ON TWO QUESTIONS OF ENTIN, KEATING, AND RUDNICK ON PRIMITIVE DIRICHLET CHARACTERS

NICHOLAS M. KATZ

Abstract. This is Part II of the paper "A question of Keating and Rudnick about primitive Dirichlet characters with squarefree conductor" [Ka-QKRPD]. Here we prove two independence results for tuples of character sums, either formed with a variable character and its powers, or formed with two characters and their product.

1. Introduction

We work over a finite field $k = \mathbb{F}_q$ inside a fixed algebraic closure $\overline{k}$. We fix a squarefree monic polynomial $f(X) \in k[X]$ of degree $n \geq 2$. We form the k-algebra

$$B := k[X]/(f(X)),$$

which is finite étale over k of degree n. We denote by $u \in B$ the image of X in B under the "reduction mod f" homomorphism $k[X] \to B$. Thus we may write this homomorphism as

$$g(X) \in k[X] \mapsto g(u) \in B.$$

We denote by $B^\times$ the multiplicative group of B, and by χ a character

$$\chi : B^\times \to \mathbb{C}^\times.$$

We extend χ to all of B by decreeing that $\chi(b) := 0$ if $b \in B$ is not invertible.

A character χ is called odd if it is nontrivial on the subgroup $k^\times$ of $B^\times$, even otherwise. A character χ is called "primitive" if the following condition holds. Factor f as a product of distinct monic irreducible polynomials, say $f = \prod_j f_j$. Then B is canonically the product of the algebras $B_j := k[X]/(f_j(X))$, and χ is the product of characters χ_j of these $B_j^\times$ factors. The primitivity requirement is that each character χ_j of each $B_j^\times$ be nontrivial. A character χ is called "totally ramified" if it is both odd and primitive.

The (possibly imprimitive) Dirichlet L-function $L(\chi, T)$ attached to a character χ of $B^\times$ is the power series in $\mathbb{C}[[T]]$ given by

$$L(\chi, T) := \sum_{\text{monic } g(X) \in k[X]} \chi(g(u)) T^{\deg(g)} = \sum_{n \geq 0} A_n T^n,$$

$$A_n := \sum_{\substack{g(X) \in k[X] \text{ monic}, \\ \deg(g)=n, \\ \gcd(f,g)=1}} \chi(g(u)).$$

If χ is nontrivial, then $L(\chi, T)$ is a polynomial in T of degree $n-1$.

If χ is totally ramified, then this L-function is "pure of weight one", i.e., in its factored form $\prod_{i=1}^{n-1}(1-\beta_i T)$, each reciprocal root β_i has complex absolute value

$$|\beta_i|_{\mathbb{C}} = \sqrt{q}.$$

Conversely, if this L-function is pure of weight one, then χ is totally ramified, cf. [Ka-QKRPD, Lemma 2.2, (2)]. For such a χ, its "unitarized" L-function $L(\chi, T/\sqrt{q})$ is the reversed characteristic polynomial $\det(1 - TA_\chi)$ of some element A_χ in the unitary group $U(n-1)$ (e.g., take $A_\chi := \mathrm{Diag}(\beta_1/\sqrt{q}, ..., \beta_{n-1}/\sqrt{q})$). In $U(n-1)$, conjugacy classes are determined by their characteristic polynomials, so $L(\chi, T/\sqrt{q})$ is $\det(1 - T\theta_\chi)$ for a well defined conjugacy class θ_χ in $U(n-1)$.

In order to keep track of the input data (k, f, χ), we denote this conjugacy class

$$\theta_{k,f,\chi}.$$

We denote by $\mathrm{TotRam}(k, f)$ the set of totally ramified characters. More generally, given an integer $d \geq 1$,we say that a character χ is "d-fold totally ramified" if each of the characters $\chi^a, 1 \leq a \leq d$, is totally ramified. We denote by $\mathrm{TotRam}_d(k, f)$ the set of d-fold totally ramified characters. [So in this notation, the set $\mathrm{TotRam}_1(k, f)$ is simply the set $\mathrm{TotRam}(k, f)$.]

