, n\}$, where we write r^S as a shorthand for $r^{\#S}$. In fact, we will show it for all S at once.

For each S, we have a disjoint union decomposition

$$W_S = \bigsqcup_{T \subseteq S} Z_T.$$

Since compact Euler characteristic is additive, we have

$$\chi(W_S, \text{down}) = \sum_{T \subseteq S} \chi(Z_T, \text{down}) \quad \text{and} \quad \chi(W_S, \text{up}) = \sum_{T \subseteq S} \chi(Z_T, \text{up}),$$

for every $S \subseteq \{1, \ldots, n\}$. By Riemann–Hurwitz, we have

$$\chi(Z_T, \text{down}) = \frac{\chi(Z_T, \text{up})}{(A+B)^T d},$$

for every $T \subseteq \{1, \ldots, n\}$.

What remains is straightforward combinatorics. For each $T \subseteq \{1, \ldots, n\}$, we have

$$\chi(W_T, \text{up}) = (-d(A+B))(1 - d(A+B))^T,$$

by the case $dA + dB = d(A+B)$ of the (dA, dB) purity theorem. This allows us to solve for the numbers $\chi(Z_T, \text{up})$, using Möbius inversion. From these, we get the numbers

$\chi(Z_T, \text{down})$ by Riemann–Hurwitz. Then we add the $\chi(Z_T, \text{down})$, over all $T \subseteq S$, to get $\chi(W_S, \text{down})$.

Writing

$$D := (A+B)d,$$

Möbius inversion upstairs gives

$$\begin{aligned}\chi(Z_S, \text{up}) &= (-1)^S \sum_{T\subseteq S} (-1)^T \chi(W_T, \text{up}) \\ &= (-1)^S \sum_{T\subseteq S} (-1)^T (-D)(1-D)^T = (-D)(-D)^S,\end{aligned}$$

the last identity being the binomial theorem for $(1+(D-1))^S$. Now using Riemann–Hurwitz, we have

$$\chi(Z_S, \text{down}) = \frac{\chi(Z_S, \text{up})}{(A+B)^S d} = \frac{(-D)(-D)^S}{(A+B)^S d} = (-A-B)(-d)^S.$$

Summing downstairs, we get the asserted formula:

$$\chi(W_S, \text{down}) = \sum_{T\subseteq S} \chi(Z_T, \text{down}) = \sum_{T\subseteq S} (-A-B)(-d)^S = (-A-B)(1-d)^S.$$

8. Twists by Multiplicative Characters, and Ramification of $\mathcal{F}$

In this section, we consider a slight variant on the set up of the (A, B) purity theorem, where we introduce a (possibly trivial) multiplicative character

$$\chi : k^\times \to \overline{\mathbb{Q}}_\ell$$

and its associated Kummer sheaf $\mathcal{L}_\chi$ on $\mathbb{G}_m/k$.

As before, we are given two integers $A \geqslant 1$, $B \geqslant 1$, such that p is prime to $AB(A+B)$. We are also given an integer $d \geqslant 1$ prime to p and an affine-Dwork-regular Deligne polynomial f of degree d in $n \geqslant 1$ variables over the finite field k, and an arbitrary polynomial g of degree $\deg(g) := e < (B/(A+B))d$. Finally, we are given a one-variable polynomial $P_B(t)$ over k of degree B. We consider the one-parameter family of Deligne polynomials

$$t \mapsto t^A f(x) + g(x) + P_B\left(\frac{1}{t}\right)$$

over $\mathbb{G}_m$. As before, we form the sheaf $\mathcal{F}$ on $\mathbb{G}_m$ given by

$$\mathcal{F} := R^n(pr_2)_! \mathcal{L}_{\psi(t^A f(x)+g(x)+P_B(1/t))},$$

which is lisse of rank $(d-1)^n$, and pure of weight n.

