ANOTHER LOOK AT THE DWORK FAMILY

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dedicated to Yuri Manin on his seventieth birthday

1. Introduction and a bit of history

After proving [Dw-Rat] the rationality of zeta functions of all algebraic varieties over finite fields nearly fifty years ago, Dwork studied in detail the zeta function of a nonsingular hypersurface in projective space, cf. [Dw-Hyp1] and [Dw-HypII]. He then developed his "deformation theory", cf. [Dw-Def], [Dw-NPI] and [Dw-NPII], in which he analyzed the way in which his theory varied in a family. One of his favorite examples of such a family, now called the Dwork family, was the one parameter (λ) family, for each degree $n \geq 2$, of degree n hypersurfaces in $\mathbb{P}^{n-1}$ given by the equation

$$\sum_{i=1}^{n} X_i^n - n\lambda \prod_{i=1}^{n} X_i = 0,$$

a family he wrote about explicitly in [Dw-Def, page 249, (i),(ii),(iv), the cases $n = 2, 3, 4$], [Dw-HypII, section 8, pp. 286-288, the case $n = 3$] and [Dw-PC, 6.25, the case $n = 3$, and 6.30, the case $n = 4$]. Dwork of course also considered the generalization of the above Dwork family consisting of single-monomial deformations of Fermat hypersurfaces of any degree and dimension. He mentioned one such example in [Dw-Def, page 249, (iii)]. In [Dw-PAA, pp. 153-154], he discussed the general single-monomial deformation of a Fermat hypersurface, and explained how such families led to generalized hypergeometric functions.

My own involvement with the Dwork family started (in all senses!) at the Woods Hole conference in the summer of 1964 with the case $n = 3$, when I managed to show in that special case that the algebraic aspects of Dwork's deformation theory amounted to what would later be called the Gauss-Manin connection on relative de Rham cohomology, but which at the time went by the more mundane name of "differentiating cohomology classes with respect to parameters".

That this article is dedicated to Manin on his seventieth birthday is particularly appropriate, because in that summer of 1964 my reference for the notion of differentiating cohomology classes with respect to parameters was his 1958 paper [Ma-ACFD]. I would also like to take this opportunity to thank, albeit belatedly, Arthur Mattuck for many helpful conversations that summer.

I discussed the Dwork family in [Ka-ASDE, 2.3.7.17-23, 2.3.8] as a "particularly beautiful family", and computed explicitly the differential equation satisfied by the cohomology class of the holomorphic $n-2$ form. It later showed up in [Ka-SE, 5.5, esp. pp. 188-190], about which more below. Ogus [Ogus-GTCC, 3.5, 3.6] used the Dwork family to show the failure in general of "strong divisibility". Stevenson, in her thesis [St-th],[St, end of section 5, page 211], discussed single-monomial deformations of Fermat hypersurfaces of any degree and dimension. Koblitz [Kob] later wrote on these same families. With mirror symmetry and the stunning work of Candelas et al [C-dlO-G-P] on the case $n = 5$, the Dwork family became widely known, especially in the physics community, though its occurence in Dwork's work was almost (not entirely, cf. [Ber], [Mus-CDPMQ]) forgotten. Recently the Dwork family turned out to play a key role in the proof of the Sato-Tate conjecture (for elliptic curves over $\mathbb{Q}$ with non-integral j-invariant), cf. [H-SB-T, section 1, pp. 5-15].

The present paper gives a new approach to computing the local system given by the cohomology of the Dwork family, and more generally of families of single-monomial deformations of Fermat hypersurfaces. This approach is based upon the surprising connection, noted in [Ka-SE, 5.5, esp. pp. 188-190], between such families and Kloosterman sums. It uses also the theory, developed later, of Kloosterman sheaves and of hypergeometric sheaves, and of their behavior under Kummer pullback followed by Fourier Transform, cf. [Ka-GKM] and [Ka-ESDE, esp. 9.2 and 9.3]. In a recent preprint, Rojas-Leon and Wan [RL-Wan] have independently implemented the same approach.

2. The situation to be studied: generalities

We fix an integer $n \geq 2$, a degree $d \geq n$, and an n-tuple $W = (w_1, ..., w_n)$ of strictly positive integers with $\sum_i w_i = d$, and with $gcd(w_1, ..., w_n) = 1$. This data (n, d, W) is now fixed. Let R be a ring in which d is invertible.

Over R we have the affine line $\mathbb{A}^1_R := Spec(R[\lambda])$. Over $\mathbb{A}^1_R$, we consider certain one parameter (namely λ) families of degree d hypersurfaces in $\mathbb{P}^{n-1}$. Given an $n+1$-tuple $(a, b) := (a_1, ..., a_n, b)$ of invertible elements in R, we consider the one parameter (namely λ) family of degree d hypersurfaces in $\mathbb{P}^{n-1}$,

$$X_\lambda(a,b) : \sum_{i=1}^{n} a_i X_i^d - b\lambda X^W = 0,$$

where we have written

$$X^W := \prod_{i=1}^{n} X_i^{w_i}.$$

More precisely, we consider the closed subscheme $\mathbb{X}(a,b)_R$ of $\mathbb{P}^{n-1}_R \times_R \mathbb{A}^1_R$ defined by the equation

$$\sum_{i=1}^{n} a_i X_i^d - b\lambda X^W = 0,$$

and denote by

$$\pi(a,b)_R : \mathbb{X}(a,b)_R \to \mathbb{A}^1_R$$

the restriction to $\mathbb{X}(a,b)_R$ of the projection of $\mathbb{P}^{n-1}_R \times_R \mathbb{A}^1_R$ onto its second factor.

Lemma 2.1. *The morphism*

$$\pi(a,b)_R : \mathbb{X}(a,b)_R \to \mathbb{A}^1_R$$

is lisse over the open set of $\mathbb{A}^1_R$ *where the function*

$$(b\lambda/d)^d \prod_i (w_i/a_i)^{w_i} - 1$$

is invertible.

Proof. Because d and the a_i are invertible in R, a Fermat hypersurface of the form

$$\sum_{i=1}^{n} a_i X_i^d = 0$$

is lisse over R. When we intersect our family with any coordinate hyperplane $X_i = 0$, we obtain a constant Fermat family in one lower dimension (because each $w_i \geq 1$). Hence any geometric point $(x, \lambda) \in \mathbb{X}$ at which π is not smooth has all coordinates X_i invertible. So the locus of nonsmoothness of π is defined by the simultaneous vanishing of all the $X_i d/dX_i$, i.e., by the simultaneous equations

$$d a_i X_i^d = b\lambda w_i X^W, \ \ for\ i = 1, ..., n.$$

Divide through by the invertible factor da_i. Then raise both sides of the i'th equation to the w_i power and multiply together right and left sides separately over i. We find that at a point of nonsmoothness we have

$$X^{dW} = (b\lambda/d)^d \prod_i (w_i/a_i)^{w_i} X^{dW}.$$

As already noted, all the X_i are invertible at any such point, and hence

$$1 = (b\lambda/d)^d \prod_i (w_i/a_i)^{w_i}$$

at any geometric point of nonsmoothness. □

In the Dwork family *per se*, all $w_i = 1$. But in a situation where there is a prime p not dividing $d\ell$ but dividing one of the w_i, then taking for R an $\mathbb{F}_p$-algebra (or more generally a ring in which p is nilpotent), we find a rather remarkable family.

Corollary 2.2. *Let p be a prime which is prime to d but which divides one of the w_i, and R a ring in which p is nilpotent. Then the morphism*

$$\pi(a,b)_R : \mathbb{X}(a,b)_R \to \mathbb{A}^1_R$$

is lisse over all of $\mathbb{A}^1_R$

Remark 2.3. Already the simplest possible example of the above situation, the family in $\mathbb{P}^1/\mathbb{F}_q$ given by

$$X^{q+1} + Y^{q+1} = \lambda XY^q,$$

is quite interesting. In dehomogenized form, we are looking at

$$x^{q+1} - \lambda x + 1$$

as polynomial over $\mathbb{F}_q(\lambda)$; its Galois group is known to be $PSL(2, \mathbb{F}_q)$, cf. [Abh-PP, bottom of p. 1643], [Car], and [Abh-GTL, Serre's Appendix]. The general consideration of "$p|w_i$ for some i" families in higher dimension would lead us too far afield, since our principal interest here is with families that "start life" over $\mathbb{C}$. We discuss briefly such "$p|w_i$ for some i" families in Appendix II. We would like to call the attention of computational number theorists to these families, with no degeneration at finite distance, as a good test case for proposed methods of computing efficiently zeta functions in entire families.

3. The particular situation to be studied: details

Recall that the data (n, d, W) is fixed. Over any ring R in which $d\prod_i w_i$ is invertible, we have the family $\pi : \mathbb{X} \to \mathbb{A}^1_R$ given by

$$X_\lambda := X_\lambda(W, d) : \sum_{i=1}^n w_i X_i^d - d\lambda X^W = 0;$$

it is proper and smooth over the open set $U := \mathbb{A}^1_R[1/(\lambda^d - 1)] \subset \mathbb{A}^1_R$ where $\lambda^d - 1$ is invertible.

The most natural choice of R, then, is $\mathbb{Z}[1/(d\prod_i w_i)]$. However, it will be more convenient to work over a somewhat larger cyclotomic ring, which contains, for each i, all the roots of unity of order dw_i. Denote by $lcm(W)$ the least common multiple of the w_i, and define $d_W := lcm(W)d$. In what follows, we will work over the ring

$$R_0 := \mathbb{Z}[1/d_W][\zeta_{d_W}] := \mathbb{Z}[1/d_W][T]/(\Phi_{d_W}(T)),$$

where $\Phi_{d_W}(T)$ denotes the d_W'th cyclotomic polynomial.

We now introduce the relevant automorphism group of our family. We denote by $\mu_d(R_0)$ the group of d'th roots of unity in R_0, by $\Gamma = \Gamma_{d,n}$ the n-fold product group $(\mu_d(R_0))^n$, by $\Gamma_W \subset \Gamma$ the subgroup consisting of all elements $(\zeta_1, ..., \zeta_n)$ with $\prod_{i=1}^n \zeta_i^{w_i} = 1$, and by $\Delta \subset \Gamma_W$ the diagonal subgroup, consisting of all elements of the form $(\zeta, ..., \zeta)$. The group Γ_W acts as automorphisms of $\mathbb{X}/\mathbb{A}^1_{R_0}$, an element $(\zeta_1, ..., \zeta_n)$ acting as

$$((X_1, ..., X_n), \lambda) \mapsto ((\zeta_1 X_1, ..., \zeta_n X_n), \lambda).$$

The diagonal subgroup Δ acts trivially.

The natural pairing

$$(\mathbb{Z}/d\mathbb{Z})^n \times \Gamma \to \mu_d(R_0) \subset R_0^\times,$$
$$(v_1, ..., v_n) \times (\zeta_1, ..., \zeta_n) \to \prod_i \zeta_i^{v_i},$$

identifies $(\mathbb{Z}/d\mathbb{Z})^n$ as the R_0-valued character group $D\Gamma := Hom_{group}(\Gamma, R_0^\times)$. The subgroup

$$(\mathbb{Z}/d\mathbb{Z})^n_0 \subset (\mathbb{Z}/d\mathbb{Z})^n$$

consisting of elements $V = (v_1, ..., v_n)$ with $\sum_i v_i = 0 \ in\ \mathbb{Z}/d\mathbb{Z}$ is then the R_0-valued character group $D(\Gamma/\Delta)$ of Γ/Δ. The quotient group $(\mathbb{Z}/d\mathbb{Z})^n_0/ < W >$ of $(\mathbb{Z}/d\mathbb{Z})^n_0$ by the subgroup generated by (the image, by reduction mod d, of) W is then the R_0-valued character group $D(\Gamma_W/\Delta)$ of Γ_W/Δ.

For G either of the groups Γ/Δ or Γ_W/Δ, an R_0-linear action of G on a sheaf of R_0-modules M gives an eigendecomposition

$$M = \oplus_{\rho \in D(G)} M(\rho).$$

If the action is by the larger group $G = \Gamma/\Delta$, then $DG = (\mathbb{Z}/d\mathbb{Z})^n_0$, and for $V \in (\mathbb{Z}/d\mathbb{Z})^n_0$ we denote by $M(V)$ the corresponding eigenspace. If the action is by the smaller group Γ_W/Δ, then DG is the quotient group $(\mathbb{Z}/d\mathbb{Z})^n_0/ < W >$; given an element $V \in (\mathbb{Z}/d\mathbb{Z})^n_0$, we denote by $V \ mod\ W$ its image in the quotient group, and we denote by $M(V \ mod\ W)$ the corresponding eigenspace.

If M is given with an action of the larger group Γ/Δ, we can decompose it for that action:

$$M = \oplus_{V \in (\mathbb{Z}/d\mathbb{Z})^n_0} M(V).$$

If we view this same M only as a representation of the sugroup Γ_W/Δ, we can decompose it for that action:

$$M = \oplus_{V \in (\mathbb{Z}/d\mathbb{Z})^n_0/<W>} M(V \ mod\ W).$$

The relation between these decompositions is this: for any element $V \in (\mathbb{Z}/d\mathbb{Z})^n_0$,

$$M(V \ mod\ W) = \oplus_{r \ mod\ d} M(V + rW).$$

We return now to our family $\pi : \mathbb{X} \to \mathbb{A}^1_{R_0}$, which we have seen is (projective and) smooth over the open set

$$U = \mathbb{A}^1_{R_0}[1/(\lambda^d - 1)].$$

We choose a prime number ℓ, and an embedding of R_0 into $\overline{\mathbb{Q}}_\ell$. [We will now need to invert ℓ, so arguably the most efficient choice is to take for ℓ a divisor of d_W.] We We form the sheaves

$$\mathcal{F}^i := R^i\pi_\star\overline{\mathbb{Q}}_\ell$$

on $\mathbb{A}^1_{R_0[1/\ell]}$. They vanish unless $0 \le i \le 2(n-2)$, and they are all lisse on $U[1/\ell]$. By the weak Lefschetz Theorem and Poincaré duality, the sheaves $\mathcal{F}^i|U[1/\ell]$ for $i \neq n-2$ are completely understood. They vanish for odd i; for even $i = 2j \le 2(n-2), i \neq n-2$, they are the Tate twists

$$\mathcal{F}^{2j}|U[1/\ell] \cong \overline{\mathbb{Q}}_\ell(-j).$$

We now turn to the lisse sheaf $\mathcal{F}^{n-2}|U[1/\ell]$. It is endowed with an autoduality pairing (cup product) toward $\overline{\mathbb{Q}}_\ell(-(n-2))$ which is symplectic if $n-2$ is odd, and orthogonal if $n-2$ is even. If $n-2$ is even, say $n-2 = 2m$, then $\mathcal{F}^{n-2}|U[1/\ell]$ contains $\overline{\mathbb{Q}}_\ell(-m)$ as a direct summand (m'th power of the hyperplane class from the ambient $\mathbb{P}$) with nonzero self-intersection. We define $Prim^{n-2}$ (as a sheaf on $U[1/\ell]$ only) to be the annihilator in $\mathcal{F}^{n-2}|U[1/\ell]$ of this $\overline{\mathbb{Q}}_\ell(-m)$ summand under the cup product pairing. So we have

$$\mathcal{F}^{n-2}|U[1/\ell] = Prim^{n-2} \oplus \overline{\mathbb{Q}}_\ell(-m),$$

when $n-2 = 2m$. When $n-2$ is odd, we define $Prim^{n-2} := \mathcal{F}^{n-2}|U[1/\ell]$, again as a sheaf on $U[1/\ell]$ only.