We say that an ordered pair of characters (χ, Λ) is a totally ramified pair if each of the three characters $\chi, \Lambda, \chi\Lambda$ is totally ramified. We denote by $\mathrm{TotRamPairs}(k, f)$ the set of totally ramified pairs.

In our earlier paper [Ka-QKRPD, Thm. 5.10], we proved the following theorem.

Theorem 1.1. *Fix an integer $n \geq 2$ and a sequence of data (k_i, f_i) with k_i a finite field (of possibly varying characteristic) and $f_i(X) \in k_i[X]$ squarefree of degree n. If $\#k_i$ tends archimedeanly to ∞, the collections of conjugacy classes*

$$\{\theta_{k_i, f_i, \chi}\}_{\chi \in \mathrm{TotRam}_1(k_i, f_i)}$$

become equidistributed in the space $U(n-1)^{\#}$ of conjugacy classes in $U(n-1)$ for its "Haar measure" of total mass one.

In this paper, we will prove the following two theorems.

Theorem 1.2. *Fix an integer $d \geq 2$, an integer $n \geq 2$, and a sequence of data (k_i, f_i) with k_i a finite field (of possibly varying characteristic p, subject only to $p > d$) and $f_i(X) \in k_i[X]$ squarefree of degree n. If $\#k_i$ tends archimedeanly to ∞, the collections of d-tuples of conjugacy classes*

$$\{(\theta_{k_i,f_i,\chi}, \theta_{k_i,f_i,\chi^2}, ..., \theta_{k_i,f_i,\chi^d})\}_{\chi \in \mathrm{TotRam}_d(k_i,f_i)}$$

become equidistributed in the space $(U(n-1)^{\#})^d$ of conjugacy classes in the d-fold product $(U(n-1))^d$.

Theorem 1.3. *Fix an integer $n \geq 2$, and a sequence of data (k_i, f_i) with k_i a finite field (of possibly varying characteristic p, subject only to $p > 2$) and $f_i(X) \in k_i[X]$ squarefree of degree n. If $\#k_i$ tends archimedeanly to ∞, the collections of triples of conjugacy classes*

$$\{(\theta_{k_i,f_i,\chi}, \theta_{k_i,f_i,\Lambda}, \theta_{k_i,f_i,\chi\Lambda})\}_{(\chi,\Lambda) \in \mathrm{TotRamPairs}(k_i,f_i)}$$

become equidistributed in the space $(U(n-1)^{\#})^3$ of conjugacy classes in the triple product $(U(n-1))^3$ for its 'Haar measure" of total mass one.

Remark 1.4. One could imagine a notion of say, totally ramified triples (χ, ρ, Λ), where various products of these characters are required to be totally ramified, e.g., one might require each of $\chi, \rho, \Lambda, \chi\rho, \chi\Lambda, \rho\Lambda$ to be totally ramified, and ask about equidistribution in $(U(n-1)^{\#})^6$. We have not pursued this line of investigation.

2. Proof of Theorem 1.2; first steps

We first recall the cohomological genesis of the L-function. We view $B^\times$ as the group of k points of the k-torus $\mathbb{B}^\times$ whose group of R-valued points, for any k-algebra R, is the group $(B \otimes_k R)^\times$. Over $\mathbb{B}^\times$ we have the Lang torsor, whose pushout by $1 - Frob_k$ is the lisse, rank one Kummer sheaf $\mathcal{L}_\chi$ on $\mathbb{B}^\times$. Denoting by u the image of X in B, we have a k-morphism $\mathbb{A}^1[1/f] \to \mathbb{B}^\times$ given by $t \mapsto u - t$. The pullback of $\mathcal{L}_\chi$ by this map is the lisse rank one sheaf $\mathcal{L}_{\chi(u-t)}$ on $\mathbb{A}^1[1/f]$. The L-function $L(\chi, T)$ is the L-function of $\mathbb{A}^1[1/f]$ with coefficients in $\mathcal{L}_{\chi(u-t)}$. In what follows, we will work with ℓ-adic cohomology, for an ℓ which is different from the characteristic p we are in. For example, we can take $\ell = 2$ throughout, because in the target theorem we require $p > d$ and $d \geq 2$.