This time, we are interested in the χ-twisted sums

$$\sum_{t\in k^\times} \sum_{x\in k^n} \chi(t)\psi\left(t^A f(x) + g(x) + P_B\left(\frac{1}{t}\right)\right).$$

It is the lisse sheaf $\mathcal{F}\otimes\mathcal{L}_\chi$ which governs these sums. Indeed, the Lefschetz trace formula tells us that

$$(-1)^n \sum_{t\in k^\times}\sum_{x\in k^n} \chi(t)\psi\left(t^A f(x)+g(x)+P_B\left(\frac{1}{t}\right)\right)$$
$$= \mathrm{Trace}(\mathrm{Frob}_E\,|H^2_c(\mathbb{G}_m\otimes_k \overline{k},\mathcal{F}\otimes\mathcal{L}_\chi)) - \mathrm{Trace}(\mathrm{Frob}_E\,|H^1_c(\mathbb{G}_m\otimes_k \overline{k},\mathcal{F}\otimes\mathcal{L}_\chi)).$$

Theorem 8.1 ((A,B,χ) purity theorem)**.** *In the above situation, we have the following three results:*

(i) $H^2_c(\mathbb{G}_m\otimes_k \overline{k},\mathcal{F}\otimes\mathcal{L}_\chi)=0$.
(ii) $\dim H^1_c(\mathbb{G}_m\otimes_k \overline{k},\mathcal{F}\otimes\mathcal{L}_\chi)=(A+B)(d-1)^n$.
(iii) $H^1_c(\mathbb{G}_m\otimes_k \overline{k},\mathcal{F}\otimes\mathcal{L}_\chi)$ *is pure of weight* $n+1$.

Proof. Denote by r the prime to p integer which is the order of the character χ, and by

$$[r]:\mathbb{G}_m\to\mathbb{G}_m$$

the r-th power map. Then under pullback we have

$$[r]^\star(\mathcal{F}\otimes\mathcal{L}_\chi)\cong[r]^\star\mathcal{F}.$$

Hence $H^i_c(\mathbb{G}_m\otimes_k \overline{k},\mathcal{F}\otimes\mathcal{L}_\chi)$ is a direct summand of $H^i_c(\mathbb{G}_m\otimes_k \overline{k},[r]^\star\mathcal{F})$. Now the sheaf $[r]^\star\mathcal{F}$ is precisely the sheaf attached to the data $(Ar,Br,d,f,g,P_B(t^r))$. So by the (Ar,Br) purity theorem, the groups $H^i_c(\mathbb{G}_m\otimes_k\overline{k},[r]^\star\mathcal{F})$ vanish for $i\neq 1$, while the group H^1_c is pure of weight $n+1$, and has dimension $(rA+rB)(d-1)^n$. This gives the vanishing and purity assertions. The dimension formula then results from Riemann–Hurwitz:

$$\chi_c(\mathbb{G}_m\otimes_k \overline{k},\mathcal{F}\otimes\mathcal{L}_\chi)=\frac{1}{r}\chi_c(\mathbb{G}_m\otimes_k \overline{k},[r]^\star(\mathcal{F}\otimes\mathcal{L}_\chi))=\frac{1}{r}\chi_c(\mathbb{G}_m\otimes_k \overline{k},[r]^\star\mathcal{F})$$
$$=\chi_c(\mathbb{G}_m\otimes_k \overline{k},\mathcal{F})=-(A+B)(d-1)^n. \qquad\square$$

Corollary 8.2. *If in the* (A,B,χ) *purity theorem we drop only the hypothesis that the Deligne polynomial* f *be affine-Dwork-regular, or we drop only the hypothesis that* $\deg(P_B)$ *be exactly* B*, then we have the following results:*

(i) *In the first case, the inertial invariants of* $\mathcal{F}\otimes\mathcal{L}_\chi$ *at zero vanish; in other words,* $(\mathcal{F}\otimes\mathcal{L}_\chi)^{I(0)}=0$. *In the second case, the inertial invariants of* $\mathcal{F}\otimes\mathcal{L}_\chi$ *at* ∞ *vanish; in other words,* $(\mathcal{F}\otimes\mathcal{L}_\chi)^{I(\infty)}=0$.
(ii) $H^2_c(\mathbb{G}_m\otimes_k \overline{k},\mathcal{F}\otimes\mathcal{L}_\chi)=0$.
(iii) $\dim H^1_c(\mathbb{G}_m\otimes_k \overline{k},\mathcal{F}\otimes\mathcal{L}_\chi)\leqslant(A+B)(d-1)^n$.
(iv) *We have the estimate*

$$\left|\sum_{t\in k^\times}\sum_{x\in k^n}\chi(t)\psi\left(t^A f(x)+g(x)+P_B\left(\frac{1}{t}\right)\right)\right|\leqslant(A+B)(d-1)^n(\#(k))^{(n+1)/2}.$$

