G_2 AND SOME EXCEPTIONAL WITT VECTOR IDENTITIES

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Abstract. We find some new one-parameter families of exponential sums in every odd characteristic whose geometric and arithmetic monodromy groups are G_2.

Introduction

In earlier work [Ka-ESDE, 9.1.1], we proved that certain very simple one-parameter families of exponential sums had the exceptional group G_2 as their (geometric and arithmetic) monodromy groups, in every finite characteristic $p \geq 17$. These sums were of the form

$$(1/g) \sum_{x \in k^\times} \chi_2(x)\psi(x^7 + tx).$$

Here k is a finite field, g is a fixed gauss sum, χ_2 is the quadratic character of $k^\times$, ψ is a nontrivial additive character of k, and $t \in k$ is the parameter. A question of Rudnick and Waxman led us to wonder if, in this construction, the polynomial x^7 inside the ψ could be replaced by other polynomials of degree seven and still yield G_2. Computer experiments suggested that the answer was indeed yes, for polynomials of the form

$$ax^7/7 + 2abx^5/5 + ab^2x^3/3,$$

any $a \neq 0$, any b. That these polynomials do indeed produce G_2 in large characteristic (see Theorem 4.3) results from certain Witt vector identities. It remains an open question if these are the only polynomials which produce G_2.

In the second half of the paper, we analyze the situation in low characteristic, especially in characteristics $3, 5, 7$, where Witt vectors reappear in order to make sense of the question, and (again) to provide the answer.

1. The exceptional identities

Fix a prime p, and consider the p-Witt vectors of length 2 as a ring scheme over $\mathbb{Z}$. The addtion law is given by

$$(x,a)+(y,b) := (x+y, a+b+(x^p+y^p-(x+y)^p)/p).$$

The multiplication law is given by

$$(x,a)(y,b) := (xy, x^p b + y^p a + pab).$$

For an odd prime p, we have

$$(x,0)+(y,0)+(-x-y,0) = (0,(x^p+y^p-(x+y)^p)/p).$$

Let us define, for odd p, the integer polynomial

$$F_p(x,y) := (x^p+y^p-(x+y)^p)/p \in \mathbb{Z}[x,y].$$

For $p=2$, we have

$$(x,0)+(y,0)+(-x-y,0) = (0, x^2+xy+y^2),$$

and we define

$$F_2(x,y) := x^2+xy+y^2 \in \mathbb{Z}[x,y].$$

Thus

$$F_3 = -xy(x+y).$$

The exceptional identities we have in mind are

$$F_5 = F_3F_2, F_7 = F_3(F_2)^2.$$

2. Basic facts about G_2

We work with algebraic groups over $\mathbb{C}$. Given a prime number p, a theorem of Gabber [Ka-ESDE, 1.6] tells us the possible connected irreducible (in the given p-dimensional representation) Zariski closed subgroups of SL_p. For $p=2$, the only possibility is SL_2. For p odd and $p \neq 7$, the possibilities are either the image of SL_2 in $Sym^{p-1}(std_2)$, SO_p, or SL_p.

For $p=7$ there is one new possibility, G_2, which sits in

$$\text{image of } \mathrm{SL}_2 \subset \mathrm{G}_2 \subset \mathrm{SO}_7 \subset \mathrm{SL}_7.$$

This new group G_2 can be determined among the four by its third and fourth moments M_3 and M_4. Recall that for a group G (given inside some $GL(V)$), its moments (with respect to the given representation V) are defined by

$$M_n(G) := M_n(G,V) := \dim((\mathrm{V}^{\otimes \mathrm{n}})^{\mathrm{G}}),$$

the dimension of the space of G-invariants in $V^{\otimes n}$. Four our four groups, M_3 is successively $1,1,0,0$, and M_4 is successively $7,4,3,2$.

In fact, in our application, we will only use M_3. Notice also that for our four possible choices, $M_3 = 1$ if and only if $M_3 > 0$.

3. The local systems

Fix a finite field k of odd characteristic p. We have the quadratic character

$$\chi_2 : k^\times \to \pm 1,$$

which we extend to all of k by defining $\chi_2(0) = 0$. Fix a nontrivial additive character

$$\psi : (k, +) \to \mu_p(\mathbb{Q}(\zeta_p)).$$

Given a polynomial $f(x) \in k[x]$ of degree $n \geq 2$ which is prime to p, we are interested in the sum

$$-\sum_{x \in k} \chi_2(x)\psi(f(x)).$$

Now fix a prime number $\ell \neq p$ and an embedding of $\mathbb{Q}(\zeta_p)$ into $\overline{\mathbb{Q}_\ell}$. Then this sum is the trace of $Frob_k$ on $H^1_c(\mathbb{A}^1_{\overline{k}}, \mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi(f(x))})$. Here $\mathcal{L}_{\chi_2(x)}$ is the Kummer sheaf (extended by 0 across $0 \in \mathbb{A}^1$) and $\mathcal{L}_{\psi(f(x))}$ is the (pullback by f of) the Artin-Schreier sheaf $\mathcal{L}_{\psi(x)}$.

If we consider these sums as we vary f by adding to it a varying linear term,

$$t \mapsto -\sum_{x \in k} \chi_2(x)\psi(f(x) + tx),$$

then we are looking at the traces, at the k-points $t \in \mathbb{A}^1(k)$, of a rank n local system on the $\mathbb{A}^1$ of t's, the Fourier Transform

$$FT(\mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi(f(x))}).$$

For a finite extension K/k, and $t \in \mathbb{A}^1(K)$, the trace is the "same" sum, now over $x \in K$, but with χ_2 replaced by $\chi_{2,K}$ the quadratic character of $K^\times$ extended by zero, and with ψ replaced by the composition $\psi \circ \mathrm{Trace}_{K/k}$.

