# $G_2$ AND HYPERGEOMETRIC SHEAVES

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Abstract. We determine, in every finite characteristic $p$, those hypergeometric sheaves of type $(7, m)$ with $7 \geq m$ whose geometric monodromy group $G_{geom}$ lies in $G_2$, cf. Theorem 3.1 and Theorem 6.1. For each of these we determine $G_{geom}$ exactly, cf. Theorem 9.1. Each of the five primitive irreducible finite subgroups of $G_2$, namely $L_2(8), U_3(3), U_3(3).2 = G_2(2), L_2(7).2 = PGL_2(7), L_2(13)$ turns out to occur as $G_{geom}$ in a single characteristic $p$, namely $p = 2, 3, 7, 7, 13$ for the groups as listed, and for essentially just one hypergeometric sheaf in that characteristic. It would be interesting to find conceptual, rather than classificational/computational, proofs of these results.

## 1. Introduction

That the exceptional group $G_2$ occurs as the monodromy group attached to certain families of exponential sums over finite fields has been known for some time. Even now, it is striking that one can conseqently obtain "random" elements of a compact form of $G_2(\mathbb{C})$ by looking over finite fields, cf. [Ka-GKM, 11.4]. As one indication of how poorly we undertand "why" $G_2$ occurs in algebraic geometry over finite fields, note that we do not know at present if any of the other exceptional groups occur in this way.

In characteristic 2, we obtain $G_2$ as the monodromy of a certain Kloosterman sheaf, cf.[Ka-GKM, 11.1]. For sufficiently large characteristics $p$, we obtain $G_2$ as the monodromy of certain hypergeometric sheaves, or of pullbacks of these, cf.[Ka-ESDE, 10.1.1 and 10.1.2 for hypergeometric sheaves, and 9.1.1 and 9.2.1 for pullbacks]. At present, no essentially different occurences of $G_2$ as the monodromy group of a lisse sheaf on an open curve over a finite field seem to be known.

The main results of this paper are, as stated in the abstract above, the complete determination of exactly which hypergeometric sheaves in which characteristics have their monodromy either $G_2$ or a subgroup of $G_2$. It turns out, very much a posteriori, that each of the five primitive irreducible finite subgroups of $G_2$ occurs as the monodromy group of

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an essentially unique hypergeometric sheaf in one single characteristic. It also turns out, again very much a posteriori, that four out of these five groups occur in their "natural" characteristics: $L_2(8)$ occurs in characteristic 2, $U_3(3)$ occurs in characteristic 3, $L_2(7).2 = PGL_2(7)$ occurs in characteristic 7, and $L_2(13)$ occurs in characteristic 13. The remaining group is $U_3(3).2 = G_2(2)$, but it does *not* occurs in characteristic 2 or 3, but rather in characteristic 7. We do not understand why the first four groups do occur in their "natural" characteristics, but we understand even less why the fifth does not.

For $G$ either $G_2$ or a classical group, one general approach to the problem of obtaining primitive irreducible finite subgroups of $G$ as monodromy groups is this. For a given $G$, one knows [Ka-ESDE, 8.11.2 and 8.11.3] the description, in large characteristic, of all those hypergeometric sheaves whose monodromy group is $G$. One then looks at "the same sorts" of hypergeometric sheaves in low characteristic, and asks which if any of the primitive irreducible finite subgroups of $G$ will occur as their monodromy groups. This is, grosso modo, the approach we follow in analyzing the $G_2$ case.

Let us now return to the subject matter proper of this paper. In the course of proving our rather technical results, we bring to bear ideas and techniques which are themselves of independent interest, and which should be of use in other contexts as well. We have in mind in particular Kubert's beautiful but unpublished approach to questions involving $p$-adic valuations of Gauss sums, which he explained to us in lectures in 1987, and which we give an account of below, in 13.3. We use it in implementing the integrality criterion (cf. Lemma 10.3) for finite monodromy.

In analyzing the question of when the monodromy is finite, we find some interesting questions which beg to be understood. It is striking that in all cases considered here, whenever the integrality (of the trace of Frobenius at every point over every finite extension $E$ of the ground field $k$) criterion fails, it already fails at a $k$-rational point. While this is certainly not a general phenomenon, it would be interesting to understand exactly when it does happen. Another aspect which we do not understand properly is that of "erasing the characters that don't make sense", cf. 14.3 and 14.6; it can sometimes be used to prove finite monodromy, and as a heuristic it seems to have good predictive power, but it remains mysterious.

## 2. Review of hypergeometric sheaves

We begin by recalling [Ka-ESDE, Chapter 8] some basic facts about hypergeometric sheaves (which include Kloosterman sheaves as a special case). Denote by $\mathbb{Q}_{ab}$ the field $\mathbb{Q}$(all roots of unity), say inside $\mathbb{C}$. Let $k$ be a finite field of characteristic $p$ and cardinality $q$, inside a fixed $\overline{\mathbb{F}_p}$, and $\psi$ a nontrivial $\mathbb{Q}_{ab}^\times$-valued additive character of $k$. Fix two non-negative integers $n$ and $m$, at least one of which is nonzero. Let $\chi_1, ..., \chi_n$ be an unordered list of $n$ $\mathbb{Q}_{ab}^\times$-valued multiplicative characters of $k^\times$, some possibly trivial, and not necessarily distinct. Let $\rho_1, ..., \rho_m$ be another such list, but of length $m$. For $E/k$ a finite extension field (inside the fixed $\overline{\mathbb{F}_p}$), denote by $\psi_E$ the nontrivial additive character of $E$ obtained from $\psi$ by composition with the trace map $Trace_{E/k}$, and denote by $\chi_{i,E}$ (resp. $\rho_{j,E}$) the multiplicative character of E obtained from $\chi_i$ (resp. $\rho_j$) by composition with the norm map $Norm_{E/k}$. For $a \in E^\times$, the hypergeometric sum $Hyp(\psi; \chi_i' s; \rho_j' s)(a, E)$ is the cyclotomic integer defined as follows. Denote by $V(n, m, a)$ the hypersurface in $(\mathbb{G}_m)^n \times (\mathbb{G}_m)^m/E$, with coordinates $x_1, ..., x_n, y_1, ..., y_m$, defined by the equation

$$\prod_i x_i = a \prod_j y_j.$$

Then

$$Hyp(\psi; \chi_i{}'s; \rho_j{}'s)(a, E) := \sum_{V(n,m,a)(E)} \psi_E(\sum_i x_i - \sum_j y_j) \prod_i \chi_{i,E}(x_i) \prod_j \overline{\rho}_{j,E}(y_j).$$

These hypergeometric sums are closely related to monomials in Gauss sums, by multiplicative Fourier transform. Recall that for $\Lambda$ a multiplicative character of $E^\times$, the Gauss sum $g(\psi_E, \Lambda)$ is defined by

$$g(\psi_E, \Lambda) := \sum_{x \in E^\times} \psi_E(x)\Lambda(x).$$

For $E/k$ a finite extension field, and for any multiplicative character $\Lambda$ of $E^\times$ we have the formula

$$\sum_{a \in E^\times} \Lambda(a) Hyp(\psi; \chi_i{}'s; \rho_j{}'s)(a, E) = \prod_i g(\psi_E, \chi_{i,E}\Lambda) \prod_j g(\overline{\psi_E}, \overline{\rho}_{j,E}\overline{\Lambda}).$$

By Fourier inversion, for each $a \in E^\times$ we have the formula

$$\sum_\Lambda \overline{\Lambda}(a) \prod_i g(\psi_E, \chi_{i,E}\Lambda) \prod_j g(\overline{\psi_E}, \overline{\rho}_{j,E}\overline{\Lambda}) = (\#E - 1) Hyp(\psi; \chi_i{}'s; \rho_j{}'s)(a, E).$$

Now make the disjointness assumption that no $\chi_i$ is a $\rho_j$. Then for every prime $\ell \neq p$, and every embedding of $\mathbb{Q}_{ab}$ into $\overline{\mathbb{Q}}_\ell$, there exists a geometrically irreducible middle extension $\overline{\mathbb{Q}}_\ell$-sheaf

$$\mathcal{H}(\psi;\chi_i\ 's;\rho_j\ 's)$$

on $\mathbb{G}_m/k$ (i.e., this sheaf, placed in degree -1, is a geometrically irreducible perverse $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{G}_m/k$), which is characterized by the following properties. It is lisse on $\mathbb{G}_m - 1$ of rank $Max(n,m)$ and pure of weight $n+m-1$. If $n \neq m$, it is lisse on and pure on all of $\mathbb{G}_m$. Its trace function on $\mathbb{G}_m - 1$ (or on $\mathbb{G}_m$, if $n \neq m$) incarnates the above hypergeometric sums, as follows. For any finite extension $E/k$ and any $a \in E^\times - 1$, (or, if $n \neq m$, for any $a \in E^\times$), we denote by $Frob_{a,E}$ the Frobenius conjugacy class in $\pi_1(\mathbb{G}_m - 1)$ (or, if $n \neq m$, in $\pi_1(\mathbb{G}_m)$) attached to $a$ as an $E$-valued point of $\mathbb{G}_m - 1$ (or, if $n \neq m$, of $\mathbb{G}_m$). Concretely, $Frob_{a,E}$ is the conjugacy class of the image in that $\pi_1$ of the geometric Frobenius generator $Frob_E$ of $\pi_1(Spec(E)) = Gal(E^{sep}/E)$ under "the" map of fundamental groups induced by the point $a$, viewed as a morphism from $Spec(E)$ to $\mathbb{G}_m - 1$ (or to $\mathbb{G}_m$ if $n \neq m$). Then we have

$$Trace(Frob_{a,E}|\mathcal{H}(\psi;\chi_i\ 's;\rho_j \acute{s})) = (-1)^{n+m-1}Hyp(\psi;\chi_i\ 's;\rho_j\ 's)(a,E).$$

In the "missing" case when $n = m$ and $a = 1$, this formula remains valid if we interpret the left hand side to mean the trace of $Frob_E$ on the stalk of $\mathcal{H}$ at 1, i.e., on the sheaf on $Spec(E)$ which is the pullback of $\mathcal{H}$ by the map 1 from $Spec(E)$ to $\mathbb{G}_m$. We call such an $\mathcal{H}$ a hypergeometric sheaf of type $(n,m)$. [A hypergeometric sheaf of type $(n,0)$ is called a Kloosterman sheaf of rank $n$.]

If $n = m$, the local monodromy of $\mathcal{H}$ at 1, i.e., its restriction to the inertia group $I_1$ at the point 1, is a tame pseudoreflection, whose determinant is $(\prod_j \rho_j)/(\prod_i \chi_i)$, viewed as a tame character of $I_1$. Here we view multiplicative characters of $k^\times$ as characters of $I_1^{tame}$ as follows. First we use additive translation to identify $I_1$ with $I_0$. Then we view multiplicative characters of $k^\times$ as characters of $I_0^{tame}$ in two steps, as follows. First use the inclusion (which is in fact an isomorphism)

$$I_0^{tame} \subset \pi_1^{tame}(\mathbb{G}_m \otimes \overline{k}),$$

and then use the isomorphism (given by the Lang torsor)

$$\pi_1^{tame}(\mathbb{G}_m \otimes \overline{k}) \cong \mathrm{liminv}_{\mathrm{E/k},\mathrm{Norm}_{\mathrm{E/k}}}\mathrm{E}^\times.$$

Under multiplicative inversion, we have

$$inv^\star\mathcal{H}(\psi;\chi_i\ 's;\rho_j\ 's)) \cong \mathcal{H}(\overline{\psi};\overline{\rho}_j\ 's;\overline{\chi}_i\ 's)).$$

The linear dual of $\mathcal{H} := \mathcal{H}(\psi;\chi_i\ 's;\rho_j\ 's)$ is given by

$$\mathcal{H}(\psi;\chi_i\ 's;\rho_j\ 's)^\vee \cong \mathcal{H}(\overline{\psi};\overline{\chi}_i\ 's;\overline{\rho}_j\ 's)(n+m-1).$$

If $((\prod_j \rho_j)/(\prod_i \chi_i))(-1) = 1$, a condition which always holds over the quadratic extension of $k$, then multiplicative translation $t \mapsto (-1)^{n-m}t$ carries $\mathcal{H}(\psi; \chi_i\ 's; \rho_j\ 's)$ into $\mathcal{H}(\overline{\psi}; \chi_i\ 's; \rho_j\ 's)$. More generally, for $A$ the constant

$$A := ((\prod_j \rho_j)/(\prod_i \chi_i))(-1) = \pm 1,$$

and $A^{deg}$ the corresponding geometrically constant lisse sheaf of rank one, we have

$$[t \mapsto (-1)^{n-m}t]^\star \mathcal{H}(\psi; \chi_i\ 's; \rho_j\ 's) \cong \mathcal{H}(\overline{\psi}; \chi_i\ 's; \rho_j\ 's) \bigotimes A^{deg}.$$

Let us now suppose in addition that $n \geq m$, a situation we can always achieve by multiplicative inversion. Then local monodromy at 0 is tame, and the action of a generator $\gamma_0$ of $I_0^{tame}$ is the action of $T$ on the $\overline{\mathbb{Q}}_\ell[T]$-module $\overline{\mathbb{Q}}_\ell[T]/(P(T))$, for $P(T)$ the polynomial

$$P(T) := \prod_i (T - \chi_i(\gamma_0)).$$

Here we view multiplicative characters $\chi_i$ of $k^\times$ as characters of $I_0^{tame}$ as explained just above. We will use later the consequence of this description that local monodromy at 0 is of finite order if and only if the $n$ characters $\chi_i$ are all distinct.

The local monodromy at $\infty$ is the direct sum of an $m$-dimensional tame summand, say $Tame_m$, and, if $n > m$, a totally wild summand, say $Wild_{n-m}$, of dimension $n - m$, Swan conductor 1, and all upper numbering breaks equal to $1/(n-m)$. On $Tame_m$, the action of a a generator $\gamma_\infty$ of $I_\infty^{tame}$ is the action of $T$ on the $\overline{\mathbb{Q}}_\ell[T]$-module $\overline{\mathbb{Q}}_\ell[T]/(Q(T))$, for $Q(T)$ the polynomial

$$Q(T) := \prod_j (T - \rho_j(\gamma_\infty)).$$

Here we use view multiplicative characters $\rho_j$ of $k^\times$ as characters of $I_\infty^{tame}$ via the inclusion, again an isomorphism,

$$I_\infty^{tame} \subset \pi_1^{tame}(\mathbb{G}_m \otimes \overline{k}).$$

If in addition $n - m \geq 2$, then $det(Wild_{n-m})$ is tame, equal to the character $(\prod_i \chi_i)/(\prod_j \rho_j)$. Moreover, the isomorphism class of $Wild_{n-m}$ as a represenation of $I_\infty$, indeed the isomorphism class of any totally wild representation of $I_\infty$ with Swan conductor 1, is determined, up to multiplicative translation on $\mathbb{G}_m \otimes \overline{k}$, by its rank $n-m$ and its determinant. It will be important later to have explicit incarnations of such wild representations.

For any $d \geq 2$, we obtain a $Wild_d$ with given determinant, say $\alpha$, as the $I_\infty$-representation attached to any hypergeometric sheaf $\mathcal{H}(\psi; \chi_1, ..., \chi_d)$ of type $(d, 0)$ with $\prod_i \chi_i = \alpha$. If $d$ is prime to the characteristic $p$, then denoting by $[d] : \mathbb{G}_m \to \mathbb{G}_m$ the $d$'th power map, and by $\mathcal{L}_\psi$ the Artin Schreier sheaf, one knows [Ka-GKM, 5.6.2] that the direct image $[d]_\star \mathcal{L}_\psi$ is geometrically isomorphic to a multiplicative translate of the hypergeometric sheaf of type $(d, 0)$, made with the $d$ characters $\chi_i$ whose $d$'th power is trivial. Its $Wild_d$ thus has trivial determinant if $d$ is odd, and has determinant the quadratic character if $d$ is even.

When $d$ is prime to the characteristic $p$, any multiplicative character $\alpha$ has, over a possibly bigger finite field, a $d$'th root. So every totally wild representation of rank $d$ is, geometrically, a multiplicative translate of the tensor product of some tame character with $[d]_\star \mathcal{L}_\psi$. As a consequence of this, when $d$ is prime to $p$, any totally wild representation of $I_\infty$ with $Swan_\infty = 1$ and rank $d$ has, up to multiplicative translation, a known and explicit restriction to the wild inertia group $P_\infty$: it is the restriction to $P_\infty$ of the direct sum of the Artin-Schreier sheaves $\mathcal{L}_{\psi(\zeta x)}$ over all $\zeta$ in the group $\mu_d(\overline{k})$ of roots of unity of order dividing $d$.

## 3. Hypergeometric sheaves whose geometric monodromy group lies in $G_2$: Statement of the theorem

Given a hypergeometric sheaf $\mathcal{H}$ on $\mathbb{G}_m/k$ of type $(n, m)$ with $n > m$ (resp. with $n = m$) its arithmetic and geometric monodromy groups $G_{arith} \triangleright G_{geom}$ are the $\overline{\mathbb{Q}}_\ell$-algebraic subgroups of $GL(n)$ which are the Zariski closures of $\pi_1(\mathbb{G}_m/k)$ and of its normal subgroup $\pi_1(\mathbb{G}_m \otimes \overline{k})$ (resp. of $\pi_1((\mathbb{G}_m - 1)/k)$ and of its normal subgroup $\pi_1((\mathbb{G}_m - 1) \otimes \overline{k})$ in the $n$-dimensional $\overline{\mathbb{Q}}_\ell$-representation which "is" $\mathcal{H}$.

Recall that $G_2$ is the automorphism group of Cayley's and Graves' octonions, cf. [Adams, 15.16], [Spr, 17.4]. We view $G_2$ as a subgroup of $SO(7)$ via its unique irreducible 7-dimensional representation (namely, its action on the "purely imaginary" octonions). We first determine all hypergeometric sheaves of type $(7, m)$ with $7 \geq m$ whose $G_{geom}$ lies in $G_2$.

**Theorem 3.1.** *The hypergeometric sheaves of type* $(7, m)$ *with* $7 \geq m$ *whose* $G_{geom}$ *lies in* $G_2$ *are the following.*

1. *In characteristic* $p = 2$*, those hypergeometrics of type* $(7, 0)$ *of the form* $\mathcal{H}(\psi; 1, \alpha, \beta, \alpha\beta, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta}; \emptyset)$*, for some finite field* $k$ *of characteristic* 2*, some nontrivial additive character* $\psi$*, and for two (possibly trivial, possibly not distinct) multiplicative characters* $\alpha$ *and* $\beta$*.*

(2) *In odd characteristic p, those hypergeometrics of type* $(7,1)$ *of the form* $\mathcal{H}(\psi; 1, \alpha, \beta, \alpha\beta, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta}; \chi_{quad})$*, for some finite field k of characteristic p, some nontrivial additive character* $\psi$*, and for two (possibly trivial, possibly not distinct) multiplicative characters* $\alpha$ *and* $\beta$ *such that none of* $\alpha$*,* $\beta$*, or* $\alpha\beta$ *is the quadratic character* $\chi_{quad}$*.*

## 4. First part of the proof of Theorem 3.1: no other candidates can work

We first observe that no hypergeometric $\mathcal{H}$ of type $(n,n)$ ever has its $G_{geom}$ inside $SO(n)$. Indeed, local monodromy at 1 is a pseudoreflection. Now a pseudoreflection lies in an orthogonal group $O(n)$ only if it is a reflection, but in that case it has determinant $-1$, so does not lie in $SO(n)$. Taking $n = 7$, and remembering that $G_2$ lies in $SO(7)$, we see that only the case $7 > m$ can occur.

We also recall that when $p$ is odd, every geometrically self-dual hypergeometric of type $(n,m)$ has $n - m$ even, cf. [Ka-ESDE, 8.8.1]. Thus for $p$ odd, we must have $m$ odd.

There are two key facts we will exploit. The first is that the seven eigenvalues of any element of $G_2(\overline{\mathbb{Q}}_\ell)$ (viewed in its 7-dimensional representation) are of the form $(1, x, y, xy, 1/x, 1/y, 1/(xy))$, for some $x$ and $y$ in $\overline{\mathbb{Q}}_\ell^\times$. The second [Asch, Theorem 5, parts (2) and (5) on page 196], [Co-Wa, page 449] is that an irreducible subgroup $G$ of $O(7)$ over $\overline{\mathbb{Q}}_\ell$ lies in (some $O(7)$-conjugate of) $G_2$ if and only if $G$ has a nonzero invariant in the third exterior power $\Lambda^3$ of the given 7-dimensional representation (in which case the space of $G$-invariants in $\Lambda^3$ is one-dimensional, and the fixer in $O(7)$ of this one-dimensional space is the $G_2$ in question).