Proof. The proof is identical to the proof of Corollary 5.2, everywhere replacing the sheaf $\mathcal{F}$ there by the sheaf $\mathcal{F}\otimes\mathcal{L}_\chi$. $\square$

Corollary 8.3. *If in the* (A,B) *purity theorem we drop only the hypothesis that the Deligne polynomial* f *be affine-Dwork-regular, i.e. if we allow* f *to be "only" a Deligne polynomial, then the sheaf* $\mathcal{F}$ *is totally wildly ramified at zero, i.e. under the wild inertia group* $P(0)\subset I(0)$, $\mathcal{F}$ *has no nonzero invariants, i.e.* $\mathcal{F}^{P(0)}=0$.

Proof. If not, then after some finite extension of the ground field, there will be a multiplicative character χ for which $(\mathcal{F}\otimes\mathcal{L}_\chi)^{I(0)} \neq 0$, *cf.* [3, paragraph 7.12]. By the previous corollary, no such χ exists. □

Corollary 8.4. *If in the (A, B) purity theorem we drop only the hypothesis that the Deligne polynomial f be affine-Dwork-regular, i.e. if we allow f to be "only" a Deligne polynomial, then the sheaf $\mathcal{F}$ has all its ∞-slopes $\leqslant A$, and all its 0-slopes $\leqslant B$. Moreover, if f is affine-Dwork-regular, then all ∞-slopes of $\mathcal{F}$ are equal to A, and all 0-slopes of $\mathcal{F}$ are equal to B.*

Proof. The statement is geometric, so we may assume that f_d is Dwork-regular. Then for all but finitely many $a \in \overline{k}$, $f + a$ is affine-Dwork-regular. And for all but a single $b \in \overline{k}$, $P_B(t) + bt^B$ still has degree B. So by the (A, B) purity theorem, applied to the situation $(A, B, d, f + a, g, P_B + bt^B)$, we conclude that we have

$$\chi_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F} \otimes \mathcal{L}_{\psi(at^A)} \otimes \mathcal{L}_{\psi(b/t^B)}) = -(A + B)(d - 1)^n = -(A + B)\operatorname{rank}(\mathcal{F})$$

for all but a finite number of pairs $(a, b) \in \overline{k}^2$. The effect of

$$\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{L}_{\psi(at^A)} \otimes \mathcal{L}_{\psi(b/t^B)}$$

on 0-slopes and on ∞-slopes is known, *cf.* [9, 1.3 and 8.5.4–5]. For all but finitely many pairs $(a, b) \in \overline{k}^2$, this operation "promotes" all ∞-slopes of $\mathcal{F}$ which are $\leqslant A$ to A, and leaves unchanged all ∞-slopes of $\mathcal{F}$ which are $> A$. Similarly, it "promotes" all 0-slopes of $\mathcal{F}$ which are $\leqslant B$ to B, and leaves unchanged all 0-slopes of $\mathcal{F}$ which are $> B$. So the Euler–Poincaré formula for the lisse sheaf $\mathcal{F} \otimes \mathcal{L}_{\psi(at^A)} \otimes \mathcal{L}_{\psi(b/t^B)}$ on $\mathbb{G}_m$ shows that

$$\begin{aligned}(A + B)\operatorname{rank}(\mathcal{F}) &= -\chi_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F} \otimes \mathcal{L}_{\psi(at^A)} \otimes \mathcal{L}_{\psi(b/t^B)}) \\ &= \operatorname{Swan}_0(\mathcal{F} \otimes \mathcal{L}_{\psi(at^A)} \otimes \mathcal{L}_{\psi(b/t^B)}) + \operatorname{Swan}_\infty(\mathcal{F} \otimes \mathcal{L}_{\psi(at^A)} \otimes \mathcal{L}_{\psi(b/t^B)}) \\ &= \sum_{\text{the rank}(\mathcal{F})\ 0\text{-slopes } \lambda_i \text{ of } \mathcal{F}} \max\{B, \lambda_i\} + \sum_{\text{the rank}(\mathcal{F})\ \infty\text{-slopes } \nu_i \text{ of } \mathcal{F}} s\max\{A, \nu_i\},\end{aligned}$$