The group Γ_W/Δ acts on our family, so on all the sheaves above. For $i \neq n-2$, it acts trivially on $\mathcal{F}^i|U[1/\ell]$. For $i = n-2 = 2m$ even, it respects the decomposition

$$\mathcal{F}^{n-2}|U[1/\ell] = Prim \oplus \overline{\mathbb{Q}}_\ell(-m),$$

and acts trivially on the second factor.

We thus decompose $Prim^{n-2}$ into eigensheaves $Prim^{n-2}(V\ mod\, W)$. The basic information on the eigensheaves $Prim^{n-2}(V\ mod\ W)$ is encoded in elementary combinatorics of the coset $V\ mod\, W$. An element $V = (v_1, ..., v_n) \in (\mathbb{Z}/d\mathbb{Z})^n_0$ is said to be totally nonzero if $v_i \neq 0$ for all i. Given a totally nonzero element $V \in (\mathbb{Z}/d\mathbb{Z})^n_0$, we define its degree, $deg(V)$ as follows. For each i, denote by $\tilde{v}_i$ the unique integer $1 \leq \tilde{v}_i \leq d-1$ which mod d gives v_i. Then $\sum_i \tilde{v}_i$ is 0 mod d, and we define

$$deg(V) := (1/d)\sum_i \tilde{v}_i.$$

Thus $deg(V)$ lies in the interval $1 \leq deg(V) \leq n-1$. The Hodge type of a totally nonzero $V \in (\mathbb{Z}/d\mathbb{Z})^n_0$ is defined to be $HdgType(V) := (n-1-deg(V), deg(V)-1)$.

We now compute the rank and the the Hodge numbers of eigensheaves $Prim^{n-2}(V\ mod\ W)$. We have already chosen an embedding of R_0 into $\overline{\mathbb{Q}}_\ell$. We now choose an embedding of $\overline{\mathbb{Q}}_\ell$ into $\mathbb{C}$. The composite embedding $R_0 \subset \mathbb{C}$ allows us to extend scalars in our family $\pi : \mathbb{X} \to \mathbb{A}^1_{R_0}$, which is projective and smooth over the open set $U_{R_0} = \mathbb{A}^1_{R_0}[1/(\lambda^d - 1)]$, to get a complex family $\pi_\mathbb{C} : \mathbb{X}_\mathbb{C} \to \mathbb{A}^1_\mathbb{C}$, which is projective and smooth over the open set $U_\mathbb{C} = \mathbb{A}^1_\mathbb{C}[1/(\lambda^d - 1)]$. Working in the classical complex topology with the corresponding analytic spaces, we can form the higher direct image sheaves $R^i\pi^{an}_\mathbb{C}\mathbb{Q}$ on $\mathbb{A}^{1,an}_\mathbb{C}$, whose restrictions to $U^{an}_\mathbb{C}$ are locally constant sheaves. We can also form the locally constant sheaf $Prim^{n-2,an}(\mathbb{Q})$ on $U^{an}_\mathbb{C}$. Extending scalars in the coefficients from $\mathbb{Q}$ to $\overline{\mathbb{Q}}_\ell$, we get the sheaf $Prim^{n-2,an}(\overline{\mathbb{Q}}_\ell)$. On the other hand, we have the lisse $\overline{\mathbb{Q}}_\ell$-sheaf $Prim^{n-2}$ on $U_{R_0[1/\ell]}$, which we can pull back, first to $U_\mathbb{C}$, and then to $U^{an}_\mathbb{C}$. By the fundamental comparison theorem, we have

$$Prim^{n-2,an}(\overline{\mathbb{Q}}_\ell) \cong Prim^{n-2}|U^{an}_\mathbb{C}.$$

Extending scalars from $\overline{\mathbb{Q}}_\ell$ to $\mathbb{C}$, we find

$$Prim^{n-2,an}(\mathbb{C}) \cong (Prim^{n-2}|U^{an}_\mathbb{C}) \otimes_{\overline{\mathbb{Q}}_\ell} \mathbb{C}.$$

This is all Γ_W/Δ-equivariant, so we have the same relation for individual eigensheaves:

$$Prim^{n-2,an}(\mathbb{C})(V\ mod\ W) \cong (Prim^{n-2}(V\ mod\ W)|U^{an}_\mathbb{C}) \otimes_{\overline{\mathbb{Q}}_\ell} \mathbb{C}.$$

If we extend scalars on $U^{an}_{\mathbb{C}}$ from the constant sheaf $\mathbb{C}$ to the sheaf $\mathcal{O}_{\mathcal{C}^\infty}$, then the resulting $\mathcal{C}^\infty$ vector bundle $Prim^{n-2,an}(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{C}^\infty}$ has a Hodge decomposition,

$$Prim^{n-2,an}(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{C}^\infty} = \bigoplus_{a\geq 0, b\geq 0, a+b=n-2} Prim^{a,b}.$$

This decomposition is respected by the action of Γ_W/Δ, so we get a Hodge decomposition of each eigensheaf:

$$Prim^{n-2,an}(\mathbb{C})(V \; mod \, W) \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{C}^\infty} = \bigoplus_{a\geq 0, b\geq 0, a+b=n-2} Prim^{a,b}(V \; mod \, W).$$

Lemma 3.1. *We have the following results.*

(1) *The rank of the lisse sheaf $Prim^{n-2}(V \; mod \; W)$ on $U_{R_0[1/\ell]}$ is given by*

$$rk(Prim^{n-2}(V \; mod \, W)) = \#\{r \in \mathbb{Z}/d\mathbb{Z} \mid V + rW \textit{ is totally nonzero}\}.$$

In particular, the eigensheaf $Prim^{n-2}(V \; mod \; W)$ vanishes if none of the W-translates $V + rW$ is totally nonzero.

(2) *For each (a, b) with $a \geq 0, b \geq 0, a + b = n - 2$, the rank of the $\mathcal{C}^\infty$ vector bundle $Prim^{a,b}(V \; mod \; W)$ on $U^{an}_{\mathbb{C}}$ is given by*

$$rk(Prim^{a,b}(V \; mod \; W))$$

$$= \#\{r \in \mathbb{Z}/d\mathbb{Z} \mid V + rW \textit{ is totally nonzero and } deg(V + rW) = b + 1\}.$$

Proof. To compute the rank of a lisse sheaf on $U_{R_0[1/\ell]}$, or the rank of a $\mathcal{C}^\infty$ vector bundle on $U^{an}_{\mathbb{C}}$, it suffices to compute its rank at a single geometric point of the base. We take the $\mathbb{C}$-point $\lambda = 0$, where we have the Fermat hypersurface. Here the larger group $(\mathbb{Z}/d\mathbb{Z})^n_0$ operates. It is well known that under the action of this larger group, the eigenspace $Prim(V)$ vanishes unless V is totally nonzero, e.g., cf. [Ka-IMH, section 6]. One knows further that if V is totally nonzero, this eigenspace is one-dimensional, and of Hodge type $HdgType(V) := (n - 1 - deg(V), deg(V) - 1)$, cf. [Grif-PCRI, 5.1 and 10.8]. ☐

The main result of this paper is to describe the eigensheaves

$$Prim^{n-2}(V \; mod \; W)$$

as lisse sheaves on $U[1/\ell]$, i.e., as representations of $\pi_1(U[1/\ell])$, and to describe the direct image sheaves $j_{U\star}(Prim^{n-2}(V \; mod \; W))$ on $\mathbb{A}^1_{R_0[1/\ell]}$, for $j_U : U[1/\ell] \subset \mathbb{A}^1_{R_0[1/\ell]}$ the inclusion. The description will be in terms of hypergeometric sheaves in the sense of [Ka-ESDE, 8.7.11].

4. Interlude: Hypergeometric sheaves

We first recall the theory in its original context of finite fields, cf. [Ka-ESDE, Chapter 8]. Let k be an $R_0[1/\ell]$-algebra which is a finite field, and $\psi : (k, +) \to \overline{\mathbb{Q}}_\ell^\times$ a nontrivial additive character. Because k is an $R_0[1/\ell]$- algebra, it contains d_W distinct d_W'th roots of unity, and the structural map gives a group isomorphism $\mu_{d_W}(R_0) \cong \mu_{d_W}(k)$. So raising to the $\#k^\times/d_W$'th power is a surjective group homomorphism

$$k^\times \to \mu_{d_W}(k) \cong \mu_{d_W}(R_0).$$

So for any character $\chi : \mu_{d_W}(R_0) \to \mu_{d_W}(R_0)$, we can and will view the composition of χ with the above surjection as defining a multiplicative character of $k^\times$, still denoted χ. Every multiplicative character of $k^\times$ of order dividing d_W is of this form. Fix two non-negative integers a and b, at least one of which is nonzero. Let $\chi_1, ..., \chi_a$ be an unordered list of a multiplicative characters of $k^\times$ of order dividing d_W, some possibly trivial, and not necessarily distinct. Let $\rho_1, ..., \rho_b$ be another such list, but of length b. Assume that these two lists are disjoint, i.e., no χ_i is a ρ_j. Attached to this data is a geometrically irreducible middle extension $\overline{\mathbb{Q}}_\ell$-sheaf

$$\mathcal{H}(\psi; \chi_i\,'s; \rho_j\,'s)$$

on $\mathbb{G}_m/k$, which is pure of weight $a+b-1$. We call it a hypergeometric sheaf of type (a, b). If $a \neq b$, this sheaf is lisse on $\mathbb{G}_m/k$; if $a = b$ it is lisse on $\mathbb{G}_m - \{1\}$, with local monodromy around 1 a tame pseudoreflection of determinant $(\prod_j \rho_j)/(\prod_i \chi_i)$.

The trace function of $\mathcal{H}(\psi; \chi_i\,'s; \rho_j\,'s)$ is given as follows. For E/k a finite extension field, denote by ψ_E the nontrivial additive character of E obtained from ψ by composition with the trace map $Trace_{E/k}$, and denote by $\chi_{i,E}$ (resp. $\rho_{j,E}$) the multiplicative character of E obtained from χ_i (resp. ρ_j) by composition with the norm map $Norm_{E/k}$. For $t \in \mathbb{G}_m(E) = E^\times$, denote by $V(a, b, t)$ the hypersurface in $(\mathbb{G}_m)^a \times (\mathbb{G}_m)^b/E$, with coordinates $x_1, ..., x_a, y_1, ..., y_b$, defined by the equation

$$\prod_i x_i = t \prod_j y_j.$$

Then

$$Trace(Frob_{t,E}|\mathcal{H}(\psi; \chi_i\,'s; \rho_j\,'s))$$

$$= (-1)^{a+b-1} \sum_{V(n,m,t)(E)} \psi_E(\sum_i x_i - \sum_j y_j) \prod_i \chi_{i,E}(x_i) \prod_j \overline{\rho}_{j,E}(y_j).$$

In studying these sheaves, we can always reduce to the case $a \geq b$, because under multiplicative inversion we have

$$inv^\star \mathcal{H}(\psi; \chi_i\ 's; \rho_j\ 's)) \cong \mathcal{H}(\overline{\psi}; \overline{\rho}_j\ 's; \overline{\chi}_i\ 's)).$$

If $a \geq b$, the local monodromy around 0 is tame, specified by the list of χ_i's: the action of a generator γ_0 of I_0^{tame} is the action of T on the $\overline{\mathbb{Q}}_\ell[T]$-module $\overline{\mathbb{Q}}_\ell[T]/(P(T))$, for $P(T)$ the polynomial

$$P(T) := \prod_i (T - \chi_i(\gamma_0)).$$

In other words, for each of the distinct characters χ on the list of the $\chi_i's$, there is a single Jordan block, whose size is the multiplicity with which χ appears on the list. The local monodromy around ∞ is the direct sum of a tame part of dimension b, and, if $a > b$, a totally wild part of dimension $a - b$, all of whose upper numbering breaks are $1/(a-b)$. The b-dimensional tame part of the local monodromy around ∞ is analogously specified by the list of ρ's: the action of a generator γ_∞ of I_∞^{tame} is the action of T on the $\overline{\mathbb{Q}}_\ell[T]$-module $\overline{\mathbb{Q}}_\ell[T]/(Q(T))$, for $Q(T)$ the polynomial

$$Q(T) := \prod_j (T - \rho_j(\gamma_0)).$$

When $a = b$, there is a canonical constant field twist of the hypergeometric sheaf $\mathcal{H} = \mathcal{H}(\psi; \chi_i\ 's; \rho_j\ 's)$ which is independent of the auxiliary choice of ψ, which we will call $\mathcal{H}^{can}$. We take for $A \in \overline{\mathbb{Q}}_\ell^\times$ the nonzero constant

$$A = (\prod_i (-g(\psi, \chi_i))(\prod_j (-g(\overline{\psi}, \overline{\rho_j})),$$

and define

$$\mathcal{H}^{can} := \mathcal{H} \otimes (1/A)^{deg}.$$

[That $\mathcal{H}^{can}$ is independent of the choice of ψ can be seen in two ways. By elementary inspection, its trace function is independent of the choice of ψ, and we appeal to Chebotarev. Or we can appeal to the rigidity of hypergeometric sheaves with given local monodromy, cf. [Ka-ESDE, 8.5.6], to infer that with given χ's and ρ's, the hypergeometric sheaves $\mathcal{H}_\psi^{can}$ with different choices of ψ are all geometrically isomorphic. Being geometrically irreducible as well, they must all be constant field twists of each other. We then use the fact that $H^1(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{H}_\psi^{can})$ is one dimensional, and that $Frob_k$ acts on it by the scalar 1, to see that the constant field twist is trivial.]

Here is the simplest example. Take $\chi \neq \rho$, and form the hypergeometric sheaf $\mathcal{H}^{can}(\psi;\chi;\rho)$. Then using the rigidity approach, we see that

$$\mathcal{H}^{can}(\psi;\chi;\rho) \cong \mathcal{L}_{\chi(x)} \otimes \mathcal{L}_{(\rho/\chi)(1-x)} \otimes (1/A)^{deg},$$

with A (minus) the Jacobi sum over k,

$$A = -J(k;\chi,\rho/\chi) := -\sum_{x \in k^\times} \chi(x)(\rho/\chi)(1-x).$$

The object

$$\mathcal{H}(\chi,\rho) := \mathcal{L}_{\chi(x)} \otimes \mathcal{L}_{(\rho/\chi)(1-x)}$$

makes perfect sense on $\mathbb{G}_m/R_0[1/\ell]$, cf. [Ka-ESDE, 8.17.6]. By [We-JS], attaching to each maximal ideal $\mathcal{P}$ of R_0 the Jacobi sum $-J(R_0/\mathcal{P};\chi,\rho/\chi)$ over its residue field is a grossencharacter, and so by [Se-ALR, Chapter 2] a $\overline{\mathbb{Q}}_\ell$-valued character, call it $\Lambda_{\chi,\rho/\chi}$, of $\pi_1(Spec(R_0[1/\ell])$. So we can form

$$\mathcal{H}^{can}(\chi,\rho) := \mathcal{H}(\chi,\rho) \otimes (1/\Lambda_{\chi,\rho/\chi})$$

on $\mathbb{G}_m/R_0[1/\ell]$. For any $R_0[1/\ell]$-algebra k which is a finite field, its pullback to $\mathbb{G}_m/k$ is $\mathcal{H}^{can}(\psi;\chi;\rho)$.