This FT is pure of weight one, thanks to Weil. Its description as an FT shows that it is geometrically irreducible. One knows from the work of Laumon [Lau-FT, 2.4.3], cf. also [Ka-ESDE, 7.3.4 (1), (2), (3)], that its I_∞-slopes are

$$\{0, n/(n-1) \text{ repeated n} - 1 \text{ times}\}.$$

Lemma 3.1. *Suppose $n \geq 5$ is prime to p, and $f(x)$ is a polynomial of degree n. Then the geometric monodromy group G_{geom} of $FT(\mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi(f(x))})$ is not contained in the image $Sym^{n-1}(SL_2)$ of SL_2 in SL_n by its irreducible representation $Sym^{n-1}(std_2)$ of dimension n.*

Proof. If G_{geom} lies in this image, then G_{geom} has a faithful representation of dimension either 2, if n is even, or 3 if n is odd (i.e., $Sym^{n-1}(std_2)$ is faithful if n is even, and factors through a faithful representation of $SL_2/\pm 1 \cong SO_3$ if n is odd). In either case, the pushout of our FT by this representation has the same highest ∞ slope as does the FT itself [Ka-ESDE, 7.2.4].The pushout has rank ≤ 3, so its highest ∞ slope has denominator one of $1, 2, 3$, whereas the original FT has highest slope $n/(n-1)$, with denominator $n-1 > 3$. □

When n is odd and f is an odd polynomial (i.e. $f(-x) = -f(x)$), then this FT is orthogonally self dual, and its G_{geom} lies in SO_n. Moreover, after we twist by an explicit Gauss sum [Ka-NG2, 1.7], our FT will be pure of weight zero, and we will have

$$G_{geom} \subset G_{arith} \subset SO_n.$$

Here is a general fact [Ka-MG, Prop. 5] about geometrically irreducible local systems $\mathcal{F}$ on $\mathbb{A}^1_k$, a consequnce of the Feit-Thompson theorem [F-T,]. If $p > 2n+1$, then $\mathcal{F}$ is Lie-irreducible, meaning that G^0_{geom} acts irreducibly.

4. Looking for local systems whose G_{geom} is G_2

Some years ago, I proved [Ka-ESDE, 9.1.1] that with $f(x) = x^7$, in any characteristic $p \geq 17$, the FT had $G_{geom} = G_2$. A question of Rudnick and Waxman made me wonder if there were other odd, degree seven polynomials $f(x)$ for which the FT would have $G_{geom} = G_2$.

Using the exceptional identities, it turned out to be a simple matter to show that $M_3 = 1$ for the (G_{geom} of the) local system $\mathcal{F}$ on $\mathbb{A}^2$ with parameters B, t whose trace function is

$$(B,t) \in k^2 \mapsto (1/g) \sum_{x \in k} \chi_2(x)\psi(x^7/7 + 2Bx^5/5 + B^2x^3/3 + tx),$$

g being the explicit Gauss sum

$$g := g(\overline{\psi}, \chi_2) = \sum_{x \in k^\times} \psi(-x)\chi_2(x) = \chi_2(-1) \sum_{x \in k^\times} \psi(x)\chi_2(x).$$

This local system is orthogonally self dual, and [Ka-NG2, 1.7] has

$$G_{geom} \subset G_{arith} \subset SO_7.$$

Theorem 4.1. *Fix a prime $p > 7$, k a finite field of characteristic p, ψ a nontrivial additive character of k, a prime number $\ell \neq p$, and an* *embeddiing of* $\mathbb{Q}(\zeta_p)$ *into* $\overline{\mathbb{Q}_\ell}$. *Consider the* $\overline{\mathbb{Q}_\ell}$ *local system* $\mathcal{F}$ *on* $\mathbb{A}^2/k$ *with coordinates* B, t *whose trace function is*

$$(B,t) \mapsto (1/g)\sum_{x\in k}\chi_2(x)\psi(x^7/7 + 2Bx^5/5 + B^2x^3/3 + tx)$$

for $(B,t) \in k^2$, g *being the above gauss sum* $g(\overline{\psi}, \chi_2)$, *with the usual variant for a finite extension* K/k *and* $(B,t) \in K^2$ *(namely the sum is over* $x \in K$, χ_2 *is replaced by* $\chi_{2,K}$ *and* ψ *is replaced by* $\psi \circ \mathrm{Trace}_{K/k}$*). Then* $M_3 = 1$.

Proof. The local system $\mathcal{F}$ is pure of weight zero. By [De-Weil II, 3.4.1 (iii)], $\mathcal{F}$ and all its tensor powers are completely reducible as representations of G_{geom}. Therefore we have

$$M_3 = \dim(H^4_c(\mathbb{A}^2 \otimes_k \overline{k}, \mathcal{F}^{\otimes 3})(2)).$$

As explained in [Ka-LFM, the idea behind the calculation], we recover M_3 as the limsup of the archimedean absolute value of the "empirical third moment sums"

$$(1/\#k)^2\sum_{B,t\in k}((1/g)\sum_{x\in k}\chi_2(x)\psi(x^7/7 + 2Bx^5/5 + B^2x^3/3 + tx))^3 =$$

$$= (1/(g^3(\#k)^2))\sum_{B,t\in k}\sum_{x,y,z\in k}\chi_2(xyz)\times$$

$$\psi((x^7+y^7+z^7)/7+2B(x^5+y^5+z^5)/5+B^2(x^3+y^3+z^3)/3+t(x+y+z)),$$

with k replaced by larger and larger finite extensions of itself. When we sum over t, we get $\#k$ times the sum over those x, y, z with $x+y+z = 0$. Substituting $z = -x-y$, the empirical sum becomes, using the exceptional identities,

$$(1/(g^3(\#k)))\sum_{B\in k}\sum_{x,y\in k}\chi_2(F_3(x,y))\psi(F_7(x,y)+2BF_5(x,y)+B^2F_3(x,y)) =$$

$$= (1/(g^3(\#k)))\sum_{B\in k}\sum_{x,y\in k}\chi_2(F_3(x,y)\psi(F_3(x,y)(B + F_2(x,y))^2) =$$

(making the change of variable $(x,y,B) \mapsto (x,y,B - F_2(x,y))$)

$$= (1/(g^3(\#k)))\sum_{x,y,B\in k}\chi_2(F_3(x,y))\psi(F_3(x,y)B^2) =$$

$$= (1/(g^3(\#k)))\sum_{x,y\in k}\chi_2(F_3(x,y))\sum_{B\in k}\psi(F_3(x,y)B^2).$$

For fixed x, y, the $\chi_2(F_3(x,y))$ factor vanishes unless $F_3(x,y) \neq 0$. For such x, y, the inner sum over B is just the Gauss sum $\chi_2(F_3(x,y))g(\psi, \chi_2)$. So the empirical sum is

$$= (1/(g^3(\#k))) \sum_{x,y \in k, F_3(x,y) \neq 0} \chi_2(F_3(x,y))\chi_2(F_3(x,y))g(\psi, \chi_2) =$$

$$= (1/(g^3(\#k))) \sum_{x,y \in k, F_3(x,y) \neq 0} g(\psi, \chi_2).$$

The number of zeros of $F_3(x,y)$ in k^2 is $3\#k - 2$, so the empirical sum is

$$\frac{(\#k-1)(\#k-2)g(\psi,\chi_2)}{g^3(\#k)}$$

Recall that $g^2 = \chi_2(-1)\#k$, hence $g^3 = \chi_2(-1)g\#k = g(\psi,\chi_2)\#k$, so the empirical sum is

$$\frac{(\#k-1)(\#k-2)g(\psi,\chi_2)}{g(\psi,\chi_2)(\#k)^2} = \frac{(\#k-1)(\#k-2)}{(\#k)^2},$$

whose limit, as $\#k$ grows, is visibly 1. □

Theorem 4.2. *In any characteristic $p > 7$, the local system $\mathcal{F}$ on $\mathbb{A}^2/k$ of the previous theorem has $G_{geom} = G_{arith} = G_2$.*