Over $\overline{k}$, our square free polynomial f factors as $f(X) = \prod_i (X - a_i)$, and $\mathcal{L}_{\chi(u-t)}$ on $\mathbb{A}^1[1/f]$ becomes $\otimes_{i=1}^n \mathcal{L}_{\chi_i(a_i-t)}$ for characters χ_i of $\pi_1^{tame}(\mathbb{G}_m/\overline{k})$. The χ_i are the "geometric components" of χ. A character χ is totally ramified if each of its geometric components is nontrivial and the product $\prod_i \chi_i$ is nontrivial. Recall from [Ka-QKRPD, §5] that a totally ramified χ is said to be generic if in addition it satisfies the following two conditions:

(1) The χ_i are all distinct.
(2) For at least one index j, $\chi_j^n \neq \prod_i \chi_i$.

We denote by $\mathrm{TotRamGen}_1(k, f)$ the set of totally ramified characters of $B^\times$ which are generic. We have the following lemma.

Lemma 2.1. *Given $n \geq 2$, take $C_n := n(2n+3)$. Then for $q \geq n+1$ we have the estimates*

$$\#\mathrm{TotRamGen}_1(k, f) > q^n - C_n q^{n-1}$$

and

$$\#(\{\text{all char's of } B^\times\} \setminus \mathrm{TotRamGen}_1(k, f)) < C_n q^{n-1}.$$

Proof. In [Ka-QKRPD, Lemma 5.5], we showed that for $q \geq n+1$ we have the estimate

$$\#\mathrm{TotRamGen}_1(k, f) \geq (q-1-n)^n + (q-2)^n - q^n + 1 - n \sum_{0 \leq i \leq n-1} q^i.$$

If we write $\#\mathrm{TotRamGen}_1(k, f)$ as $q^n - 1 - E$, then

$$E \leq (q^n - (q-1-n)^n) + (q^n - (q-2)^n) - 1 + n \sum_{0 \leq i \leq n-1} q^i.$$

Estimate (brutally) the sum $\sum_{0 \leq i \leq n-1} q^i$ by nq^{n-1}, and estimate each of the differences by the mean value theorem:

$$q^n - (q-1-n)^n \leq nq^{n-1}(n+1), \quad q^n - (q-2)^n \leq nq^{n-1}2,$$

to get the first assertion.

The number of characters of $B^\times$ is $\#B^\times \geq \#B - 1 = q^n - 1$. So the number of characters not in $\mathrm{TotRamGen}_1(k, f)$ is

$$< q^n - 1 - (q^n - C_n q^{n-1}) < C_n q^{n-1}.$$

□

We now make another ad hoc definition. Given an integer $d \geq 1$ denote by $\mathrm{TotRamGen}_d(k, f)$ the set of those characters χ such that $\chi^{d!}$ is totally ramified and generic. Notice that if χ lies in $\mathrm{TotRamGen}_d(k, f)$, then for every divisor a of $d!$, χ^a lies in $\mathrm{TotRamGen}_1(k, f)$.

Lemma 2.2. *Given $n \geq 2$ and $d \geq 1$, define the constant $C_{n,d} := (d!)^n C_n$. Then for $q \geq n+1$, we have the estimates*

$$\#(\{\text{all char's of } B^\times\} \setminus \mathrm{TotRamGen}_d(k,f)) \leq C_{n,d}q^{n-1}$$

and

$$\#\mathrm{TotRamGen}_d(k,f) > (q-1)^n - C_{n,d}q^{n-1}.$$

Proof. A character χ fails to lie in $\mathrm{TotRamGen}_d(k,f)$ if $\chi^{d!}$ fails to lie in $\mathrm{TotRamGen}_1(k,f)$. Let us denote by $E(k,f)$ the set

$$E(k,f) := \{\text{all characters of } B^\times\} \setminus \mathrm{TotRamGen}_d(k,f).$$

By the previous lemma, $\#E(k,f) \leq C_n q^{n-1}$.