This in turn allows us to perform the following global construction. Suppose we are given an integer $a > 0$, and two unordered lists of characters,$\chi_1, ..., \chi_a$ and $\rho_1, ..., \rho_a$, of the group $\mu_{d_W}(R_0)$ with values in that same group. Assume that the lists are disjoint. For a fixed choice of orderings of the lists, we can form the sheaves $\mathcal{H}^{can}(\chi_i,\rho_i)$, $i = 1, ..., a$ on $\mathbb{G}_m/R_0[1/\ell]$. We can then define, as in [Ka-ESDE, 8.17.11], the ! multiplicative convolution

$$\mathcal{H}^{can}(\chi_1,\rho_1)[1] \star_! \mathcal{H}^{can}(\chi_2,\rho_2)[1] \star_! ... \star_! \mathcal{H}^{can}(\chi_a,\rho_a)[1],$$

which will be of the form $\mathcal{F}[1]$ for some sheaf $\mathcal{F}$ on $\mathbb{G}_m/R_0[1/\ell]$ which is "tame and adapted to the unit section". This sheaf $\mathcal{F}$ we call $\mathcal{H}^{can}(\chi_i\,'s;\rho_j\,'s)$. For any $R_0[1/\ell]$-algebra k which is a finite field, its pullback to $\mathbb{G}_m/k$ is $\mathcal{H}^{can}(\psi;\chi_i\,'s;\rho_j\,'s)$. By Chebotarev, the sheaf $\mathcal{H}^{can}(\psi;\chi_i\,'s;\rho_j\,'s)$ is, up to isomorphism, independent of the orderings that went into its definition as an interated convolution. This canonical choice (as opposed to, say, the ad hoc construction given in [Ka-ESDE, 8.17.11], which *did depend* on the orderings) has the property that, denoting by

$$f : \mathbb{G}_m/R_0[1/\ell] \to Spec(R_0[1/\ell])$$

the structural map, the sheaf $R^1 f_! \mathcal{H}^{can}(\chi_i\,'s;\rho_j\,'s)$ on $Spec(R_0[1/\ell])$ is the constant sheaf, i.e., it is the trivial one-dimensional representation of $\pi_1(Spec(R_0[1/\ell]))$.

If the unordered lists $\chi_1, ..., \chi_a$ and $\rho_1, ..., \rho_b$ are not disjoint, but not identical, then we can "cancel" the terms in common, getting shorter disjoint lists. The hypergeometric sheaf we form with these shorter, disjoint "cancelled" lists we denote $\mathcal{H}(\psi; \mathbf{Cancel}(\chi_i\ 's; \rho_j\ 's))$, cf. [Ka-ESDE, 9.3.1], where this was denoted $\mathbf{Cancel}\mathcal{H}(\psi; \chi_i\ 's; \rho_j\ 's)$. If $a = b$, then after cancellation the shorter disjoint lists still have the same common length, and so we can form the constant field twist $\mathcal{H}^{can}(\psi; \mathbf{Cancel}(\chi_i\ 's; \rho_j\ 's))$. And in the global setting, we can form the object $\mathcal{H}^{can}(\mathbf{Cancel}(\chi_i\ 's; \rho_j\ 's))$ on $\mathbb{G}_m/R_0[1/\ell]$.

5. Statement of the main theorem

We continue to work with the fixed data (n, d, W). Given an element $V = (v_1, ..., v_n) \in (\mathbb{Z}/d\mathbb{Z})^n_0$, we attach to it an unordered list $List(V, W)$ of $d = \sum_i w_i$ multiplicative characters of $\mu_{d_W}(R_0)$, by the following procedure. For each index i, denote by χ_{v_i} the character of $\mu_{d_W}(R_0)$ given by

$$\zeta \mapsto \zeta^{(v_i/d)d_W}.$$

Because w_i divides d_W/d, this characterχ_{v_i} has w_i distinct w_i'th roots. We then define

$$List(V, W) = \{all\ w'_1th\ roots\ of\ \chi_{v_1}, ..., all\ w'_nth\ roots\ of\ \chi_{v_n}\}.$$

We will also need the same list, but for $-V$, and the list

$$List(all\ d) := \{all\ characters\ of\ order\ dividing\ d\}.$$

So long as the two lists $List(-V, W)$ and $List(all\ d)$ are not identical, we can apply the **Cancel** operation, and form the hypergeometric sheaf

$$\mathcal{H}_{V,W} := \mathcal{H}^{can}(\mathbf{Cancel}(List(all\ d); List(-V, W)))$$

on $\mathbb{G}_m/R_0[1/\ell]$.

Lemma 5.1. *If $Prim^{n-2}(V\ mod\ W)$ is nonzero, then the unordered lists $List(-V, W)$ and $List(all\ d)$ are not identical.*

Proof. If $Prim^{n-2}(V\ mod\ W)$ is nontrivial, then at least one choice of V in the coset $V\ mod\ W$ is totally nonzero. For such a totally nonzero V, the trivial character is absent from $List(-V, W)$. If we choose another representative of the same coset, say $V - rW$, then denoting by χ_r the character of order dividing d of $\mu_{d_W}(R_0)$ given by $\zeta \mapsto \zeta^{(r/d)d_W}$, we see easily that $List(-(V - rW), W) = \chi_r List(-V, W)$. Hence the character χ_r is absent from $List(-V + rW, W)$. $\square$

Lemma 5.2. *Suppose that $Prim^{n-2}(V\ mod\ W)$ is nonzero. Then $Prim^{n-2}(V\ mod\ W)$ and $[d]^\star\mathcal{H}_{V,W}$ have the same rank on $U_{R_0[1/\ell]}$.*

Proof. Choose V in the coset $V\ mod\, W$. The rank of $Prim^{n-2}(V\ mod\, W)$ is the number of $r \in \mathbb{Z}/d\mathbb{Z}$ such that $V + rW$ is totally nonzero. Equivalently, this rank is $d - \delta$, for δ the number of $r \in \mathbb{Z}/d\mathbb{Z}$ such that $V + rW$ fails to be totally nonzero. On the other hand, the rank of $\mathcal{H}_{V,W}$ is $d - \epsilon$, for ϵ the number of elements in $List(all\ d)$ which also appear in $List(-V, W)$. Now a given character χ_r in $List(all\ d)$ appears in $List(-V, W)$ if and only if there exists an index i such that χ_r is a w_i'th root of χ_{-v_i}, i.e., such that $\chi_r^{w_i} = \chi_{-v_i}$, i.e., such that $rw_i \equiv -v_i\ mod\ d$. □

Theorem 5.3. *Suppose that $Prim^{n-2}(V\ mod\ W)$ is nonzero. Denote by $j_1 : U_{R_0[1/\ell]} \subset \mathbb{A}^1_{R_0[1/\ell]}$ and $j_2 : \mathbb{G}_{m,R_0[1/\ell]} \subset \mathbb{A}^1_{R_0[1/\ell]}$ the inclusions, and by $[d] : \mathbb{G}_{m,R_0[1/\ell]} \to \mathbb{G}_{m,R_0[1/\ell]}$ the d'th power map. Then for any choice of V in the coset $V\ mod\ W$, there exists a continuous character $\Lambda_{V,W} : \pi_1(Spec(R_0[1/\ell])) \to \overline{\mathbb{Q}}_\ell^\times$ and an isomorphism of sheaves on $\mathbb{A}^1_{R_0[1/\ell]}$,*

$$j_{1\star}Prim^{n-2}(V\ mod\ W) \cong j_{2\star}[d]^\star\mathcal{H}_{V,W} \otimes \Lambda_{V,W}.$$

Remark 5.4. What happens if we change the choice of V in the coset $V\ mod\ W$, say to $V - rW$? As noted above, $List(-(V - rW), W) = \chi_r List(-V, W)$. As $List(all\ d) = \chi_r List(all\ d)$ is stable by multiplication by any character of order dividing d, we find [Ka-ESDE, 8.2.5] that $\mathcal{H}_{V-rW,W} \cong \mathcal{L}_{\chi_r} \otimes \mathcal{H}_{V,W} \otimes \Lambda$, for some continuous character $\Lambda : \pi_1(Spec(R_0[1/\ell])) \to \overline{\mathbb{Q}}_\ell^\times$. Therefore the pullback $[d]^\star\mathcal{H}_{V,W}$ is, up to tensoring with a character Λ of $\pi_1(Spec(R_0[1/\ell]))$, independent of the particular choice of V in the coset $V\ mod\, W$. Thus the truth of the theorem is independent of the particular choice of V.

Question 5.5. There should be a universal recipe for the character $\Lambda_{V,W}$ which occurs in Theorem 5.3. For example, if we look at the Γ_W/Δ-invariant part, both $Prim^{n-2}(0\ mod\ W)$ and $\mathcal{H}_{0,W}$ are pure of the same weight $n - 2$, and both have traces (on Frobenii) in $\mathbb{Q}$. So the character $\Lambda_{0,W}$ must take $\mathbb{Q}$-values of weight zero on Frobenii in large characteristic. [To make this argument legitimate, we need to be sure that over every sufficiently large finite field k which is an $R_0[1/\ell]$-algebra, the sheaf $Prim^{n-2}(0\ mod\ W)$ has nonzero trace at some k-point. This is in fact true, in virtue of Corollary 8.7 and a standard equidistribution argument.] But the only rational numbers of weight zero are ± 1. So $\Lambda_{0,W}^2$ trivial. Is $\Lambda_{0,W}$ itself trivial?

6. Proof of the main theorem: the strategy

Let us admit for a moment the truth of the following characteristic p theorem, which will be proven in the next section.

Theorem 6.1. *Let k be an $R_0[1/\ell]$-algebra which is a finite field, and $\psi : (k,+) \to \overline{\mathbb{Q}}_\ell^\times$ a nontrivial additive character. Suppose that $Prim^{n-2}(V \bmod W)$ is nonzero. Denote by $j_{1,k} : U_k \subset \mathbb{A}^1_k$ and $j_{2,k} : \mathbb{G}_{m,k} \subset \mathbb{A}^1_k$ the inclusions. Choose V in the coset $V \bmod W$, and put*

$$\mathcal{H}_{V,W,k} := \mathcal{H}^{can}(\psi; \mathbf{Cancel}(; List(all\ d); List(-V, W))).$$

Then on $\mathbb{A}^1_k$ the sheaves $j_{1,k\star}Prim^{n-2}(V \bmod W)$ and $j_{2,k\star}[d]^\star\mathcal{H}_{V,W,k}$ are geometrically isomorphic, i.e., they become isomorphic on $\mathbb{A}^1_{\overline{k}}$.

We now explain how to deduce the main theorem. The restriction to $U_{R_0} - \{0\} = \mathbb{G}_{m,R_0} - \mu_d$ of our family

$$X_\lambda : \sum_{i=1}^n w_i X_i^d = d\lambda X^W$$

is the pullback, through the d'th power map, of a projective smooth family over $\mathbb{G}_m - \{1\}$, in a number of ways. Here is one way to write down such a descent $\pi_{desc} : \mathbb{Y} \to \mathbb{G}_m - \{1\}$. Use the fact that $gcd(w_1, ..., w_n) = 1$ to choose integers $(b_1, ..., b_n)$ with $\sum_i b_i w_i = 1$. Then in the new variables

$$Y_i := \lambda^{b_i} X_i$$

the equation of X_λ becomes

$$\sum_{i=1}^n w_i \lambda^{-db_i} Y_i^d = dY^W.$$

Then the family

$$Y_\lambda : \sum_{i=1}^n w_i \lambda^{-b_i} Y_i^d = dY^W$$

is such a descent. The same group Γ_W/Δ acts on this family. On the base $\mathbb{G}_m - \{1\}$, we have the lisse sheaf $Prim^{n-2}_{desc}$ for this family, and its eigensheaves $Prim^{n-2}_{desc}(V \bmod W)$, whose pullbacks $[d]^\star Prim^{n-2}_{desc}(V \bmod W)$ are the sheaves $Prim^{n-2}(V \bmod W)|(\mathbb{G}_{m,R_0} - \mu_d)$.

Lemma 6.2. *Let k be an $R_0[1/\ell]$-algebra which is a finite field. Suppose $Prim^{n-2}_{desc}(V \bmod W)$ is nonzero. Then there exists a choice of V in the coset $V \bmod W$ such that the lisse sheaves $Prim^{n-2}_{desc}(V \bmod W)$ and*

$\mathcal{H}_{V,W,k}$ *on* $\mathbb{G}_{m,k} - \{1\}$ *are geometrically isomorphic, i.e., isomorphic on* $\mathbb{G}_{m,\overline{k}} - \{1\}$.

Proof. Fix a choice of V in the coset $V\ mod\ W$. By Theorem 6.1, the lisse sheaves $[d]^\star Prim^{n-2}_{desc}(V\ mod\ W)$ and $[d]^\star \mathcal{H}_{V,W,k}$ are isomorphic on $\mathbb{G}_{m,\overline{k}} - \mu_d$. Taking direct image by $[d]$ and using the projection formula, we find an isomorphism

$$\bigoplus_{\chi\ with\ \chi^d\ trivial} \mathcal{L}_\chi \otimes Prim^{n-2}_{desc}(V\ mod\ W) \cong \bigoplus_{\chi\ with\ \chi^d\ trivial} \mathcal{L}_\chi \otimes \mathcal{H}_{V,W,k}$$

of lisse sheaves $\mathbb{G}_{m,\overline{k}} - \{1\}$. The right hand side is completely reducible, being the sum of d irreducibles. Therefore the left hand side is completely reducible, and each of its d nonzero summands $\mathcal{L}_\chi \otimes Prim^{n-2}_{desc}(V\ mod\ W)$ must be irreducible (otherwise the left hand side is the sum of more than d irreducibles). By Jordan-Hölder, the summand $Prim^{n-2}_{desc}(V\ mod\ W)$ on the left is isomorphic to one of the summands $\mathcal{L}_\chi \otimes \mathcal{H}_{V,W,k}$ on the right, say to the summand $\mathcal{L}_{\chi_r} \otimes \mathcal{H}_{V,W,k}$. As explained in Remark 5.3, this summand is geometrically isomorphic to $\mathcal{H}_{V-rW,W,k}$. □

Lemma 6.3. *Suppose* $Prim^{n-2}_{desc}(V\ mod\ W)$ *is nonzero. Choose an* $R_0[1/\ell]$*-algebra* k *which is a finite field, and choose* V *in the coset* $V\ mod\ W$ *such that the lisse sheaves* $Prim^{n-2}_{desc}(V\ mod\ W)$ *and* $\mathcal{H}_{V,W,k}$ *on* $\mathbb{G}_{m,k} - \{1\}$ *are geometrically isomorphic. Then there exists a continuous character* $\Lambda_{V,W} : \pi_1(Spec(R_0[1/\ell])) \to \overline{\mathbb{Q}}_\ell^\times$ *and an isomorphism of lisse sheaves on* $\mathbb{G}_{m,R_0[1/\ell]} - \{1\}$,

$$Prim^{n-2}_{desc}(V\ mod\ W) \cong \mathcal{H}_{V,W} \otimes \Lambda_{V,W}.$$

This is an instance of the following general phenomenon, which is well known to the specialists. In our application, the S below is $Spec(R_0[1/\ell])$, C is $\mathbb{P}^1$, and D is the union of the three everywhere disjoint sections $0, 1, \infty$. We will also use it a bit later when D is the union of the $d+2$ everywhere disjoint sections $0, \mu_d, \infty$.