Proof. We will show that $\mathcal{F}$ is Lie-irreducible. Admitting this temporarily, we argue as follows. We know that

$$G_{geom} \subset G_{arith} \subset SO_7.$$

We have already shown that G_{geom} has $M_3 = 1$. Therefore its identity component has a larger $M_3 \geq 1$. But as already observed, among connected irreducible subgroups of SL_7, $M_3 \geq 1$ implies $M_3 = 1$. Therefore G^0_{geom} has $M_3 = 1$, so by Gabber's theorem G^0_{geom} is either G_2 or the image of SL_2 in SO_7. Both of these groups are their own normalizers in SO_7, so we either have

$$G_{geom} = G_{arith} = \text{ the image in } SO_7 \text{ of } SL_2$$

or we have

$$G_{geom} = G_{arith} = G_2.$$

The SL_2 case is ruled out by Lemma 3.1.

It remains to show that $\mathcal{F}$ is Lie-irreducible. Consider a pullback $\mathcal{F}_{B=b_0}$ to a line $B = b_0$ in $\mathbb{A}^2$. Its G_{geom} is a subgroup of the G_{geom} for $\mathcal{F}$, so it suffices to exhibit such a pullfback which is Lie-irreducible. If $p \geq 17$, then any such pullback will be Lie-irreducible.This follows from the fact that a geometrically irreducible local system on $\mathbb{A}^1/\overline{\mathbb{F}_p}$ of rank n is Lie-irreducible if $p > 2n+1$, cf. [Ka-MG, Prop. 5], applied to our rank 7 pullback.

For $p = 11$ or $p = 13$, we first reduce to the case when $k = \mathbb{F}_p$. Fix a nontrivial additive character $\psi_{\mathbb{F}_p}$ of $\mathbb{F}_p$, and denote by $\psi_{k/\mathbb{F}_p} := \psi_{\mathbb{F}_p} \circ \mathrm{Trace}_{k/\mathbb{F}_p}$. Then $\psi(x)$ is of the form

$$\psi_{k/\mathbb{F}_p, A_0}(x) := \psi_{k/\mathbb{F}_p}(A_0 x)$$

for some $A_0 \in k^\times$. Extending scalars from k to a finite extension, we may assume A_0 is a seventh power, say $A_0 = A^7$. Our sums, for fixed b_0, are then

$$(1/g(\overline{\psi_{k/\mathbb{F}_p, A^7}}, \chi_2)) \sum_{x \in k} \chi_2(x) \psi_{k/\mathbb{F}_p}(A^7(x^7/7 + 2b_0 x^5/5 + b_0^2 x^3/3 + tx)).$$

Making the change of variable $x \mapsto x/A$, our sums becomes

$$(1/g(\overline{\psi_{k/\mathbb{F}_p}}, \chi_2)) \sum_{x \in k} \chi_2(x) \psi_{k/\mathbb{F}_p}(x^7/7 + A^2 b_0 x^5/5 + A^4 b_0^2 x^3/3 + A^6 tx)).$$

Now make the choice $b_0 = 1/A^2$. Then our sums become

$$(1/g(\overline{\psi_{k/\mathbb{F}_p}}, \chi_2)) \sum_{x \in k} \chi_2(x) \psi_{k/\mathbb{F}_p}(x^7/7 + 2x^5/5 + x^3/3 + A^6 tx)).$$

So we are looking at the multiplicative translate (by $t \mapsto A^6 t$) of the pullback from $\mathbb{A}^1/\mathbb{F}_p$ to $\mathbb{A}^1/k$ of the Fourier Transform of $(-1/g)^{deg} \mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi_{\mathbb{F}_p}(x^7/7 + 2x^5/5 + x^3/3)}$ on $\mathbb{A}^1\mathbb{F}_p$. So we are reduced to proving that this Fourier Transform is Lie-irreducible.

We apply [Ka-NG2, Lemma 3.5] to know that our Fourier Transform is either Lie-irreducible or has **finite** G_{geom}. We then apply the "low ordinal" criterion, [Ka-WVQKR, text before Lemma 7.2] and [Ka-ESDE, 8.14.3], according to which its G_{geom} cannot be finite if the single sum (the value at $t = 0$)

$$\sum_{x \in \mathbb{F}_p^\times} \chi_2(x) \psi(x^7/7 + 2x^5/5 + x^3/3)$$

has $ord_p < 1/2$. In fact, for $p = 13$, this sum has $ord_p = 2/(p-1)$, and for $p = 11$ this sum has $ord_p = 1/(p-1)$.

To see this, we calculate in the ring $\mathbb{Z}[\zeta_p]$. Define $\pi \in \mathbb{Z}[\zeta_p]$ by

$$1 + \pi = \zeta_p.$$

Then $ord_p(\pi) = 1/(p-1)$, and modulo $p\mathbb{Z}[\zeta_p]$ this sum is congruent to

$$\sum_{x \in \mathbb{F}_p^\times} x^{(p-1)/2} (1+\pi)^{x^7/7 + 2x^5/5 + x^3/3}.$$

Expanding by the binomial theorem, this sum is congruent mod π^3 to

$$\sum_{x\in\mathbb{F}_p^\times} x^{(p-1)/2}[1+(x^7/7+2x^5/5+x^3/3)\pi+Binom(x^7/7+2x^5/5+x^3/3,2)\pi^2].$$

The sum $\sum_{x\in\mathbb{F}_p^\times} x^{(p-1)/2}$ vanishes in $\mathbb{F}_p$.

If $p=13$ the coefficient of π is

$$\sum_{x\in\mathbb{F}_p^\times} x^{(p-1)/2}(x^7/7+2x^5/5+x^3/3)=$$

$$=\sum_{x\in\mathbb{F}_{13}^\times} x^6(x^7/7+2x^5/5+x^3/3)=\sum_{x\in\mathbb{F}_{13}^\times}(x^{13}/7+x^{11}/5+x^9/3),$$

which vanishes in $\mathbb{F}_p$, since each of the exponents $13,11,9$ is nonzero mod $p-1=12$. So mod π^3, our sum is

$$\pi^2\sum_{x\in\mathbb{F}_p^\times} x^{(p-1)/2}(x^7/7+2x^5/5+x^3/3)^2/2=$$

$$=\pi^2\sum_{x\in\mathbb{F}_p^\times}(x^{12}/18+2x^{14}/15+67x^{16}/525+2x^{18}/35+x^{20}/98).$$

Of the exponents $12,14,16,18,20$, only 12 is zero mod $p-1=12$, so mod π^3 our sum is

$$\pi^2\sum_{x\in\mathbb{F}_p^\times}(1/18)=5\pi^2.$$

Thus for $p=13$, our sum has $ord_p=2/(p-1)=1/6$.