It follows from the first key fact that if a hypergeometric $\mathcal{H}$ of type $(7,m)$ with $7 > m$ has $G_{geom}$ inside $G_2$, then it must be of the form $\mathcal{H}(\psi; 1, \alpha, \beta, \alpha\beta, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta}; \rho_1, ..., \rho_m)$ for two (possibly trivial, possibly not distinct) multiplicative characters $\alpha$ and $\beta$, and for some list of $m < 7$ multiplicative characters $(\rho_1, ..., \rho_m)$, none of which is on the list $(1, \alpha, \beta, \alpha\beta, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta})$. If such an $\mathcal{H}$ has $G_{geom}$ inside $G_2$, and hence inside $SO(7)$, it is geometrically self-dual, and hence the list $(\rho_1, ..., \rho_m)$ must be stable under complex conjugation. As none of the $\rho_i$ is trivial, either $m$ is even, say $m = 2d$, and our list consists of $d$ pairs of inverse characters, or $m$ is odd, say $m = 2d + 1$, and our list consists $d$ pairs of inverse characters together with a single real-valued character which is non-trivial. Such a character exists only if $p$ is odd, in which case it is the quadratic character $\chi_{quad}$.

Suppose first that $p = 2$. Then $m$ is even. We will show that $m = 0$. We proceed case by case.

Suppose first that $m = 6$. Then the wild part $Wild_1$ of the $I_\infty$-representation is a non-tame character. So there exists an element $\gamma_{wild}$ in the wild interia group $P_\infty$ on which $Wild_1$ is non-trivial. Replacing $\gamma_{wild}$ by a power of itself, and remembering that $p = 2$, we may assume that $Wild_1$ takes the value $-1$ on $\gamma_{wild}$. As the six $\rho_j$ are all tame, they are trivial on $\gamma_{wild}$. So the seven eigenvalues of $\gamma_{wild}$ in the $I_\infty$-representation are $(1, 1, 1, 1, 1, 1, -1)$, and hence $\gamma_{wild}$ has determinant $-1$, which is not possible for an element of $G_2$.

Suppose next that $m = 4$. Then the wild part $Wild_3$ of the $I_\infty$-representation is, on the wild interia group $P_\infty$, isomorphic to the direct sum over the cube roots of unity, i.e., over element $\zeta \in \mathbb{F}_4^\times$, of the Artin Schrier sheaves $\mathcal{L}_{\psi(\zeta x)}$. Since $\mathbb{F}_2(\zeta_3) = \mathbb{F}_{2^2}$, the cyclotomic polynomial $1 + X + X^2$ is irreducible over $\mathbb{F}_2$. In other words,

$$\sum_{\zeta \in \mathbb{F}_4^\times} \zeta = 0,$$

but any two are linearly independent over $\mathbb{F}_2$. So the $\mathcal{L}_{\psi(\zeta x)}$ are three nontrivial characters of $P_\infty$, whose product is trivial, but no product of two of them is trivial. So any two of them map $P_\infty$ onto the product group $(\pm 1) \times (\pm 1)$. Taking an element $\gamma_{wild} \in P_\infty$ which maps to $(-1, -1)$ this way, we see that its eigenvalues in $Wild_3$ are $(-1, -1, 1)$. As the four $\rho_j$ are all tame, they are trivial on $\gamma_{wild}$. So its seven eigenvalues in the $I_\infty$-representation are $(1, 1, 1, 1, 1, -1, -1)$. But this eigenvalue pattern does not occur in $G_2$: indeed, for any $z \neq 1$ in $\overline{\mathbb{Q}}_\ell^\times$, the eigenvalue pattern $(1, 1, 1, 1, 1, z, 1/z)$ does not occur in $G_2$.

Suppose finally that $m = 2$. Then the wild part $Wild_5$ of the $I_\infty$-representation is, on the wild interia group $P_\infty$, isomorphic to the direct sum over the fifth roots of unity in $\mathbb{F}_{2^4}$ of the Artin Schrier sheaves $\mathcal{L}_{\psi(\zeta x)}$. Since $\mathbb{F}_2(\zeta_5) = \mathbb{F}_{2^4}$,the cyclotomic polynomial $1 + X + X^2 + X^3 + x^4$ is irreducible over $\mathbb{F}_2$. In other words,

$$\sum_{\zeta \in \mu_5(\mathbb{F}_{16})} \zeta = 0,$$

but any four are linearly independent over $\mathbb{F}_2$. So any four of them map $P_\infty$ onto the product group $(\pm 1) \times (\pm 1) \times (\pm 1) \times (\pm 1)$. Taking an element $\gamma_{wild} \in P_\infty$ which maps to $(-1, -1, 1, 1)$ this way, we see that its eigenvalues in $Wild_5$ are $(-1, -1, 1, 1, 1)$. As the two $\rho_j$ are both tame, they are trivial on $\gamma_{wild}$. So its seven eigenvalues in the $I_\infty$-representation are $(1, 1, 1, 1, 1, -1, -1)$. As noted above, this eigenvalue pattern does not occur in $G_2$.

We next show that if $p$ is odd, then $m = 1$. We have already noted that $m$ is odd, and that $m < 7$. Once again, we argue case by case.

Suppose first that $m = 5$. Then the wild part $Wild_2$ of the $I_\infty$- representation is, on the wild interia group $P_\infty$, isomorphic to the direct sum $\mathcal{L}_{\psi(x)} \bigoplus \mathcal{L}_{\psi(-x)}$. So for an element $\gamma_{wild} \in P_\infty$ on which $\mathcal{L}_{\psi(2x)}$ is nontrivial, its eigenvalues in $Wild_2$ are $(\zeta_p, 1/\zeta_p)$, for some nontrivial $p$'th root of unity $\zeta_p$. As the five $\rho_j$ are all tame, they are trivial on $\gamma_{wild}$. So its seven eigenvalues in the $I_\infty$-representation are $(1, 1, 1, 1, 1, \zeta_p, 1/\zeta_p)$. As noted above, this eigenvalue pattern does not occur in $G_2$.

Suppose next that $m = 3$. Then the wild part $Wild_4$ of the $I_\infty$-representation is, on the wild interia group $P_\infty$, isomorphic to the direct sum $\mathcal{L}_{\psi(x)} \bigoplus \mathcal{L}_{\psi(ix)} \bigoplus \mathcal{L}_{\psi(-x)} \bigoplus \mathcal{L}_{\psi(-ix)}$, where $i$ denotes a chosen primitive fourth root of unity in $\overline{\mathbb{F}}_p$.

Suppose first that 1 and $i$ are linearly independent over $\mathbb{F}_p$, i.e., suppose that $p \equiv 3 \bmod 4$. Then the direct sum $\mathcal{L}_{\psi(x)} \bigoplus \mathcal{L}_{\psi(ix)}$ maps $P_\infty$ onto the product group $\mu_p(\overline{\mathbb{Q}}_\ell) \times \mu_p(\overline{\mathbb{Q}}_\ell)$. Pick an element $\gamma_{wild} \in P_\infty$ which maps to $(\zeta_p, 1)$ this way, for some nontrivial $p$'th root of unity $\zeta_p$. Then its eigenvalues in $Wild_4$ are $(\zeta_p, 1, 1/\zeta_p, 1)$. As the three $\rho_j$ are all tame, they are trivial on $\gamma_{wild}$. So its seven eigenvalues in the $I_\infty$-representation are $(1, 1, 1, 1, 1, \zeta_p, 1/\zeta_p)$. As noted above, this eigenvalue pattern does not occur in $G_2$.

Suppose next that $i$ lies in $\mathbb{F}_p$, i.e., suppose that $p \equiv 1 \bmod 4$. Let us pick a more neutral name, say $a$, for an element of order 4 in $\mathbb{F}_p^\times$. Then for an element $\gamma_{wild} \in P_\infty$ on which $Wild_4$ is nontrivial, its eigenvalues in $Wild_4$ are $(\zeta_p, \zeta_p^a, 1/\zeta_p, 1/\zeta_p^a)$, and so its seven eigenvalues in the $I_\infty$-representation are $(1, 1, 1, \zeta_p, \zeta_p^a, 1/\zeta_p, 1/\zeta_p^a)$. Because $a$ is neither 1 nor $-1$, this eigenvalue pattern does not occur in $G_2$. Indeed, if $(1, 1, 1, \zeta_p, \zeta_p^a, 1/\zeta_p, 1/\zeta_p^a)$ as unordered list is of the form $(1, x, y, xy, 1/x, 1/y, 1/(xy))$, then at least two of the six elements $(x, y, xy, 1/x, 1/y, 1/(xy))$ must be 1. If $xy$ (or, equivalently $1/xy$) is 1, our list is of the form $(1, , 1, 1, x, x, 1/x, 1/x)$. Otherwise at least one of $x$ or $y$ is 1, say $y = 1$. Again our list is of the form $(1, , 1, 1, x, x, 1/x, 1/x)$. But $(1, 1, 1, \zeta_p, \zeta_p^a, 1/\zeta_p, 1/\zeta_p^a)$ is not of this form, precisely because $a$ is neither 1 nor $-1$.

Thus for $p$ odd, we must have $m = 1$. By autoduality, the single $\rho$ must be equal to its complex conjugate, so must be either trivial or the quadratic character $\chi_{quad}$. It cannot be trivial, since the trivial character already occurs on the list $(1, \alpha, \beta, \alpha\beta, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta})$.

## 5. Final part of the proof of Theorem 3.1: all our candidates do work

To show that all of our candidates work, we use the second key fact. Each candidate $\mathcal{H}$ is geometrically irreducible and geometrically self-dual of rank 7, so its $G_{geom}$ is certainly an irreducible subgroup of $O(7)$. [In fact, each $G_{geom}$ lies in $SO(7)$. Indeed, since $n - m \geq 2$ in each case, $det(\mathcal{H})$ is lisse of rank 1 on $\mathbb{G}_m$ and tame at both 0 and $\infty$, so geometrically isomorphic to a Kummer sheaf $\mathcal{L}_\chi$. Looking at the $I_0$-representation, we see that $\chi$ is trivial.] So to show that $G_{geom}$ lies in $G_2$, it suffices to show that $G_{geom}$ has a nonzero invariant in $\Lambda^3$ of its given 7-dimensional representation, i.e., that $\pi_1(\mathbb{G}_m \otimes \overline{k})$ acting on $\Lambda^3$ of the 7-dimensional representation which "is" $\mathcal{H}$, has a nonzero invariant, or equivalently that $H^0(\mathbb{G}_m \otimes \overline{k}, \Lambda^3(\mathcal{H}))$ is nonzero. Since $\mathcal{H}$ is $\pi_1(\mathbb{G}_m \otimes \overline{k})$-irreducible, $\Lambda^3(\mathcal{H})$ is completely reducible as a $\pi_1(\mathbb{G}_m \otimes \overline{k})$-representation, so it suffices to show instead that the space of $\pi_1(\mathbb{G}_m \otimes \overline{k})$-coinvariants is nonzero, or equivalently that the group $H^2_c(\mathbb{G}_m \otimes \overline{k}, \Lambda^3(\mathcal{H}))$ is nonzero. Denote by

$$j : \mathbb{G}_m \to \mathbb{P}^1$$

the inclusion. Then we have natural isomorphisms

$$H^0(\mathbb{G}_m \otimes \overline{k}, \Lambda^3(\mathcal{H})) \cong H^0(\mathbb{P}^1 \otimes \overline{k}, j_\star\Lambda^3(\mathcal{H}))$$

and

$$H^2_c(\mathbb{G}_m \otimes \overline{k}, \Lambda^3(\mathcal{H})) \cong H^2(\mathbb{P}^1 \otimes \overline{k}, j_\star\Lambda^3(\mathcal{H})).$$

[The first is tautological, the second is the birational invariance of $H^2_c$.] Thus it suffices finally to show the dimension inequality

$$h^0(\mathbb{P}^1 \otimes \overline{k}, j_\star\Lambda^3(\mathcal{H})) + h^2(\mathbb{P}^1 \otimes \overline{k}, j_\star\Lambda^3(\mathcal{H})) > 0.$$

We will show something stronger, namely that the Euler-Poincare characteristic satisfies the inequality

$$\chi(\mathbb{P}^1 \otimes \overline{k}, j_\star\Lambda^3(\mathcal{H})) > 0.$$

For this, we argue as follows. The sheaf $\mathcal{H}$ is lisse on $\mathbb{G}_m$, and tame at 0, so also the sheaf $\Lambda^3(\mathcal{H})$. By the Euler- Poincare formula, we have

$$\chi(\mathbb{P}^1 \otimes \overline{k}, j_\star\Lambda^3(\mathcal{H})) = dim((\Lambda^3(\mathcal{H}))^{I_0}) + dim((\Lambda^3(\mathcal{H}))^{I_\infty}) - Swan_\infty(\Lambda^3(\mathcal{H})).$$

From the shape of the $I_0$-representation, one sees easily that

$$dim((\Lambda^3(\mathcal{H}))^{I_0}) \geq 5,$$

cf. [Ka-ESDE, page 125].

To conclude the proof when $p = 2$, it suffices to show that we have the equality

$$Swan_\infty(\Lambda^3(\mathcal{H})) = 4.$$

To see this, we use the fact that $Wild_7$ is geometrically isomorphic to a multiplicative translate of the direct image $[7]_\star\mathcal{L}_\psi$ of $\mathcal{L}_\psi$ by the 7-th power endomorphism $[7]$ of $G_m$. Since 7 is prime to the characteristic $p = 2$, it is the same to show that

$$Swan_\infty([7]^\star\Lambda^3(\mathcal{H})) = 28.$$

But we have

$$[7]^\star\Lambda^3(\mathcal{H}) \cong \Lambda^3([7]^\star(\mathcal{H})) \cong \Lambda^3([7]^\star[7]_\star\mathcal{L}_\psi) \cong \Lambda^3(\bigoplus_{\zeta\in\mu_7(\overline{\mathbb{F}}_2)} \mathcal{L}_{\psi(\zeta x)}).$$

But $\mu_7(\overline{\mathbb{F}}_2)$ is just $\mathbb{F}_8^\times$, so this last object is

$$\Lambda^3(\bigoplus_{a\in\mathbb{F}_8^\times} \mathcal{L}_{\psi(ax)})$$

$$\cong \bigoplus_{\text{unordered triples } (a_1,a_2,a_3) \text{ of distinct elements } \in\mathbb{F}_8^\times} \mathcal{L}_{\psi((a_1+a_2+a_3)x)}.$$

So $Swan_\infty([7]^\star\Lambda^3(\mathcal{H}))$ is the number of unordered triples $(a_1, a_2, a_3)$ of distinct elements of $\mathbb{F}_8^\times$ whose sum is nonzero, simply because $Swan_\infty\mathcal{L}_{\psi(bx)}$ is 1 if $b \neq 0$ , and is 0 if $b = 0$. So it is equivalent to see that there are precisely 7 unordered triples $(a_1, a_2, a_3)$ of distinct elements of $\mathbb{F}_8^\times$ whose sum is zero. Such a triple is simply the list of the nonzero elements in a 2-dimensional $\mathbb{F}_2$-subspace of the 3-dimensional $\mathbb{F}_2$-space $\mathbb{F}_8$. These 2-dimensional subspaces are just the $\mathbb{F}_2$-rational points of $\mathbb{P}^2$, of which there are $1 + 2 + 2^2 = 7$. This concludes the proof in characteristic $p = 2$ that

$$\chi(\mathbb{P}^1 \otimes \overline{k}, j_\star\Lambda^3(\mathcal{H})) > 0.$$

We now turn to proving this inequality when $p$ is odd. We first observe that we have the upper bound

$$Swan_\infty(\Lambda^3(\mathcal{H})) \leq 5.$$

Indeed, all the $\infty$-slopes of $\mathcal{H}$ are either 1/6 or 0, so all are $\leq 1/6$. Therefore all the $\infty$-slopes of $\Lambda^3(\mathcal{H})$ are $\leq 1/6$, so we have the inequality

$$Swan_\infty(\Lambda^3(\mathcal{H})) \leq (1/6)rank(\Lambda^3(\mathcal{H})) = 35/6.$$

But Swan conductors are integers, so we have the asserted upper bound.

To conclude the proof for $p$ odd, it suffices to establish the inequality

$$dim((\Lambda^3(\mathcal{H}))^{I_\infty}) \geq 1.$$

As $I_\infty$-representation, $\mathcal{H}$ is the direct sum $Wild_6 \bigoplus \mathcal{L}_{\chi_{quad}}$, and $Wild_6$ has determinant $\chi_{quad}$. So as $I_\infty$-representation, $\Lambda^3(\mathcal{H})$ is the direct sum

$$\Lambda^3(\mathcal{H}) \cong \Lambda^3(Wild_6) \bigoplus \mathcal{L}_{\chi_{quad}} \otimes \Lambda^2(Wild_6).$$

So it suffices to establish the inequality

$$dim((\Lambda^3(Wild_6)^{I_\infty}) \geq 1.$$

We will do this by a global argument. Namely, we will construct a hypergeometric sheaf $\mathcal{G}$ of type $(6,0)$ whose $I_\infty$-representation is a multiplicative translate of $Wild_6$, and show that $\Lambda^3(\mathcal{G})$, as a representation of the entire geometric fundamental group $\pi_1(\mathbb{G}_m \otimes \overline{k})$, has a nonzero invariant. We know that $Wild_6$ has determinant $\chi_{quad}$ as an $I_\infty$-representation, so any $\mathcal{G}$ of type $(6,0)$ whose characters satisfy $\prod_i \chi_i = \chi_{quad}$ has the desired $I_\infty$-representation. We will choose a particular $\mathcal{G}$ as follows. Choose a multiplicative character $\alpha$ which is nontrivial and of odd order, e.g. of order an odd prime different from $p$. This is always possible after enlarging the finite field $k$. Then take

$$\mathcal{G} := \mathcal{H}(\psi; 1, \chi_{quad}, \alpha, \overline{\alpha}, \chi_{quad}\alpha, \chi_{quad}\overline{\alpha}).$$

The key virtue of this choice of $\mathcal{G}$ is that the six characters

$$(1, \chi_{quad}, \alpha, \overline{\alpha}, \chi_{quad}\alpha, \chi_{quad}\overline{\alpha})$$

are all distinct, and among them are four subsets of three, namely $(1, \alpha, \overline{\alpha})$, $(1, \chi_{quad}\alpha, \chi_{quad}\overline{\alpha})$, $(\chi_{quad}, \alpha, \chi_{quad}\overline{\alpha})$, and $(\chi_{quad}, \chi_{quad}\alpha, \overline{\alpha})$, where the product of the named characters is trivial.

Exactly as above, to show that $\Lambda^3(\mathcal{G})$, as a representation of the entire geometric fundamental group $\pi_1(\mathbb{G}_m \otimes \overline{k})$, has a nonzero invariant, it suffices to establish the inequality

$$\chi(\mathbb{P}^1 \otimes \overline{k}, j_\star \Lambda^3(\mathcal{G})) > 0.$$

As $\mathcal{G}$ is lisse on $\mathbb{G}_m$ and tame at $\infty$, we have

$$\chi(\mathbb{P}^1 \otimes \overline{k}, j_\star \Lambda^3(\mathcal{G})) = dim((\Lambda^3(\mathcal{G}))^{I_0}) + dim((\Lambda^3(\mathcal{G}))^{I_\infty}) - Swan_\infty(\Lambda^3(\mathcal{G})).$$

From the shape of the $I_0$-representation, it is obvious that

$$dim((\Lambda^3(\mathcal{G}))^{I_0}) \geq 4.$$

So to conclude the proof, it suffices to observe that we have the inequality

$$Swan_\infty(\Lambda^3(\mathcal{G})) \leq 3.$$

But this is obvious, as all the $\infty$-slopes of $\mathcal{G}$ are $1/6$, and $\Lambda^3(\mathcal{G})$ has rank 20. Thus $Swan_\infty(\Lambda^3(\mathcal{G})) \leq 20/6$, but Swan conductors are integers. This concludes the proof in the case when $p$ is odd.

## 6. Exact determination of $G_{geom}$; statement of a preliminary result

We need to recall some basic terminology before we state the preliminary result we have in mind. An irreducible, finite-dimensional $\overline{\mathbb{Q}}_\ell$-representation $(V, \pi)$ of an algebraic group $G$ over $\overline{\mathbb{Q}}_\ell$ is said to be *Lie-irreducible* if $Lie(G)$ acts irreducibly on $V$, or equivalently if the identity component $G^0$ acts irreducibly on $V$. It is said to be *imprimitive*, or *induced*, if there exists a direct sum decomposition of $V$ as a direct sum of at least 2 nonzero subspaces, say $V = \bigoplus_i V_i$, which are permuted by $G$. It is said to be *primitive*, or *non-induced*, if there exists no such direct sum decomposition.

A geometrically irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$, say of rank $n$, on an open smooth connected curve $V$ over an algebraically closed field $\overline{k}$ in which $\ell$ is invertible, is said to be *Lie-irreducible*, or *induced*, or *primitive*, if the corresponding $n$-dimensional representation of its $G_{geom}$ has that property. To say that $\mathcal{F}$ as above is induced is to say that it is of the form $f_\star \mathcal{G}$ for $f : U \to V$ a finite etale $\overline{k}$-map of degree $d \geq 2$ with connected source $U$, and $\mathcal{G}$ a lisse sheaf on $U$.