Theorem 6.4. *Let* S *be a reduced and irreducible normal noetherian* $\mathbb{Z}[1/\ell]$*-scheme whose generic point has characteristic zero. Let* $\overline{s}$ *be a chosen geometric point of* S. *Let* C/S *be a proper smooth curve with geometrically connected fibres, and let* $D \subset C$ *be a Cartier divisor which is finite étale over* S. *Let* $\mathcal{F}$ *and* $\mathcal{G}$ *be lisse* $\overline{\mathbb{Q}}_\ell$*-sheaves on* $C - D$. *Then we have the following results.*

1. *Denote by* $j : C - D \subset C$ *and* $i : D \subset C$ *the inclusions. Then the formation of* $j_\star\mathcal{F}$ *on* C *commutes with arbitrary change of base* $T \to S$, *and* $i^\star j_\star\mathcal{F}$ *is a lisse sheaf on* D.

(2) *Denoting by* $f : C - D \to S$ *the structural map, the sheaves* $R^i f_! \mathcal{F}$ *on* S *are lisse.*
(3) *The sheaves* $R^i f_\star \mathcal{F}$ *on* S *are lisse, and their formation commutes with arbitrary change of base* $T \to S$.
(4) *Consider the pullbacks* $\mathcal{F}_{\overline{s}}$ *and* $\mathcal{G}_{\overline{s}}$ *of* $\mathcal{F}$ *and of* $\mathcal{G}$ *to* $C_{\overline{s}} - D_{\overline{s}}$. *Suppose that* $\mathcal{F}_{\overline{s}} \cong \mathcal{G}_{\overline{s}}$, *and that* $\mathcal{G}_{\overline{s}}$ *(and hence also* $\mathcal{F}_{\overline{s}}$*) are irreducible. Then there exists a continuous character* $\Lambda : \pi_1(S) \to \overline{\mathbb{Q}}_\ell^\times$ *an isomorphism of lisse sheaves on* $C - D$,

$$\mathcal{G} \otimes \Lambda \cong \mathcal{F}.$$

Proof. The key point is that because the base S has generic characteristic zero, any lisse sheaf on $C - D$ is automatically tamely ramified along the divisor D; this results from Abhyankar's Lemma. See [Ka-SE, 4.7] for assertions (1) and (2). Assertion (3) results from (2) by Poincaré duality, cf. [De-CEPD, Corollaire, p. 72].

To prove assertion (4), we argue as follows. By the Tame Specialization Theorem [Ka-ESDE, 8.17.13], the geometric monodromy group attached to the sheaf $\mathcal{F}_{\overline{s}}$ is, up to conjugacy in the ambient $GL(rk(\mathcal{F}), \overline{\mathbb{Q}}_\ell)$, independent of the choice of geometric point $\overline{s}$ of S. Since $\mathcal{F}_{\overline{s}}$ is irreducible, it follows that $\mathcal{F}_{\overline{s_1}}$ is irreducible, for every geometric point $\overline{s_1}$ of S. Similarly, $\mathcal{G}_{\overline{s_1}}$ is irreducible, for every geometric point $\overline{s_1}$ of S. Now consider the lisse sheaf $\underline{Hom}(\mathcal{G}, \mathcal{F}) \cong \mathcal{F} \otimes \mathcal{G}^\vee$ on $C - D$. By assertion (3), the sheaf $f_\star \underline{Hom}(\mathcal{G}, \mathcal{F})$ is lisse on S, and its stalk at a geometric point $\overline{s_1}$ of S is the group $\mathrm{Hom}(\mathcal{G}_{\overline{s_1}}, \mathcal{F}_{\overline{s_1}})$. At the chosen geometric point $\overline{s}$, this Hom group is one-dimensional, by hypothesis. Therefore the lisse sheaf $f_\star \underline{Hom}(\mathcal{G}, \mathcal{F})$ on S has rank one. So at every geometric point $\overline{s_1}$, $\mathrm{Hom}(\mathcal{G}_{\overline{s_1}}, \mathcal{F}_{\overline{s_1}})$ is one-dimensional. As source and target are irreducible, any nonzero element of this Hom group is an isomorphism, and the canonical map

$$\mathcal{G}_{\overline{s_1}} \otimes \mathrm{Hom}(\mathcal{G}_{\overline{s_1}}, \mathcal{F}_{\overline{s_1}}) \to \mathcal{F}_{\overline{s_1}}$$

is an isomorphism. Therefore the canonical map of lisse sheaves on $C - D$

$$\mathcal{G} \otimes f^\star f_\star \underline{Hom}(\mathcal{G}, \mathcal{F}) \to \mathcal{F}$$

is an isomorphism, as we see looking stalkwise. Interpreting the lisse sheaf $f_\star \underline{Hom}(\mathcal{G}, \mathcal{F})$ on S as a character Λ of $\pi_1(S)$, we get the asserted isomorphism. ☐

Applying this result, we get Lemma 6.3. Now pull back the isomorphism of that lemma by the d'th power map, to get an isomorphism

$$Prim^{n-2}(V \ mod \ W) \cong [d]^\star \mathcal{H}_{V,W} \otimes \Lambda_{V,W}$$

of lisse sheaves on $\mathbb{G}_{m,R_0[1/\ell]} - \mu_d$. Then extend by direct image to $\mathbb{A}^1_{R_0[1/\ell]}$ to get the isomorphism asserted in Theorem 5.3.

7. Proof of Theorem 6.1

Let us recall the situation. Over the ground ring $R_0[1/\ell]$, we have the family $\pi : \mathbb{X} \to \mathbb{A}^1$ given by

$$X_\lambda := X_\lambda(W,d) : \sum_{i=1}^{n} w_i X_i^d - d\lambda X^W = 0,$$

which is projective and smooth over $U = \mathbb{A}^1 - \mu_d$. We denote by $V \subset \mathbb{X}$ the open set where X^W is invertible, and by $Z \subset \mathbb{X}$ the complementary reduced closed set, defined by the vanishing of X^W. As scheme over $\mathbb{A}^1$, Z/A^1 is the constant scheme with fibre

$$(X^W = 0) \cap (\sum_i w_i X_i^d = 0).$$

The group Γ_W/Δ, acting as $\mathbb{A}^1$-automorphisms of $\mathbb{X}$, preserves both the open set V and its closed complement Z. In the following discussion, we will repeatedly invoke the following general principle, which we state here before proceeding with the analysis of our particular situation.

Lemma 7.1. *Let S be a noetherian $\mathbb{Z}[1/\ell]$-scheme, and $f : X \to S$ a separated morphism of finite type. Suppose that a finite group G acts admissibly (:= every point lies in a G-stable affine open set) as S-automorphisms of X. Then in $D^b_c(S, \overline{\mathbb{Q}}_\ell)$, we have a direct sum decomposition of $Rf_!\overline{\mathbb{Q}}_\ell$ into G-isotypical components*

$$Rf_!\overline{\mathbb{Q}}_\ell = \bigoplus_{\text{irred. } \overline{\mathbb{Q}}_\ell \text{ rep.'s } \rho \text{ of } G} Rf_!\overline{\mathbb{Q}}_\ell(\rho).$$

Proof. Denote by $h : X \to Y := X/G$ the projection onto the quotient, and denote by $m : Y \to S$ the structural morphism of Y/S. Then $Rh_!\overline{\mathbb{Q}}_\ell = h_\star\overline{\mathbb{Q}}_\ell$ is a constructible sheaf of $\overline{\mathbb{Q}}_\ell[G]$ modules on Y, so has a G-isotypical decomposition

$$Rh_!\overline{\mathbb{Q}}_\ell = h_\star\overline{\mathbb{Q}}_\ell = \bigoplus_{\text{irred. } \overline{\mathbb{Q}}_\ell \text{ rep.'s } \rho \text{ of } G} h_\star\overline{\mathbb{Q}}_\ell(\rho).$$

Applying $Rm_!$ to this decomposition gives the asserted decomposition of $Rf_!\overline{\mathbb{Q}}_\ell$. □

We now return to our particular situation. We are given a $R_0[1/\ell]$-algebra k which is a finite field, and a nontrivial additive character

$\psi : (k,+) \to \overline{\mathbb{Q}}_\ell^\times$. We denote by

$$\pi_k : \mathbb{X}_k \to \mathbb{A}^1_k$$

the base change to k of our family. Recall that the Fourier Transform FT_ψ is the endomorphism of the derived category $D^b_c(\mathbb{A}^1_k, \overline{\mathbb{Q}}_\ell)$ defined by looking at the two projections pr_1, pr_2 of $\mathbb{A}^2_k$ onto $\mathbb{A}^1_k$, and at the "kernel" $\mathcal{L}_{\psi(xy)}$ on $\mathbb{A}^2_k$, and putting

$$FT_\psi(K) := R(pr_2)_!(\mathcal{L}_{\psi(xy)} \otimes pr_1^\star K[1]),$$

cf. [Lau-TFCEF, 1.2]. One knows that FT_ψ is essentially involutive,

$$FT_\psi(FT_\psi(K)) \cong [x \mapsto -x]^\star K(-1),$$

or equivalently

$$FT_{\overline{\psi}}(FT_\psi(K)) \cong K(-1),$$

that FT_ψ maps perverse sheaves to perverse sheaves and induces an exact autoequivalence of the category of perverse sheaves with itself.

We denote by $K(\mathbb{A}^1_k, \overline{\mathbb{Q}}_\ell)$ the Grothendieck group of $D^b_c(\mathbb{A}^1_k, \overline{\mathbb{Q}}_\ell)$. One knows that K is the free abelian group on the isomorphism classes of irreducible perverse sheaves, cf. [Lau-TFCEF, 0.7, 0.8]. We also denote by FT_ψ the endomorphism of $K(\mathbb{A}^1_k, \overline{\mathbb{Q}}_\ell)$ induced by FT_ψ on $D^b_c(\mathbb{A}^1_k, \overline{\mathbb{Q}}_\ell)$.

The key fact for us is the following, proven in [Ka-ESDE, 9.3.2], cf. also [Ka-ESDE, 8.7.2 and line -4, p.327].

Theorem 7.2. *Denote by $\psi_{-1/d}$ the additive character $x \mapsto \psi(-x/d)$, and denote by $j : \mathbb{G}_{m,k} \subset \mathbb{A}^1_k$ the inclusion. Denote by $\Lambda_1, ..., \Lambda_d$ the list $List(all\ d)$ of all the multiplicative characters of $k^\times$ of order dividing d. For any unordered list of d multiplicative characters $\rho_1, ...\rho_d$ of $k^\times$ which is different from $List(all\ d)$, the perverse sheaf*

$$FT_\psi j_\star [d]^\star \mathcal{H}(\psi_{-1/d}; \rho_1, ...\rho_d; \emptyset)[1]$$

on $\mathbb{A}^1_k$ is geometrically isomorphic to the perverse sheaf

$$j_\star [d]^\star \mathcal{H}(\psi; \mathbf{Cancel}(List(all\ d); \overline{\rho_1}, ..., \overline{\rho_d}))[1].$$

Before we can apply this result, we need some preliminaries. We first calculate the Fourier Transform of $R\pi_{k,!}\overline{\mathbb{Q}}_\ell$, or more precisely its restriction to $\mathbb{G}_{m,k}$, in a Γ_W/Δ-equivariant way. Recall that $V_k \subset \mathbb{X}_k$ is the open set where X^W is invertible, and $Z_k \subset \mathbb{X}_k$ is its closed complement. We denote by

$$f := \pi_k | V_k : V_k \to \mathbb{A}^1_k$$

the restriction to V_k of π_k. Concretely, V_k is the open set $\mathbb{P}^{n-1}_k[1/X^W]$ of $\mathbb{P}^{n-1}_k$ (with homogeneous coordinates $(X_1, ..., X_n)$) where X^W is invertible, and f is the map

$$(X_1, ..., X_n) \mapsto \sum_i (w_i/d) X_i^d / X^W.$$

Lemma 7.3. *For any character V mod W of Γ_W/Δ, the canonical map of ρ-isotypical components $Rf_!\overline{\mathbb{Q}}_\ell(V \bmod W) \to R\pi_{k,!}\overline{\mathbb{Q}}_\ell(V \bmod W)$ induced by the $\mathbb{A}^1_k$-linear open immersion $V_k \subset \mathbb{X}_k$ induces an isomorphism in $D^b_c(\mathbb{G}_{m,k}, \overline{\mathbb{Q}}_\ell)$,*

$$(FT_\psi Rf_!\overline{\mathbb{Q}}_\ell)(V \bmod W)|\mathbb{G}_{m,k} \cong (FT_\psi R\pi_{k,!}\overline{\mathbb{Q}}_\ell)(V \bmod W)|\mathbb{G}_{m,k}.$$

Proof. We have an "excision sequence" distinguished triangle

$$Rf_!\overline{\mathbb{Q}}_\ell(V \bmod W) \to R\pi_{k,!}\overline{\mathbb{Q}}_\ell(V \bmod W) \to R(\pi|Z)_{k,!}\overline{\mathbb{Q}}_\ell(V \bmod W) \to .$$

The third term is constant, i.e., the pullback to $\mathbb{A}^1_k$ of a an object on $Spec(k)$, so its FT_ψ is supported at the origin. Applying FT_ψ to this distinguished triangle gives a distinguished triange

$$FT_\psi Rf_!\overline{\mathbb{Q}}_\ell(V \bmod W) \to FT_\psi R\pi_{k,!}\overline{\mathbb{Q}}_\ell(V \bmod W)$$

$$\to FT_\psi R(\pi|Z)_{k,!}\overline{\mathbb{Q}}_\ell(V \bmod W) \to .$$

Restricting to $\mathbb{G}_{m,k}$, the third term vanishes. □

We next compute $(FT_\psi Rf_!\overline{\mathbb{Q}}_\ell)|\mathbb{G}_{m,k}$ in a Γ_W/Δ-equivariant way. We do this by working upstairs, on V_k with its Γ_W/Δ-action.

Denote by $T_W \subset \mathbb{G}^n_{m,k}$ the connected (because $gcd(w_1, ...w_n) = 1$) torus of dimension $n-1$ in $\mathbb{G}^n_{m,k}$, with coordinates $x_i, i = 1, ..., n$, defined by the equation $x^W = 1$. Denote by $\mathbb{P}^{n-1}_k[1/X^W] \subset \mathbb{P}^{n-1}_k$ the open set of $\mathbb{P}^{n-1}_k$ (with homogeneous coordinates $(X_1, ..., X_n)$) where X^W is invertible. Our group Γ_W is precisely the group $T_W[d]$ of points of order dividing d in T_W. And the subgroup $\Delta \subset \Gamma_W$ is just the intersection of T_W with the diagonal in the ambient $\mathbb{G}^n_{m,k}$. We have a surjective map

$$g : T_W \to \mathbb{P}^{n-1}_k[1/X^W], (x_1, ..., x_n) \mapsto (x_1, ..., x_n).$$

This map g makes T_W a finite étale galois covering of $\mathbb{P}^{n-1}_k[1/X^W]$ with group Δ. The d'th power map $[d] : T_W \to T_W$ makes T_W into a finite étale galois covering of itself, with group Γ_W. We have a beautiful factorization of $[d]$ as $h \circ g$, for

$$h : \mathbb{P}^{n-1}_k[1/X^W] \to T_W, (X_1, ..., X_n) \mapsto (X_1^d/X^W, ..., X_n^d/X^W).$$

This map h makes $\mathbb{P}_k^{n-1}[1/X^W]$ a finite étale galois covering of T_W with group Γ_W/Δ. Denote by m the map

$$m : T_W \to A_k^1, (x_1, ..., x_n) \mapsto \sum_i (w_i/d)x_i.$$

Let us state explicitly the tautology which underlies our computation.