If $p=11$, already the coefficient of π is

$$\sum_{x\in\mathbb{F}_p^\times} x^{(p-1)/2}(x^7/7+2x^5/5+x^3/3)=$$

$$=\sum_{x\in\mathbb{F}_{11}^\times} x^5(x^7/7+2x^5/5+x^3/3)=\sum_{x\in\mathbb{F}_{11}^\times}(x^{12/}7+2x^{10}/5+x^8/3),$$

and here, of the exponents $12,10,8$ only 10 is zero mod $p-1=10$, so mod π^2 our sum is

$$\pi\sum_{x\in\mathbb{F}_{11}^\times}(2/5)=4\pi.$$

Thus for $p=11$, our sum has $ord_p=1/(p-1)=1/10$.

This conclude the proof that $\mathcal{F}$ is Lie-irreducible. $\square$

Theorem 4.3. *Suppose that either* $p \geq 17$ *or* $p = 11$. *Then for any finite field* k *of characteristic* p, *any nontrivial additive character* ψ *of* k, *and any* $b \in k$, *the local system* $FT((-1/g)^{deg}\mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi(x^7/7+2bx^5/5+b^2x^3/3)})$, *whose trace function is*

$$t \in k \mapsto (1/g)\sum_{x \in k} \chi_2(x)\psi(x^7/7 + 2bx^5/5 + b^2x^3/3 + tx),$$

has $G_{geom} = G_{arith} = G_2$.

Proof. For $p \geq 17$, our FT is Lie-irreduclble (by the "$p > 2n + 1$" argument) and, as a pullback of $\mathcal{F}$, has $G_{geom} \subset G_{arith} \subset G_2$. Then G^0_{geom} is a connected irreducible subgroup of G_2. By Gabber's theorem, it is either G_2 or it is the image SO_3 of SL_2 in G_2 by $Sym^6(std_2)$. As both these candidates are their own normalizers in G_2, G_{geom} is either G_2 or the image of SL_2. The SL_2 case is ruled out by Lemma 3.1.

For $p = 11$, our pullback is either Lie-irreducible or has finite G_{geom} [Ka-NG2, 3.5]. which is then a finite irreducible (in the ambient seven-dimensional representation) subgroup of G_2. Moreover it is a primitive subgroup, simply because in characteristic $11 > 7$, $\mathbb{A}^1/\overline{\mathbb{F}_p}$ has no connected finite etale coverings of degree 7. Because our pullback has some strictly positive I_∞-slopes, the wild inertia group P_∞ acts nontrivially, and hence

$$11 | \#G_{geom}.$$

But the primitive finite irreducible subgroups of G_2 have been classified by Cohen-Wales [C-W, Theorem page 449], and none of them has order divisible by 11. ☐

5. Sawin's analysis of the situation in characteristic 13

The situation in characteristic $p = 13$ is more subtle, because we know that when $b = 0$, the FT in question has finite $G_{geom} = PSL(2, \mathbb{F}_{13})$, [Ka-NG2, 4.13]. However Will Sawin has proven the following theorem.

Theorem 5.1. *(***Sawin***) For any finite field* k *of characteristic* 13, *any nontrivial additive character* ψ *of* k, *and any* **nonzero** $b \in k^\times$, *the local system*

$$FT((-1/g)^{deg}\mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi(x^7/7+2bx^5/5+b^2x^3/3)}),$$

whose trace function is

$$t \in k \mapsto (1/g)\sum_{x \in k} \chi_2(x)\psi(x^7/7 + 2bx^5/5 + b^2x^3/3 + tx),$$

has $G_{geom} = G_{arith} = G_2$.

Proof. Fix $b \in k^\times$. Exactly as in the proof of Theorem 4.3 above, it suffices to show that our FT is Lie-irreducible. If not, then its G_{geom} is a finite primitive (because $13 > 7$) irreducible subroup, call it Γ, of G_2. Because our pullback has some strictly positive I_∞-slopes, the wild inertia group P_∞ acts nontrivially, and hence

$$13 | \#G_{geom}.$$

By the classification of Cohen-Wales, the only possibility for Γ is $PSL(2, \mathbb{F}_{13})$. The key point is that the order of $PSL(2, \mathbb{F}_{13})$ is not divisible by 13^2. Sawin shows that, because $b \neq 0$, the order of the image of P_∞ is divisible by 13^2. This is a special case of the following theorem of his, applied with $n = 7$ and $p = 13$. □

Theorem 5.2. *(***Sawin***)Let n be an integer $n \geq 3$, k a finite field of characteristic $p > n$, and ψ a nontrivial additive character of k. Let $f(x) \in k[x]$ be a polynomial of degree n with $f(0) = 0$ which is not of the form $\alpha x^n + \beta x$. Let χ be a (possibly trivial) multiplicative character of $k^\times$. Then the image of P_∞ in the I_∞ representation of $FT(\mathcal{L}_{\psi(f(x))} \otimes \mathcal{L}_{\chi(x)})$ has order divisible by p^2.*

Proof. At the expense of replacing f by a $k^\times$ multiple of itself, we may assume ψ comes from (by composition with the trace) a nontrivial additive character of $\mathbb{F}_p$. Let us write

$$f(x) = a_n x^n + a_{n-t} x^{n-t} + \text{lower terms},$$

with $1 \leq t \leq n-2$ and $a_{n-t} \neq 0$. Passing to a finite extension of k, we may take the n'th root of $-na_n$, say

$$-na_n = \lambda^n.$$

Making the change of variable $x \mapsto x/\lambda$, we are reduced to the case when f has the form

$$f(x) = -x^n/n - a_{n-t} x^{n-t} + \text{lower terms},$$

with some new nonzero a_{n-t} We then apply a result of Lei Fu, [Fu, part (ii) of Theorem 0.1] (his $\alpha(t)$ is our $f(x)$ and his (s, r) are our $(n, 1)$) according to which the wild part of the I_∞-representation of this FT is an explicit direct image by $-\frac{d}{dx}(f(x))$, namely it is

$$[-\frac{d}{dx}(f(x))]_\star(\mathcal{L}_{\psi(f(x) - x\frac{d}{dx}(f(x)))} \otimes \mathcal{L}_{\chi(x)} \otimes \mathcal{L}_{\chi(-(n-1)x^n/2)}).$$