An irreducible subgroup $G$ of $GL(n, \overline{\mathbb{Q}}_\ell)$ is said to be Lie-irreducible, or induced, or primitive, if the given $n$-dimensional representation of $G$ has that property.

**Theorem 6.1.** *Let $\mathcal{H}$ be a hypergeometric of type $(7, m)$, $7 > m$, whose $G_{geom}$ lies in $G_2$. Then we have the following results.*

1. *If either $p > 13$ or if $p$ is 11 or 5, then $G_{geom} = G_2$.*
2. *If $p$ is 3, 7, or 13, then $G_{geom}$ is either $G_2$ or it is a finite irreducible primitive (viewed in the ambient $GL(7)$) subgroup of $G_2$.*
3. *If $p$ is 2, then with one exception $G_{geom}$ is either $G_2$ or it is a finite irreducible primitive subgroup of $G_2$. The exceptional case is when $\mathcal{H}$ is geometrically isomorphic to $[7]_\star \mathcal{L}_\psi$ for some $\psi$, in which case $G_{geom}$ is isomorphic to the $ax + b$ group of the field $\mathbb{F}_8$. In fact, for the sheaf $[7]_\star \mathcal{L}_\psi$ on $\mathbb{G}_m$ over a field containing $\mathbb{F}_8$, both $G_{geom}$ and $G_{arith}$ are this $ax + b$ group.*

## 7. proof of Theorem 6.1

Let us recall [Ka-MG] the following fundamental trichotomy for a geometrically irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$, say of rank $n$, on an open smooth connected curve $V$ over an algebraically closed field $\overline{k}$ in which $\ell$ is invertible. Either $\mathcal{F}$ is Lie-irreducible, or it is induced, or there exists a divisor $d \geq 2$ of $n$ such that $\mathcal{F}$ is the tensor product of a Lie irreducible lisse sheaf of rank $n/d$ and an irreducible lisse sheaf of rank $d$ whose $G_{geom}$ is finite and primitive. In the special case when $n$ is prime and when $det(\mathcal{F})$ is of finite order, the third case means precisely that the $G_{geom}$ of $\mathcal{F}$ is finite and primitive.

Let us apply this to a hypergeometric $\mathcal{H}$ of type $(7, m)$, $m \leq 1$, whose $G_{geom}$ lies in $G_2$. Suppose first that $\mathcal{H}$ is Lie-irreducible. Then we claim that $G_{geom} = G_2$. To see this, we argue as follows. We have $G_{geom} \subset G_2 \subset SO(7) \subset SL(7)$, so $G^0_{geom}$ is a connected irreducible subgroup of $SL(7)$, hence is semisimple, cf. [Ka-GKM, 11.5.2.2]. By the classification of prime-dimensional representations [Ka-ESDE, Theorem 1.6], the only connected irreducible semi-simple subgroups of $G_2$ are $G_2$ itself and $SL(2)/\pm 1$ (acting in $Sym^6$ of the standard representation of $SL(2)$). If $G^0_{geom}$ is $G_2$, then we trivially have $G_{geom} = G_2$. If $G^0_{geom}$ is $SL(2)/\pm 1$, then we observe first that $G_{geom}$ is $SL(2)/\pm 1$, because $SL(2)/\pm 1$ is its own normalizer in $G_2$ (indeed in $SO(7)$, simply because every automorphism of $SL(2)/\pm 1$ is inner, and $SO(7)$ contains no nontrivial scalars), while $G_{geom}$ lies in this normalizer. But $SL(2)/\pm 1 \cong SO(3)$ has a faithful 3-dimensional representation, say $\pi$, so the "pushout" $\pi(\mathcal{H})$ is a geometrically irreducible lisse sheaf on $\mathbb{G}_m$ of rank 3 which is tame at 0 (indeed, $\mathcal{H}$ is tame at 0, so the wild inertia subgroup $P_0$ of $I_0$ dies in $G_{geom}$, so it dies in any homomorphic image), and all of whose $\infty$-slopes are at most 1/6 (indeed, $\mathcal{H}$ has all its nonzero $\infty$-slopes either 1/7, if $p = 2$, or 1/6 otherwise, so always at most 1/6, so for any real $y > 1/6$, the upper numbering subgroup $I^y_\infty$ of $I_\infty$ dies in $G_{geom}$, so dies in any homomorphic image). Thus $\pi(\mathcal{H})$ is tame at both 0 and $\infty$ ( its Swan conductor at $\infty$ is at most 3/6, but is an integer), yet is geometrically irreducible of rank 3: this is impossible, because $\pi_1^{tame}(\mathbb{G}_m \otimes \overline{k})$ is abelian.

Now suppose that $\mathcal{H}$ is induced. We apply to it the following proposition.

**Proposition 7.1.** *Let $\mathcal{F}$ be a geometrically irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ of rank $n$ on $\mathbb{G}_m$ over an algebraically closed field $\overline{k}$, of positive characteristic $p$, in which $\ell$ is invertible. Suppose that $\mathcal{F}$ is tame at* 0 *and that $Swan_\infty(\mathcal{F}) = 1$. Suppose that $\mathcal{F}$ is induced. Then we are in one of the following three cases.*

(1) *$\mathcal{F}$ is Kummer induced, i.e., of the form $[d]_\star\mathcal{G}$ for some divisor $d \geq 2$ of $n$ which is invertible in $\overline{k}$, and some lisse $\mathcal{G}$ on $\mathbb{G}_m$ of rank $n/d$, which is itself geometrically irreducible, tame at* 0*, and has $Swan_\infty(\mathcal{G}) = 1$.*

(2) *We have $p|n$, and $\mathcal{F}$ is Belyi induced from an everywhere tame, lisse sheaf $\mathcal{G}$ on $\mathbb{P}^1 - \{0, 1, \infty\}$ of rank one. More precisely, $\mathcal{F}$*

*is* $f_\star\mathcal{G}$ *for a finite etale map*

$$f : \mathbb{P}^1 - \{0, 1, \infty\} \to \mathbb{G}_m$$

*given by a Belyi polynomial*

$$f(X) = \lambda X^a(1 - X)^b$$

*with* $\lambda \in \overline{k}^\times$ *and with* $a$ *and* $b$ *strictly positive, prime to* $p$*, integers with* $a + b = n$*.*

(3) *We have* $p \not| n$*, and* $\mathcal{F}$ *is inverse-Belyi induced from an everywhere tame, lisse sheaf* $\mathcal{G}$ *on* $\mathbb{P}^1 - \{0, 1, \infty\}$ *of rank one. More precisely,* $\mathcal{F}$ *is* $f_\star\mathcal{G}$ *for a finite etale map*

$$f : \mathbb{P}^1 - \{0, 1, \infty\} \to \mathbb{G}_m$$

*given by the reciprocal of a Belyi polynomial*

$$f(X) = 1/(\lambda X^a(1 - X)^b)$$

*with* $\lambda \in \overline{k}^\times$ *and with* $a$ *and* $b$ *strictly positive integers with* $a + b = n$*, such that precisely one of* $a$ *or* $b$ *is divisible by* $p$*.*

*Proof.* As $\mathcal{F}$ is induced, it is of the form $f_\star\mathcal{G}$ for $f : U \to \mathbb{G}_m$ a finite etale $\overline{k}$-morphism of degree $d \geq 2$ with connected source $U$, and $\mathcal{G}$ a lisse sheaf on $U$. We have

$$\chi(U, \mathcal{G}) = \chi(\mathbb{G}_m, f_\star\mathcal{G}) = \chi(\mathbb{G}_m, \mathcal{F}) = -1,$$

the last equality by the Euler-Poincare formula on $\mathbb{G}_m$. Denote by $X$ the complete nonsingular model of $U$. Then $f : U \to \mathbb{G}_m$ extends to a finite flat map $\overline{f} : X \to \mathbb{P}^1$, which is $f$ over $\mathbb{G}_m$. Denote by $X_0$ and $X_\infty$ the fibres of $\overline{f}$ over 0 and $\infty$ respectively, and define $d_0 := \#X_0(\overline{k})$, $d_\infty := \#X_\infty(\overline{k})$. The Euler-Poincare formula on $U$ gives

$$\chi(U, \mathcal{G}) = \chi(U, \overline{\mathbb{Q}}_\ell)\mathrm{rank}(\mathcal{G}) - \sum_{\mathrm{x_0 \in X_0(\overline{k})}} \mathrm{Swan}_{\mathrm{x_0}}(\mathcal{G}) - \sum_{\mathrm{x_\infty \in X_\infty(\overline{k})}} \mathrm{Swan}_{\mathrm{x_\infty}}(\mathcal{G}).$$

Thus we have

$$-1 = \chi(U, \overline{\mathbb{Q}}_\ell)\mathrm{rank}(\mathcal{G}) - \sum_{\mathrm{x_0 \in X_0(\overline{k})}} \mathrm{Swan}_{\mathrm{x_0}}(\mathcal{G}) - \sum_{\mathrm{x_\infty \in X_\infty(\overline{k})}} \mathrm{Swan}_{\mathrm{x_\infty}}(\mathcal{G}).$$

Both $d_0$ and $d_\infty$ are strictly positive integers, so we have the inequality

$$\chi(U, \overline{\mathbb{Q}}_\ell) = \chi(X, \overline{\mathbb{Q}}_\ell) - d_0 - d_\infty \leq \chi(X, \overline{\mathbb{Q}}_\ell) - 2 \leq 0,$$

with equality only in the case when $X = \mathbb{P}^1$ and $d_0 = d_\infty = 1$.

Suppose first that $\chi(U, \overline{\mathbb{Q}}_\ell) = 0$, i.e., suppose that $X = \mathbb{P}^1$ and $d_0 = d_\infty = 1$. Then we may take coordinates on $X = \mathbb{P}^1$ so that 0 is the unique point over 0, and $\infty$ is the unique point over $\infty$. Then $\overline{f}$ maps $X - \{0\} = \mathbb{A}^1$ to $\mathbb{A}^1$, so $f$ is a polynomial of degree $d$. As this polynomial $f$ has 0 as its unique zero, we have $f = ax^d$ for some $a \in \overline{k}^\times$. Making a multiplicative translation on the source, i.e. rescaling $x$, we have $f = x^d$. As $f$ is finite etale over $\mathbb{G}_m$, $d$ is is invertible in $\overline{k}$. This is precisely the Kummer-induced case. Because $[d]_\star\mathcal{G} = \mathcal{F}$ is geometrically irreducible, $\mathcal{G}$ must itself be geometrically irreducible. Since $Swan_0$ and $Swan_\infty$ are preserved by $[d]_\star$, we see that $\mathcal{G}$ is tame at 0, and has $Swan_\infty(\mathcal{G}) = 1$. This is case (1).

Suppose now that $\chi(U, \overline{\mathbb{Q}}_\ell) \leq -1$. Then $\chi(U, \overline{\mathbb{Q}}_\ell)\mathrm{rank}(\mathcal{G})$ is a strictly negative integer. But Swan conductors are non-negative integers, so from the formula

$$-1 = \chi(U, \overline{\mathbb{Q}}_\ell)\mathrm{rank}(\mathcal{G}) - \sum_{\mathrm{x_0 \in X_0(\overline{k})}} \mathrm{Swan}_{\mathrm{x_0}}(\mathcal{G}) - \sum_{\mathrm{x_\infty \in X_\infty(\overline{k})}} \mathrm{Swan}_{\mathrm{x_\infty}}(\mathcal{G})$$

we infer that $\chi(U, \overline{\mathbb{Q}}_\ell) = -1$, that $\mathrm{rank}(\mathcal{G}) = 1$, and that $\mathcal{G}$ is tamely ramified at each point of $X - U$. Once we know that $\mathrm{rank}(\mathcal{G}) = 1$, we see that $f$ is finite etale of degree $n$. Since $\chi(U, \overline{\mathbb{Q}}_\ell) = -1$, either $X$ is an elliptic curve $E$ and $U$ is $E - \{\text{a single point}\}$, or $X = \mathbb{P}^1$ and $U$ is $\mathbb{P}^1 - \{3 \text{ points}\}$. The elliptic curve case is impossible, because there are $d_0 + d_\infty \geq 2$ missing points. Thus $X = \mathbb{P}^1$, and $U$ is $\mathbb{P}^1 - \{3 \text{ points}\}$. [That such a $U$ be a finite etale covering of $\mathbb{G}_m$ is possible because we are in positive characteristic $p$, otherwise it would violate Riemann-Hurwitz.]

The three missing points are all the points in the disjoint union of the two nonempty sets $X_0(\overline{k}) \sqcup X_\infty(\overline{k})$. Suppose first that precisely one of the missing points lies in $X_\infty(\overline{k})$. Then we may choose coordinates on $\mathbb{P}^1$ so that $X_\infty(\overline{k}) = \{\infty\}$ and $X_0(\overline{k}) = \{0, 1\}$. Then $f$ is a polynomial, whose only zeros are $\{0, 1\}$. So we have

$$f(X) = \lambda X^a(1 - X)^b$$

with $\lambda \in \overline{k}^\times$ and with $a$ and $b$ strictly positive integers with $a + b = n$. Since $\mathcal{G}$ is tame at $\infty$, $n$ cannot be prime to $p$, otherwise $f_\star\mathcal{G}$ would be tame at $\infty$. Thus $p|n$. Since $f$ is finite etale, it cannot be a $p$'th power, hence both $a$ and $b$ are prime to $p$. This is case (2).

Suppose finally that precisely two of the missing points lie in $X_\infty(\overline{k})$. Then may choose coordinates on $\mathbb{P}^1$ so that $X_\infty(\overline{k}) = \{0, 1\}$ and $X_0(\overline{k}) = \{\infty\}$. In this case, $1/f$ is a polynomial whose only zeros are $\{0, 1\}$. So we have

$$1/f(X) = \lambda X^a(1 - X)^b$$

with $\lambda \in \overline{k}^\times$ and with $a$ and $b$ strictly positive integers with $a + b = n$. Since $f_\star\mathcal{G}$ is tame at 0, and $\infty$ is the only point lying over 0, $n$ must be prime to p. At least one of $a$ or $b$ must be divisible by $p$, otherwise $f_\star\mathcal{G}$ would be tame at $\infty$. Since $a+b=n$ is prime to $p$, precisely one of $a$ or $b$ is divisible by $p$. This is case (3). □

We now return to the proof of the theorem. Thus $\mathcal{H}$ is a hypergeometric of type $(7,m)$, $7>m$, whose $G_{geom}$ lies in $G_2$. In view of Theorem 3.1, $\mathcal{H}$ has type $(7,0)$ if we are in characteristic 2, and it has type $(7,1)$ if we are in odd characteristic.

We first show that we cannot be in case (3) of Proposition 7.1, i.e. that $\mathcal{H}$ cannot be inverse-Belyi induced from an everywhere tame, rank one lisse sheaf $\mathcal{G}$ on such $\mathbb{P}^1-\{0,1,\infty\}$. Indeed no hypergeometric sheaf of type $(n,m)$ with $n>m$ and $m\le 1$ can be so induced. To see this, we argue as follows. The sheaf $\mathcal{G}$ is (geometrically) of the form

$$\mathcal{G}\cong\mathcal{L}_{\chi_0(x)}\otimes\mathcal{L}_{\chi_1(1-x)},$$

for some multiplicative characters $\chi_0$ and $\chi_1$, and

$$f(x):=1/(\lambda x^a(1-x)^b).$$

But from the projection formula applied locally we see that the $I_\infty$-representation of $f_\star\mathcal{G}$ at $\infty$ contains all the $a$'th roots of $1/\chi_0$ and all the $b$'th roots of $1/\chi_1$, and so contains a tame part of dimension at least two, which is impossible because $m\le 1$.

We next show that we cannot be in case (2) of Proposition 7.1. Suppose not. Then we are in characteristic $p=7$, $\mathcal{H}$ is of the form

$$\mathcal{H}=\mathcal{H}(\psi;1,\alpha,\beta,\alpha\beta,\overline{\alpha},\overline{\beta},\overline{\alpha\beta};\chi_{quad}),$$

and there exist strictly positive integers $a$ and $b$, $a+b=7$, $\lambda\in\overline{k}^\times$, and an everywhere tame, lisse rank 1 sheaf $\mathcal{G}$ on such $\mathbb{P}^1-\{0,1,\infty\}$ such that putting

$$f(x):=\lambda x^a(1-x)^b$$

we have a geometric isomorphism

$$\mathcal{H}\cong f_\star\mathcal{G}.$$

The sheaf $\mathcal{G}$ is (geometrically) of the form

$$\mathcal{G}\cong\mathcal{L}_{\chi_0(x)}\otimes\mathcal{L}_{\chi_1(1-x)},$$

for some multiplicative characters $\chi_0$ and $\chi_1$. Put

$$\chi_\infty:=\chi_0\chi_1.$$

We first claim that

$$\chi_\infty=\chi_{quad}.$$

Indeed, as $I_\infty$-representation, $\mathcal{G}$ agrees with $\mathcal{L}_{\chi_\infty(x)}$. Denote by $\Lambda$ the unique (remember $p=7$) multiplicative character whose 7'th power is $\chi_\infty$ . Then $f^\star(\mathcal{L}_{\Lambda(x)})$ gives the $I_\infty$-representation of $\mathcal{G}$. By the projection formula, the $I_\infty$-representation of $\mathcal{H}$ is

$$f_\star f^\star(\mathcal{L}_{\Lambda(x)}) \cong \mathcal{L}_{\Lambda(x)} \otimes f_\star(\overline{\mathbb{Q}}_\ell).$$

But $f_\star(\overline{\mathbb{Q}}_\ell)$ contains $\overline{\mathbb{Q}}_\ell$ as a direct factor, and hence $\mathcal{L}_{\Lambda(x)}$ occurs as a direct factor of the $I_\infty$-representation of $\mathcal{H}$. As $\mathcal{H}$ is

$$\mathcal{H} = \mathcal{H}(\psi; 1, \alpha, \beta, \alpha\beta, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta}; \chi_{quad}),$$

we indeed have

$$\chi_\infty = \chi_{quad}.$$

We next claim that either $\{\chi_0 = 1 \text{ and } \chi_1 = \chi_{\rm quad}\}$ or $\{\chi_0 = \chi_{quad} \text{ and } \chi_1 = 1\}$. The point is that the local monodromy at 0 of $\mathcal{H} \cong f_\star(\mathcal{L}_{\chi_0(x)} \otimes \mathcal{L}_{\chi_1(1-x)})$ consists of all the $a$'th roots of $\chi_0$, together will all the $b$'th roots of $\chi_1$. But the local monodromy of $\mathcal{H}$ at 0 contains the trivial character 1, which is therefore either an $a$'th root of $\chi_0$ or a $b$'th root of $\chi_1$. So either $\chi_0$ or $\chi_1$ is trivial. As their product is $\chi_\infty = \chi_{quad}$, the claim is established.

Therefore the local monodromy of $\mathcal{H}$ at 0 consists either of

$$\{\text{all a}'\text{th roots of } 1\} \cup \{\text{all b}'\text{th roots of } \chi_{\rm quad}\}$$

or of

$$\{\text{all a}'\text{th roots of } \chi_{\rm quad}\} \cup \{\text{all b}'\text{th roots of } 1\}.$$

We will arrive at a contradiction. Interchanging $a$ and $b$, it suffices to deal with the first case, when the local monodromy of $\mathcal{H}$ at 0 consists of

$$\{\text{all a}'\text{th roots of } 1\} \cup \{\text{all b}'\text{th roots of } \chi_{\rm quad}\}.$$

None of these characters can be $\chi_{quad}$ (which occurs at $\infty$). Therefore $a$ must odd, and $b$ must be even. But one checks easily that none of the three resulting lists of eigenvalues

$$(1, \text{all sixth roots of } -1); \text{ the case a} = 1, \text{b} = 6,$$

$$(\text{all cube roots of } 1, \text{all fourth roots of } -1); \text{ the case a} = 3, \text{b} = 4,$$

$$(\text{all fifth roots of } 1, \text{both square roots of } -1); \text{ the case a} = 5, \text{b} = 2,$$

are the list of eigenvalues of an element of $G_2$. Thus the Belyi-induced case does not occur for hypergeometrics whose $G_{geom}$ lies in $G_2$.