Lemma 7.4. *The map* $f : V_k = \mathbb{P}_k^{n-1}[1/X^W] \to A_k^1$ *is the composition*

$$f = m \circ h : \mathbb{P}_k^{n-1}[1/X^W] \xrightarrow{h} T_W \xrightarrow{m} \mathbb{A}_k^1.$$

Because h is a a finite étale galois covering of T_W with group Γ_W/Δ, we have a direct sum decomposition on T_W,

$$Rh_!\overline{\mathbb{Q}}_\ell = h_\star\overline{\mathbb{Q}}_\ell = \bigoplus_{char's\ V\ mod\ W\ of\ \Gamma_W/\Delta} \mathcal{L}_{V\ mod\ W}.$$

More precisely, any V in the coset $V\ mod\ W$ is a character of Γ/Δ, hence of Γ, so we have the Kummer sheaf $\mathcal{L}_V$ on the ambient torus $\mathbb{G}_{m,k}^n$. In the standard coordinates $(x_1, ..., x_n)$ on $\mathbb{G}_{m,k}^n$, this Kummer sheaf $\mathcal{L}_V$ is $\mathcal{L}_{\prod_i \chi_{v_i}(x_i)}$. The restriction of $\mathcal{L}_V$ to the subtorus T_W is independent of the choice of V in the coset $V\ mod\ W$; it is the sheaf denoted $\mathcal{L}_{V\ mod\ W}$ in the above decomposition.

Now apply $Rm_!$ to the above decomposition. We get a direct sum decomposition

$$Rf_!\overline{\mathbb{Q}}_\ell = Rm_!h_\star\overline{\mathbb{Q}}_\ell = \bigoplus_{char's\ V\ mod\ W\ of\ \Gamma_W/\Delta} Rm_!\mathcal{L}_{V\ mod\ W}$$

into eigenobjects for the action of Γ_W/Δ.

Apply now $FT_{\overline{\psi}}$. We get a direct sum decomposition

$$FT_{\overline{\psi}}Rf_!\overline{\mathbb{Q}}_\ell = \bigoplus_{char's\ V\ mod\ W\ of\ \Gamma_W/\Delta} FT_{\overline{\psi}}Rm_!\mathcal{L}_{V\ mod\ W}$$

into eigenobjects for the action of Γ_W/Δ; we have

$$(FT_{\overline{\psi}}Rf_!\overline{\mathbb{Q}}_\ell)(V\ mod\ W) = FT_{\overline{\psi}}Rm_!\mathcal{L}_{V\ mod\ W}$$

for each character $V\ mod\ W$ of Γ_W/Δ.

Theorem 7.5. *Given a character* $V\ mod\ W$ *of* Γ_W/Δ, *pick* V *in the coset* $V\ mod\ W$. *We have a geometric isomorphism*

$$(FT_{\overline{\psi}}Rf_!\overline{\mathbb{Q}}_\ell)(V\ mod\ W)|\mathbb{G}_{m,k}$$

$$\cong [d]^\star\mathcal{H}(\psi_{-1/d}; List(V, W); \emptyset)[2-n].$$

Proof. By the definition of $FT_{\overline{\psi}}$, and proper base change for $Rm_!$, we see that $FT_{\overline{\psi}}Rm_!\mathcal{L}_{V\ mod\ W}$ is obtained as follows. Choose V in the coset $V\ mod\ W$. Endow the product $T_W \times \mathbb{A}^1_k$, with coordinates $(x = (x_1, ..., x_n); t)$ from the ambient $\mathbb{G}^n_{m,k} \times \mathbb{A}^1_k$. The product has projections pr_1, pr_2 onto T_W and $\mathbb{A}^1_k$ respectively. On the product we have the lisse sheaf $\mathcal{L}_{\overline{\psi}(t\sum_i (w_i/d)x_i)} \otimes pr_1^\star \mathcal{L}_V$. By definition, we have

$$FT_{\overline{\psi}}Rm_!\mathcal{L}_{V\ mod\ W} = Rpr_{2,!}(\mathcal{L}_{\overline{\psi}(t\sum_i (w_i/d)x_i)} \otimes pr_1^\star \mathcal{L}_{\prod_i \chi_{v_i}(x_i)})[1].$$

If we pull back to $\mathbb{G}_{m,k} \subset \mathbb{A}^1_k$, then the source becomes $T_W \times \mathbb{G}_{m,k}$. This source is isomorphic to the subtorus Z of $\mathbb{G}^{n+1}_{m,k}$, with coordinates $(x = (x_1, ..., x_n); t)$, defined by

$$x^W = t^d,$$

by the map

$$(x = (x_1, ..., x_n); t) \mapsto (tx = (tx_1, ..., tx_n); t).$$

On this subtorus Z, our sheaf becomes $\mathcal{L}_{\overline{\psi}(\sum_i (w_i/d)x_i)} \otimes pr_1^\star \mathcal{L}_{\prod_i \chi_{v_i}(x_i)}[1]$.[Remember that V has $\sum_i v_i = 0$, so $\mathcal{L}_{\prod_i \chi_{v_i}(x_i)}$ is invariant by $x \mapsto tx$.] Thus we have

$$FT_{\overline{\psi}}Rm_!\mathcal{L}_{V\ mod\ W}|\mathbb{G}_{m,k} = Rpr_{n+1,!}(\mathcal{L}_{\overline{\psi}(\sum_i (w_i/d)x_i)} \otimes pr_1^\star \mathcal{L}_{\prod_i \chi_{v_i}(x_i)}[1]).$$

This situation,

$$\mathcal{L}_{\overline{\psi}(\sum_i (w_i/d)x_i)} \otimes pr_1^\star \mathcal{L}_{\prod_i \chi_{v_i}(x_i)}[1]\ on\ Z := (x^W = t^d) \overset{pr_{n+1}}{\to} \mathbb{G}_{m,k},$$

is the pullback by the d'th power map on the base of the situation

$$\mathcal{L}_{\overline{\psi}(\sum_i (w_i/d)x_i)} \otimes pr_1^\star \mathcal{L}_{\prod_i \chi_{v_i}(x_i)}[1]\ on\ \mathbb{G}^n_{m,k} \overset{x^W}{\to} \mathbb{G}_{m,k}.$$

Therefore we have

$$FT_{\overline{\psi}}Rm_!\mathcal{L}_{V\ mod\ W}|\mathbb{G}_{m,k} \cong [d]^\star R(x^W)_!(\mathcal{L}_{\overline{\psi}(\sum_i (w_i/d)x_i)} \otimes pr_1^\star \mathcal{L}_{\prod_i \chi_{v_i}(x_i)}[1]).$$

According to [Ka-GKM, 4.0,4.1, 5.5],

$$R^a(x^W)_!(\mathcal{L}_{\overline{\psi}(\sum_i (w_i/d)x_i)} \otimes pr_1^\star \mathcal{L}_{\prod_i \chi_{v_i}(x_i)})$$

vanishes for $a \neq n-1$, and for $a = n-1$ is the multiple multiplicative ! convolution

$$Kl(\psi_{-w_1/d}; \chi_{v_1}, w_1) \star_! Kl(\psi_{-w_2/d}; \chi_{v_2}, w_2) \star_! ... \star_! Kl(\psi_{-w_n/d}; \chi_{v_n}, w_n).$$

By [Ka-GKM, 4.3,5.6.2], for each convolvee we have geometric isomorphisms

$$Kl(\psi_{-w_i/d}; \chi_{v_i}, w_i) = [w_i]_\star Kl(\psi_{-w_i/d}; \chi_{v_i}) \cong Kl(\psi_{-1/d}; all\ w_i'th\ roots\ of\ \chi_{v_i}).$$

So the above multiple convolution is the Kloosterman sheaf

$$Kl(\psi_{-1/d}; all\ w_1'th\ roots\ of\ \chi_{v_1}, ..., all\ w_n'th\ roots\ of\ \chi_{v_n})$$

$:= \mathcal{H}(\psi_{-1/d}; all\ w'_1 th\ roots\ of\ \chi_{v_1}, ..., all\ w'_n th\ roots\ of\ \chi_{v_n}; \emptyset).$

Recall that by definition

$$List(V, W) := (all\ w'_1 th\ roots\ of\ \chi_{v_1}, ..., all\ w'_n th\ roots\ of\ \chi_{v_n}).$$

Putting this all together, we find the asserted geometric isomorphism

$$(FT_{\overline{\psi}} Rf_! \overline{\mathbb{Q}}_\ell)(V\ mod\ W)|\mathbb{G}_{m,k}$$
$$\cong [d]^\star \mathcal{H}(\psi_{-1/d}; List(V, W); \emptyset)[2-n].$$

□

We are now ready for the final step in the proof of Theorem 6.1. Recall that $j_{1,k} : U_k := \mathbb{A}^1_k - \mu_d \subset \mathbb{A}^1_k$, and $j_{2,k} : \mathbb{G}_{m,k} \subset \mathbb{A}^1_k$ are the inclusions. We must prove

Theorem 7.6. *(***Restatement of 6.1***) Let $V\ mod\ W$ be a character of Γ_W/Δ for which $Prim^{n-2}(V\ mod\ W)$ is nonzero. Pick V in the coset $V\ mod\ W$. Then we have a geometric isomorphism of perverse sheaves on $\mathbb{A}^1_k$*

$$j_{1,k,\star} Prim^{n-2}(V\ mod\ W)[1] \cong j_{2,k,\star}[d]^\star \mathcal{H}_{V,W,k}[1].$$

Proof. Over the open set U_k, we have seen that sheaves $R^i\pi_{k,\star}\overline{\mathbb{Q}}_\ell|U_k$ are geometrically constant for $i \neq n-2$, and that $R^{n-2}\pi_{k,\star}\overline{\mathbb{Q}}_\ell|U_k$ is the direct sum of $Prim^{n-2}$ and a geometrically constant sheaf. The same is true for the Γ_W/Δ-isotypical components. Thus in $K(U_k, \overline{\mathbb{Q}}_\ell)$, we have

$$R\pi_{k,\star}\overline{\mathbb{Q}}_\ell(V\ mod\ W)|U_k := \sum_i (-1)^i R^i\pi_{k,\star}\overline{\mathbb{Q}}_\ell(V\ mod\ W)|U_k$$

$$= (-1)^{n-2} Prim^{n-2}(V\ mod\ W) + (geom.\ const.).$$

Comparing this with the situation on all of $\mathbb{A}^1_k$, we don't know what happens at the d missing points of μ_d, but in any case we will have

$$R\pi_{k,\star}\overline{\mathbb{Q}}_\ell(V\ mod\ W) = (-1)^{n-2} j_{1,k,\star} Prim^{n-2}(V\ mod\ W)$$
$$+ (geom.\ const.) + (punctual,\ supported\ in\ \mu_d)$$

in $K(\mathbb{A}^1_k, \overline{\mathbb{Q}}_\ell)$.

Taking Fourier Transform, we get

$$FT_{\overline{\psi}} j_{1,k,\star} Prim^{n-2}(V\ mod\ W) =$$

$$(-1)^{n-2} FT_{\overline{\psi}} R\pi_{k,\star}\overline{\mathbb{Q}}_\ell(V\ mod\ W) + (punctual,\ supported\ at\ 0) + (sum\ of\ \mathcal{L}_{\psi_\zeta}\ 's)$$

in $K(\mathbb{A}^1_k, \overline{\mathbb{Q}}_\ell)$.

By Lemma 7.3 , we have

$$(FT_\psi R\pi_{k,!}\overline{\mathbb{Q}}_\ell)(V\ mod\ W)|\mathbb{G}_{m,k} \cong FT_\psi Rf_!\overline{\mathbb{Q}}_\ell(V\ mod\ W)|\mathbb{G}_{m,k},$$

so we have

$$FT_{\overline{\psi}} j_{1,k,\star} Prim^{n-2}(V\ mod\ W) =$$
$$(-1)^{n-2} FT_{\psi} Rf_! \overline{\mathbb{Q}}_\ell(V\ mod\, W) + (punctual,\ supported\, at\, 0) + (sum\ of\ \mathcal{L}_{\psi_\zeta}\ 's)$$

in $K(\mathbb{A}^1_k, \overline{\mathbb{Q}}_\ell)$.

By the previous theorem, we have

$$(FT_{\overline{\psi}} Rf_! \overline{\mathbb{Q}}_\ell)(V\ mod\ W)|\mathbb{G}_{m,k}$$
$$= (-1)^{n-2}[d]^\star \mathcal{H}(\psi_{-1/d}; List(V,W); \emptyset)$$

in $K(\mathbb{G}_{m,\overline{k}}, \overline{\mathbb{Q}}_\ell)$. We don't know what happens at the origin, but in any case we have

$$(FT_{\overline{\psi}} Rf_! \overline{\mathbb{Q}}_\ell)(V\ mod\ W) =$$
$$(-1)^{n-2} j_{2,k,\star}[d]^\star \mathcal{H}(\psi_{-1/d}; List(V,W); \emptyset) + (punctual,\ supported\ at\ 0)$$

in $K(\mathbb{A}^1_{\overline{k}}, \overline{\mathbb{Q}}_\ell)$. So we find

$$FT_{\overline{\psi}} j_{1,k,\star} Prim^{n-2}(V\ mod\ W) =$$
$$j_{2,k,\star}[d]^\star \mathcal{H}(\psi_{-1/d}; List(V,W); \emptyset) +$$
$$(punctual,\ supported\ at\ 0) + (sum\ of\ \mathcal{L}_{\psi_\zeta}\ 's)$$

in $K(\mathbb{A}^1_{\overline{k}}, \overline{\mathbb{Q}}_\ell)$. Now apply the inverse Fourier Transform FT_ψ. By Theorem 7.2, we obtain an equality

$$j_{1,k,\star} Prim^{n-2}(V\ mod\ W)[1] =$$
$$j_{2,k,\star}[d]^\star \mathcal{H}_{V,W,k}[1] + (geom.\ constant) + (punctual)$$

in the group $K(\mathbb{A}^1_{\overline{k}}, \overline{\mathbb{Q}}_\ell)$. This is the free abelian group on isomorphism classes of irreducible perverse sheaves on $\mathbb{A}^1_{\overline{k}}$. So in any equality of elements in this group, we can delete all occurrences of any particular isomorphism class, and still have an equality.