Now we try to write $-\frac{d}{dx}(f(x))$ as a $n-1$'st power. We have

$$-\frac{d}{dx}(f(x)) = x^{n-1} + (n-t)a_{n-t}x^{n-1-t} + \text{lower terms} =$$

$$= x^{n-1}(1 + \frac{(n-t)a_{n-t}}{x^t} + \text{higher terms in } 1/x).$$

We wish to find a new formal parameter $1/w$ at ∞, with

$$w^{n-1} = \frac{d}{dx}(f(x)) = x^{n-1}(1 + \frac{(n-t)a_{n-t}}{x^t} + \text{higher terms in } 1/x).$$

We simply take the $n-1$'st root:

$$w := x(1 + \frac{(n-t)a_{n-t}/(n-1)}{x^t} + \text{higher terms in } 1/x).$$

In terms of w, we have

$$x = w(1 - \frac{(n-t)a_{n-t}/(n-1)}{w^t} + \text{higher terms in } 1/w).$$

We now write $f(x) - x\frac{d}{dx}(f(x))$ in terms of w. We have

$$f(x) - x\frac{d}{dx}(f(x)) = \frac{(n-1)x^n}{n} + (n-t-1)a_{n-t}x^{n-t} + \text{lower terms},$$

which in terms of w is

$$\frac{(n-1)w^n}{n}[1 - \frac{n(n-t)a_{n-t}/(n-1)}{w^t} + \text{higher terms in } 1/w] + (n-t-1)a_{n-t}w^{n-t} + \ldots$$

$$= \frac{(n-1)w^n}{n} - a_{n-t}w^{n-t} + \text{less polar at } \infty.$$

The key point is that this is of the form

$$\alpha x^n + \beta x^{n-t} + \text{less polar at } \infty$$

with both α, β nonzero.

In terms of w, then, the wild part of the I_∞-representation is (denoting by $[n-1]$ the $n-1$'st power map),

$$[n-1]_\star(\mathcal{L}_{\psi(\alpha w^n + \beta w^{n-t} + \text{less polar at } \infty)} \otimes (\text{rank one and tame at } \infty))$$

with both α, β nonzero. The image of P_∞ does not change if we pass to the $[n-1]$ pullback, which, restricted to P_∞, is the direct sum

$$\bigoplus_{\zeta \in \mu_{n-1}(\overline{k})} \mathcal{L}_{\psi(\alpha(\zeta w)^n + \beta(\zeta w)^{n-t} + \text{less polar at } \infty)}.$$

For the image of P_∞ to have order p, the polynomials $\alpha(\zeta w)^n + \beta(\zeta w)^{n-t}$, indexed by $\zeta \in \mu_{n-1}(\overline{k})$, would each need to be $\mathbb{F}_p$ multiples of $\alpha w^n + \beta w^{n-t}$. But as $1 \le t \le n-2$, if we take for ζ a primitive $n-1$'st root of unity, the two polynomials

$$\alpha w^n + \beta w^{n-t} \text{ and } \zeta^n \alpha w^n + \zeta^{n-t}\beta w^{n-t}$$

are not $\overline{k}$-proportional (simply because $\zeta^t \neq 1$). ∎

6. The situation in characteristic $p = 7, 5, 3$

For p one of $7, 5, 3$, denote by W_2 the ring scheme of p-Witt vectors of length 2. Let k be a finite field of characteristic p, and

$$\psi_2 : W_2(k) \twoheadrightarrow \mu_{p^2}(\mathbb{Z}[\zeta_{p^2}]).$$

a character of order p^2 of the additive group of $W_2(k)$. Then

$$x \in k \mapsto \psi_2(0, x) := \psi(x)$$

is a nontrivial additive character of k (and every nontrivial additive character of k is of this form).

For $p = 7$, we have the local system $\mathcal{F}$ on $\mathbb{A}^2/k$ with coordinates B, t whose trace function is

$$(B, t) \mapsto (1/g) \sum_{x \in k} \chi_2(x)\psi_2(x, 0)\psi(2Bx^5/5 + B^2x^3/3 + tx)$$

for $(B, t) \in k^2$, g being the above gauss sum $g(\overline{\psi}, \chi_2)$, with the usual variant for a finite extension K/k and $(B, t) \in K^2$ (namely the sum is over $x \in K$, χ_2 is replaced by $\chi_{2,K}$ and ψ_2 , respectively ψ are replaced by their compositions with $\mathrm{Trace}_{K/k}$ from $W_2(K)$ to $W_2(k)$, respectively from K to k).

This local system $\mathcal{F}$ is pure of weight zero, geometrically irreducible and self dual (its trace is $\mathbb{R}$-valued). As its rank, 7, is odd, the autoduality is orthogonal, and hence

$$G_{geom} \subset G_{arith} \subset O_7.$$

Theorem 6.1. *In characteristic* 7*, the local system* $\mathcal{F}$ *has* $M_3 = 1$*, and* $Frob_k$ *acts on* $H^4_c(\mathbb{A}^2 \otimes_k \overline{k}, \mathcal{F}^{\otimes 3})(2)$ *as* 1*.*

Proof. The proof that $M_3 = 1$ is identical to the proof of Theorem 4.1(the first one), using the exceptional identities. Once $M_3 = 1$, then the H^4 has dimension one, so $Frob_k$ acts on it as a unitary scalar. This scalar lies in $\mathbb{Q}(\zeta_{p^2})$ (Galois invariance of the L-function, and isolation of its highest weight part) and is an λ-adic unit for all places λ of $\mathbb{Q}(\zeta_{p^2})$ not over p. So by the product formula for $\mathbb{Q}(\zeta_{p^2})$, it is be a unit in $\mathbb{Z}[\zeta_{p^2}]$ all of whose archimedean absolute values are 1, hence is a root of unity of order dividing $2p^2$. So we can recover it as the archimedean limit of the empirical M_3 calculated over those extensions of k whose degrees over k are congruent to 1 modulo $2p^2$. The calculation of the empirical M_3 shows that this limit is 1. □

Theorem 6.2. *In characteristic* 7*, the local system* $\mathcal{F}$ *has*

$$G_{geom} = G_{arith} = G_2.$$

Proof. Suppose first that $\mathcal{F}$ is Lie-irreducible. Then (as in the proof of Theorem4.2) by Gabber's theorem, G^0_{geom} is either G_2 or $Sym^6(SL_2)$:= the image of SL_2 in SO_7. The normalizer of either of these groups G in O_7 is $\pm G$. So G_{geom} is either G_2 or $\pm G_2$, or $Sym^6(SL_2)$ or $\pm Sym^6(SL_2)$. Of these four groups, only G_2 and $Sym^6(SL_2)$ have $M_3 = 1$, the other two have $M_3 = 0$. Since $M_3 = 1$ for G_{arith}, the same argument shows that G_{arith} is either G_2 or $Sym^6(SL_2)$. Because G_{geom} is a normal subgroup of G_{arith}, we have the same dichotomy as in Theorem earlier, either

$$G_{geom} = G_{arith} = \text{ the image in } SO_7 \text{ of } SL_2$$

or we have

$$G_{geom} = G_{arith} = G_2.$$

We rule out the SL_2 case by Lemma 3.1.