and so we get the asserted inequalities on the slopes of $\mathcal{F}$ at both 0 and ∞: all $\max\{B, \lambda_i\} = B$, and all $\max\{A, \nu_i\} = A$. In the case when f is affine-Dwork-regular, we know that

$$\begin{aligned}(A + B)\operatorname{rank}(\mathcal{F}) &= -\chi_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}) \\ &= \sum_{\text{the rank}(\mathcal{F})\ 0\text{-slopes } \lambda_i \text{ of } \mathcal{F}} \lambda_i + \sum_{\text{the rank}(\mathcal{F})\ \infty\text{-slopes } \nu_i \text{ of } \mathcal{F}} \nu_i,\end{aligned}$$

so the equalities are forced: all $\lambda_i = B$, and all $\nu_i = A$. □

References

[1] T.D. Browning, D.R. Heath-Brown. Integral points on cubic hypersurfaces. *Analytic Number Theory: Essays in Honour of Klaus Roth* (W.W.L. Chen, W.T.Gowers, H. Halberstam, W.M. Schmidt, R.C. Vaughan, eds.), pp. 75–90 (Cambridge University Press, 2008).

[2] P. Deligne. La conjecture de Weil. *Publ. Math. Inst. Hautes Études Sci.*, 43 (1974), 273–307.

[3] P. Deligne. Applications de la formule des traces aux sommes trigonométriques. *Cohomologie Etale – Séminaire de Géométrie Algébrique du Bois Marie (SGA 4 1/2)*, pp. 168–232 (Lecture Notes in Mathematics 569, Springer-Verlag, 1977).

[4] P. Deligne. La conjecture de Weil II. *Publ. Math. Inst. Hautes Études Sci.*, 52 (1981), 313–428.

[5] B. Dwork. On the zeta function of a hypersurface. *Publ. Math. Inst. Hautes Études Sci.*, 12 (1962), 5–68.

[6] A. Grothendieck. Formule de Lefschetz et rationalité des fonctions *L*. *Séminaire Bourbaki 9 (1964–1966)*, Exposé 279, pp. 41–55 (Société Mathématiques de France, 1995).

[7] H. Hasse. Theorie der relativ-zyklischen algebraischen Funktionenkörper, insbesondere bei endlichem Konstantenkörper. *J. reine angew. Math.*, 172 (1934), 37–54.

[8] N.M. Katz. *Sommes exponentielles* (Astérisque 79, Société Mathématiques de France, 1980).

[9] N.M. Katz. *Gauss sums, Kloosterman sums, and monodromy groups* (Annals of Mathematics Studies 116, Princeton University Press, 1988).

[10] N.M. Katz. *Exponential sums and differential equations* (Annals of Mathematics Studies 124, Princeton University Press, 1990).

[11] N.M. Katz. Estimates for "singular" exponential sums. *Int. Math. Res. Not.*, (1999), 875–899.

[12] N.M. Katz. A semicontinuity result for monodromy under degeneration. *Forum Math.*, 15 (2003), 191–200.

[13] N.M. Katz. *Moments, monodromy, and perversity: a Diophantine perspective* (Annals of Mathematics Studies 159, Princeton University Press, 2005).

[14] A. Weil. On some exponential sums. *Proc. Natl. Acad. Sci. USA*, 34 (1948), 204–207.

[15] *Revêtements Étales et Groupe Fondamental – Séminaire de Géométrie Algébrique du Bois Marie (SGA 1)* (A. Grothendieck, dir.) (Lecture Notes in Mathematics 224, Springer-Verlag, 1971).

[16] *Théorie des Topos et Cohomologie Etale des Schémas – Séminaire de Géométrie Algébrique du Bois Marie (SGA 4)* (M. Artin, A. Grothendieck, J.L. Verdier, dir.) (Lecture Notes in Mathematics 305, Springer-Verlag, 1973).

[17] *Cohomologie Etale – Séminaire de Géométrie Algébrique du Bois Marie (SGA 4 1/2)* (P. Deligne, avec la collaboration de J.F. Boutot, A. Grothendieck, L. Illusie, J.L. Verdier) (Lecture Notes in Mathematics 569, Springer-Verlag, 1977).

[18] *Groupes de Monodromie en Géométrie Algébrique – Séminaire de Géométrie Algébrique du Bois Marie (SGA 7)* (P. Deligne, N. Katz, dir.) (Lecture Notes in Mathematics 340, Springer-Verlag, 1973).

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