On the open set U_k, the lisse sheaves $Prim^{n-2}(V\ mod\ W)$ and $[d]^\star \mathcal{H}_{V,W,k}$ are both pure, hence completely reducible on $U_{\overline{k}}$ by [De-Weil II, 3.4.1 (iii)]. So both of the perverse sheaves $j_{1,k,\star} Prim^{n-2}(V\ mod\ W)[1]$ and $j_{2,k,\star}[d]^\star \mathcal{H}_{V,W,k}[1]$ on $(\mathbb{A}^1_{\overline{k}}$ are direct sums of perverse irreducibles which are middle extensions from $U_{\overline{k}}$, and hence have no punctual constituents. So we may cancel the punctual terms, and conclude that we have

$$j_{1,k,\star} Prim^{n-2}(V\ mod\ W)[1] - j_{2,k,\star}[d]^\star \mathcal{H}_{V,W,k}[1] = (geom.\ constant)$$

in the group $K(\mathbb{A}^1_{\overline{k}}, \overline{\mathbb{Q}}_\ell)$. By Lemma 5.2, the left hand side has generic rank zero, so there can be no geometrically constant virtual summand. Thus we have an equality of perverse sheaves

$$j_{1,k,\star} Prim^{n-2}(V\ mod\ W)[1] = j_{2,k,\star}[d]^\star \mathcal{H}_{V,W,k}[1]$$

in the group $K(\mathbb{A}^1_{\overline{k}}, \overline{\mathbb{Q}}_\ell)$. Therefore the two perverse sheaves have geometrically isomorphic semisimplifications. But by purity, both are geometrically semisimple. This concludes the proof of Theorem 6.1, and so also the proof of Theorem 5.3 □

8. Appendix I: The transcendental approach

In this appendix, we continue to work with the fixed data (n, d, W), but now over the groundring $\mathbb{C}$. We give a transcendental proof of Theorem 5.3, but only for the Γ_W/Δ-invariant part $Prim^{n-2}(0 \ mod \ W)$. Our proof is essentially a slight simplification of an argument that Shepherd-Barron gave in a November, 2006 lecture at MSRI, where he presented a variant of [H-SB-T, pages 5-22]. We do not know how to treat the other eigensheaves $Prim^{n-2}(V \ mod \ W)$, with $V \ mod \ W$ a nontrivial character of Γ_W/Δ, in an analogous fashion.

First, let us recall the bare definition of hypergeometric D-modules. We work on $\mathbb{G}_m$ (always over $\mathbb{C}$), with coordinate λ. We write $D := \lambda d/d\lambda$. We denote by $\mathcal{D} := \mathbb{C}[\lambda, 1/\lambda][D]$ the ring of differential operators on $\mathbb{G}_m$. Fix nonnegative integers a and b, not both 0. Suppose we are given an unordered list of a complex numbers $\alpha_1, ..., \alpha_a$,not necessarily distinct. Let $\beta_1, ..., \beta_b$ be a second such list, but of length b. We denote by $Hyp(\alpha_i' s; \beta_j' s)$ the differential operator

$$Hyp(\alpha_i' s; \beta_j' s) := \prod_i (D - \alpha_i) - \lambda \prod_j (D - \beta_j)$$

and by $\mathcal{H}(\alpha_i' s; \beta_j' s)$ the holonomic left D-module

$$\mathcal{H}(\alpha_i' s; \beta_j' s) := \mathcal{D}/\mathcal{D}Hyp(\alpha_i' s; \beta_j' s).$$

We say that $\mathcal{H}(\alpha_i' s; \beta_j' s)$ is a hypergeometric of type (a, b).

One knows [Ka-ESDE, 3.2.1] that this $\mathcal{H}$ is an irreducible $\mathcal{D}$-module on $\mathbb{G}_m$, and remains irreducible when restricted to any dense open set $U \subset \mathbb{G}_m$, if and only if the two lists are disjoint "mod $\mathbb{Z}$", i.e., for all i, j, $\alpha_i - \beta_j$ is not an integer. [If we are given two lists $List_1$ and $List_2$ which are not identical mod $\mathbb{Z}$, but possibly not disjoint mod $\mathbb{Z}$, we can "cancel" the common (mod $\mathbb{Z}$) entries, and get an irreducible hypergeometric $\mathcal{H}(\mathbf{Cancel}(List_1, List_2))$.]

We will assume henceforth that this disjointness mod $\mathbb{Z}$ condition is satisfied, and that $a = b$. Then $\mathcal{H}(\alpha_i' s; \beta_j' s)$ has regular singular points at $0, 1, \infty$. If all the α_i and β_j all lie in $\mathbb{Q}$, pick a common denominator N, and denote by χ_{α_i} the character of $\mu_N(\mathbb{C})$ given by

$$\chi_{\alpha_i}(\zeta) := \zeta^{\alpha_i N}.$$

Similarly for χ_{β_j}. For any prime number ℓ, the Riemann-Hilbert partner of $\mathcal{H}(\alpha_i's;\beta_j's)$ is the $\overline{\mathbb{Q}}_\ell$ perverse sheaf $\mathcal{H}^{can}(\chi_{\alpha_i}{}'s;\chi_{\beta_j}{}'s)[1]$ on $\mathbb{G}_m$, cf. [Ka-ESDE, 8.17.11].

We denote by $\mathcal{D}_\eta := \mathbb{C}(\lambda)[D]$ the ring of differential operators at the generic point. Although this ring is not quite commutative, it is near enough to being a one-variable polynomial ring over a field that it is left (and right) Euclidean, for the obvious notion of long division. So every nonzero left ideal in $\mathcal{D}_\eta$ is principal, generated by the monic (in D) operator in it of lowest order. Given a left $\mathcal{D}_\eta$-module M, and an element $m \in M$, we denote by $Ann(m,M)$ the left ideal in $\mathcal{D}_\eta$ defined as

$$Ann(m,M) := \{operators\ L \in \mathcal{D}_\eta | L(m) = 0\ in\ M\}.$$

If $Ann(m,M) \neq 0$, we define $L_{m,M} \in \mathcal{D}_\eta$ to be the lowest order monic operator in $Ann(m,M)$.

We have the following elementary lemma, whose proof is left to the reader.

Lemma 8.1. *Let N and M be left $\mathcal{D}_\eta$-modules, $f: M \to N$ a horizontal (:= $\mathcal{D}_\eta$-linear) map, and $m \in M$. Suppose that $Ann(m,M) \neq 0$. Then $Ann(m,M) \subset Ann(f(m),N)$, and $L_{m,M}$ is right-divisible by $L_{f(m),N}$.*

We now turn to our complex family $\pi : \mathbb{X} \to \mathbb{A}^1$, given by

$$X_\lambda := X_\lambda(W,d) : \sum_{i=1}^{n} w_i X_i^d - d\lambda X^W = 0.$$

We pull it back to $U := \mathbb{G}_m - \mu_d \subset \mathbb{A}^1$, over which it is proper and smooth, and form the de Rham incarnation of $Prim^{n-2}$, which we denote $Prim^{n-2}_{dR}$. We also have the relative de Rham cohomolgy of $(\mathbb{P}^{n-1} \times U - \mathbb{X}_U)/U$ over the base U in degree $n-1$, which we denote simply $H^{n-1}_{dR}((\mathbb{P} - \mathbb{X})/U)$. Both are $\mathcal{O}$-locally free D-modules (Gauss-Manin connection) on U, endowed with a horizontal action of Γ_W/Δ. The Poincaré residue map gives a horizontal, Γ_W/Δ-equivariant isomorphism

$$Res : H^{n-1}_{dR}((\mathbb{P} - \mathbb{X})/U) \cong Prim^{n-2}_{dR}.$$

Exactly as in the discussion beginning section 6, we write $1 = \sum_i b_i w_i$ to obtain a descent of our family through the d'power map: the family $\pi_{desc} : \mathbb{Y} \to \mathbb{G}_m$ given by

$$Y_\lambda : \sum_{i=1}^{n} w_i \lambda^{-b_i} Y_i^d = dX^W.$$

The same group Γ_W/Δ acts on this family, which is projective and smooth over $\mathbb{G}_m - \{1\}$. So on $\mathbb{G}_m - \{1\}$, we have $Prim^{n-2}_{dR,desc}$ for this family, and its fixed part $Prim^{n-2}_{dR,desc}(0 \ mod \ W)$, whose pullback $[d]^\star Prim^{n-2}_{dR,desc}(0 \ mod \ W)$ is the sheaf $Prim^{n-2}_{dR}(0 \ mod \ W)|(\mathbb{G}_m - \mu_d)$.

Our next step is to pull back further, to a small analytic disk. Choose a real constant $C > 4$. Pull back the descended family to a small disc $\mathcal{U}_{an,C}$ around C. We take the disc small enough that for $\lambda \in \mathcal{U}_{an,C}$, we have $|C/\lambda|^{b_i} < 2$ for all i. The extension of scalars map

$$H^{n-1}_{dR}((\mathbb{P}-\mathbb{Y})/(\mathbb{G}_m-\{1\})) \mapsto H^{n-1}_{dR}((\mathbb{P}-\mathbb{Y})/(\mathbb{G}_m-\{1\}))\otimes_{\mathcal{O}_{\mathbb{G}_m-\{1\}}}\mathcal{O}_{\mathcal{U}_{an,C}}$$

is a horizontal map; we view both source and target as $\mathcal{D}$ -modules.

Over this disc, the $\mathcal{C}^\infty$ closed immersion

$$\gamma : (S^1)^n/Diagonal \to \mathbb{P}^{n-1}, (z_1, ..., z_n) \mapsto (C^{b_1/d}z_1, ..., C^{b_{n-1}/d}z_{n-1}, C^{b_n/d}z_n)$$

lands entirely in $\mathbb{P} - \mathbb{Y}$: its image is an $n-1$-torus $Z \subset \mathbb{P}^{n-1}$ which is disjoint from Y_λ for $\lambda \in \mathcal{U}_{an,C}$. In Restricting to the Γ_W/Δ-invariant part $H^{n-1}_{dR}((\mathbb{P} - \mathbb{Y})/(\mathbb{G}_m - \{1\}))(0 \ mod \ W)$, we get a horizontal map

$$H^{n-1}_{dR}((\mathbb{P} - \mathbb{Y})/(\mathbb{G}_m - \{1\}))(0 \ mod \ W) \to H^0(\mathcal{U}_{an,C}, \mathcal{O}_{\mathcal{U}_{an,C}}), \omega \mapsto \int_Z \omega.$$

Write $y_i := Y_i/Y_n$ for $i = 1, ..., n-1$. Denote by

$$\omega \in H^{n-1}_{dR}((\mathbb{P} - \mathbb{Y})/(\mathbb{G}_m - \{1\}))(0 \ mod \ W)$$

the (cohomology class of the) holomorphic $n-1$-form

$$\omega := (1/2\pi i)^{n-1}\left(\frac{dY^W}{dY^W - \sum_{i=1}^n w_i\lambda^{-b_i}Y_i^d}\right)\prod_{i=1}^{n-1} dy_i/y_i.$$

Our next task is to compute the integral

$$\int_Z \omega.$$

The computation will involve the Pochammer symbol. For $\alpha \in \mathbb{C}$, and $k \geq 1$ a positive integer, the Pochammer symbol $(\alpha)_k$ is defined by

$$(\alpha)_k := \Gamma(\alpha + k)/\Gamma(\alpha) = \prod_{i=0}^{k-1}(\alpha + i).$$

We state for ease of later reference the following elementary identity.

Lemma 8.2. *For integers $k \geq 1$ and $r \geq 1$, we have*

$$(kr)!/r^{kr} = \prod_{i=1}^{r}(i/r)_k.$$

Lemma 8.3. *We have the formula*

$$\int_Z \omega = 1 + \sum_{k\geq 1} \Big(\frac{\prod_{i=1}^{d}(i/d)_k}{\prod_{i=1}^{n}\prod_{j=1}^{w_i}(j/w_i)_k}\Big)(1/\lambda)^k.$$

Proof. Divide top and bottom by dY^W, expand the geometric series, and integrate term by term. This is legitimate because at a point $z \in Z$, the function $\sum_{i=1}^{n}(w_i/d)\lambda^{-b_i}Y_i^d/Y^W$ has the value

$$\sum_{i=1}^{n}(w_i/d)\lambda^{-b_i}C^{b_i}z_i^d/Cz^W = \sum_{i=1}^{n}(w_i/d)(C/\lambda)^{b_i}z_i^d/Cz^W,$$

which has absolute value $\leq 2(\sum_{i=1}^{n}(w_i/d))/C = 2/C \leq 1/2$. Because each term in the geometric series is homogeneous of degree zero, the integral of the k'th term in the geometric series is the coefficient of z^{kW} in $(\sum_{i=1}^{n}(w_i/d)(\lambda)^{-b_i}z_i^d)^k$. This coefficient vanishes unless k is a multiple of d (because $gcd(w_1, ..., w_n) = 1$). The integral of the dk'th term is the coefficient of z^{kdW} in $(\sum_{i=1}^{n}(w_i/d)(\lambda)^{-b_i}z_i^d)^{dk}$, i.e., the coefficient of z^{kW} in $(\sum_{i=1}^{n}(w_i/d)(\lambda)^{-b_i}z_i)^{dk}$. Expanding by the multinomial theorem, this coefficient is

$$(dk)!\prod_{i=1}^{n}(((w_i/d)\lambda^{-b_i})^{kw_i}/(kw_i)!) = (\lambda)^{-k}((dk)!/d^{dk})/\prod_{i=1}^{n}((kw_i)!/w_i^{kw_i}),$$

which, by the previous lemma, is as asserted. ☐

This function

$$F(\lambda) : \int_Z \omega = 1 + \sum_{k\geq 1} \Big(\frac{\prod_{i=1}^{d}(i/d)_k}{\prod_{i=1}^{n}\prod_{j=1}^{w_i}(j/w_i)_k}\Big)(1/\lambda)^k$$

is annihilated by the following differential operator. Consider the two lists of length d.

$$List(all\ d) := \{1/d, 2/d, ..., d/d\},$$

$$List(0, W) := \{1/w_1, 2/w_1, ..., w_1/w_1, ..., 1/w_n, 2/w_n, ..., w_n/w_n\}.$$

These lists are certainly not identical mod $\mathbb{Z}$; the second one contains 0 with multiplicity n, while the first contains only a single integer. Let us denote the cancelled lists, whose common length we call a,

$$\mathbf{Cancel}(List(all\ d); List(0, W)) = (\alpha_1, ..., \alpha_a); (\beta_1, ..., \beta_a).$$

So we have

$$F(\lambda) : \int_Z \omega = 1 + \sum_{k\geq 1} \Big(\frac{\prod_{i=1}^{a}(\alpha_i)_k}{\prod_{i=1}^{a}(\beta_i)_k}\Big)(1/\lambda)^k,$$

which one readily checks is annihilated by the differential operator

$$Hyp_{0,W} := Hyp(\alpha_i's; \beta_i - 1's) := \prod_{i=1}^{a}(D - \alpha_i) - \lambda \prod_{i=1}^{a}(D - (\beta_i - 1)).$$

Theorem 8.4. *We have an isomorphism of $\mathcal{D}$-modules on $\mathbb{G}_m - \{1\}$,*

$$H^{n-1}_{dR}((\mathbb{P}-\mathbb{Y})/(\mathbb{G}_m-\{1\})) \cong \mathcal{H}_{0,W}|(\mathbb{G}_m-\{1\}) := \mathcal{H}(\alpha_i's; \beta_i-1's)|(\mathbb{G}_m-\{1\}).$$

Proof. Both sides of the alleged isomorphism are $\mathcal{O}$-coherent $\mathcal{D}$-modules on $\mathbb{G}_m - \{1\}$, so each is the "middle extension" of its restriction to any Zariski dense open set in $\mathbb{G}_m - \{1\}$. So it suffices to show that both sides become isomorphic over the function field of $\mathbb{G}_m - \{1\}$, i.e., that they give rise to isomorphic $\mathcal{D}_\eta$-modules. For this, we argue as follows. Denote by $\mathcal{A}$ the ring

$$\mathcal{A} := H^0(\mathcal{U}_{an,C}, \mathcal{O}_{\mathcal{U}_{an,C}}) \otimes_{\mathcal{O}_{\mathbb{G}_m-\{1\}}} \mathbb{C}(\lambda),$$

which we view as a $\mathcal{D}_\eta$-module. We have the horizontal map

$$H^{n-1}_{dR}((\mathbb{P} - \mathbb{Y})/(\mathbb{G}_m - \{1\}))(0 \mod W) \xrightarrow{\int_Z} H^0(\mathcal{U}_{an,C}, \mathcal{O}_{\mathcal{U}_{an,C}}).$$