It remains to show that $\mathcal{F}$ is Lie-irreducible. For this it suffices to find a pullback $\mathcal{F}_{B=b_0}$ which is Lie-irreducible. We will use the "low ordinal" method to show that $\mathcal{F}_{B=0}$ is Lie-irreducible. For this we first reduce to the case when k is $\mathbb{F}_p$. Fix a character $\psi_{2,\mathbb{F}_p}$ of $W_2(\mathbb{F}_p) \cong \mathbb{Z}/p^2\mathbb{Z}$ of order p^2, so of the form

$$x \in \mathbb{Z}/p^2\mathbb{Z} \mapsto \zeta_{p^2}^x$$

for a fixed primitive p^2'th root of unity ζ_{p^2}. We denote by $\psi_{\mathbb{F}_p}$ the attached addtive character of $\mathbb{F}_p$,

$$\psi_{\mathbb{F}_p}(x) := \psi_{2,\mathbb{F}_p}(0, x)$$

which is just $x \mapsto \zeta_p^x$ for $\zeta_p := \zeta_{p^2}^p$. We denote by $\psi_{k,\mathbb{F}_p}$ the character $\psi_{\mathbb{F}_p} \circ \mathrm{Trace}_{k/\mathbb{F}_p}$ of k.

We denote by $\psi_{2,k,\mathbb{F}_p}$ the character of $W_2(k)$ which is $\psi_{2,\mathbb{F}_p} \circ \mathrm{Trace}_{k/\mathbb{F}_p}$. For a unique element $(\alpha, \beta) \in W_2(k)^\times$, the character ψ_2 is of the form

$$(x, y) \mapsto \psi_{2,k,\mathbb{F}_p}((\alpha, \beta)(x, y)).$$

In Witt vector multiplication, we have

$$(\alpha, \beta)(x, y) = (\alpha x, \beta x^p + \alpha^p y).$$

The trace function of the pullback sheaf $\mathcal{F}_{B=0}$ is

$$t \in k \mapsto (1/g) \sum_{x \in k} \chi_2(x)\psi_2(x, 0)\psi(tx) = (1/g) \sum_{x \in k} \chi_2(x)\psi_2(x, tx) =$$

$$= (1/g) \sum_{x \in k} \chi_2(x)\psi_{2,k,\mathbb{F}_p}((\alpha, \beta)(x, tx)) = (1/g) \sum_{x \in k} \chi_2(x)\psi_{2,k,\mathbb{F}_p}(\alpha x, \beta x^p + \alpha^p tx).$$

Replacing k if necessary by its quadratic extension, we may further assume that α is a square in $k^\times$. After the change of variable $x \mapsto x/\alpha$, the trace function becomes

$$(1/g)\sum_{x\in k}\chi_2(x/\alpha)\psi_{2,k,\mathbb{F}_p}(x,(\beta/\alpha^p)x^p+\alpha^{p-1}tx) =$$

$$= (1/g)\sum_{x\in k}\chi_2(x)\psi_{2,k,\mathbb{F}_p}(x,0)\psi_{2,k,\mathbb{F}_p}(0,(\beta^{1/p}/\alpha)+\alpha^{p-1}t)x),$$

which is the the pullback by an affine transformation on the t-line of

$$t \mapsto (1/g)\sum_{x\in k}\chi_2(x)\psi_{2,k,\mathbb{F}_p}(x,0)\psi_{2,k,\mathbb{F}_p}(0,tx).$$

This is the trace function of the pullback to $\mathbb{A}^1/k$ of the corresponding Fourier Transform on $\mathbb{A}^1/\mathbb{F}_p$.

When k is $\mathbb{F}_p$, we use the "low ordinal" method. It suffices to show that the sum

$$\sum_{x\in\mathbb{F}_p}\chi_2(x)\psi_{2,\mathbb{F}_p}(x,0)$$

has $ord_p < 1/2$. This sum, the "Gauss-Heilbron sum", is

$$\sum_{x=1}^{p-1}\chi_2(x)\zeta_{p^2}^{x^p}.$$

If we write

$$\zeta_{p^2} = 1+\pi_{p^2}, \quad \zeta_{p^2}^p = 1+\pi_p,$$

then our sum is congruent, modulo $\pi_p\mathbb{Z}[\zeta_{p^2}]$, to

$$\sum_{x=1}^{p-1}x^{(p-1)/2}(1+\pi_{p^2})^x.$$

Expanding $(1+\pi_{p^2})^x$ by the binomial theorem, we see that this last sum, modulo $p\mathbb{Z}_p[\zeta_{p^2}]$, starts in degree $(p-1)/2$ as a series in π_{p^2}, so has $ord_p = 1/(2p) < 1/2$. This concludes the proof that $\mathcal{F}$ is Lie-irreducible in characteristic $p = 7$. □

Theorem 6.3. *Let k be a finite field of characteristic $p = 7$, and ψ_2 an additive character of $W_2(k)$ of order p^2. For any $b \in k$, the pullback local system $\mathcal{F}_{B=b}$ on $\mathbb{A}^1/k$, whose trace function is*

$$t \in k \mapsto (1/g)\sum_{x\in k}\chi_2(x)\psi_2(x,0)\psi(2bx^5/5+b^2x^3/3+tx),$$

has

$$G_{geom} = G_{arith} = G_2.$$

Proof. This pullback, being itself a Fourier transform, is geometrically irreducible, and its ∞-slopes are

$$\{0, 7/6 \text{ repeated 6 times}\},$$

cf. the discussion preceeding Lemma 3.1. Being a pullback of $\mathcal{F}$, it has

$$G_{geom} \subset G_{arith} \subset G_2.$$

Admit for the moment that this pullback is Lie-irreducible. Then by Gabber's theorem, G_{geom} is either G_2 or it is $Sym^6(SL_2)$. The second possibility is ruled out by Lemma 3.1.