Suppose now that we are in case (1) of Proposition 7.1. Thus $\mathcal{H}$ is a hypergeometric of type $(7, m)$, $7 > m$, whose $G_{geom}$ lies in $G_2$, and which is Kummer induced. So we are not in characteristic 7, and we have a geometric isomorphism $\mathcal{H} \cong [7]_\star \mathcal{G}$, with $\mathcal{G}$ lisse of rank 1 on $\mathbb{G}_m$, tame at 0 and with $Swan_\infty(\mathcal{G}) = 1$. Any such $\mathcal{G}$ is geometrically of the form $\mathcal{L}_\chi \otimes \mathcal{L}_{\psi_7}$ for some multiplicative character $\chi$ and some additive character $\psi$, where we put $\psi_7(x) := \psi(7x)$. Then $\mathcal{H}$ is geometrically isomorphic to

$$\mathcal{H}(\psi; \text{all seventh roots of } \chi; \emptyset),$$

of type $(7,0)$. Therefore $p = 2$. Since the trivial character occurs at 0 in $\mathcal{H}$, $\chi$ must be trivial. So we are dealing with

$$\mathcal{H} \cong [7]_\star \mathcal{L}_{\psi_7} \cong \mathcal{H}(\psi; \text{all characters of order dividing } 7; \emptyset).$$

In fact, for this $\mathcal{H}$, we have an arithmetic isomorphism

$$\mathcal{H}(3) \cong [7]_\star \mathcal{L}_{\psi_7},$$

cf. [Ka-GKM, 5.6.2]. Now

$$[7]^\star \mathcal{H} \cong \bigoplus_{\zeta \in \mathbb{F}_8^\times} \mathcal{L}_{\psi(7\zeta x)}.$$

The characters of $\pi_1$ given by the various $\mathcal{L}_{\psi(7\zeta x)}$ are all the nonzero elements in an $\mathbb{F}_8$ of $(\pm 1)$-valued characters of $\pi_1$, and the covering group $\mu_7(\mathbb{F}_8) = \mathbb{F}_8^\times$ acts on this $\mathbb{F}_8$ by multiplicative translation. So in characteristic $p = 2$, $G_{geom}$ and $G_{arith}$ for $[7]_\star \mathcal{L}_{\psi_7}$ are both the $ax + b$ group over $\mathbb{F}_8$, as asserted.

Having now dealt with the induced cases, we turn to the remaining case, that in which $G_{geom}$ is a finite irreducible primitive subgroup of $G_2$. Fortunately, such groups have been classified, by Cohen and Wales, cf. [Co-Wa]. There are five possible groups. Two of these groups have more than one faithful, irreducible 7-dimensional representations which lands it in $G_2$, but in both cases all such representations happen to be Galois-conjugate. Here is a list giving the group (in ATLAS [CCNPW-Atlas] notation), its order, the number of Galois-conjugate representations into $G_2$, and the field of character values of any such representation.

$$L_2(13),\ 2^2 \cdot 3 \cdot 7 \cdot 13 = 1092,\ 2,\ \mathbb{Q}(\sqrt{13}),$$

$$L_2(8),\ 2^3 \cdot 3^2 \cdot 7 = 504,\ 3,\ \mathbb{Q}(\zeta_9)^+ := \text{the real subfield of } \mathbb{Q}(\zeta_9),$$

$$L_2(7).2 = L_3(2).2,\ 2^4 \cdot 3 \cdot 7 = 336,\ 1,\ \mathbb{Q},$$

$$U_3(3),\ 2^5 \cdot 3^3 \cdot 7 = 6048,\ 1,\ \mathbb{Q},$$

$$U_3(3).2 = G_2(2),\ 2^6 \cdot 3^3 \cdot 7 = 12096,\ 1,\ \mathbb{Q}.$$

The relevant observation is that in characteristic $p$, $G_{geom}$ for any hypergeometric of any type $(n,m)$ with $n > m$ contains elements of order $p$, because $Swan_\infty = 1 > 0$, and hence $P_\infty$ has image a nontrivial $p$-group in $G_{geom}$. Therefore the case when $G_{geom}$ is a finite irreducible primitive subgroup of $G_2$ can arise only in characteristic $p$ a prime which divides the order of one of the five listed groups, i.e., only possibly in characteristics $2, 3, 7$, or 13. This concludes the proof of Theorem 6.1.

## 8. Interlude: an arithmetic determinant lemma

**Lemma 8.1.** *Let $k$ be a finite field, and $\mathcal{H}$ a hypergeometric sheaf on $\mathbb{G}_m/k$ of type $(7, m)$ with $7 > m$, formed using the additive character $\psi$ of $k$, whose $G_{geom}$ lies in $G_2$.*

(1) *If $p := char(k)$ is 2, the Tate-twisted sheaf $\mathcal{F} := \mathcal{H}(3)$, whose trace function is related to that of $\mathcal{H}$ by*

$$Trace(Frob_{a,E}|\mathcal{H}(3)) = (\#E)^{-3}Trace(Frob_{a,E}|\mathcal{H}),$$

*has $G_{arith}$ in $G_2$. If $G_{geom}$ is finite (resp. primitive), then so is $G_{arith}$.*

(2) *If $p$ is odd, denote by $A$ the constant in $\mathbb{Q}(\zeta_p)$ given by the negative of the quadratic Gauss sum:*

$$A := -g(\psi, \chi_{quad}) := -\sum_{x \in k^\times} \psi(x)\chi_{quad}(x).$$

*Then the $A^{-7}$-twisted sheaf $\mathcal{F} := \mathcal{H} \otimes (A^{-7})^{deg}$, whose trace function is related to that of $\mathcal{H}$ by*

$$Trace(Frob_{a,E}|(\mathcal{H} \otimes (A^{-7})^{deg}) = (A^{-7})^{deg(E/k)}Trace(Frob_{a,E}|\mathcal{H}),$$

*has $G_{arith}$ in $G_2$. If $G_{geom}$ is finite (resp. primitive), then so is $G_{arith}$.*

*Proof.* In both cases, the twisted sheaf in question is pure of weight 0, and it results from [Ka-ESDE, 8.12.2] that its determinant is arithmetically trivial. From the explicit formula for the traces of $\mathcal{H}$ in terms of hypergeometric sums, one sees easily that the twisted sheaf in question has real-valued traces, so is self-dual. As the rank is odd, the autoduality is orthogonal. Thus we have an inclusion $G_{arith} \subset SO(7)$, and so $G_{arith}$ lies in the normalizer of $G_{geom}$ in $SO(7)$. The proof is then completed by the following lemma. □

**Lemma 8.2.** *Over $\overline{\mathbb{Q}}_\ell$, let $G$ be an irreducible subgroup of $G_2$. Then the normalizer of $G$ in $SO(7)$ lies in $G_2$. If in addition $G$ is finite (resp. primitive), then so is the normalizer of $G$ in $SO(7)$.*

*Proof.* Denote by $V$ the 7-dimensional orthogonally self-dual space of our $SO(7)$. Because $G$ is an irreducible subgroup of $G_2$, we know ([Asch, Theorem 5, parts (2) and (5) on page 196], [Co-Wa, page 449]) that when we decompose $\Lambda^3(V)$, the space $(\Lambda^3(V))^G$ is one-dimensional. But $\Lambda^3(V)$ is an orthogonally self-dual space, and $G$ acts orthogonally on it, so its decomposition into $G$-isotypical components is an orthogonal direct sum decomposition. In particular, the inner product is nondegenerate on the one-dimensional space $(\Lambda^3(V))^G$. So any element of $SO(7) = SO(V)$ that, acting on $\Lambda^3(V)$, maps $(\Lambda^3(V))^G$ to itself, acts on $(\Lambda^3(V))^G$ orthogonally, hence by a scalar which is $\pm 1$. Now any $g \in SO(V)$ that normalizes $G$ maps $(\Lambda^3(V))^G$ to itself. We must show that any such $g$ acts by the scalar $+1$ rather than $-1$ on this space, for then it lies in $G_2$, which is the fixer of $(\Lambda^3(V))^G$ in $O(V)$ ([Asch, Theorem 5, parts (2) and (5) on page 196], [Co-Wa, page 449]). But if $g$ acts on $(\Lambda^3(V))^G$ by the scalar $-1$, then the element $-g$ fixes $(\Lambda^3(V))^G$ and lies in $O(V)$, hence lies in $G_2$. But this is impossible, since $det(-g) = -1$, and every element of $G_2$ has determinant 1. Therefore $g$ acts on $(\Lambda^3(V))^G$ by the scalar $+1$, as required.

If $G$ is primitive, the larger group $N_{SO(7)}(G)$ which is its normalizer in $SO(7)$ is all the more primitive. If $G$ is finite, and $\gamma \in SO(7)$ normalizes $G$, then putting $C := \#Aut(G)$, $\gamma^C$ centralizes $G$, so by Schur's Lemma is a scalar in $SO(7)$, so $\gamma^C = 1$. So $N_{SO(7)}(G)$ is an algebraic group over $\overline{\mathbb{Q}}_\ell$ whose Lie algebra $Lie(N_{SO(7)}(G))$ is killed by $C$, hence vanishes, and so $N_{SO(7)}(G)$ is a finite group. ☐

## 9. Exact determination of $G_{geom}$; Statement of the result

**Theorem 9.1.** *Let $k$ be a finite field, and $\mathcal{H}$ a hypergeometric sheaf on $\mathbb{G}_m/k$ of type $(7, m)$ with $7 > m$, formed using the additive character $\psi$ of $k$, whose $G_{geom}$ lies in $G_2$. Form the sheaf $\mathcal{F}$ as in Lemma 8.1 of the last section. Consider the groups $G_{geom}$ and $G_{arith}$ for $\mathcal{F}$. We have*

$$G_{geom} = G_{arith} = G_2,$$

*except in the following six cases.*

(1) $p = 13$, *and*

$$\mathcal{H} = \mathcal{H}(\psi; \text{all char's of order dividing } 7; \chi_{\text{quad}}).$$

*In this case, $\mathcal{F}$ has $G_{geom} = G_{arith} = L_2(13)$.*

(2) $p = 3$, *and*

$$\mathcal{H} = \mathcal{H}(\psi; \text{all char's of order dividing } 7; \chi_{\text{quad}}).$$

*In this case, $\mathcal{F}$ has $G_{geom} = G_{arith} = U_3(3)$.*

(3) $p = 2$, *and*

$$\mathcal{H} = \mathcal{H}(\psi; \text{all char's of order dividing } 7; \emptyset).$$

*In this case, $\mathcal{F}$ has $G_{geom} = G_{arith} =$ the $ax + b$ group over $\mathbb{F}_8$.*

(4) $p = 7$, *and*

$$\mathcal{H} = \mathcal{H}(\psi; \text{all char's of order dividing } 8 \text{ except } \chi_{\text{quad}}; \chi_{\text{quad}}).$$

*In this case, $\mathcal{F}$ has $G_{geom} = G_{arith} = L_2(7).2$.*

(5) $p = 2$, *and*

$$\mathcal{H} = \mathcal{H}(\psi; \text{all char's of order dividing } 9 \text{ except two inverse char's of order } 9; \emptyset).$$

*In this case,* $\mathcal{F}$ *has* $G_{geom} = G_{arith} = L_2(8)$.

(6) $p = 7$, *and*

$$\mathcal{H} = \mathcal{H}(\psi; \text{all characters of orders } 12, 3, \text{and } 1; \chi_{\text{quad}}).$$

*In this case,* $\mathcal{F}$ *has* $G_{geom} = G_{arith} = U_3(3).2 = G_2(2)$.

## 10. Exact determination of $G_{geom}$; first steps in the proof of Theorem 9.1

Consider a hypergeometric $\mathcal{H}$ of type $(7, m)$, $m \le 1$, whose $G_{geom}$ lies in $G_2$. We have seen that with the exception of the single case $p = 2$ and $\mathcal{H} = \mathcal{H}(\psi; \text{all characters of order dividing } 7; \emptyset)$, $G_{geom}$ is either $G_2$ or it is a finite irreducible primitive subgroup of $G_2$, and that the latter case can arise only in characteristic $2, 3, 7$ or $13$. The only remaining question is to determine, in each of these characteristics, exactly when the finite primitive irreducible case occurs. We have already remarked that $G_{geom}$ contains elements of order $p$. So when $p = 13$, the only possibility is $L_2(13)$. But for $p = 2$, $p = 3$, and $p = 7$, all of the five groups $L_2(13)$, $L_2(8)$, $L_2(7).2 = L_3(2).2$, $U_3(3)$, and $U_3(3).2 = G_2(2)$ have order divisible by $p$.

The first step is to make a (short, finite) list of all $\mathcal{H}$ in each characteristic that could possibly have $G_{geom}$ one of these five groups. Begin with an $\mathcal{H}$ of the form

$$\mathcal{H}(\psi; 1, \alpha, \beta, \alpha\beta, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta}; \chi_2),\ p \text{ odd},$$

$$\mathcal{H}(\psi; 1, \alpha, \beta, \alpha\beta, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta}; \emptyset),\ p = 2.$$

for some $\alpha$ and $\beta$. If its $G_{geom}$ is one of the five listed groups, then the action of local monodromy at 0, i.e., the action of a generator $\gamma_0$ of $I_0^{tame}$, is of finite order $N$, and $N$ is both prime to $p$ and is the order of an element in one of the five groups. Hence the seven characters $(1, \alpha, \beta, \alpha\beta, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta})$ are all distinct, none of them is $\chi_{quad}$ when $p$ is odd, and the *l.c.m.* of their orders, namely $N$, is the order of an element in at least one of the five groups. So we have an a priori inequality

$$N \ge 7,$$

simply because we cannot have seven distinct characters all of whose orders divide any smaller integer. To exploit the fact that $N$ must be the order of an element in one of the five groups, and prime to $p$, here is a table giving for each group the list of the orders of its elements.

$$L_2(13),\ \{1, 2, 3, 6, 7, 13\}$$

$$L_2(8),\ \{1,2,3,7,9\}$$

$$L_2(7).2 = L_3(2).2,\ \{1,2,3,4,6,7,8\}$$

$$U_3(3),\ \{1,2,3,4,6,7,8,12\}$$

$$U_3(3).2 = G_2(2),\ \{1,2,3,4,6,7,8,12\}.$$

Thus we have the following list of possible $N$ in each characteristic $p$:

$$p = 13,\ N = 7,\ \text{only } \mathrm{L}_2(13) \text{ is possible},$$

$$p = 7,\ N = 13, 12, 9, 8,$$

$$p = 3,\ N = 13, 8, 7,$$

$$p = 2,\ N = 13, 9, 7.$$

We then get the desired short, finite list of all possible candidates by listing, in each allowed characteristic $p = 13, 7, 3, 2$ and for each allowed $N$ in that characteristic, all unordered lists of 6 distinct non-trivial characters $(\chi_1, \chi_2, ..., \chi_6)$, none of which is $\chi_{quad}$ if $p$ is odd, all of which have order dividing $N$ and such that $N$ is the *l.c.m.* of their orders, such that the unordered 7-tuple $(1, \chi_1, \chi_2, ..., \chi_6)$ is of the form $(1, \alpha, \beta, \alpha\beta, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta})$ for some pair $(\alpha, \beta)$ of characters.

Before continuing with the general analysis, let us first treat the case of characteristic $p = 13$, where, as we have noted above, the only possible finite group is $L_2(13)$, and the only possible $N$ is 7. Since $(1, \alpha, \beta, \alpha\beta, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta})$ are all distinct, and all have order dividing 7, the only possible candidate is

$$\mathcal{H} := \mathcal{H}(\psi; \text{all characters of order dividing } 7; \chi_{\mathrm{quad}}).$$

Of course, this $\mathcal{H}$ makes sense in any characteristic $p$ other than 2 or 7. One knows [Ka-ESDE, 9.2.2] that in any such characteristic $p$, its Kummer pullback $[7]^\star\mathcal{H}$ is geometrically isomorphic to a multiplicative translate of (the restriction to $\mathbb{G}_m$ of)

$$\mathcal{F}_7 := NFT(\mathcal{L}_{\chi_{quad}} \otimes \mathcal{L}_{\psi(x^7)}),$$

the lisse sheaf on $\mathbb{A}^1/k$ whose trace function is given as follows: for any finite extension $E/k$, and for any $t \in \mathbb{A}^1(E)$,

$$Trace(Frob_{t,E}|\mathcal{F}_7) = -\sum_{x \in \mathbb{A}^1(E)} \chi_{quad,E}(x)\psi_E(x^7 + tx).$$

It is proven in [Ka-NG2, 4.13] that $\mathcal{F}_7$ has $G_{geom} = L_2(13)$ in characteristic $p = 13$, and that it has $G_{geom} = U_3(3)$ in characteristic $p = 3$.

**Lemma 10.1.** *Let $k$ be a finite field of characteristic $p = 13$ which contains the 7'th roots of unity (i.e., $k$ is an even degree extension of $\mathbb{F}_{13}$), and $\psi$ a nontrivial $\overline{\mathbb{Q}}^{\times}$-valued additive character of $k$. For any prime $\ell \neq 13$, and for any embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_\ell$, consider the lisse $\overline{\mathbb{Q}}_\ell$-sheaf*

$$\mathcal{H} := \mathcal{H}(\psi; \text{all characters of order dividing } 7; \chi_{\text{quad}})$$

*on $\mathbb{G}_m/k$, and the twisted sheaf*

$$\mathcal{F} := \mathcal{H} \otimes (A^{-7})^{deg},$$

*for $A$ the constant*

$$A := -g(\psi, \chi_{quad}).$$

*Then $\mathcal{F}$ has $G_{geom} = G_{arith} = L_2(13)$.*

*Proof.* Since $[7]^\star\mathcal{H}$ has $G_{geom} = L_2(13)$, and $[7]$ is a Galois covering of $\mathbb{G}_m \otimes \overline{k}$ by itself, the $G_{geom}$ for $\mathcal{H}$ (lies in $G_2$ and) contains $L_2(13)$ as a normal subgroup. But $L_2(13)$ is its own normalizer in $G_2$; indeed, its normalizer is a finite primitive irreducible subgroup of $G_2$ which contains $L_2(13)$, so has order divisible by 13, so by classification must be $L_2(13)$. Thus $\mathcal{H}$ has $G_{geom} = L_2(13)$. So also the twist $\mathcal{F}$ has $G_{geom} = L_2(13)$. Its $G_{arith}$, which lies in $SO(7)$ and normalizes $G_{arith}$, is then a finite primitive subgroup of $G_2$ which contains $L_2(13)$, so again by classification must itself be $L_2(13)$. □

We also have an analogous result in characteristic 3.

**Lemma 10.2.** *Let $k$ be a finite field of characteristic $p = 3$ which contains the 7'th roots of unity (i.e., $k$ is extension of $\mathbb{F}_{3^6}$), and $\psi$ a nontrivial $\overline{\mathbb{Q}}^{\times}$-valued additive character of $k$. For any prime $\ell \neq 3$, and for any embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_\ell$, consider the lisse $\overline{\mathbb{Q}}_\ell$-sheaf*

$$\mathcal{H} := \mathcal{H}(\psi; \text{all characters of order dividing } 7; \chi_{\text{quad}})$$

*on $\mathbb{G}_m/k$, and the twisted sheaf*

$$\mathcal{F} := \mathcal{H} \otimes (A^{-7})^{deg},$$

*for $A$ the constant*

$$A := -g(\psi, \chi_{quad}).$$

*Then $\mathcal{F}$ has $G_{geom} = G_{arith} = U_3(3)$.*

*Proof.* Since $[7]^\star\mathcal{H}$ has $G_{geom} = U_3(3)$, and $[7]$ is a Galois covering of $\mathbb{G}_m \otimes \overline{k}$ by itself, the $G_{geom}$ for $\mathcal{H}$ (lies in $G_2$ and) contains $U_3(3)$ as a normal subgroup of index dividing 7. But the normalizer of $U_3(3)$ in $G_2$ is $U_3(3).2$; indeed, the normalizer is a finite primitive irreducible subgroup of $G_2$ which certainly contains $U_3(3).2$, so by classification must be $U_3(3).2$. Thus we have

$$U_3(3) \subset G_{geom} \subset U_3(3).2.$$

As the index of $U_3(3)$ in $G_{geom}$ divides 7, it follows that $G_{geom} = U_3(3)$. So also the twist $\mathcal{F}$ has $G_{geom} = U_3(3)$. Its $G_{arith}$, which lies in $SO(7)$ and normalizes $G_{arith}$, is then a finite primitive subgroup of $G_2$ which contains $U_3(3)$, hence has order divisible by 6048, and so by classification must itself be either $U_3(3)$ or $U_3(3).2$. To prove that $G_{arith}$ is in fact $U_3(3)$, we resort to the following ad hoc argument. We first observe that $G_{arith}$ is independent of the choice of the additive character $\psi$ of $k$ used to define $\mathcal{H}, A := -g(\psi, \chi_{quad})$, and $\mathcal{F}$. Indeed, temporarily denote them $\mathcal{H}_\psi, A_\psi$, and $\mathcal{F}_\psi$. Now any other nontrivial additive character of $k$ is of the form $\psi_a(x) := \psi(ax)$, for some $a \in k^\times$. One checks easily that on Frobenius elements we have, for any finite extension $E/k$, and for any $t \in \mathbb{G}_m(E)$,

$$Trace(Frob_{t,E}|\mathcal{F}_{\psi_a}) = Trace(Frob_{a^6 t,E}|\mathcal{F}_\psi).$$

Thus by Chebotarev we have an arithmetic isomorphism

$$\mathcal{F}_{\psi_a} \cong [t \mapsto a^6 t]^\star \mathcal{F}_\psi,$$

and hence $\mathcal{F}_{\psi_a}$ and $\mathcal{F}_\psi$ have the same $G_{geom}$ as each other, and the same $G_{arith}$ as each other.