Tensoring over $\mathcal{O}_{\mathbb{G}_m-\{1\}}$ with $\mathbb{C}(\lambda)$, we obtain a horizontal map

$$H^{n-1}_{dR}((\mathbb{P} - \mathbb{Y})/\mathbb{C}(\lambda))(0 \mod W) \xrightarrow{\int_Z} \mathcal{A}.$$

By (the Hyp analogue of) Lemma 5.2, we know that the source has $\mathbb{C}(\lambda)$-dimension a:= the order of $Hyp(\alpha_i's; \beta_i - 1's)$. So the element ω in the source is annihilated by some operator in $\mathcal{D}_\eta$ of order at most a, simply because ω and its first a derivatives must be linearly dependent over $\mathbb{C}(\lambda)$. So the lowest order operator annihilating ω in $H^{n-1}_{dR}((\mathbb{P} - \mathbb{Y})/\mathbb{C}(\lambda))(0 \mod W)$, call it $L_{\omega,H_{dR}}$, has order at most a. On the other hand, the irreducible operator $Hyp(\alpha_i's; \beta_i - 1's)$ annihilates $\int_Z \omega \in \mathcal{A}$. But $\int_Z \omega \neq 0$, so $Ann(\int_Z \omega, \mathcal{A})$ is a proper left ideal in $\mathcal{D}_\eta$, and hence is generated by the irreducible monic operator $(1/(1-\lambda))Hyp(\alpha_i's; \beta_i - 1's)$. By Lemma 8.2, we know that $L_{\omega,H_{dR}}$ is divisible by $(1/(1-\lambda))Hyp(\alpha_i's; \beta_i - 1's)$. But $L_{\omega,H_{dR}}$ has order at most a, the order of $Hyp(\alpha_i's; \beta_i - 1's)$, so we conclude that $L_{\omega,H_{dR}} = (1/(1-\lambda))Hyp(\alpha_i's; \beta_i - 1's)$. Thus the $\mathbb{D}_\eta$-span of ω in $H^{n-1}_{dR}((\mathbb{P} - \mathbb{Y})/\mathbb{C}(\lambda))(0 \mod W)$ is $\mathcal{D}_\eta/\mathcal{D}_\eta Hyp(\alpha_i's; \beta_i - 1's)$. Comparing dimensions, we see that this $\mathcal{D}_\eta$-span must be all of $H^{n-1}_{dR}((\mathbb{P} - \mathbb{Y})/\mathbb{C}(\lambda))(0 \mod W)$. $\square$

Corollary 8.5. *For the family*

$$X_\lambda := X_\lambda(W, d) : \sum_{i=1}^{n} w_i X_i^d - d\lambda X^W = 0,$$

its $Prim^{n-2}_{dR}(0\ mod\ W)$ *as* D*-module on* $\mathbb{A}^1 - \mu_d$ *is related to the* $\mathcal{D}$*-module* $[d]^\star(\mathcal{H}_{0,W}|(\mathbb{G}_m - \{1\}))$ *on* $\mathbb{G}_m - \mu_d$ *as follows.*

(1) *We have an isomorphism of* $\mathcal{D}$*-modules on* $\mathbb{G}_m - \mu_d$,

$$Prim^{n-2}_{dR}(0\ mod\ W)|(\mathbb{G}_m - \mu_d) \cong [d]^\star(\mathcal{H}_{0,W}|(\mathbb{G}_m - \{1\})).$$

(2) *Denote by* $j_1 : \mathbb{A}^1 - \mu_d \subset \mathbb{A}^1$ *and* $j_2 : \mathbb{G}_m - \mu_d \subset \mathbb{A}^1$ *the inclusions. The we have an isomorphism of* D*-modules on* $\mathbb{A}^1$ *of the middle extensions*

$$j_{1,!,\star}(Prim^{n-2}_{dR}(0\ mod\ W)) \cong j_{2,!,\star}([d]^\star(\mathcal{H}_{0,W}|(\mathbb{G}_m - \{1\}))).$$

Proof. The first isomorphism is the pullback by d'th power of the isomorphism of the theorem above. We obtain the second isomorphism as follows. Denote by $j_3 : \mathbb{G}_m - \mu_d \subset \mathbb{A}^1 - \mu_d$ the inclusion. Because $Prim^{n-2}_{dR}(0\ mod\ W)$ is an $\mathcal{O}$-coherent D-module on $\mathbb{A}^1 - \mu_d$, it is the middle extension $j_{3,!,\star}(Prim^{n-2}_{dR}(0\ mod\ W)|(\mathbb{G}_m - \mu_d))$. Because $j_2 = j_1 \circ j_3$, we obtain the second isomorphism by applying $j_{2,!,\star}$ to the first isomorphism. □

Theorem 8.6. *Suppose* $n \geq 3$. *For either the family*

$$X_\lambda := X_\lambda(W,d) : \sum_{i=1}^{n} w_i X_i^d - d\lambda X^W = 0,$$

over $\mathbb{A}^1 - \mu_d$, *or the descended family*

$$Y_\lambda : \sum_{i=1}^{n} w_i \lambda^{-b_i} Y_i^d = dX^W$$

over $\mathbb{G}_m - \{1\}$, *consider its* $Prim^{n-2}_{dR}(0\ mod\ W)$ *(resp. consider its* $Prim^{n-2}_{dR,desc}(0\ mod\ W)$*) as* D*-module, and denote by* a *its rank. For either family, its differential galois group* G_{gal} *(which here is the Zariski closure of its monodromy group) is the symplectic group* $Sp(a)$ *if* $n-2$ *is odd, and the orthogonal group* $O(a)$ *if* $n-2$ *is even.*

Proof. Poincaré duality induces on $Prim^{n-2}_{dR}(0\ mod\ W)$ (resp. on $Prim^{n-2}_{dR,desc}(0\ mod\ W)$) an autoduality which is symplectic if $n-2$ is odd, and orthogonal if $n-2$ is even. So we have a priori inclusions $G_{gal} \subset Sp(a)$ if $n-2$ is odd, $G_{gal} \subset O(a)$ if $n-2$ is even. It suffices to prove the theorem for the descended family. This is obvious in the Sp case, since the identity component of G_{gal} is invariant under finite pullback. In the O case, we must rule out the possibility that the pullback has group $SO(a)$ rather than $O(a)$. For this, we observe that an orthogonally autodual hypergeometric of type (a,a) has a true reflection as local monodromy around 1 (since in any case an irreducible hypergeometric of type (a,a) has as local monodromy around 1 a pseudoreflection, and the only pseudoreflection in an orthogonal group is a true reflection). As the d'th power map is finite étale over 1, the pullback has a true reflection as local monodromy around each $\zeta \in \mu_d$. So the group for the pullback contains true reflections, so must be $O(a)$.

We now consider the descended family. So we are dealing with $\mathcal{H}_{0,W} := \mathcal{H}(\alpha_i's; \beta_i - 1's)$. From the definition of $\mathcal{H}_{0,W}$, we see that $\beta = 1 \mod \mathbb{Z}$ occurs among the β_i precisely $n-1$ times ($n-1$ times and not n times because of a single cancellation with $List(all\ d)$). Because $n - 1 \geq 2$ by hypothesis, local monodromy around ∞ is not semisimple [Ka-ESDE, 3.2.2] and hence $\mathcal{H}(\alpha_i's; \beta_j's)$ is not Belyi induced or inverse Belyi induced, cf. [Ka-ESDE, 3.5], nor is its $G^{0,der}$ trivial.

We next show that $\mathcal{H}_{0,W}$ is not Kummer induced of any degree $r \geq 2$. Suppose not. As the α_i all have order dividing d in $\mathbb{C}/\mathbb{Z}$, r must divide d, since $1/r \mod \mathbb{Z}$ is a difference of two α_i's, cf. [3.5.6]Ka-ESDE. But the $\beta_j \mod \mathbb{Z}$ are also stable by $x \mapsto x + 1/r$, so we would find that $1/r \mod \mathbb{Z}$ occurs with the same multiplicity $n-1$ as $0 \mod \mathbb{Z}$ among the $\beta_j \mod \mathbb{Z}$. So r must divide at least $n-1$ of the w_i; it cannot divide all the w_i because $gcd(w_1, ..., w_n) = 1$. But this $1/r$ cannot cancel with $List(all\ d)$, otherwise its multiplicity would be at most $n-2$. This lack of cancellation means that r does not divide d, contradiction.

Now we appeal to [Ka-ESDE, 3.5.8]: let $\mathcal{H}(\alpha_i's; \beta_j's)$ be an irreducible hypergeometric of type (a,a) which is neither Belyi induced nor inverse Belyi induced not Kummer induced. Denote by G its differential galois group G_{gal}, G^0 its identity component, and $G^{0,der}$ the derived group (:= commutator subgroup) of G^0. Then $G^{0,der}$ is either trivial or it is one of $SL(a)$ or $SO(a)$ or, if a is even, possibly $Sp(a)$.

In the case of $\mathcal{H}_{0,W}$, we have already seen that $G_{gal}^{0,der}$ is not trivial. Given that G_{gal} lies in either $Sp(a)$ or $O(a)$, depending on the parity of $n-2$, the only possibility is that $G_{gal} = Sp(a)$ for $n-2$ odd, and that $G_{gal} = O(a)$ or $SO(a)$ if $n-2$ is even. In the even case, the presence of a true reflection in G_{gal} rules out the SO case. □

Corollary 8.7. *In the context of Theorem 5.3, on each geometric fibre of $U_{R_0[1/\ell]}/Spec(R_0[1/\ell])$, the geometric monodromy group G_{geom} of $Prim^{n-2}(0 \mod W)$ is the full symplectic group $Sp(a)$ if $n-2$ is odd, and is the full orthogonal group $O(a)$ if $n-2$ is even.*

Proof. On a $\mathbb{C}$-fibre, this is just the translation through Riemann-Hilbert of the theorem above. The passage to other geometric fibres is done by the Tame Specialization Theorem [Ka-ESDE, 8.17.3]. □

When does it happen that $Prim^{n-2}_{dR}(0\ mod\ W)$ has rank $n-1$ and all Hodge numbers 1?

Lemma 8.8. *The following are equivalent.*

(1) *$Prim^{n-2}_{dR}(0\ mod\ W)$ has rank $n-1$.*
(2) *Every w_i divides d, and for all $i \neq j$, $gcd(w_i, w_j) = 1$.*
(3) *Local monodromy at ∞ is a single unipotent Jordan block.*
(4) *Local monodromy at ∞ is a single Jordan block.*
(5) *All the Hodge numbers $Prim^{a,b}_{dR}(0\ mod\ W)_{a+b=n-2}$ are 1.*

Proof. (1)⇒(2) The rank is at least $n-1$, as this is the multiplicity of 0 mod $\mathbb{Z}$ as a β in $\mathcal{H}_{0,W}$. If the rank is no higher, then each w_i must divide d, so that the elements $1/w_i, ..., (w_i-1)/w_i$ mod $\mathbb{Z}$ can cancel with $List(all\ d)$. And the w_i must be pairwise relatively prime, for if a fraction $1/r$ mod $\mathbb{Z}$ with $r \geq 2$ appeared among both $1/w_i, ..., (w_i - 1)/w_i$ and $1/w_j, ..., (w_j-1)/w_j$, only one of its occurrences at most can cancel with $List(all\ d)$.

(2)⇒(1) If all w_i divide d, and if the w_i are pairwise relatively prime, then after cancellation we find that $\mathcal{H}_{0,W}$ has rank $n-1$.

(1)⇒(3) If (1) holds, then the β_i's are all 0 mod $\mathbb{Z}$, and there are $n-1$ of them. This forces $\mathcal{H}_{0,W}$ and also $[d]^\star\mathcal{H}_{0,W}$ to have its local monodromy around ∞, call it T, unipotent, with a single Jordan block, cf. [Ka-ESDE, 3.2.2].

(3)⇒(4) is obvious.

(4)⇒(3) Although d'th power pullback may change the eigenvalues of local monodromy at ∞, it does not change the number of distinct Jordan blocks. But there is always one unipotent Jordan block of size $n-1$, cf. the proof of (1)⇒(2).

(3)⇒(5) If not all the $n-1$ Hodge numbers are 1, then some Hodge number vanishes, and at most $n-2$ Hodge numbers are nonzero. But by [Ka-NCMT, 14.1] [strictly speaking, by projecting its proof onto Γ_W/Δ-isotypical components] any local monodromy is quasiunipotent of exponent of nilpotence $\leq h:=$ the number of nonzero Hodge numbers. So our local monodromy T around ∞, already unipotent, would satisfy $(T-1)^{n-2} = 0$. But as we have already remarked, $\mathcal{H}_{0,W}$ always has unipotent Jordan block of size $n-1$. Therefore all the Hodge numbers are nonzero, and hence each is 1.

(5)⇒(1) is obvious. □

Remark 8.9. Four particular $n = 5$ cases where condition (2) is satisfied, namely $W = (1, 1, 1, 1, 1)$, $W = (1, 1, 1, 1, 2)$, $W = (1, 1, 1, 1, 4)$, and $W = (1, 1, 1, 2, 5)$, were looked at in detain in the early days of mirror symmetry, cf. [Mor, Section 4, Table 1].

Whatever the rank of $Prim_{dR}^{n-2}(0\ mod\ W)$, we have:

Lemma 8.10. *All the Hodge numbers* $Prim_{dR}^{a,b}(0\ mod\ W)_{a+b=n-2}$ *are nonzero.*

Proof. Repeat the proof of (3)⇒(5). □

9. Appendix II: The situation in characteristic p, when p divides some w_i

We continue to work with the fixed data (n, d, W). In this appendix, we indicate briefly what happens in a prime-to-d characteristic p which divides one of the w_i. For each i, we denote by w_i° the prime-to-p part of w_i, i.e.,

$$w_i = w_i^\circ \times (\text{a power of } p),$$

and we define

$$W^\circ := (w_1^\circ, ..., w_n^\circ).$$

We denote by d_{W° the integer

$$d_{W^\circ} := lcm(w_1^\circ, ..., w_n^\circ)d,$$

and define

$$d' := \sum_i w_i^\circ.$$

For each i, we have $w_i \equiv w_i^\circ \bmod p-1$, so we have the congruence, which will be used later,

$$d \equiv d' \bmod p-1.$$

We work over a finite field k of characteristic p prime to d which contains the d_{W°'th roots of unity. We take for ψ a nontrivial additive character of k which is of the form $\psi_{\mathbb{F}_p} \circ Trace_{k/\mathbb{F}_p}$, for some nontrivial additive character $\psi_{\mathbb{F}_p}$ of $\mathbb{F}_p$. The signifigance of this choice of ψ is that for $q = p^e, e \geq 1$, any power of p, under the q'th power map we have

$$[q]_\star \mathcal{L}_\psi = \mathcal{L}_\psi, [q]^\star \mathcal{L}_\psi = \mathcal{L}_\psi$$

on $\mathbb{A}^1_k$.