It remains to show that our pullback is Lie-irreducible. If not, its G_{geom} is a finite irreducible subgroup of G_2, whose order must be divisible by 7 (because it has some ∞-slopes which are > 0). From the Cohen-Wales classification, we see that there are no finite irreducible subgroup of G_2 whose order is divisible by 7^2. So it suffices to show that the image of the wild inertia group P_∞ has order divisible by 7^2. To see this, denote by M the wild part of the I_∞-representation of our pullback. We apply [Ka-GKM, 1.14] with its $(a, n) = (7, 6)$ in characteristic 7 to conclude that

$$M = [n]_\star V$$

for a one-dimensional representation V of I_∞ whose Swan conductor is $p = 7$. In characteristic p, for any one-dimensional representation of I_∞ of Swan conductor p, its restriction to P_∞ has order p^2 (and, more generally, if the Swan conductor is strictly positive and has $ord_p(\text{Swan}) = r$, then its restriction to P_∞ has order p^{r+1}). Therefore V is a direct summand of

$$[n]^\star M = [n]^\star [n]_\star V = \bigoplus_{\zeta \in \mu_n(\overline{k})} [x \mapsto \zeta x]^\star V.$$

But the image of P_∞ on M is the same as its image on $[n]^\star M$. This last image has order divisible by p^2, this already being true for the direct factor V. □

We now turn to the situation in characteristic $p = 5$. We fix a finite field k of characrteristic $p = 5$, and a character ψ_2 of order p^2 of the additive group of $W_2(k)$. We denote by $\mathcal{F}$ the local system on $\mathbb{A}^2/k$ with coordinates (B, t) whose trace function is given by

$$(B, t) \in \mathbb{A}^2(k) \mapsto (1/g) \sum_{x \in k} \chi_2(x) \psi(x^7/7 + B^{2p} x^3/3 + tx) \psi_2(2Bx, 0).$$

Theorem 6.4. *In characteristic $p = 5$, the local system $\mathcal{F}$ has*

$$G_{geom} = G_{arith} = G_2.$$

Proof. We first observe that we have $G_{arith} \subset SO_7$. Indeed, $\mathcal{F}$ is geometrically irreducible (becase any pullback $\mathcal{F}_{B=b_0}$ is, being a Fourier Transform), orthogonally self dual (real trace, odd rank), so its determinant, being lisse on $\mathbb{A}^2/k$ of order two, must be geometrically constant. So it suffices to check for the pullback $\mathcal{F}_{B=0}$, and here we invoke [Ka-NG2, 1.7]. We then show that $M_3 = 1$. This results from the exceptional identities, with the slight diffference that what previously had been the term $(B + F_2(x, y))^2$ here becomes $(B^p + F_2(x, y))^2$, In the sum over (B, x, y), we can replace B^p by B, and proceed as in the proof of Theorem 4.1.

It then remains only to show that $\mathcal{F}$ is Lie-irreducible. For this, it suffices to show that the pullback $\mathcal{F}_{B=0}$ is Lie-irreducible. This is shown in [Ka-NG2, 4.12]. □

Theorem 6.5. *In characteristic $p = 5$, for any $b \in k$, the pullback local system $\mathcal{F}_{B=b}$ on $\mathbb{A}^1/k$, whose trace function is*

$$t \in k \mapsto (1/g) \sum_{x \in k} \chi_2(x)\psi(x^7/7 + b^{2p}x^3/3 + tx)\psi_2(2bx, 0).$$

has

$$G_{geom} = G_{arith} = G_2.$$

Proof. Exactly as in the proof of Theorem 6.3 (the p=7 case), we need only rule out the possibility that G_{geom} is a finite irreducible subgroup of G_2. From the wild inertia at ∞, this finite irreducible subgroup of G_2 would have order divisible by $p = 5$, The Cohen-Wales classification shows there are no such subgroups. □

We now turn to the situation in characteristic $p = 3$. We fix a finite field k pf characteristic $p = 3$, and a character ψ_2 of order p^2 of the additive group of $W_2(k)$. We denote by $\mathcal{F}$ the local system on $\mathbb{A}^2/k$ with coordinates (B, t) whose trace function is given by

$$(B, t) \in \mathbb{A}^2(k) \mapsto (1/g) \sum_{x \in k} \chi_2(x)\psi(x^7/7 + 2B^p x^5/5 + tx)\psi_2(B^2 x, 0).$$

Just as in the proof of Theorem 6.4, we see that $G_{arith} \subset SO_7$, and, using the exceptional identities, that $M_3 = 1$.

Theorem 6.6. *For $p = 3$,the local system $\mathcal{F}$ on $\mathbb{A}^2/k$ has*

$$G_{geom} = G_{arith} = G_2.$$

Proof. As we have seen above, it suffices to show that $\mathcal{F}$ is Lie-irreducible. For this, it suffices to exhibit a pullback which is Lie-irreducible. For this, we first reduce to the case when k is the prime field $\mathbb{F}_3$. Just as in the proof of Theorem 6.2, we choose a character $\psi_{2,\mathbb{F}_p}$ of order p^2 of the additive group of $W_2(\mathbb{F}_p) \cong \mathbb{Z}/9\mathbb{Z}$. We denote by $\psi_{2,k,\mathbb{F}_3}$ the character of $W_2(k)$ obtained by composition with the trace. Similarly, we denote by $\psi_{k,\mathbb{F}_p}$ the additive character of k obtained from $x \mapsto \psi_{2,\mathbb{F}_p}(0,x)$ by composition with the trace.

For a unique element $(\alpha_0, \beta) \in W_2(k)^\times$, the given character ψ_2 is of form

$$\psi_2(x,y) = \psi_{2,k,\mathbb{F}_p}((\alpha_0,\beta)(x,y)) = \psi_{2,k,\mathbb{F}_p}(\alpha_0 x, \alpha_0^p y + \beta x^p).$$

At the expense of replacing k by a finite extension, we may assume that α_0 is itself a seventh power, say

$$\alpha_0 = \alpha^7,$$

and that α itself is a square in $k^\times$. Then our local system has trace function

$$(B,t) \mapsto (1/g)\sum_{x\in k} \chi_2(x)\psi_{2,\mathbb{F}_p,k}((\alpha^7,\beta)(B^2x, x^7/7 + 2B^p x^5/5 + tx) =$$

$$= (1/g)\sum_{x\in k} \chi_2(x)\psi_{2,\mathbb{F}_p,k}(\alpha^7 B^2 x, 0)\psi_{2,k,\mathbb{F}_p}(0, \alpha^{7p}(x^7/7{+}2B^p x^5/5{+}tx){+}\beta B^{2p}x^p).$$

After the change of variable $x \mapsto x/\alpha^p = x/\alpha^3$, this sum becomes

$$(1/g)\sum_{x\in k} \chi_2(x)\psi_{2,\mathbb{F}_p,k}(\alpha^4 B^2 x, 0)\psi_{k,\mathbb{F}_p}(x^7/7{+}2\alpha^6 B^3 x^5/5{+}((\alpha^{6p}t{+}\beta^{1/p}B^2)x).$$

Now choose $B = 1/\alpha^2$. Then we have the pullback, by the affine linear transformation $t \mapsto \alpha^{6p}t + \beta^{1/p}B^2$ of t, of the Fourier Transform of the pullback from $\mathbb{A}^1/\mathbb{F}_3$ to $\mathbb{A}^1/k$ of

$$(-1/g)^{deg}\mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi_{2,\mathbb{F}_3}(x,0)} \otimes \mathcal{L}_{\psi_{\mathbb{F}_3}(x^7/7+2x^5/5)}.$$