So it suffices to treat the case when the additive character $\psi$ of $k$ "comes from the prime field $\mathbb{F}_3$", i.e., is of the form $\psi = \psi_1 \circ Trace_{k/\mathbb{F}_3}$ for some nontrivial additive character $\psi_1$ of $\mathbb{F}_3$. In that case, the resulting $\mathcal{H}$ has a natural descent to a lisse sheaf, say $\mathcal{H}_1$, on $\mathbb{G}_m/\mathbb{F}_3$. [First take the obvious descent to $\mathbb{F}_{3^6} = \mathbb{F}_3(\zeta_7)$, where all characters of order dividing 7 are defined, using the additive character $\psi_{1,6} := \psi_1 \circ Trace_{\mathbb{F}_{3^6}/\mathbb{F}_3}$. To descend from $\mathbb{F}_{3^6}$ to $\mathbb{F}_3$, we apply [Ka-GKM, 8.8.7] to descend the auxiliary sheaf

$$\mathcal{H}(\psi_{1,6}; \text{all nontrivial characters of order dividing } 7; \emptyset)$$

from $\mathbb{F}_{3^6}$ to $\mathbb{F}_3$, and then form its ! multiplicative convolution, on $\mathbb{G}_m/\mathbb{F}_3$, with $\mathcal{H}(\psi_1; 1; \chi_{quad})$ to obtain $\mathcal{H}_1$.] Since $\mathcal{H}_1$ becomes isomorphic to $\mathcal{H}$ after extension of scalars from $\mathbb{F}_3$ to $k$, the two sheaves have the same $G_{geom}$, which we already know to be $U_3(3)$.

We next claim that there is a unique scalar twist $\mathcal{F}_1 := \mathcal{H}_1 \otimes B^{deg}$ whose $G_{arith}$ lies in $U_3(3).2$. Indeed, every automorphism of $U_3(3)$ is induced by conjugation by an element of $U_3(3).2$. Since $U_3(3).2$, viewed in $G_2 \subset SO(7) \subset GL(7)$, contains no nontrivial scalars, it follows that the normalizer of $U_3(3)$ in $GL(7)$ is the product group $\mathbb{G}_m \times U_3(3).2$. Now $G_{arith}$ for $\mathcal{H}_1$ lies in this normalizer, so we have an inclusion

$$G_{arith} \subset \mathbb{G}_m \times U_3(3).2.$$

Then projection to the first factor defines a homomorphism

$$\rho : G_{arith} \to \mathbb{G}_m,$$

which is trivial on $G_{geom}$. So viewed as a character of $\pi_1^{arith}$, $\rho$ is geometrically trivial, so of the form $B_0^{deg}$. Taking $B := 1/B_0$ gives the desired twist (which is unique because $U_3(3).2$, viewed in $GL(7)$, contains no nontrivial scalars). This uniqueness, applied after extension of scalars from $\mathbb{F}_3$ to $k$, shows that $B^{deg(k/\mathbb{F}_3)} = A_\psi^{-7}$, i.e., that $\mathcal{F}_1$ becomes $\mathcal{F}_\psi$ after this extension of scalars.

Now we consider $G_{arith}$ for $\mathcal{F}_1$ on $\mathbb{G}_m/\mathbb{F}_3$. It is either $U_3(3)$ or $U_3(3).2$, and it is the former if and only if a certain geometrically trivial character of order dividing 2, namely the canonical map of $\pi_1^{arith}$ to $G_{arith}/G_{geom}$, is trivial. This character, viewed as having values in the group $\pm 1$, is of the form $\epsilon^{deg}$, for an $\epsilon$ which is either 1 or $-1$. In either case, this character becomes trivial if we extend scalars to any even degree extension of $\mathbb{F}_3$. As $k$ is such an extension, having degree divisible by 6, we obtain the asserted equality $G_{geom} = G_{arith} = U_3(3)$ for $\mathcal{F}$. ☐

Having dealt entirely with the situation in characteristic $p = 13$, we now return to some general considerations which will be particularly useful in characteristics $7, 3$, and 2. We can further shorten the list of candidate 7-tuples by taking into account $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugation. We first recall the relevant general fact, cf. [Ka-ESDE, 8.14.5], which we will apply to $\mathcal{H}(3)$ if $p = 2$, and to $\mathcal{H} \otimes (A^{-7})^{deg}$ if $p$ is odd.

**Lemma 10.3.** *Suppose we are given a middle extension $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on a smooth, geometrically connected curve $C/k$, $k$ a finite field of characteristic $p \neq \ell$. Suppose that $\mathcal{F}$ is lisse on a dense open set $U \subset C$, and that $\mathcal{F}|U$ is geometrically irreducible. Suppose further that $det(\mathcal{F}|U)$ is arithmetically of finite order, and that, for every embedding of $\overline{\mathbb{Q}}_\ell$ into $\mathbb{C}$, $\mathcal{F}|U$ is pure of weight zero. Then the following conditions on $\mathcal{F}$ are equivalent.*

1. *$G_{geom}$ is finite.*
2. *$G_{arith}$ is finite.*
3. *For every finite extension $E/k$, and for every point $t \in C(E)$, $Trace(Frob_{t,E}|\mathcal{F})$ is an algebraic integer.*

To apply this result, let us first record explicitly the effect of Galois conjugation on hypergeometric sums. The proof, entirely straightforward, is left to the reader.

**Lemma 10.4.** *Given a hypergeometric $\mathcal{H} := \mathcal{H}(\psi; \chi_i\,'s; \rho_j\,'s)$ on $\mathbb{G}_m/k$ of type $(n,m)$, and given $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, let us define the hypergeometric $\mathcal{H}^\sigma$ by*

$$\mathcal{H}^\sigma := \mathcal{H}(\psi; \chi_i^\sigma\,'s; \rho_j^\sigma\,'s),$$

*formed with the same $\psi$ but with the $\overline{\mathbb{Q}}^\times$-valued characters $\chi_i^\sigma := \sigma \circ \chi_i$ and $\rho_j^\sigma := \sigma \circ \rho_j$. The trace functions of these two sheaves are related as follows. Denote by $a = a_\sigma \in \mathbb{F}_p^\times$ the scalar such that $\psi^\sigma = \psi_a := [x \mapsto \psi(ax)]$. For every finite extension $E/k$, and for every point $t \in \mathbb{G}_m(E)$,*

$$\sigma(Trace(Frob_{t,E}|\mathcal{H})) = Trace(Frob_{a^{n-m}t,E}|\mathcal{H}^\sigma)\cdot\sigma(\prod_j \rho_{j,E}(a)/\prod_i \chi_{i,E}(a)).$$

Applying this lemma to the twisted hypergeometric sheaves we are interested in, we find the following.

**Lemma 10.5.** *Let $k$ be a finite field, and $\mathcal{H}$ a hypergeometric sheaf on $\mathbb{G}_m/k$ of type $(7,m)$ with $7 > m$, formed using the additive character $\psi$ of $k$, whose $G_{geom}$ lies in $G_2$. For any $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, with auxiliary constant $a = a_\sigma \in \mathbb{F}_p^\times$, we have the following results.*

1. *If $p := char(k)$ is 2, consider the Tate-twisted sheaf $\mathcal{H}(3)$. For every finite extension $E/k$, and for every point $t \in \mathbb{G}_m(E)$, we have*
$$\sigma(Trace(Frob_{t,E}|\mathcal{H}(3))) = Trace(Frob_{a^7t,E}|\mathcal{H}^\sigma(3)).$$
2. *If $p$ is odd, consider the $A^{-7}$-twisted sheaf $\mathcal{H} \otimes (A^{-7})^{deg}$. For every finite extension $E/k$, and for every point $t \in \mathbb{G}_m(E)$, we have*

$$\sigma(Trace(Frob_{t,E}|\mathcal{H} \otimes (A^{-7})^{deg})) = Trace(Frob_{a^6t,E}|\mathcal{H}^\sigma \otimes (A^{-7})^{deg})).$$

*Proof.* Whatever the value of $p$, $\prod_i \chi_i$ is trivial. If $p = 2$, there are no $\rho_j$, and the Tate twist is invariant under $\sigma$, so the assertion is clear. If $p$ is odd, then $\prod_i \chi_i$ is trivial, $\prod_j \rho_j = \chi_{quad}$, and the factor $\sigma(\prod_j \rho_{j,E}(a)/\prod_i \chi_{i,E}(a))$, which reduces to $\chi_{quad,E}(a) = \pm 1$, is exactly cancelled by its seventh power, which comes from the effect of $\sigma$ on the quadratic Gauss sum A. So the assertion is clear in this case as well. □

**Corollary 10.6.** *In the situation of the above lemma, suppose in addition that $p$ is $2, 3$, or $7$. For any $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, we have the following results.*

(1) *If $p = 2$, the trace function of $\mathcal{H}^\sigma(3)$ on Frobenius elements is obtained from that of $\mathcal{H}(3)$ by applying $\sigma$.*
(2) *If $p = 3$ or $p = 7$, the trace function of $\mathcal{H}^\sigma \otimes (A^{-7})^{deg}$ on Frobenius elements is obtained from that of $\mathcal{H} \otimes (A^{-7})^{deg}$ by applying $\sigma$.*

*Proof.* Indeed, if $p = 2$, then $a = 1$, so $a^7 = 1$. If $p$ is 3 or 7, then $a^6 = 1$ for every $\in \mathbb{F}_p^\times$. □

This result in turn has the following corollary.

**Corollary 10.7.** *Let $k$ be a finite field whose characteristic $p$ is $2, 3$, or $7$, and $\mathcal{H}$ a hypergeometric sheaf on $\mathbb{G}_m/k$ of type $(7, m)$ with $7 > m$, formed using the additive character $\psi$ of $k$, whose $G_{geom}$ lies in $G_2$. Let $K/\mathbb{Q}$ be a finite extension $K$ of $\mathbb{Q}$ (inside $\overline{\mathbb{Q}}$).*

(1) *If $p = 2$, consider the Tate-twisted sheaf $\mathcal{H}(3)$.*
 (1a) *Suppose the trace function of $\mathcal{H}(3)$ takes values, on all Frobenius elements, in $K$. Then for every $\sigma \in Gal(\overline{\mathbb{Q}}/K)$, the sheaves $\mathcal{H}^\sigma(3)$ and $\mathcal{H}(3)$ are isomorphic on $\mathbb{G}_m/k$.*
 (1b) *Conversely, if for every $\sigma \in Gal(\overline{\mathbb{Q}}/K)$, the sheaves $\mathcal{H}^\sigma(3)$ and $\mathcal{H}(3)$ are isomorphic on $\mathbb{G}_m/k$, then their common trace function takes values, on all Frobenius elements, in $K$.*
(2) *If $p$ is 3 or 7, consider the $A^{-7}$-twisted sheaf $\mathcal{H} \otimes (A^{-7})^{deg}$.*
 (2a) *Suppose the trace function of $\mathcal{H} \otimes (A^{-7})^{deg}$ takes values, on all Frobenius elements, in $K$. Then for every $\sigma \in Gal(\overline{\mathbb{Q}}/K)$, the sheaves $\mathcal{H}^\sigma \otimes (A^{-7})^{deg}$ and $\mathcal{H} \otimes (A^{-7})^{deg}$ are isomorphic on $\mathbb{G}_m/k$.*
 (2b) *Conversely, if for every $\sigma \in Gal(\overline{\mathbb{Q}}/K)$, the two sheaves $\mathcal{H}^\sigma \otimes (A^{-7})^{deg}$ and $\mathcal{H} \otimes (A^{-7})^{deg}$ are isomorphic on $\mathbb{G}_m/k$, then their common trace function takes values, on all Frobenius elements, in $K$.*

*Proof.* To prove (1a) and (2a), fix $\sigma \in Gal(\overline{\mathbb{Q}}/K)$. By the previous corollary, the two sheaves in question have the same trace on every Frobenius. As both are geometrically and hence arithmetically irreducible, they are isomorphic by Chebotarev. Assertions (1b) and (2b) are immediate from the previous corollary, together with Galois theory applied to the trace of each Frobenius element. □

This last result gives very strong restrictions on the possible candidates for finite monodromy in characteristics $2, 3$ and $7$, the only characteristics we have yet to treat.

**Corollary 10.8.** *Let $k$ be a finite field whose characteristic $p$ is $2, 3$, or $7$, and $\mathcal{H}$ a hypergeometric sheaf on $\mathbb{G}_m/k$ of type $(7, m)$ with $7 > m$. formed using the additive character $\psi$ of $k$, whose $G_{geom}$ is a finite primitive irreducible subgroup of $G_2$. Write $\mathcal{H}$ explicitly as*

$$\mathcal{H}(\psi; 1, \chi_1, \chi_2, ..., \chi_6),\ if\ p = 2,$$

$$\mathcal{H}(\psi; 1, \chi_1, \chi_2, ..., \chi_6; \chi_{quad}),\ if\ p = 3 \text{ or } 7.$$

*Then we have the following results.*

(1) *If $G_{geom}$ is one of the groups $L_2(7).2$, $U_3(3)$, or $U_3(3).2$, then the undordered list of the six characters $\chi_i$ is fixed by every $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.*
(2) *If $G_{geom}$ is the group $L_2(13)$, then the undordered list of the six characters $\chi_i$ is fixed by every element $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{13}))$, and by no other element in $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.*
(3) *If $G_{geom}$ is the group $L_2(8)$, then the undordered list of the six characters $\chi_i$ is fixed by every element $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_9)^+)$, and by no other element in $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.*

*Proof.* Put $\mathcal{F} := \mathcal{H}(3)$ if $p = 2$, and put $\mathcal{F} := \mathcal{H} \otimes (A^{-7})^{deg}$ otherwise. Then $\mathcal{F}$ has finite primitive irreducible $G_{arith}$ in $G_2$ which normalizes $G_{geom}$. The normalizers of the possible $G_{geom}$'s in $G_2$ are just themselves, except for $U_3(3)$, whose normalizer is $U_3(3).2$. Now for the groups $L_2(7).2$, $U_3(3)$, or $U_3(3).2$, there is a unique 7-dimensional irreducible representation which puts it in $G_2$, and the trace of that representation has values in $\mathbb{Q}$. For the group $L_2(13)$, there are precisely two 7-dimensional irreducible representations which put it in $G_2$, their trace have values in $\mathbb{Q}(\sqrt{13})$, and their traces are $Gal(\mathbb{Q}(\sqrt{13})/\mathbb{Q})$-conjugate. For the group $L_2(8)$, there are precisely three 7-dimensional irreducible representations which put it in $G_2$, their trace have values in $\mathbb{Q}(\zeta_9)^+$, and their traces are $Gal(\mathbb{Q}(\zeta_9)^+/\mathbb{Q})$- conjugate. With this group-theoretic information at hand, the result is immediate from the previous result, applied to $\mathcal{F}$, since by Chebotarev every conjugacy class in the finite group $G_{arith}$ is the image of a Frobenius class, so all traces of elements of $G_{arith}$ occur as traces of Frobenius elements. □

We now return to the general case, and give another corollary of the previous lemma.

**Corollary 10.9.** *Let $k$ be a finite field, and let $\mathcal{H}$ a hypergeometric sheaf on $\mathbb{G}_m/k$ of type $(7, m)$ with $7 > m$, formed using the additive character $\psi$ of $k$, whose $G_{geom}$ lies in $G_2$. Suppose that $\mathcal{H}$ has finite $G_{geom}$, say the finite group $G$. Then for every $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, $\mathcal{H}^\sigma$*

*has finite $G_{geom}$, which as an abstract group is isomorphic to $G$. In addition, we have the following more precise results.*

(1) *If $p := char(k)$ is 2, consider the Tate-twisted sheaf $\mathcal{H}(3)$. Then for every $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, $\mathcal{H}^\sigma(3)$ has finite $G_{arith}$, whose isomorphism class as an abstract group is independent of $\sigma$.*

(2) *If $p$ is odd, consider the $A^{-7}$-twisted sheaf $\mathcal{H} \otimes (A^{-7})^{deg}$. Then for every $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, $\mathcal{H}^\sigma \otimes (A^{-7})^{deg}$ has finite $G_{arith}$, whose isomorphism class as an abstract group is independent of $\sigma$.*

*Proof.* We first prove the finiteness statements. The twisted sheaf $\mathcal{H}(3)$, if $p = 2$, [respectively $\mathcal{H} \otimes (A^{-7})^{deg}$, if $p$ is odd], which has the same finite $G_{geom}$ as $\mathcal{H}$, is geometrically irreducible, pure of weight zero for every embedding of $\overline{\mathbb{Q}}_\ell$ into $\mathbb{C}$, and its determinant is arithmetically trivial. By Lemma 10.3 above, the twisted sheaf has finite $G_{arith}$ (because it has finite $G_{geom}$). Therefore the twisted sheaf has algebraic integer traces, again by Lemma 10.3. This property is invariant under conjugation by any $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. So by the previous lemma, the sheaf $\mathcal{H}^\sigma(3)$, if $p = 2$ [respectively $\mathcal{H}^\sigma \otimes (A^{-7})^{deg}$, if $p$ is odd] has algebraic integer traces, and hence (again by Lemma 10.3) has finite $G_{arith}$, and so finite $G_{geom}$ as well. We next prove that the isomorphism class of $G_{arith}$ as a finite group is independent of $\sigma$. Fix one $\sigma$, and consider the two sheaves

$$\mathcal{F}_1 := \mathcal{H}(3), \mathcal{F}_2 := \mathcal{H}^\sigma(3), \text{ if p} = 2,$$

$$\mathcal{F}_1 := \mathcal{H} \otimes (A^{-7})^{deg}, \mathcal{F}_2 := \mathcal{H}^\sigma \otimes (A^{-7})^{deg}, \text{ if p is odd.}$$

Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are two lisse sheaves of the same rank, here 7, with finite $G_{arith}$'s, say $G_1$ and $G_2$, whose trace functions, which take algebraic integer values on all Frobenius elements, are $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugate on Frobenius elements.

Denote by $\rho_1$ and $\rho_2$ the representations of $\pi_1(\mathbb{G}_m/k)$ in $GL(7, \overline{\mathbb{Q}}_\ell)$ corresponding to $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively. We have, for $i = 1, 2$, a tautological isomorphism of finite groups,

$$G_i \cong \pi_1(\mathbb{G}_m/k)/Ker(\rho_i).$$

So it suffices to show that $Ker(\rho_1) = Ker(\rho_2)$. But in a finite subgroup of $GL(7, \overline{\mathbb{Q}}_\ell)$, the identity is the unique element whose trace is 7. So to conclude it suffices to show that the traces of the representations $\rho_1$ and $\rho_2$ are related by

$$Trace(\rho_2(\gamma)) = \sigma(Trace(\rho_1(\gamma)))$$

for all elements $\gamma \in \pi_1(\mathbb{G}_m/k)$, and not just for Frobenius elements. But as the direct sum representation $\rho_1 \oplus \rho_2$ also has finite image, say $\Gamma \subset G_1 \times G_2$, it follows by Chebotarev that for any chosen $\gamma \in$

$\pi_1(\mathbb{G}_m/k)$, its image under $\rho_1 \oplus \rho_2$ is conjugate in $\Gamma$, and hence in $G_1 \times G_2$, to the image of a Frobenius element, say $F$. Projecting onto each factor $G_i$ of $G_1 \times G_2$, we see that $\rho_i(\gamma)$ is conjugate in $G_i$ to $\rho_i(F)$. Thus we have

$$Trace(\rho_2(\gamma)) = Trace(\rho_2(F)) = \sigma(Trace(\rho_1(F))) = \sigma(Trace(\rho_1(\gamma))),$$

as required, and so $G_1$ and $G_2$ are isomorphic.

Once we know that

$$Trace(\rho_2(\gamma)) = \sigma(Trace(\rho_1(\gamma)))$$

for all elements $\gamma \in \pi_1(\mathbb{G}_m/k)$, we can show that the groups $G_{geom}$ for $\mathcal{F}_1$ and for $\mathcal{F}_2$ are isomorphic. Indeed, these finite groups are, for $i = 1, 2$ respectively, the quotient groups $\pi_1^{geom}/Ker(\rho_i|\pi_1^{geom})$. Just as above the two representations $\rho_i|\pi_1^{geom}$ for $i = 1, 2$ have the same kernel, namely the elements in $\pi_1^{geom}$ with trace 7. □

## 11. The final list of candidates

In the previous section, we compiled the list of possible $N$ for each $p$. We also completely analyzed two cases: the case $p = 13$, where only $N = 7$ (with its unique list of characters of order dividing 7) and the group $L_2(13)$ were possible, and the case $N = 7$, with its unique list of characters, which led to the group $L_2(13)$ for $p = 13$, to the group $U_3(3)$ for $p = 3$, and to the (imprimitive, but irreducible) $ax + b$ group over $\mathbb{F}_8$ for $p = 2$.