The family we study in this situation is $\pi : \mathbb{X} \to \mathbb{A}^1$,

$$X_\lambda := X_\lambda(W, d) : \sum_{i=1}^n w_i^\circ X_i^d - d\lambda X^W = 0.$$

The novelty is that, because p divides some w_i, this family is projective and smooth over all of $\mathbb{A}^1$.

The group Γ_W/Δ operates on this family. Given a character $V\ mod\ W$ of this group, the rank of the eigensheaf $Prim^{n-2}(V\ mod\ W)$ is still given by the same recipe as in Lemma 3.1(1), because at $\lambda = 0$ we have a smooth Fermat hypersurface of degree d.

Given an element $V = (v_1, ..., v_n) \in (\mathbb{Z}/d\mathbb{Z})^n_0$, we attach to it an unordered list $List(V, W)$ of $d' = \sum_i w_i^\circ$ multiplicative characters of $k^\times$, by the following procedure. For each index i, we denoted by χ_{v_i} the character of $k^\times$ given by

$$\zeta \mapsto \zeta^{(v_i/d)\#k^\times}.$$

This characterχ_{v_i} has w_i° (as opposed to w_i) distinct w_i'th roots. We then define

$$List(V, W) = \{all\ w_1\ 'th\ roots\ of\ \chi_{v_1}, ..., all\ w_n\ 'th\ roots\ of\ \chi_{v_n}\}.$$

We will also need the same list, but for $-V$, and the list

$$List(all\ d) := \{all\ characters\ of\ order\ dividing\ d\}.$$

The two lists $List(-V, W)$ and $List(all\ d)$ are not identical, as they have different lengths d' and d respectively, so we can apply the **Cancel** operation, and form the hypergeometric sheaf

$$\mathcal{H}_{V,W} := \mathcal{H}^{can}(\mathbf{Cancel}(List(all\ d); List(-V, W)))$$

on $\mathbb{G}_{m,k}$. Exactly as in Lemma 5.2, if $Prim^{n-2}(V\ mod\ W)$ is nonzero, its rank is the rank of $\mathcal{H}_{V,W}$.

An important technical fact in this situation is the following variant of Theorem 7.2, cf. [Ka-ESDE, 9.3.2], which "works" because $\mathbb{F}_p^\times$ has order $p - 1$.

Theorem 9.1. *Denote by $\psi_{-1/d}$ the additive character $x \mapsto \psi(-x/d)$, and denote by $j : \mathbb{G}_{m,k} \subset \mathbb{A}^1_k$ the inclusion. Denote by $\Lambda_1, ..., \Lambda_d$ the list $List(all\ d)$ of all the multiplicative characters of $k^\times$ of order dividing d. Let d' be a strictly positive integer with $d' \equiv d \bmod p - 1$. For any unordered list of d' multiplicative characters $\rho_1, ...\rho_{d'}$ of $k^\times$ which is not identical to $List(all\ d)$, the perverse sheaf*

$$FT_\psi j_\star [d]^\star \mathcal{H}(\psi_{-1/d}; \rho_1, ...\rho_{d'}; \emptyset)[1]$$

on $\mathbb{A}^1_k$ is geometrically isomorphic to the perverse sheaf

$$j_\star [d]^\star \mathcal{H}(\psi; \mathbf{Cancel}(List(all\ d); \overline{\rho_1}, ..., \overline{\rho_{d'}}))[1].$$

The main result is the following.

Theorem 9.2. *Suppose $Prim^{n-2}(V\ mod\ W)$ is nonzero. Denote by $j : \mathbb{G}_m \subset \mathbb{A}^1$ the inclusion. Choose V in the coset $V\ mod\ W$. There exists a constant $A_{V,W} \in \overline{\mathbb{Q}}_{ell}^\times$ and an isomorphism of lisse sheaves on $\mathbb{A}^1_k$,*

$$Prim^{n-2}(V\ mod\ W) \cong j_\star [d]^\star \mathcal{H}_{V,W} \otimes (A_{V,W})^{deg}.$$

Proof. Because our family is projective and smooth over all of $\mathbb{A}^1$, Deligne's degeneration theorem [De-TLCD, 2.4] gives a decomposition

$$R\pi_\star\overline{\mathbb{Q}}_\ell \cong Prim^{n-2}[2-n]\bigoplus(geom.\ constant).$$

So applying Fourier Transform, we get

$$FT_{\overline{\psi}}R\pi_\star\overline{\mathbb{Q}}_\ell(V\ mod\ W)|\mathbb{G}_m \cong FT_{\overline{\psi}}Prim^{n-2}(V\ mod\ W)[2-n]|\mathbb{G}_m.$$

On the open set $V \subset \mathbb{X}$ where X^W is invertible, the restriction of π becomes the map f, now given by

$$(X_1, ..., X_n) \mapsto \sum_i (w_i^\circ/d)X_i^d/X^W.$$

Then the argument of Lemma 7.3 gives

$$FT_{\overline{\psi}}Prim^{n-2}(V\ mod\ W)[2-n]|\mathbb{G}_m \cong FT_{\overline{\psi}}Rf_!\overline{\mathbb{Q}}_\ell(V\ mod\ W)|\mathbb{G}_m.$$

Theorem 7.5 remains correct as stated. [In its proof, the only modification needed is the analysis now of the sheaves $Kl(\psi_{-w_i^\circ/d}; \chi_{v_i}, w_i)$. Pick for each i a w_i'th root ρ_i of χ_{v_i}. We have geometric isomorphisms

$$Kl(\psi_{-w_i^\circ/d}; \chi_{v_i}, w_i) = [w_i]_\star Kl(\psi_{-w_i^\circ/d}; \chi_{v_i}) = \mathcal{L}_{\rho_i} \otimes [w_i]_\star\mathcal{L}_{\psi_{-w_i^\circ/d}}$$

$$= \mathcal{L}_{\rho_i}\otimes[w_i^\circ]_\star\mathcal{L}_{\psi_{-w_i^\circ/d}} \cong \mathcal{L}_{\rho_i}\otimes Kl(\psi_{-1/d}; all\ the\ w_i^\circ\ char's\ of\ order\ dividing\ w_i)$$

$$\cong Kl(\psi_{-1/d}; all\ the\ w_i^\circ\ w_i'th\ roots\ of\ \chi_{v_i}).]$$

At this point, we have a geometric isomorphism

$$FT_{\overline{\psi}}Prim^{n-2}(V\ mod\ W)[2-n]|\mathbb{G}_m$$

$$\cong [d]^\star\mathcal{H}(\psi_{-1/d}; List(V, W); \emptyset)[2-n].$$

So in the Grothendieck group $K(\mathbb{A}^1_k, \overline{\mathbb{Q}}_\ell)$, we have

$$FT_{\overline{\psi}}Prim^{n-2}(V\ mod\ W)$$

$$= j_\star[d]^\star\mathcal{H}(\psi_{-1/d}; List(V, W); \emptyset) + (punctual,\ supported\ at\ 0).$$

Applying the inverse Fourier Transform, we find that in $K(\mathbb{A}^1_k, \overline{\mathbb{Q}}_\ell)$ we have

$$Prim^{n-2}(V\ mod\ W) = j_\star[d]^\star\mathcal{H}_{V,W} + (geom.\ constant).$$

As before, the fact that $Prim^{n-2}(V\ mod\ W)$ and $j_\star[d]^\star\mathcal{H}_{V,W}$ have the same generic rank shows that there is no geometically constant term, so we have an equality of perverse sheaves in $K(\mathbb{A}^1_k, \overline{\mathbb{Q}}_\ell)$,

$$Prim^{n-2}(V\ mod\ W) = j_\star[d]^\star\mathcal{H}_{V,W}.$$

So these two perverse sheaves have isomorphic semisimplifications. Again by purity, both are geometrically semisimple. So the two sides are geometrically isomorphic. To produce the constant field twist, we repeat the descent argument of Lemma 6.2 to reduce to the case when both descended sides are geometrically irreducible and geometrically isomorphic, hence constant field twists of each other. □

10. Appendix III: Interesting pieces in the original Dwork family

In this appendix, we consider the case $n = d, W = (1, 1, ..., 1)$. We are interested in those eigensheaves $Prim^{n-2}(V\ mod\ W)$ that have unipotent local monodromy at ∞ with a single Jordan block. In view of the explicit description of $Prim^{n-2}(V\ mod\ W)|(\mathbb{G}_m - \mu_d)$ as $[d]^\star\mathcal{H}_{V,W}$, and the known local monodromy of hypergeometric sheaves, as recalled in section 4, we have the following characterization.

Lemma 10.1. *In the case $n = d, W = (1, 1, ..., 1)$, let $V\ mod\ W$ be a character of Γ_W/Δ such that $Prim^{n-2}(V\ mod\ W)$ is nonzero. The following are equivalent.*

1. *Local monodromy at ∞ on $Prim^{n-2}(V\ mod\ W)$ has a single Jordan block.*
2. *Local monodromy at ∞ on $Prim^{n-2}(V\ mod\ W)$ is unipotent with a single Jordan block.*
3. *Every $V = (v_1, ..., v_n)$ in the coset $V\ mod\ W$ has the following property: there is at most one v_i which occurs more than once, i.e., there is at most one $a \in \mathbb{Z}/dZ$ for which the number of indices i with $v_i = a$ exceeds 1.*
4. *A unique $V = (v_1, ..., v_n)$ in the coset $V\ mod\ W$ has the following property: the value $0 \in \mathbb{Z}/d\mathbb{Z}$ occurs more than once among the v_i, and no other value $a \in \mathbb{Z}/dZ$ does.*

Proof. In order for $Prim^{n-2}(V\ mod\ W)$ to be nonzero, the list $List(-V, W)$ must differ from $List(all\ d)$. In this $n = d$ case, that means precisely that $List(-V, W)$ must have at least one value repeated. Adding a suitable multiple of $W = (1, 1, ..., 1)$, we may assume that the value 0 occurs at least twice among the v_i. So (3) ⇔ (4).

For a hypergeometric $\mathcal{H}^{can}(\chi_i's; \rho_j's)$ of type (a, a), local monodromy at ∞ has a single Jordan block if and only if all the ρ_j's coincide, in which case the common value of all the ρ_j's is the eigenvalue in that Jordan block. And $[d]^\star\mathcal{H}^{can}(\chi_i's; \rho_j's)$'s local monodromy at at ∞ has the same number of Jordan blocks (possibly with different eigenvalues) as that of $\mathcal{H}^{can}(\chi_i's; \rho_j's)$. In our situation, if we denote by $(\chi_1, ..., \chi_d)$ all the characters of order dividing d, and by $(\chi_{-v_1}, ..., \chi_{-v_d})$ the list $List(-V, W)$, then

$$\mathcal{H}_{V,W} = \mathcal{H}^{can}(\mathbf{Cancel}((\chi_1, ..., \chi_d); (\chi_{-v_1}, ..., \chi_{-v_d}))).$$

So in order for local monodromy at ∞ to have a single Jordan block, we need all but one of the characters that occur among the χ_{v_i} to cancel into $List(all\ d)$. But those that cancel are precisely those which occur with multiplicity 1. So (1) $\Leftrightarrow$ (3). Now (2) $\Rightarrow$ (1) is trivial, and (2) $\Rightarrow$ (4) by the explicit description of local monodromy at ∞ in terms of the ρ_j's. □

Lemma 10.2. *Suppose the equivalent conditions of Lemma 10.1 hold. Denote by a the rank of* $Prim^{n-2}(V \bmod W)$. *Then on any geometric fibre of* $(\mathbb{A}^1 - \mu_d)/Spec(\mathbb{Z}[\zeta_d][1/d\ell])$, *the geometric monodromy group* G_{geom} *attached to* $Prim^{n-2}(V \bmod W)$ *has identity component either* $SL(a)$ *or* $SO(a)$ *or, if a is even, possibly* $Sp(a)$.

Proof. By the Tame Specialization Theorem [Ka-ESDE, 8.17.13], the group is the same on all geometric fibres. So it suffices to look in some characteristic $p > a$. Because on our geometric fibre $\mathcal{H}_{V,W}$ began life over a finite field, and is geometrically irreducible, G^0_{geom} is semisimple. The case $a = 1$ is trivial. Suppose $a \geq 2$. Because its local monodromy at ∞ is a single unipotent block, the hypergeometric $\mathcal{H}_{V,W}$ is not Belyi induced, or inverse Belyi induced, or Kummer induced, and $G^{0,der}_{geom}$ is nontrivial. The result now follows from [Ka-ESDE, 8.11.2]. □

Lemma 10.3. *Suppose the equivalent conditions of Lemma 10.1 hold. Denote by a the rank of* $Prim^{n-2}(V \bmod W)$. *Suppose* $a \geq 2$. *Denote by V the unique element in the coset V mod W in which* $0 \in \mathbb{Z}/dZ$ *occurs with multiplicity* $a + 1$, *while no other value occurs more than once. Then we have the following results.*

(1) *Suppose that* $-V$ *is not a permutation of V. Then* $G_{geom} = SL(a)$ *if* $n - 2$ *is odd, and* $G_{geom} = \{A \in GL(a) | det(A) = \pm 1\}$ *if* $n - 2$ *is even.*
(2) *If* $-V$ *is a permutation of V and* $n - 2$ *is odd, then a is even and* $G_{geom} = Sp(a)$.
(3) *If* $-V$ *is a permutation of V and* $n - 2$ *is even, then a is odd and* $G_{geom} = O(a)$.

Proof. That these results hold for $\mathcal{H}_{V,W}$ results from [Ka-ESDE, 8.11.5, 8.8.1,8.8.2]. In applying those results, one must remember that $\sum_i v_i = 0 \in \mathbb{Z}/d\mathbb{Z}$, which implies that ("even after cancellation") local monodromy at ∞ has determinant one. Thus in turn implies that when d, or equivalently $n - 2$, is even, then ("even after cancellation") local monodromy at 0 has determinant the quadratic character, and hence local monodromy at 1 also has determinant the quadratic character. So in the cases where the group does not have determinant one, it is because local monodromy at 1 is a true reflection. After $[d]^{\star}$, which is finite étale over 1, we get a true reflection at each point in μ_d. □

Lemma 10.4. *If the equivalent conditions of the previous lemma hold, then over* $\mathbb{C}$ *the Hodge numbers of* $Prim^{n-2}(V\ mod\ W)$ *form an unbroken string of* 1*'s, i.e., the nonzero among the* $Prim^{b,n-2-b}(V\ mod\ W)$ *are all* 1*, and the* b *for which* $Prim^{b,n-2-b}(V\ mod\ W)$ *is nonzero form (the integers in) an interval* $[A, A-1+a]$ *for some* A*.*

Proof. From the explicit determination of G_{geom}, we see in particular that $Prim^{n-2}(V\ mod\ W)$ is an irreducible local system. Looking in a $\mathbb{C}$-fibre of $(\mathbb{A}^1 - \mu_d)/Spec(\mathbb{Z}[\zeta_d][1/d\ell])$ and applying Riemann-Hilbert, we get that the D-module $Prim^{n-2}_{dR}(V\ mod\ W)$ is irreducible. By Griffiths transversality, this irreducibility implies that the b for which $Prim^{b,n-2-b}(V\ mod\ W)$ is nonzero form (the integers in) an interval. The fact that local monodromy at ∞ is unipotent with a single Jordan block implies that the number of nonzero Hodge groups $Prim^{b,n-2-b}(V\ mod\ W)$ is at least a, cf. the proof of Lemma 8.8, (3) $\Leftrightarrow$ (5). □

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