To see that this is Lie-irreducible, we use the "low ordinal" method. It suffices to show that at $t = 0$, our sum

$$\sum_{x\in\mathbb{F}_3} \chi_2(x)\psi(x^7/7 + 2x^5/5)\psi_2(x,0)$$

has $ord_p < 1/2$. This sum has only two terms: it is

$$\chi_2(1)\psi(1/7+2/5)\psi_2(1,0) \quad + \quad \chi_2(-1)\psi(-1/7-2/5)\psi_2(-1,0) =$$
$$= \zeta_3^2\zeta_9 - \zeta_3^{-2}\zeta_9^{-1} =$$
$$= \zeta_9^7 - \zeta_9^{-7} = \zeta_9^7 - \zeta_9^2 = -\zeta_9^2(1-\zeta_9^5),$$

which has $ord_3 = 1/6 < 1/2$. □

It is proven in [Ka-NG2, 4.15] that for for $b = 0$, the pullback $\mathcal{F}_{B=0}$ on$\mathbb{A}^1/k$ has finite $G_{geom} = U_3(3)$ in Atlas [CCNPW-Atlas] notation.

Theorem 6.7. *For any finite field k of characteristic $p = 3$, any additive character ψ_2 of $W_2(k)$, and any* **nonzero** *$b \neq 0$ in $k^\times$, the pullback sheaf $\mathcal{F}_{B=b}$, whose trace function is*

$$t \in k \mapsto (1/g) \sum_{x \in k} \chi_2(x) \psi(x^7/7 + 2b^3x^5/5 + tx) \psi_2(b^2x, 0),$$

has

$$G_{geom} = G_{arith} = G_2.$$

Proof. As above, it suffices to prove that such a pullback is Lie-irreducible. If not, its G_{geom} is a finite irreducible subgroup of G_2. The wild part of its I_∞-representation has rank six, with all six slopes $= 7/6$. Because $p = 3$ divides the rank 6, the restriction to the wild inertia group P_∞ is the direct sum of two three-dimensional irreducible representations of P_∞, cf. [Ka-GKM, 1.14]. The image of P_∞ in either of these representations is a p-group, whose order must be at least p^3, simply because groups of order p or p^2 are abelian. Therefore if G_{geom} is finite, its order is divisible by $p^3 = 3^3$. In the Cohen-Wales classification of finite irreducible subroups of G_2, only $U_3(3)$ and $G_2(2)$ have orders divisible by 3^3. The group $G_2(2)$ cannot occur, because it contains $U_3(3)$ as a normal subgroup of index 2, so admits a surjective homomorphism to $\mathbb{Z}/2Z$. By pre-composing with the surjection of $\pi_1(\mathbb{A}^1/\overline{\mathbb{F}_3})$ onto G_{geom}, we would obtain $\mathbb{Z}/2\mathbb{Z}$ as a quotient of $\pi_1(\mathbb{A}^1/\overline{\mathbb{F}_3})s$, which is nonsense. Thus if G_{geom} is finite, it is $U_3(3)$. Moreover, the normalizer of $U_3(3)$ in G_2 is $G_2(2)$, so if G_{geom} is finite, then G_{arith} is either $U_3(3)$ or it is $G_2(2)$.

The unique orthogonal seven-dimensional irreducible representation of $U_3(3)$ has integer traces, as do both orthogonal seven-dimensional irreducible representations of $G_2(2)$. So if G_{geom} is finite, then all the traces of our pullback are integers. In particular, they all lie in $\mathbb{Q}(\zeta_3)$ (rather than in the larger field $\mathbb{Q}(\zeta_9)$ which obviously contains them). This will lead to a congradiction, as follows.

The galois group of $Q(\zeta_9)/\mathbb{Q}(\zeta_3)$ is the cyclic group of order three generated by $\zeta_9 \mapsto \zeta_9^4$. In $W_2(\mathbb{F}_3) \cong \mathbb{Z}/9\mathbb{Z}$, the element $4 \in \mathbb{Z}/9\mathbb{Z}$ is the Witt vector $(1, 1)$. So the image of the trace at time $t \in k$,

$$(1/g) \sum_{x \in k} \chi_2(x) \psi(x^7/7 + 2b^3x^5/5 + tx) \psi_2(b^2x, 0),$$

under the automorphism $\zeta_9 \mapsto \zeta_9^4$ is simultaneously equal to itself (because it lies in $\mathbb{Q}(\zeta_3)$) and equal to

$$(1/g) \sum_{x \in k} \chi_2(x) \psi(x^7/7 + 2b^3x^5/5 + tx) \psi_2((1, 1)(b^2x, 0)) =$$

$$= (1/g) \sum_{x \in k} \chi_2(x) \psi(x^7/7 + 2b^3x^5/5 + tx) \psi_2(b^2x, b^6x^3) =$$

$$= (1/g) \sum_{x \in k} \chi_2(x) \psi(x^7/7 + 2b^3x^5/5 + tx + b^6x^3) \psi_2(b^2x, 0) =$$

$$= (1/g) \sum_{x \in k} \chi_2(x) \psi(x^7/7 + 2b^3x^5/5 + (t + b^2)x) \psi_2(b^2x, 0).$$

This is the trace function of the additive translation $t \mapsto t + b^2$ of our pullback. By Chebotarev, this pullback, being arithmetically irreducible, is isomorphic to it additive translate by $t \mapsto t + b^2$. In particular, this pullback is geometrically isomorphic to its additive translate by $t \mapsto t + b^2$. On the Fourier Transform side, this says that

$$\mathcal{K} := \mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi(x^7/7+2b^3x^5/5)} \otimes \mathcal{L}_{\psi_2(b^2x,0)}$$

is geometrically isomorphic on $\mathbb{G}_m/\overline{\mathbb{F}_3}$ to

$$\mathcal{K} \otimes \mathcal{L}_{\psi(b^2x)} = \mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi(x^7/7+2b^3x^5/5+b^2x)} \otimes_{\psi_2(b^2x,0))} .$$

This says that $\mathcal{L}_{\psi(b^2x)}$ is geometrically constant on $\mathbb{G}_m/\overline{\mathbb{F}_3}$, which is nonsense, as it has Swan conductor one at ∞. □

7. An Open Question

In characteristic $p \geq 17$, suppose $f_{B,C}(x) := x^7/7 + 2Bx^5/5 + Cx^3/3$ is a polynomial such that the G_{geom} of the Fourier Transform of $\mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi(f_{B,C}(x))}$ is G_2. Is it true that $C = B^2$?

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