The remaining cases are

$$N = 13, \text{ in characteristic } 2, 3, 7$$

$$N = 12, \text{ in characteristic } 7$$

$$N = 9, \text{ in characteristic } 2, 7$$

$$N = 8, \text{ in characteristic } 3, 7.$$

If $N = 13$, $p$ is any of 2, 3, or 7, and the only possible group is $L_2(13)$. So we are looking for unordered lists $(\chi_1, ..., \chi_6)$ of distinct nontrivial characters of order 13 which are fixed, as unordered lists, by precisely the elements of $Gal(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{13}))$, and which are of the correct form

$$(\alpha, \beta, \alpha\beta, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta}).$$

The Galois action factors through $Gal(\mathbb{Q}(\zeta_{13})/\mathbb{Q}(\sqrt{13}))$, the subgroup of squares in $\mathbb{F}_{13}^\times$. The squares are $(1, 3, 4, 9, 10, 12)$ in $\mathbb{F}_{13}^\times$, and the non-squares are $(2, 5, 6, 7, 8, 11)$. So the unordered lists we have to consider are those of the form

$$(\chi, \chi^3, \chi^4, \chi^9, \chi^{10}, \chi^{12}),$$

for some choice of nontrivial character $\chi$ of order 13. There are exactly two such: for a fixed choice of $\chi$, the other, obtained by choosing $\chi^2$ instead of $\chi$, is

$$(\chi^2, \chi^5, \chi^6, \chi^7, \chi^8, \chi^{11}).$$

These lists are conjugate by $Gal(\mathbb{Q}(\sqrt{13})/\mathbb{Q})$. Both are of the correct form

$$(\alpha, \beta, \alpha\beta, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta}),$$

with $(\alpha, \beta) = (\chi, \chi^3)$ and $(\alpha, \beta) = (\chi^2, \chi^6)$ respectively.

If $N = 12$, then $p$ is 7, and the only possible groups are $U_3(3)$ and $U_3(3).2$. So we are looking for unordered lists $(\chi_1, ..., \chi_6)$ of distinct nontrivial characters of order dividing 12, none of order 2, which are fixed, as unordered lists, by $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, and which are of the correct form

$$(\alpha, \beta, \alpha\beta, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta}).$$

The $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-orbits in the set of all characters of order dividing 12 but not dividing 2 are the sets of those characters having orders 3, 4, 6, and 12. So the only $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-fixed lists are

$$(\text{orders } 3, 4, 6),$$

$$(\text{orders } 3, 12),$$

$$(\text{orders } 4, 12),$$

$$(\text{orders } 6, 12).$$

Of these, one easily checks that only the last is of the correct form

$$(\alpha, \beta, \alpha\beta, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta}).$$

In terms of a chosen character $\chi$ having order 12, this last list is

$$(\chi, \chi^4, \chi^5, \chi^7, \chi^8, \chi^{11}),$$

for which we can take $(\alpha, \beta) = (\chi, \chi^4)$.

If $N = 9$, then $p$ is 2 or 7, and the only possible group is $L_2(8)$. So we are looking for unordered lists $(\chi_1, ..., \chi_6)$ of distinct nontrivial characters of order dividing 9, which are fixed, as unordered lists, by complex conjugation (the effect on $\mathbb{Q}(\zeta_9)$ of any element of $Gal(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_9)^+)$ which acts nontrivially on $\mathbb{Q}(\zeta_9)$), but which are not fixed by $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, and which are of the correct form

$$(\alpha, \beta, \alpha\beta, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta}).$$

The $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-orbits in the set of all characters of order dividing 9 are the sets of those characters of orders 1, 3 and 9. The set of characters of order 9 breaks into 3 orbits under $\{1,\ complex\ conjugation\}$. So the only Galois-allowed lists are the three lists of the form

$$(\text{order } 3, \text{all but 2 complex conjugate characters of order } 9).$$

These three lists are $Gal(\mathbb{Q}(\zeta_9)^+/\mathbb{Q})$-conjugate to each other. If we choose a character $\chi$ having order 9, these three lists are

$$(\chi, \chi^2, \chi^3, \chi^6, \chi^7, \chi^8),$$
$$(\chi^2, \chi^3, \chi^4, \chi^5, \chi^6, \chi^7),$$
$$(\chi, \chi^3, \chi^4, \chi^5, \chi^6, \chi^8).$$

Each is of the correct form, with $(\alpha, \beta)$ successively $(\chi, \chi^2)$, $(\chi^2, \chi^4)$, and $(\chi^4, \chi^8)$. We obtain the second and third lists from the first by choosing successively $\chi^2$ and $\chi^4$ instead of $\chi$ as our character having order 9.

If $N = 8$, then $p$ is 3 or 7, and the only possible groups are $L_2(7).2$, $U_3(3)$, or $U_3(3).2$. So so we are looking for unordered lists $(\chi_1, ..., \chi_6)$ of distinct nontrivial characters of order dividing 8, none of order 2, which are fixed, as unordered lists, by $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, and which are of the correct form

$$(\alpha, \beta, \alpha\beta, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta}).$$

The $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-orbits in the set of all characters of order dividing 8 but not dividing 2 are the sets of those characters having orders 4 and 8. So the only Galois-allowed list is

$$(\text{orders } 4, 8).$$

If we choose a character $\chi$ having order 8, this list is

$$(\chi, \chi^2, \chi^3, \chi^5, \chi^6, \chi^7),$$

which is of the correct form, with $(\alpha, \beta) = (\chi, \chi^2)$.

## 12. Case by case analysis of the candidates: preparations

We have already seen that Galois-conjugate lists $(\chi_1, ..., \chi_6)$ give rise to the same $G_{geom}$ as each other, and the same $G_{arith}$ as each other. For each of the 8 allowed $(N, p)$, namely

$$(N, p) = (13, 7), (13, 3), (13, 2), (12, 7), (9, 7), (9, 2), (8, 7), (8, 3),$$

there is either only one possible list, or all the possible lists are Galois-conjugate. So for each listed $(N, p)$, it suffices to choose just one list, and treat it. Its characters all begin life as characters of $\mathbb{F}_p(\zeta_N)^\times$. Using a nontrivial additive character $\psi$ of $\mathbb{F}_p(\zeta_N)$ which is obtained from a nontrivial additive character of $\mathbb{F}_p$ by composition with the trace, we form the relevant hypergeometric sheaf

$$\mathcal{H} := \mathcal{H}(\psi; 1, \chi_1, ..., \chi_6; \emptyset) \text{ if p} = 2,$$

$$\mathcal{H} := \mathcal{H}(\psi; 1, \chi_1, ..., \chi_6; \chi_{quad}) \text{ if p} = 3, 7,$$

and the corresponding twist

$$\mathcal{F} := \mathcal{H}(3) \text{ if p} = 2,$$

$$\mathcal{F} := \mathcal{H} \otimes (A^{-7})^{deg} \text{ if p} = 3, 7.$$

We know that $G_{geom}$ for $\mathcal{H}$ is finite if and only if $G_{arith}$ for $\mathcal{F}$ is finite, otherwise both groups are $G_2$. In the finite case, we know further that $G_{geom} \lhd G_{arith}$, that each is one of the groups $L_2(7).2$, $U_3(3)$, $U_3(3).2$, $L_2(8)$, or $L_2(13)$, viewed inside $G_2$, and that the order of $G_{geom}$ is divisible by the characteristic $p$.

The main task, then, is to decide in each case whether or not $G_{geom}$ is finite. We will do this by using the p-adic criterion for finite monodromy in its Stickelberger incarnation [Ka-ESDE, 8.16.8], and a beautiful result of Kubert. We will find that, of our eight cases,

$$(N, p) = (13, 7), (13, 3), (13, 2), (12, 7), (9, 7), (9, 2), (8, 7), (8, 3),$$

it is only the three cases

$$(N, p) = (12, 7), (9, 2), (8, 7)$$

which have finite $G_{geom}$.

## 13. Interlude: review of the Stickelberger criterion for finite monodromy of hypergeometrics, and a result of Kubert

Consider a hypergeometric $\mathcal{H} := \mathcal{H}(\psi; \chi_i\ 's; \rho_j\ 's)$ of type $(n, m)$, $n \geq m$, on $\mathbb{G}_m/k$. For $A$ satisfying

$$A^n := q^{n(n-1)/2} \prod_{i,j} (-g(\overline{\psi}, \overline{\rho_j}/\overline{\chi_i}),$$

the twisted sheaf

$$\mathcal{F} := \mathcal{H} \otimes A^{-deg}$$

is pure of weight zero, and its determinant is arithmetically of finite order, cf. [Ka-ESDE, 8.12.2]. As recalled in Lemma 10.3 above, $\mathcal{H}$ has finite $G_{geom}$ if and only if $\mathcal{F}$ has finite $G_{arith}$, if and only if all the traces of $\mathcal{F}$ are algebraic integers.

**Proposition 13.1.** *All traces of $\mathcal{F}$ are algebraic integers if and only if the following conditions are satisfied. For every field embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}_p}$, for every finite extension field $E/k$, and for every multiplicative character $\Lambda$ of $E^\times$, we have*

$$ord(A^{-deg(E/k)} \prod_i g(\psi_E, \chi_{i,E}\Lambda) \prod_j g(\overline{\psi_E}, \overline{\rho}_{j,E}\overline{\Lambda})) \geq 0.$$

*More precisely, for each finite extension field $E/k$, the following conditions are equivalent.*

1. *For every $a \in E^\times$, $Trace(Frob_{a,E}|\mathcal{F})$ is an algebraic integer.*
2. *For every field embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}_p}$, and for every multiplicative character $\Lambda$ of $E^\times$, we have*

$$ord(A^{-deg(E/k)} \prod_i g(\psi_E, \chi_{i,E}\Lambda) \prod_j g(\overline{\psi_E}, \overline{\rho}_{j,E}\overline{\Lambda})) \geq 0.$$

*Proof.* The traces of

$$\mathcal{F} := \mathcal{H} \otimes A^{-deg}$$

are given by

$$Trace(Frob_{a,E}|\mathcal{F}) = A^{-deg(E/k)}(-1)^{n+m-1} Hyp(\psi; \chi_i\,'s; \rho_j\,'s)(a, E).$$

The hypergeometric sums are cyclotomic integers. The quantity $A$ is an algebraic integer which, being a product of Gauss sums, is a unit at all finite places not lying over the characteristic $p$. So the traces of $\mathcal{F}$ are, a priori, integers except possibly at places over $p$. The relation of hypergeometric sums to monomials in Gauss sums gives, for each $\Lambda$,

$$-\sum_{a \in E^\times} \Lambda(a) Trace(Frob_{a,E}|\mathcal{F})$$

$$= (-1)^{n+m} A^{-deg(E/k)} \prod_i g(\psi_E, \chi_{i,E}\Lambda) \prod_j g(\overline{\psi_E}, \overline{\rho}_{j,E}\overline{\Lambda}).$$

Thus if $\mathcal{F}$ has integral traces, the quantities

$$A^{-deg(E/k)} \prod_i g(\psi_E, \chi_{i,E}\Lambda) \prod_j g(\overline{\psi_E}, \overline{\rho}_{j,E}\overline{\Lambda})$$

are algebraic integers, so have non-negative ord's. Conversely, if all the ord conditions are satisfied, then multiplicative Fourier inversion shows that $(\#E - 1) Trace(Frob_{a,E}|\mathcal{F})$ is integral at all places over $p$. But $(\#E - 1)$ is a unit at places over $p$, so $Trace(Frob_{a,E}|\mathcal{F})$ is itself integral at all places over $p$. Since this trace is always integral outside of $p$, it is an algebraic integer. ∎

Fix a field embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}_p}$, and an isomorphism of our fixed $\overline{\mathbb{F}_p}$ with the residue field of $\overline{\mathbb{Q}_p}$. Then for each finite subfield $E$ of $\overline{\mathbb{F}_p}$, we have isomorphisms of character groups

$$Hom(E^\times, \overline{\mathbb{Q}}^\times) \cong Hom(E^\times, \mu_{\#E-1}(\overline{\mathbb{Q}}))$$
$$\cong Hom(E^\times, \mu_{\#E-1}(\overline{\mathbb{Q}_p})) \cong Hom(E^\times, E^\times),$$

the last map being reduction to the residue field. Thus the character group has a canonical generator, the Teichmuller character $Teich_E$, corresponding to the identity map $x \mapsto x$ of $E^\times$. Using this canonical generator, we get a (slightly nonstandard) isomorphism

$$(1/(\#E-1))\mathbb{Z}/\mathbb{Z} \cong Hom(E^\times, \overline{\mathbb{Q}}^\times),$$
$$x \mapsto \Lambda_{x,E} := Teich_E^{-x(\#E-1)}.$$

These isomorphisms are "norm-compatible" in the sense that if $E/k$ is a finite extension, the inclusion of $(1/(\#k-1))\mathbb{Z}/\mathbb{Z}$ into $(1/(\#E-1))\mathbb{Z}/\mathbb{Z}$ is the inclusion of $Hom(k^\times, \overline{\mathbb{Q}}^\times)$ into $Hom(E^\times, \overline{\mathbb{Q}}^\times)$ given by $\chi \mapsto \chi_E$. We define a ("valuation of the corresponding Gauss sum") function on $(1/(\#E-1))\mathbb{Z}/\mathbb{Z}$ with values in the interval $[0, 1)$,

$$x \mapsto V(x),$$

as follows. Denote by $ord_{\#E}$ the ord function on $\overline{\mathbb{Q}_p}$, normalized by $ord_{\#E}(\#E) = 1$. Pick any nontrivial additive character $\psi_E$ of $E$. Then we define

$$V(x) := ord_{\#E}(g(\psi_E, \Lambda_{x,E})).$$

This function does not depend on the auxiliary choice of $\psi_E$, because changing that choice only changes the Gauss sum by multiplication by a root of unity. For $L/E$ a finite extension, and $x \in (1/(\#E-1))\mathbb{Z}/\mathbb{Z}$, the Hasse-Davenport relation [Dav-Ha]

$$(-g(\psi_E, \Lambda_{x,E}))^{deg(L/E)} = -g(\psi_{E_L}, \Lambda_{x,L})$$

shows that these functions are the restrictions to finite subgroups of the source of a single function, still denoted $x \mapsto V(x)$, on the group $(\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$ of elements of prime-to-p order in $\mathbb{Q}/\mathbb{Z}$,

$$V : (\mathbb{Q}/\mathbb{Z})_{prime\ to\ p} \to [0, 1).$$

This function does not depend on the particular isomorphism we choose between our fixed $\overline{\mathbb{F}_p}$ and the residue field of $\overline{\mathbb{Q}_p}$. Indeed, the choices are principal homogeneous under the automorphism group of the residue field. But all such automorphisms are induced by automorphisms of $\overline{\mathbb{Q}_p}$, and all of these are isometries, and so leave the ord function invariant.

What happens if we change the field embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}_p}$? From the definition, we see that () depends only on the restriction of this embedding to the field $\mathbb{Q}(all\ \zeta_N,\ N\ prime\ to\ p)$, which is Galois over $\mathbb{Q}$ with group $Aut((\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}) = \prod_{\ell\ not\ p} \mathbb{Z}_\ell^\times$. As the embeddings of this field into $\overline{\mathbb{Q}_p}$ are principal homogeneous under this Galois group, we see that the effect of changing the embedding is to replace the function $x \mapsto V(x)$ by a function of the form $x \mapsto V(\alpha x)$, for some $\alpha \in \prod_{\ell\ not\ p} \mathbb{Z}_\ell^\times$.

Following Kubert, the standard properties of Gauss sums lead to the following properties of this function.

(1) $V(x) = 0$ if and only if $x = 0$ in $(\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$.
(2) For $x$ nonzero in $(\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$, $V(x) + V(-x) = 1$.
(3) $V(1/2) = 1/2$.
(4) For any $x$ in $(\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$, $V(x) = V(px)$.
(5) For any $x$ and $y$ in $(\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$, $V(x) + V(y) \geq V(x + y)$.
(6) For any $x$ in $(\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$, and any integer $N \geq 1$ prime to $p$, $\sum_{i\ mod\ N} V(x + i/N) = V(Nx) + (N - 1)/2$.

The first two of these properties reflect the known absolute value of Gauss sums, and the third is the special case $x = 1/2$ of the second. The fourth is obvious if we use an additive character coming from the prime field, for then the two Gauss sums in question are equal. [Alternatively, one can see the fourth as an incarnation of the fact that any automorphism of $\overline{\mathbb{Q}_p}$, in particular any that induces the absolute Frobenius automorphism $x \mapsto x^p$ of the residue field, leaves the ord function invariant.] The fifth is the integrality of Jacobi sums. The sixth is the following Hasse-Davenport relation. Pick $E$ so that $N$ divides $\#E - 1$, pick a nontrivial $\psi_E$, and put $\psi_{E,N} := t \mapsto \psi_E(Nt)$. Then for any multiplicative character $\chi$ of $E$, we have

$$-g(\psi_{E,N}, \chi^N) = \prod_{\rho\ of\ order\ dividing\ N} (-g(\psi_E, \rho\chi)/ - g(\psi_E, \rho)).$$

It is now a simple matter to restate the previous proposition in terms of the function $x \mapsto V(x)$.

**Proposition 13.2.** *Given the hypergeometric sheaf $\mathcal{H} := \mathcal{H}(\psi; \chi_i{}'s; \rho_j{}'s)$ on $\mathbb{G}_m/k$ and its twist $\mathcal{F}$, pick any multiplicative character $Teich_k$ of $k^\times$ which is faithful, i.e., has order $\#k - 1$. Define a list of $n+m$ elements $(a_1, ..., a_n, b_1, ..., b_m)$ of $(1/(\#k - 1))\mathbb{Z}/\mathbb{Z}$ by*

$$\chi_i = Teich_k^{-a_i(\#k-1)},\ \rho_j = Teich_k^{-b_j(\#k-1)}.$$

*Then $\mathcal{F}$ has finite $G_{arith}$ if and only if the following conditions hold. For every $N \in (\mathbb{Z}/(\#k - 1)\mathbb{Z})^\times$, and for every $x \in (\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$, we* *have the inequality*

$$\sum_i V(Na_i+x)+\sum_j V(-Nb_j-x) \geq (n-1)/2+(1/n)\sum_{i,j} V(Na_i-Nb_j).$$

*Proof.* For some embedding of $\mathbb{Q}(\zeta_{\#k-1})$ into $\overline{\mathbb{Q}_p}$, and some embedding of $k$ into the residue field, our chosen $Teich_k$ is indeed the Teichmuller character. So the condition of p-integrality for (any embedding of $\overline{\mathbb{Q}}$ extending) this embedding is that for every $x \in (\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$, we have the inequality

$$\sum_i V(a_i+x)+\sum_j V(-b_j-x) \geq (n-1)/2+(1/n)\sum_{i,j} V(a_i-b_j).$$

The condition of p-integrality for an arbitrary embedding is that for every fixed $\alpha \in \prod_{\ell\ not\ p} \mathbb{Z}_\ell^\times$, and for every $x \in (\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$, we have the inequality

$$\sum_i V(\alpha a_i+\alpha x)+\sum_j V(-\alpha b_j-\alpha x) \geq (n-1)/2+(1/n)\sum_{i,j} V(\alpha a_i-\alpha b_j).$$

For fixed $\alpha$, we make the change of variable $x \mapsto \alpha x$, and the condition becomes

$$\sum_i V(\alpha a_i+x)+\sum_j V(-\alpha b_j-x) \geq (n-1)/2+(1/n)\sum_{i,j} V(\alpha a_i-\alpha b_j).$$

Since the $a_i, b_j$ all have denominator cleared by $\#k-1$, $\alpha$ only enters through its value mod $\#k-1$. □

We now give a beautiful unpublished result of Kubert [Ku].