The map $\chi \mapsto \chi^{d!}$ has a kernel of cardinality $\#\mu_{d!}(B^\times)$, for$\mu_a(B^\times) := \{\zeta \in B^\times | \zeta^{d!} = 1\}$. Now $B^\times$ is a product of multiplicative groups of fields $k_{d_i}^\times$ with $\sum_i d_i = n$, where the d_i are the degrees of the irreducible factors of f, and k_{d_i}/k has degree d_i. Thus in all cases $\#\mu_{d!}(B^\times) \leq (d!)^n$. So the map $\chi \mapsto \chi^{d!}$ is at most $(d!)^n$ to one. Hence there are at most $(d!)^n C_n q^{n-1}$ characters χ such that χ^a lies in $E(k,f)$. This gives the first asserted estimate.

For the second, we argue as follows. The description above of $B^\times$ is a product of multiplicative groups of fields shows that $\#B^\times = \prod_i (q^{d_i}-1)$ is $\geq (q-1)^n$. So the number of characters of $B^\times$, namely $\#B^\times$, is $\geq (q-1)^n$, so the second assertion follows from the first. ☐

In our earlier paper, we proved the following theorem [Ka-QKRPD, Thm. 4.1, with λ there taken to be $(-1)^n$].

Theorem 2.3. *Suppose $n \geq 2$. Let χ be a character of $B^\times$ which is totally ramified and generic. Form the perverse sheaf*

$$N(\chi) := [(-1)^n f]_\star(\mathcal{L}_{\chi(u-t)})(1/2)[1]$$

on $\mathbb{G}_m/k$. Then we have the following results.

1. *$N(\chi)$ is geometrically irreducible, pure of weight zero, and lies in the Tannakian category $\mathcal{P}_{arith}$ in the sense of* [Ka-CE]. *It has generic rank n, Tannakian "dimension" $n-1$, and it has, geometrically, at most $2n$ bad characters (i.e., characters ρ of $\pi_1^{tame}(\mathbb{G}_m/\overline{k})$ such that for $j : \mathbb{G}_m/\overline{k} \subset \mathbb{P}^1/\overline{k}$ the inclusion, the "forget supports" map $Rj_!(N(\chi)\otimes\mathcal{L}_\rho) \to Rj_\star(N(\chi)\otimes\mathcal{L}_\rho)$ fails to be an isomorphism).*
2. *$N(\chi)$ has $G_{geom} = G_{arith} = GL(n-1)$.*

Our key technical result, which will be proven in Section 3, is this.

Theorem 2.4. *Suppose $n \geq 2$, $d \geq 2$, and $p > d$. Suppose χ is a character of $B^\times$ which lies in* $\mathrm{TotRamGen}_d(k, f)$. *For each integer $1 \leq a \leq d$, denote by $[a]$ the a-th power map $x \mapsto x^a$ of $\mathbb{G}_m$ to itself. Then the direct sum object*

$$\mathcal{F}_d(\chi) := \bigoplus_{a=1}^{d} [a]_\star(N(\chi^a))$$

is pure of weight zero and lies in the Tannakian category $\mathcal{P}_{arith}$ in the sense of [Ka-CE]. *It has generic rank $nd(d+1)/2$, Tannakian "dimension" $(n-1)d$, it has at most $d(d+1)n$ bad characters, and it has*

$$G_{geom} = G_{arith} = (GL(n-1))^d.$$

To apply it, we need some background material.

Lemma 2.5. *Let ρ be a character of $k^\times$ which is good for $\mathcal{F}_d(\chi)$. Denote by ρ_{Norm} the character of $B^\times$ given by $\rho \circ \mathrm{Norm}_{B/k}$. For each a with $1 \leq a \leq d$, the product character $\chi^a \rho^a_{\mathrm{Norm}}$ is totally ramified, and the conjugacy class $\theta_{k,f,\chi^a\rho^a_{\mathrm{Norm}}}$ is given by*

$$\det(1 - T\theta_{k,f,\chi^a\rho^a_{\mathrm{Norm}}}) = \det(1 - TFrob_k | H^0_c(\mathbb{G}_m/\overline{k}, [a]_\star(N(\chi)) \otimes \mathcal{L}_\rho)),$$

cf. [Ka-QKRPD, end of §4].