**Theorem 13.3.** *Consider the function $x \mapsto V(x)$ on $(\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$. Let $q$ be a power of $p$.*

(1) *Let $(a_1, ..., a_q)$ be all but one of the fractions $\{n/(q+1)\}_{n\ mod\ q+1}$. Then for any $x \in (\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$, we have the inequality*

$$\sum_i V(a_i+x) \geq (q-1)/2.$$

(2) *Let $(a_1, ..., a_{q-1})$ be all but two of the fractions $\{n/(q+1)\}_{n\ mod\ q+1}$. Then for any $x \in (\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$, we have the inequality*

$$\sum_i V(a_i+x) \geq (q-2)/2.$$

(3) *Suppose $p$ is odd. Let $(b_1, ..., b_{(q-1)/2})$ be all but one of the fractions $\{n/((q+1)/2)\}_{n \ mod \ (q+1)/2}$. Then for any $x \in (\mathbb{Q}/\mathbb{Z})_{prime \ to \ p}$, we have the inequality*

$$\sum_i V(b_i + x) \geq (q-3)/4.$$

*Proof.* For the first assertion, we argue as follows. By an additive translation, we reduce to the case when the missing fraction is 0. Then we we use property (6) of the $V$ function to write

$$\sum_i V(a_i+x) = \sum_{n \ mod \ q+1} V(x+n/(q+1)) - V(x) = V((q+1)x) + q/2 - V(x).$$

Thus we must show that

$$V((q+1)x) + 1/2 \geq V(x).$$

If $x = 0$, or indeed if $V(x) \leq 1/2$, we are done. Suppose now that $V(x) > 1/2$. Add $V(-x)$ to both sides, so the assertion becomes

$$V((q+1)x) + V(-x) + 1/2 \geq V(x) + V(-x).$$

But $V(x) + V(-x) = 1$ (remember $x \neq 0$, since $V(x) > 1/2$), so we must show

$$V((q+1)x) + V(-x) \geq 1/2.$$

By properties (5) and (4) of the $V$ function, we have

$$V((q+1)x) + V(-x) \geq V(qx) = V(x),$$

and by assumption $V(x) > 1/2$.

For the second assertion, denote by $a_q$ and $a_{q+1}$ the two missing fractions. Fix an $x$. At least one of $a_q + x$ or $a_{q+1} + x$ is nonzero, since they are different. So at least one of the values $V(a_q+x)$ or $V(a_{q+1}+x)$ is nonzero. So if

$$V(a_q + x) \geq V(a_{q+1} + x),$$

as we may assume by renumbering, then certainly $a_q + x$ is nonzero. Now write

$$\sum_{i=1}^{q-1} V(a_i + x) = \sum_{i=1}^{q+1} V(a_i + x) - V(a_q + x) - V(a_{q+1} + x)$$
$$= V((q+1)x) + q/2 - V(a_q + x) - V(a_{q+1} + x)$$
$$= V((q+1)(a_q + x)) + q/2 - V(a_q + x) - V(a_{q+1} + x).$$

We must show that

$$V((q+1)(a_q + x)) + 1 \geq V(a_q + x) + V(a_{q+1} + x).$$

Now add $V(-a_q - x)$ to both sides, to find the equivalent inequality

$$V((q+1)(a_q+x)) + V(-a_q-x) + 1 \geq 1 + V(a_{q+1}+x),$$

i.e.,

$$V((q+1)(a_q+x)) + V(-a_q-x) \geq V(a_{q+1}+x).$$

By property (5), we have

$$V((q+1)(a_q+x)) + V(-a_q-x) \geq V(q(a_q+x)) = V(a_q+x),$$

and by hypothesis we have $V(a_q+x) \geq V(a_{q+1}+x)$.

The third assertion results immediately from the second one, by means of property (6), applied in the form

$$V(x) + V(x+1/2) = V(2x) + 1/2.$$

Indeed, if $(b_1, ..., b_{(q-1)/2})$ are all but one of the fractions $\{n/((q+1)/2)\}_{n \ mod \ (q+1)/2}$, then $(b_1/2, b_1/2+1/2, ..., b_{(q-1)/2}/2, b_{(q-1)/2}/2+1/2)$ are all but two of the fractions $\{n/(q+1)\}_{n \ mod \ q+1}$. Since $p$ is odd, $x \mapsto 2x$ maps $(\mathbb{Q}/\mathbb{Z})_{prime \ to \ p}$ onto itself. So it suffices to show that for any $x \in (\mathbb{Q}/\mathbb{Z})_{prime \ to \ p}$, we have

$$\sum_i^{(q-1)/2} V(2x+b_i) \geq (q-3)/4,$$

i.e.,

$$\sum_i^{(q-1)/2} [V(x+b_i/2) + V(x+b_i/2+1/2) - 1/2] \geq (q-3)/4,$$

which is just a trivial rearrangement of the second assertion. □

To end this section, we recall that Stickelberger gives an exact formula for the function $V$, in terms of the $[0,1)$-valued "fractional part" function $x \mapsto < x >$ on $\mathbb{R}/\mathbb{Z}$, which assigns to each $x$ in the quotient group $\mathbb{R}/\mathbb{Z}$ its unique representative in $[0,1)$.

**Theorem 13.4** (Stickelberger)**.** *Given $x \in (\mathbb{Q}/\mathbb{Z})_{prime \ to \ p}$, pick any integer $k \geq 1$ such that $x$ lies in $(1/(p^k-1))\mathbb{Z}/\mathbb{Z}$. Then we have*

$$V(x) = (1/k)\sum_{j=0}^{k-1} < p^j x > .$$

Here is a concrete application of Stickelberger's theorem.

**Lemma 13.5.** *Suppose we are given a list $(a_1, ..., a_n, b_1, ..., b_m)$ of elements of $(\mathbb{Q}/\mathbb{Z})_{prime \ to \ p}$, a list of integers $(N_1, ..., N_n, M_1, ..., M_n)$, and a real number $T \geq 0$. Let $k \geq 1$ be the least integer such that $p^k - 1$ is a common denominator for all the $a_i$ and all the $b_j$.*

(1) *Suppose that for each integer $e \in [0, k-1]$, we have the inequality*

$$\sum_i < p^e a_i + N_i x > + \sum_j < -p^e b_j - M_j x > \ \geq T,$$

*for every $x \in \mathbb{R}/\mathbb{Z}$. Then for every $x \in (\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$, we have the inequality*

$$\sum_i V(a_i + N_i x) + \sum_j V(-b_j - M_j x) \geq T.$$

(2) *Fix an integer $e \in [0, k-1]$. The inequality*

$$\sum_i < p^e a_i + N_i x > + \sum_j < -p^e b_j - M_j x > \ \geq T$$

*hods for every $x \in \mathbb{R}/\mathbb{Z}$ if and only if it holds at the finitely many points of $\mathbb{R}/\mathbb{Z}$ at which one of the arguments $p^e a_i + N_i x$ or $-p^e b_j - M_j x$ vanishes in $\mathbb{R}/\mathbb{Z}$.*

*Proof.* For the first assertion, we argue as follows. Given $x \in (\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$, pick a multiple $f$ of $k$ such that $(p^f - 1)x = 0$. Then by Stickelberger we have

$$\sum_i V(a_i + N_i x) + \sum_j V(-b_j - M_j x) =$$

$$(1/f) \sum_{e\ mod\ f} [\sum_i < p^e(a_i + N_i x) > + \sum_j < p^e(-b_j - M_j x) >].$$

By hypothesis, each of the terms in the $\sum_{e\ mod\ f}$ summation is separately $\geq T$.

For the second assertion, notice that any function on $\mathbb{R}$ of the form $f(x) =< Ax + B >$, with $A$ a nonzero integer, is continuous precisely on the complement in $\mathbb{R}$ of the points $\{-B/A + n/A\}_{n \in \mathbb{Z}}$ where its argument vanishes mod $\mathbb{Z}$. In the open interval between two successive discontinuities, it is linear. At any point $x_0 \in \mathbb{R}$, the one-sided limits $f^-(x_0)$ and $f^+(x_0)$ both exist, and satisfy

$$f^-(x_0) \geq f(x_0), f^+(x_0) \geq f(x_0).$$

Therefore any finite sum $F(x)$ of such functions, e.g.,

$$F(x) := \sum_i < p^e a_i + N_i x > + \sum_j < -p^e b_j - M_j x >,$$

has the same properties, i.e., it is continuous on the complement in $\mathbb{R}$ of the points where any of the arguments $p^e a_i + N_i x$ or $-p^e b_j - M_j x$ vanishes mod $\mathbb{Z}$, it is linear in the open interval between two successive discontinuities, and at any point $x_0 \in \mathbb{R}$, the one-sided limits $f^-(x_0)$ and $f^+(x_0)$ both exist, and satisfy

$$f^-(x_0) \geq f(x_0), f^+(x_0) \geq f(x_0).$$

So an inequality

$$\sum_i < p^e a_i + N_i x > + \sum_j < -p^e b_j - M_j x > \ \geq T$$

holds for all $x \in \mathbb{R}/\mathbb{Z}$ if and only if it holds at the finitely many points of discontinuity in $\mathbb{R}/\mathbb{Z}$. □

The following Theorem gives a very fast algorithm, via its condition (6), for deciding if a hypergeometric sheaf $\mathcal{H}$ defined over the prime field $\mathbb{F}_p$ has finite $G_{geom}$. Although it will not be used below, we include it here for ease of future reference.

**Theorem 13.6.** *Hypotheses and notations as in Proposition 13.2, suppose in addition that $k$ is the prime field $\mathbb{F}_p$. Then the following conditions are equivalent.*

(1) *For every finite extension $E/\mathbb{F}_p$, and for every point $a \in \mathbb{G}_m(E)$, $Trace(Frob_{a,E}|\mathcal{F})$ is an algebraic integer.*
(2) *$\mathcal{H}$ has finite $G_{geom}$.*
(3) *$\mathcal{F}$ has finite $G_{arith}$.*
(4) *For every $N \in (\mathbb{Z}/(p-1)\mathbb{Z})^\times$, and for every $x \in (\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$, we have the inequality*

$$\sum_i V(Na_i+x)+\sum_j V(-Nb_j-x) \ \ \geq \ \ (n-1)/2+(1/n)\sum_{i,j} V(Na_i-Nb_j).$$

(5) *For every $N \in (\mathbb{Z}/(p-1)\mathbb{Z})^\times$, and for every $x \in (1/(p-1))\mathbb{Z}/\mathbb{Z}$, we have the inequality*

$$\sum_i < Na_i+x > + \sum_j < -Nb_j-x > \ \geq \ \ (n-1)/2+(1/n)\sum_{i,j} < Na_i-Nb_j > .$$

(6) *At every point $a \in \mathbb{F}_p^\times$, $Trace(Frob_{a,\mathbb{F}_p}|\mathcal{F})$ is an algebraic integer.*

*Proof.* The equivalence of the first three conditions is Lemma 10.3. That (3) ⇔ (4) is Proposition 13.2. That (4) ⇒ (5) is a special case of Stickelberger's Theorem 13.4: for $x \in (1/(p-1))\mathbb{Z}/\mathbb{Z}, (x) =< x >$. The implication (5) ⇒ (4) is given by the previous Lemma. That (5) ⇔ (6) is just Proposition 13.1, together with Stickelberger's Theorem 13.4. □

## 14. Return to the case by case analysis of the candidates

14.1. **The case $N = 13$.** Here the allowed characteristics are $2, 3, 7$, and there are two Galois-conjugate candidates. For one, the six nontrivial $\chi_i$ are the $(1, 3, 4, 9, 10, 12)$'th powers of a fixed character of order 13, and for the other they are the same powers of the square of that character, or, what is the same, the $(2, 5, 6, 7, 8, 11)$'th powers of the original character. Thus the criterion for finite $G_{geom}$ is as follows.

If $p = 2$, we are to have, for every $x \in (\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$, both

$$V(x) + V(x + 1/13) + V(x + 3/13) + V(x + 4/13) + V(x + 9/13) \\ +V(x + 10/13) + V(x + 12/13) \geq 3$$

and

$$V(x) + V(x + 2/13) + V(x + 5/13) + V(x + 6/13) + V(x + 7/13) \\ +V(x + 8/13) + V(x + 11/13) \geq 3.$$

Computer calculation shows that the first inequality fails to hold for $x = 5/(2^{12} - 1)$, and hence this candidate does not have finite $G_{geom}$.

If $p = 3$ or $p = 7$, we are to have, for every $x \in (\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$, both

$$V(x) + V(x + 1/13) + V(x + 3/13) + V(x + 4/13) + V(x + 9/13) \\ +V(x + 10/13) + V(x + 12/13) + V(1/2 - x) \geq 7/2$$

and

$$V(x) + V(x + 2/13) + V(x + 5/13) + V(x + 6/13) + V(x + 7/13) \\ +V(x + 8/13) + V(x + 11/13) + V(1/2 - x) \geq 7/2.$$

Hand calculation for $p = 3$ shows that the first inequality fails for $x = 1/26$, and hence this candidate does not have finite $G_{geom}$. And computer calculation for $p = 7$ shows that the first inequality fails for $x = 212/(7^{12} - 1)$, and hence this candidate does not have finite $G_{geom}$.

14.2. **The case $N = 12$.** Here the only allowed characteristic is $p = 7$, and there is just one candidate, whose six nontrivial $\chi_i$ are the four characters of order 12 and the two characters of order 3. We will show that in this case we have finite $G_{geom}$. The criterion is that for every $x \in (\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$, we are to have

$$V(x) + V(x + 1/12) + V(x + 4/12) + V(x + 5/12) + V(x + 7/12) \\ +V(x + 8/12) + V(x + 11/12) + V(1/2 - x) \geq 7/2.$$

Using the relations

$$V(x + 1/12) + V(x + 7/12) = V(2x + 1/6) + 1/2,$$
$$V(x + 5/12) + V(x + 11/12) = V(2x + 5/6) + 1/2$$

and

$$V(x) + V(x + 4/12) + V(x + 8/12) = V(3x) + 1,$$

the inequality becomes

$$V(3x) + V(2x + 1/6) + V(2x + 5/6) + V(1/2 - x) \geq 3/2.$$

Because we are in characteristic $p = 7$, we have

$$V(1/2 - x) = V(7/2 - 7x) = V(1/2 - 7x),$$

so we can rewrite the inequality as

$$V(3x) + V(2x + 1/6) + V(2x + 5/6) + V(1/2 - 7x) \geq 3/2.$$

We now apply Lemma 13.5 to this sum. As $p = 7$, we have $k = 1$ in the notations of that lemma, so it suffices to show that for every $x \in \mathbb{R}/\mathbb{Z}$, we have the inequality

$$< 3x > + < 2x + 1/6 > + < 2x + 5/6 > + < 1/2 - 7x > \ \geq \ 3/2.$$

For this, it suffices to check at each of the fourteen points of discontinuity in $\mathbb{R}/\mathbb{Z}$, namely the points

$$0, 1/3, 2/3,$$

the points

$$-1/12, -1/12 + 1/2, 5/12, 5/12 + 1/2,$$

and the points

$$1/14, 3/14, 5/14, 7/14, 9/14, 11/14, 13/14.$$

We leave this verification to the patient reader.

14.3. **Another approach to case $N = 12$; "erasing".** There is another approach to this case, which is both more illuminating and more mysterious. Let us return to the inequality

$$< 3x > + < 2x + 1/6 > + < 2x + 5/6 > + < 1/2 - 7x > \ \geq \ 3/2,$$

which we rewrite as

$$< 3x > + < 2(x+1/12) > + < 2(x+5/12) > + < 7(1/14-x) > \ \geq \ 3/2,$$

For any integer $d \geq 1$, the function $< x >$ satisfies the identity

$$\sum_{i \ mod \ d} < x + i/d > = < dx > + (d-1)/2.$$

[Translate $x$ by a multiple of $1/d$ until $x \in [0, 1/d)$, at which point the identity is obvious.] Applying this to the the four terms on the left, with $d$ successively $3, 2, 2, 7$, we may rewrite the needed inequality as

$$< x > + < x+1/12 > + < x+4/12 > + < x+5/12 > + < x+7/12 >$$

$$+ < x+8/12 > + < x+11/12 > + \sum_{n \ mod \ 7} < 1/14+n/7-x > \ \geq 13/2.$$

As explained in [Ka-ESDE, 8.17.2 and 8.17.2.1], this last inequality holds because the two lists of fractions

$$(0, 1/12, 4/12, 5/12, 7/12, 8/12, 11/12)$$

and

$$(1/14, 3/14, 5/14, 7/14, 9/14, 11/14, 13/14)$$

give rise, by the map $x \mapsto e^{2\pi ix}$, to Galois-stable sets of roots of unity which are *intertwined* on the unit circle. Now this intertwining is precisely the condition that the hypergeometric differential equation of type $(7, 7)$ with these parameters has its differential Galois group $G_{gal}$ finite, cf. [B-H, 4.8] and [Ka-ESDE, 5.5.3]. And this intertwining implies that in any characteristic $p$ other than $2, 3, 7$, the corresponding hypergeometric sheaf of type $(7, 7)$,

$$\mathcal{H}(\psi, all\ char's\ of\ order\ 1, 3,\ or\ 12;\ all\ char's\ of\ order\ 2\ or\ 14)$$

has finite $G_{geom}$, cf.[Ka-ESDE, 8.17.15]. What is striking here is that if, in the above prescription of the characters $\chi_i$ and $\rho_j$ for this hypergeometric, we simply erase those whose order is divisible by 7, we obtain the hypergeometric sheaf

$$\mathcal{H}(\psi, all\ char's\ of\ order\ 1, 3,\ or\ 12; \chi_{quad})$$

with which we are concerned in characteristic $p = 7$.

What happens in characteristics 2 and 3 if we do this "erasing of characters that don't make sense"? In characteristic 3, we will get

$$\mathcal{H}(\psi, 1;\ all\ char's\ of\ order\ 2\ or\ 14) \cong$$

$$\mathcal{L}_{\chi_{quad}} \otimes [x \mapsto 1/x]^{\star}\mathcal{H}(\overline{\psi}, all\ char's\ of\ order\ dividing\ 7; \chi_{quad}).$$

As we have seen in the earlier discussion of the $N = 7$ case, the sheaf

$$\mathcal{H}(\overline{\psi}, all\ char's\ of\ order\ dividing\ 7; \chi_{quad})$$

in characteristic $p = 3$ has finite $G_{geom} = U_3(3)$.

If we do the erasing in characteristic $p = 2$, we are left with $\mathcal{L}_{\psi}$, not a very convincing case. But if we first tensor with $\mathcal{L}_{\chi_{quad}}$, our input sheaf of type $(7, 7)$ becomes

$$\mathcal{H}(\psi, all\ char's\ of\ order\ 2, 6,\ or\ 12;\ all\ char's\ of\ order\ dividing\ 7).$$

If we now erase all the characters which fail to have odd order, we are left with

$$\mathcal{H}(\psi, \emptyset;\ all\ char's\ of\ order\ dividing\ 7) \cong$$

$$[x \mapsto 1/x]^{\star}\mathcal{H}(\overline{\psi}, all\ char's\ of\ order\ dividing\ 7; \emptyset).$$

As we have seen in the earlier discussion of the $N = 7$ case, this last sheaf in characteristic $p = 2$ has finite $G_{geom}$ the $ax + b$ group over $\mathbb{F}_8$.

14.4. **The case** $N = 9$**.** Here the allowed characteristics are $2, 7, 13$.

In characteristic $p = 2$, we have $9 = q + 1$ with $q = 8$, so it is immediate from the "all but two" case of Kubert's Theorem 13.3 above that we have finite $G_{geom}$ in this case.

If $p = 7$ or $p = 13$, the criterion for finite $G_{geom}$ is that we are to have, for every $x \in (\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$, all three of the inequalities

$$\begin{aligned} &V(x) + V(x + 1/9) + V(x + 3/9) + V(x + 4/9) + V(x + 5/9) \\ &\qquad + V(x + 6/9) + V(x + 8/9) + V(1/2 - x) \quad \geq 7/2, \\ &V(x) + V(x + 1/9) + V(x + 2/9) + V(x + 3/9) + V(x + 6/9) \\ &\qquad + V(x + 7/9) + V(x + 8/9) + V(1/2 - x) \quad \geq 7/2, \end{aligned}$$

and

$$\begin{aligned} &V(x) + V(x + 2/9) + V(x + 3/9) + V(x + 4/9) + V(x + 5/9) \\ &\qquad + V(x + 6/9) + V(x + 7/9) + V(1/2 - x) \quad \geq 7/2. \end{aligned}$$

Computer calculation for $p = 7$ shows that the first inequality fails for $x = 66/(7^3 - 1)$, and hence this candidate does not have finite $G_{geom}$. And computer calculation for $p = 13$ shows that the first inequality fails for $x = 3/(13^6 - 1)$, and hence this candidate does not have finite $G_{geom}$.

14.5. **The case** $N = 8$**.** Here the allowed characteristics $p$ are $3, 7, 13$. For each, the criterion for finite $G_{geom}$ is that we are to have, for every $x \in (\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$,

$$\begin{aligned} &V(x) + V(x + 1/8) + V(x + 2/8) + V(x + 3/8) + V(x + 5/8) \\ &\qquad + V(x + 6/8) + V(x + 7/8) + V(1/2 - x) \quad \geq 7/2. \end{aligned}$$

Hand calculation for $p = 13$ shows that the inequality fails for $x = 5/168$, and hence this candidate does not have finite $G_{geom}$.

Hand calculation for $p = 3$ shows that the inequality fails for $x = 1/8$, and hence this candidate does not have finite $G_{geom}$.

The remaining case, $p = 7$, turns out to have finite $G_{geom}$. Because $p = 7$, we have

$$V(1/2 - x) = V(7(1/2 - x)) = V(1/2 - 7x),$$

so we may rewrite the needed inequality as

$$\begin{aligned} &V(x) + V(x + 1/8) + V(x + 2/8) + V(x + 3/8) + V(x + 5/8) \\ &\qquad + V(x + 6/8) + V(x + 7/8) + V(1/2 - 7x) \quad \geq 7/2. \end{aligned}$$

The unordered list $(0, 1/8, 2/8, 3/8, 5/8, 6/8, 7/8)$ of elements of $(\mathbb{Q}/\mathbb{Z})_{prime\ to\ p}$ are all the elements of order dividing 8 save for the unique element of order 2, so this list is invariant under multiplication by $p = 7$. So it suffices to prove that for every $x \in \mathbb{R}/\mathbb{Z}$ we have

$$< x > + < x + 1/8 > + < x + 2/8 > + < x + 3/8 > + < x + 5/8 >$$

$$+ < x + 6/8 > + < x + 7/8 > + < 1/2 - 7x > \ \geq 7/2.$$

Exactly as in the case $N = 12, p = 7$ above, we may now rewrite this as

$$< x > + < x + 1/8 > + < x + 2/8 > + < x + 3/8 > + < x + 5/8 >$$

$$+ < x + 6/8 > + < x + 7/8 > + \sum_{n\ mod\ 7} < 1/14 + n/7 - x > \ \geq 13/2.$$

And once again this holds because the two lists of fractions

$$(0, 1/8, 2/8, 3/8, 5/8, 6/8, 7/8)$$

and

$$(1/14, 3/14, 5/14, 7/14, 9/14, 11/14, 13/14)$$

give rise, by the map $x \mapsto e^{2\pi i x}$, to Galois-stable sets of roots of unity which are intertwined on the unit circle.

14.6. **Another remark on "erasing".** Just as in the case $N = 12, p = 7$, the above intertwining implies that in any characteristic $p$ other than $2, 7$, the corresponding hypergeometric sheaf of type $(7, 7)$,

$$\mathcal{H}(\psi, all\ char's\ of\ order\ dividing\ 8\ save\ \chi_{quad};\ all\ char's\ of\ order\ 2\ or\ 14)$$

has finite $G_{geom}$. And our sheaf in characteristic $p = 7$ is again obtained from this one by "erasing the characters that don't make sense".

In characteristic $p = 2$, this erasing again leaves us with $\mathcal{L}_\psi$. But if we first tensor with $\mathcal{L}_{\chi_{quad}}$, our input sheaf of type $(7, 7)$ becomes

$$\mathcal{H}(\psi, all\ nontriv,\ char's\ of\ order\ dividing\ 8;\ all\ char's\ of\ order\ dividing\ 7).$$

If we now erase all the characters which fail to have odd order, we are again left in characteristic $p = 2$ with the finite-$G_{geom}$ sheaf

$$\mathcal{H}(\psi, \emptyset;\ all\ char's\ of\ order\ dividing\ 7) \cong$$

$$[x \mapsto 1/x]^\star \mathcal{H}(\overline{\psi}, all\ char's\ of\ order\ dividing\ 7; \emptyset),$$

exactly as in the $N = 12$ discussion of erasing.

14.7. **Another approach to the case $N = 7$.** Here the allowed characteristics are $2, 3, 13$. We will give proofs, not using [Ka-NG2], that we have finite $G_{geom}$ here.

In characteristic $p = 2$, the criterion for finite monodromy is

$$\sum_{i \bmod 7} V(x + i/7) \geq 3,$$

which is obvious, thanks to the identity

$$\sum_{i \bmod 7} V(x + i/7) = V(7x) + 3.$$

In characteristics $p = 3$ and $p = 13$, the criterion is

$$\sum_{i \bmod 7} V(x + i/7) + V(1/2 - x) \geq 7/2,$$

which in view of the above identity we may rewrite as

$$V(7x) + V(1/2 - x) \geq 1/2.$$

Suppose first $p = 3$. Then we have

$$V(7x) + V(1/2 - x) \geq V(1/2 + 6x) = V(1/2 + 2x),$$

the last equality because $p = 3$. Again because $p = 3$, we have the equality $V(1/2 - x) = V(1/2 - 9x)$, so we have

$$V(7x) + V(1/2 - x) = V(7x) + V(1/2 - 9x) \geq V(1/2 - 2x).$$

Adding these inequalities, we obtain

$$2[V(7x) + V(1/2 - x)] \geq V(1/2 + 2x) + V(1/2 - 2x).$$

But $1/2 - 2x$ and $1/2 + 2x$ are negatives of each other mod $\mathbb{Z}$, so if $x$ is neither $1/4$ nor $3/4$ mod $\mathbb{Z}$, then $V(1/2 + 2x) + V(1/2 - 2x) = 1$, and we are done. For $x$ either of the excluded values, we check directly that

$$V(7x) + V(1/2 - x) = V(1/4) + V(3/4) = 1 \geq 1/2.$$

Suppose now $p = 13$. Then we have

$$V(7x) + V(1/2 - x) \geq V(1/2 + 6x).$$

Because $p = 13$, we have $V(1/2 - x) = V(1/2 - 13x)$, so we have

$$V(7x) + V(1/2 - x) = V(7x) + V(1/2 - 13x) \geq V(1/2 - 6x).$$

Adding these inequalities, we obtain

$$2[V(7x) + V(1/2 - x)] \geq V(1/2 + 6x) + V(1/2 - 6x).$$

But $1/2 - 6x$ and $1/2 + 6x$ are negatives of each other mod $\mathbb{Z}$, so if $1/2 - 6x$ is nonzero mod $\mathbb{Z}$, then $V(1/2 + 6x) + V(1/2 - 6x) = 1$, and we are done. If $1/2 - 6x = 0 \bmod \mathbb{Z}$, then $7x$ and $1/2 - x$ are negatives of each other mod $\mathbb{Z}$, and neither lies in $\mathbb{Z}$, so for these $x$ we have

$$V(7x) + V(1/2 - x) = V(7x) + V(-7x) = 1 \geq 1/2.$$

## 15. Conclusion of the proof of Theorem 9.1: exact results in the cases $(N, p) = (12, 7), (9, 2), (8, 7)$

In this section, we determine the groups $G_{geom}$ and $G_{arith}$ in the three named cases, and thus finish the proof of Theorem 9.1.

15.1. **The case** $(N, p) = (9, 2)$**.** Here we have $G_{geom} = G_{arith} = L_2(8)$, simply because $L_2(8)$ is the only candidate group which contains an element of order $N = 9$.

15.2. **The case** $(N, p) = (12, 7)$**.** In this case, two of the candidate groups, namely $U_3(3)$ and $U_3(3).2 = G_2(2)$, have elements of order $N = 12$. So we are in one of the following three cases.

(1) $G_{geom} = G_{arith} = U_3(2)$,
(2) $G_{geom} = U_3(2), G_{arith} = U_3(2).2$,
(3) $G_{geom} = G_{arith} = U_3(2).2$.

We will show that we have the third case. Suppose not. Then we have $G_{geom} = U_3(2)$. We will show that this leads to a contradiction.

Recall that the sheaves in question live on $\mathbb{G}_m/\mathbb{F}_{49}$. Choose a multiplicative character $\chi_{12}$ of order 12 of $\mathbb{F}_{49}^\times$, and choose a nontrivial additive character $\psi$ of $\mathbb{F}_7$. Then our sheaves are

$$\mathcal{H} := \mathcal{H}(\psi_{\mathbb{F}_{49}}, 1, \chi_{12}, \chi_{12}^4, \chi_{12}^5, \chi_{12}^7, \chi_{12}^8, \chi_{12}^{11}; \chi_{quad})$$

and

$$\mathcal{F} := \mathcal{H} \otimes (A^{-7})^{deg},$$

for $A$ the negative of the quadratic Gauss sum over $\mathbb{F}_{49}$,

$$A := -g(\psi_{\mathbb{F}_{49}}, \chi_{quad}).$$

The key observation is that the sheaf $\mathcal{H}$ has a descent $\mathcal{H}_0$ to a lisse sheaf on $\mathbb{G}_m/\mathbb{F}_7$. To construct this descent, we recall from [Ka-ESDE, 8.2.3] that the hypergeometric sheaves are built out of simple ingredients by multiplicative convolution with compact support on $\mathbb{G}_m$, which we will denote here $\star_!$. For us, the relevant convolution formula is

$$\mathcal{H}(\psi_{\mathbb{F}_{49}}, 1, \chi_{12}, \chi_{12}^4, \chi_{12}^5, \chi_{12}^7, \chi_{12}^8, \chi_{12}^{11}; \chi_{quad}) \cong$$

$$\mathcal{H}(\psi_{\mathbb{F}_{49}}, \chi_{12}, \chi_{12}^5, \chi_{12}^7, \chi_{12}^{11}; \emptyset) \star_! \mathcal{H}(\psi_{\mathbb{F}_{49}}, 1, \chi_{12}^4, \chi_{12}^8; \chi_{quad}).$$

Denote by $\psi_2$ the additive character $x \mapsto \psi(2x)$, denote by $\chi_6$ a multiplicative character of order 6 of $\mathbb{F}_7^\times$, and denote by $A_0$ the negative of the quadratic Gauss sum over $\mathbb{F}_7$,

$$A_0 := -g(\psi, \chi_{quad}).$$

By the direct image formula [Ka-GKM, 5.6.2] for Kloosterman sheaves, the first convolvee descends to $\mathbb{G}_m/\mathbb{F}_7$ as

$$[2]_\star(\mathcal{H}(\psi_2, 1, \chi_6, \chi_6^5; \emptyset)) \otimes (A_0^2)^{deg}.$$

The second convolvee descends to $\mathbb{G}_m/\mathbb{F}_7$ as

$$\mathcal{H}(\psi, 1, \chi_6^2, \chi_6^4; \chi_{quad}).$$

The desired descent $\mathcal{H}_0$ is then given by

$$\mathcal{H}_0 := \mathcal{H}(\psi, 1, \chi_6^2, \chi_6^4; \chi_{quad}) \star_! [2]_\star(\mathcal{H}(\psi_2, 1, \chi_6, \chi_6^5; \emptyset)) \otimes (A_0^2)^{deg}.$$

We next descend the sheaf $\mathcal{F}$ to a lisse sheaf $\mathcal{F}_0$ on $\mathbb{G}_m/\mathbb{F}_7$. As a first attempt, consider the "obvious" descent

$$\mathcal{F}_1 := \mathcal{H}_0 \otimes (A_0^{-7})^{deg}.$$

Its traces are real, and it is pure of weight zero, so it is isomorphic to its dual. Being irreducible, it is orthogonally self dual, i.e., its $G_{arith}$ lies in $O(7)$. After extension of scalars, we recover $\mathcal{F}$, whose $G_{arith}$ lies in $G_2 \subset SO(7)$. Therefore for some choice of sign $\epsilon = \pm$, the twisted sheaf

$$\mathcal{F}_0 := \mathcal{F}_1 \otimes (\epsilon)^{deg}$$

has its $G_{arith}$, call it $G_{0,arith}$ lying in $SO(7)$. Now $\mathcal{F}_0$ is a descent of $\mathcal{F}$, so its $G_{geom}$, equal to that of $\mathcal{H}$, is the finite primitive irreducible subgroup $U_3(3)$ of $G_2$. As $G_{0,arith}$ lies in the normalizer in $SO(7)$ of $G_{geom}$, it follows from Lemma 7.2 that $G_{0,arith}$ is itself a finite primitive irreducible subgroup of $G_2$ which contains $U_3(3)$, and hence contains an element of order $N = 12$. Therefore $G_{0,arith}$ is either $U_3(3)$ or $U_3(3).2$. In either case, the quotient $G_{0,arith}/G_{geom}$ has order dividing 2, so becomes trivial after the constant field extension of scalars from $G_m/\mathbb{F}_7$ to $\mathbb{G}_m/\mathbb{F}_{49}$. Therefore our original sheaf $\mathcal{F}$ on $\mathbb{G}_m/\mathbb{F}_{49}$ has $G_{arith} = G_{geom}$, and hence $G_{arith} = U_3(3)$ (under the ongoing assumption that $G_{geom} = U_3(3)$).

An examination of the ATLAS [CCNPW-Atlas] character tables of $U_3(3)$ and of $U_3(3).2$ shows that, in the unique seven-dimensional representation that puts both groups inside $G_2$, there are elements $\gamma$ in $U_3(3).2$ with the property that

$$Trace(\gamma) = -1, Trace(\gamma^2) = 1.$$

It also shows that all such elements lie in a single conjugacy class ($6B$ in ATLAS notation) of $U_3(3).2$, and that no such elements lie in $U_3(3)$. But computer calculation shows that there are eight points $a \in \mathbb{G}_m(\mathbb{F}_{49})$ for which the conjugacy class of $Frob_{a,\mathbb{F}_{49}}|\mathcal{F}$ in $G_{arith}$ is precisely the class of such an element $\gamma$, i.e., for these eight points $a \in \mathbb{G}_m(\mathbb{F}_{49})$ we have

$$Trace(Frob_{a,\mathbb{F}_{49}}|\mathcal{F}) = -1, Trace(Frob_{a,\mathbb{F}_{49^2}}|\mathcal{F}) = 1.$$

[These eight points $a$ are the roots of the following four irreducible quadratic polynomials over $\mathbb{F}_7$:

$$X^2 + X - 1, X^2 - X - 1, X^2 - 2X - 2, X^2 - X - 4.]$$

This contradiction shows that in fact we have $G_{geom} = G_{arith} = U_3(2).2$.

15.3. **The case** $(N, p) = (8, 7)$**.** In this case, three of the candidate groups, namely $L_2(7).2$, $U_3(3)$, and $U_3(3).2 = G_2(2)$, have elements of order $N = 8$. So both $G_{geom}$ and $G_{arith}$ are among these three groups. Now $G_{geom}$ is a normal subgroup of $G_{arith}$, but $L_2(7).2$ is not a normal subgroup of either of the other groups. So we are in one of the following four cases.

(1) $G_{geom} = G_{arith} = L_2(7).2$,
(2) $G_{geom} = G_{arith} = U_3(2)$,
(3) $G_{geom} = U_3(2), G_{arith} = U_3(2).2$,
(4) $G_{geom} = G_{arith} = U_3(2).2$.

We will show that we are in the first case. We will do this by a consideration of higher moments.

Let us briefly review the general theory. Our sheaf $\mathcal{F}$ on $\mathbb{G}_m/k$, $k = \mathbb{F}_{49}$, is lisse, and pure of weight zero. It is geometrically irreducible, and hence it is arithmetically irreducible. Thus it is irreducible representation, say $V$, of the unknown group $G_{arith}$, and its restriction to the unknown normal subgroup $G_{geom}$ remains irreducible. The higher moments $M_n, n \geq 1$, attached to this situation are defined as dimensions of spaces of tensor invariants:

$$M_n := dim(V^{\otimes n})^{G_{geom}} = dim H^0(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}^{\otimes n}) = dim H^2_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{F}^{\otimes n}).$$

For any finite extension $E/k$, we have the "empirical moments" $M_n(E)$ attached to this situation, defined as the sums

$$M_n(E) := (1/\#\mathbb{G}_m(E)) \sum_{a \in \mathbb{G}_m(E)} Trace(Frob_{a,E}|\mathcal{F}^{\otimes n})$$

$$= (1/\#\mathbb{G}_m(E)) \sum_{a \in \mathbb{G}_m(E)} (Trace(Frob_{a,E}|\mathcal{F}))^n.$$

**Lemma 15.1.** *In the general situation above, denote by $\lambda_0$ and by $\lambda_\infty$ respectively the largest slopes of the $I_0$ and $I_\infty$-representations of $\mathcal{F}$. We have the inequality*

$$M_n \geq |M_n(E)| - (\lambda_0 + \lambda_\infty) rank(\mathcal{F})^n / (\sqrt{\#E} \ - \ 1/\sqrt{\#E}).$$

*Proof.* The representation $V$ of $G_{arith}$ is irreducible, so it is completely reducible. Therefore all its tensor powers $V^{\otimes n}$ are completely reducible. As $G_{geom}$ is a normal subgroup of $G_{arith}$, the space $(V^{\otimes n})^{G_{geom}}$ is a $G_{arith}$-subrepresentation of $V^{\otimes n}$. By the complete reducibility of $V^{\otimes n}$, we have a direct sum decomposition of $G_{arith}$-representations

$$V^{\otimes n} = (V^{\otimes n})_{ginv} \bigoplus (V^{\otimes n})_{nginv},$$

in which

$$(V^{\otimes n})_{ginv} := (V^{\otimes n})^{G_{geom}}$$

is the space of invariants under $G_{geom}$ ("ginv" for "geometric invariants"), and in which $(V^{\otimes n})_{nginv}$ has no nonzero $G_{geom}$-invariants ("nginv" for "no geometric invariants"). In terms of lisse sheaves on $\mathbb{G}_m/k$, we have a direct sum decomposition

$$\mathcal{F}^{\otimes n} = (\mathcal{F}^{\otimes n})_{ginv} \bigoplus (\mathcal{F}^{\otimes n})_{nginv},$$

in which $(\mathcal{F}^{\otimes n})_{ginv}$ is geometrically constant of rank $M_n$, and in which

$$H^2_c(\mathbb{G}_m \otimes_k \overline{k}, (\mathcal{F}^{\otimes n})_{nginv}) = 0.$$

Each summand is pure of weight zero. In each summand, the largest largest slopes of the $I_0$ and $I_\infty$-representations are bounded above by $\lambda_0$ and $\lambda_\infty$ respectively, since this holds for $\mathcal{F}^{\otimes n}$ itself. Thus we have

$$M_n(E) = S_1 + S_2,$$

$$S_1 = (1/\#\mathbb{G}_m(E)) \sum_{a \in \mathbb{G}_m(E)} Trace(Frob_{a,E} | (\mathcal{F}^{\otimes n})_{ginv})$$

$$S_2 = (1/\#\mathbb{G}_m(E)) \sum_{a \in \mathbb{G}_m(E)} Trace(Frob_{a,E} | (\mathcal{F}^{\otimes n})_{nginv}).$$

In $S_1$, the fact that $(\mathcal{F}^{\otimes n})_{ginv}$ is pure of weight zero and has rank $M_n$ shows that for each summand we have the trivial estimate

$$|Trace(Frob_{a,E} | (\mathcal{F}^{\otimes n})_{ginv})| \leq M_n.$$

Thus for $S_1$ we have the estimate

$$|S_1| \leq M_n.$$

For $S_2$, we apply the Lefshetz Trace formula for $(\mathcal{F}^{\otimes n})_{nginv}$. Since this sheaf has vanishing $H^2_c$, (and also vanishing $H^0_c$, being lisse on an open curve), we find

$$S_2 = -(1/\#\mathbb{G}_m(E))Trace(Frob_E|H^1_c(\mathbb{G}_m \otimes_k \overline{k}, (\mathcal{F}^{\otimes n})_{nginv})).$$

The sheaf $(\mathcal{F}^{\otimes n})_{nginv}$ is pure of weight zero, so by Deligne's fundamental result [De-Weil II, 3.3] we have

$$|S_2| \leq (1/\#\mathbb{G}_m(E))|\chi(\mathbb{G}_m \otimes_k \overline{k}, (\mathcal{F}^{\otimes n})_{nginv})|\sqrt{\#E}.$$

By the Euler-Poincare formula on $\mathbb{G}_m$, we have

$$-\chi(\mathbb{G}_m \otimes_k \overline{k}, (\mathcal{F}^{\otimes n})_{nginv}) = Swan_0((\mathcal{F}^{\otimes n})_{nginv}) + Swan_\infty((\mathcal{F}^{\otimes n})_{nginv}).$$

Since we have upper bounds for the biggest slopes at 0 and $\infty$, we have

$$|\chi(\mathbb{G}_m \otimes_k \overline{k}, (\mathcal{F}^{\otimes n})_{nginv})| \leq (\lambda_0+\lambda_\infty)rank((\mathcal{F}^{\otimes n})_{nginv}) \leq (\lambda_0+\lambda_\infty)rank(\mathcal{F})^n.$$

Thus we find

$$|S_2| \leq (\lambda_0 + \lambda_\infty)rank(\mathcal{F})^n/(\sqrt{\#E} - 1/\sqrt{\#E}).$$

Since $M_n(E) = S_1 + S_2$, with $|S_1| \leq M_n$ and $S_2$ estimated above, we get the asserted inequality. ☐

We now apply this lemma to our $\mathcal{F}$, which has $\lambda_0 = 0, \lambda_\infty = 1/6$. Computer calculation over $\mathbb{F}_{7^6}$ gives $M_4(\mathbb{F}_{7^6}) = 7.99$. So by the above lemma, we have

$$M_4 \geq 7.99 - (1/6)7^4/(7^3 - 1/7^3) = 6.82.$$

Using the ALTAS tables in GAP [GAP], we find that for the unique seven-dimensional representation which puts each of our candidate groups into $G_2$, the fourth moment is 4 for $U_3(3)$, 4 for $U_3(3).2$, and 8 for $L_2(7).2$. Therefore we have $G_{geom} = G_{arith} = L_2(7).2$, as asserted.

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