Proof. Indeed, given an integer $a \geq 1$ prime to p, ρ is good for $[a]_\star(N(\chi^a))$ if and only if $H^0_c(\mathbb{G}_m/\overline{k}, [a]_\star(N(\chi^a)) \otimes \mathcal{L}_\rho)$ is pure of weight zero. This cohomology group is, by the Leray spectral sequence for $[a]_\star$, the group

$$H^0_c(\mathbb{G}_m/\overline{k}, N(\chi^a) \otimes \mathcal{L}_{\rho^a}).$$

By the Leray spectral sequence for $f_\star$, this group is in turn equal to the group

$$H^0_c(\mathbb{A}^1[1/f]/\overline{k}, \mathcal{L}_{(\chi^a\rho^a_{\mathrm{Norm}})(u-t)}).$$

□

The general equidistribution theorem [Ka-CE, Remark 7.5 and the proof of Theorem 28.1] gives the following corollary of the theorem.

Corollary 2.6. *Suppose $n \geq 2$, $d \geq 2$, $p > d$, and χ a character of $B^\times$ which lies in* $\mathrm{TotRamGen}_d(k, f)$. *Let Ξ be an irreducible nontrivial representation of the d-fold product $(U(n-1))^d$ which occurs in $\mathbb{V}^{\otimes r} \otimes (\mathbb{V}^\vee)^{\otimes s}$ for $\mathbb{V} := \oplus_{i=1}^d std_{n-1,i}$, for $std_{n-1,i}$ the standard representation of the i'th factor $U(n-1)$ in the d-fold product $(U(n-1))^d$. Define the constant*

$$C := d(d+1)n.$$

Denote by $\mathrm{Good}(\mathcal{F}_d(\chi), k)$ *the set of characters* ρ *which are good for* $\mathcal{F}_d(\chi)$*. For each good* ρ*, denote by* $\theta_d(k, \chi, \rho)$ *the conjugacy class in* $(U(n-1))^d$ *given by*

$$\theta_d(k, \chi, \rho) := (\theta_{k,f,\chi\rho_{\mathrm{Norm}}}, \theta_{k,f,\chi^2\rho^2_{\mathrm{Norm}}}, ..., \theta_{k,f,\chi^d\rho^d_{\mathrm{Norm}}}).$$

Then for $\#k > (1+C)^2$*, we have the estimate*

$$|\sum_{\rho \in \mathrm{Good}(\mathcal{F}_d(\chi),k)} \mathrm{Trace}(\Xi(\theta_d(k,\chi,\rho)))| \leq \frac{\#\mathrm{Good}(\mathcal{F}_d(\chi),k) 2(r+s+1)C^{r+s}}{\sqrt{\#k}}.$$

Let us now explain how our target Theorem 1.2 follows from a variant of this corollary. Given χ a character of $B^\times$ which lies in $\mathrm{TotRamGen}_d(k, f)$, we form the auxiliary object

$$\mathcal{G}_d(\chi) := [d!]_\star(N(\chi^{d!})).$$

This object has dimension $n-1$, generic rank $n(d!)$, and so has at most $2n(!)$ bad characters.

Lemma 2.7. *Suppose* χ *lies in* $\mathrm{TotRamGen}_d(k, f)$*. Then a character* ρ *is good for* $\mathcal{G}_d(\chi)$ *if and only if the product* $\chi\rho_{\mathrm{Norm}}$ *lies in* $\mathrm{TotRamGen}_d(k, f)$*.*

Proof. If the product $\chi\rho_{\mathrm{Norm}}$ lies in $\mathrm{TotRamGen}_d(k, f)$, then in particular each power $\chi^{d!}\rho^{d!}_{\mathrm{Norm}}$, is totally ramified, which is to say that ρ is good for $\mathcal{G}_d(\chi)$. Conversely, suppose that $\chi^{d!}\rho^{d!}_{\mathrm{Norm}}$ is totally ramified. By hypothesis, $\chi^{d!}$ is both totally ramified and generic. Denote by $\chi_i, i = 1, ..., n$ the geometric components of χ. So the $\chi_i^{d!}$ are pairwise distinct, and there exists an index j such that $\chi_j^{n(d!)} \neq \prod_i \chi_i^{d!}$. Then for **any** character σ of $\pi_1^{tame}(\mathbb{G}_m/\overline{k})$, the geometric components of $\chi^{d!}\sigma_{\mathrm{Norm}}$ are the $\chi_i^{d!}\sigma$. So these are automatically pairwise distinct. And for the same index j, $(\chi_j^{d!}\sigma)^n \neq \prod_i(\chi_i^{d!}\sigma)$. □

Here is our variant corollary.

Corollary 2.8. *Suppose* $n \geq 2$*,* $d \geq 2$*,* $p > d$*, and* χ *a character of* $B^\times$ *which lies in* $\mathrm{TotRamGen}_d(k, f)$*. Let* Ξ *be an irreducible nontrivial representation of the d-fold product* $(U(n-1))^d$ *which occurs in* $\mathbb{V}^{\otimes r} \otimes (\mathbb{V}^\vee)^{\otimes s}$ *for* $\mathbb{V} := \oplus_{i=1}^d std_{n-1,i}$*, for* $std_{n-1,i}$ *the standard representation of the i'th factor* $U(n-1)$ *in the d-fold product* $(U(n-1))^d$*. Define the constant* $C := 2n(d!)$*. Then for* $\#k > (1+C)^2$*, we have the estimate*

$$|\sum_{\rho \in \mathrm{Good}(\mathcal{G}_d(\chi),k)} \mathrm{Trace}(\Xi(\theta_d(k,\chi,\rho)))| \leq \frac{\#\mathrm{Good}(\mathcal{G}_d(\chi),k) 2(r+s+1)C^{r+s}}{\sqrt{\#k}}.$$

Proof. In the estimate of [Ka-CE, Remark 7.5 and the proof of Theorem 28.1] used above, one can replace the summation over $\mathrm{Good}(\mathcal{F}_d(\chi), k)$ by the sum over any subset of $\mathrm{Good}(\mathcal{F}_d(\chi), k)$, here the subset $\mathrm{Good}(\mathcal{G}_d(\chi), k)$, so long as the constant C exceeds the dimension (here $d(n-1)$), the generic rank (here $nd(d+1)/2$) and the number of characters which are omitted in the sum (here the set $Bad(\mathcal{G}_d(\chi), k)$, whose cardinality is at most the C we have chosen. □

Given this variant, we complete the proof of Theorem 1.2 as follows. On the set $\mathrm{TotRamGen}_d(k, f)$, we have an equivalence relation $\chi \cong \Lambda$ if $\chi = \Lambda\rho_{\mathrm{Norm}}$ for some ρ. For χ in $\mathrm{TotRamGen}_d(k, f)$, we denote by $\theta_d(k, \chi) := \theta_d(k, \chi, \mathbb{1})$ the d-tuple of conjugacy classes

$$\theta_d(k, \chi) := (\theta_{k,f,\chi}, \theta_{k,f,\chi^2}, ..., \theta_{k,f,\chi^d}).$$

The sums in Corollary 2.8 are precisely the sums over the equivalence classes. Adding these all up, we find that $\#k > (1 + C)^2$, we have the estimate

$$|\sum_{\chi \in \mathrm{TotRamGen}_d(k,f)} \mathrm{Trace}(\Xi(\theta_d(\chi, \rho)))| \leq \frac{\#\mathrm{TotRamGen}_d(k, f) 2(r + s + 1)C^{r+s}}{\sqrt{\#k}}.$$

However, it is the sum

$$\sum_{\chi \in \mathrm{TotRam}_d(k,f)} \mathrm{Trace}(\Xi(\theta_d(\chi, \rho)))$$

we need to estimate. It differs from the $\sum_{\chi \in \mathrm{TotRamGen}_d(k,f)}$ by at most $\#Bad(\mathcal{G}_d(\chi), k) \leq C$ terms, each of which is bounded in absolute value by $(r + s)C^{r+s}$. So the $\sum_{\chi \in \mathrm{TotRam}_d(k,f)}$ is bounded in absolute value by

$$\frac{\#\mathrm{TotRamGen}_d(k, f) 2(r + s + 1)C^{r+s}}{\sqrt{\#k}} + C(r + s)C^{r+s} \leq$$

$$\leq \frac{\#\mathrm{TotRam}_d(k, f) 2(r + s + 1)C^{r+s}}{\sqrt{\#k}} + C(r + s)C^{r+s}.$$

On the other hand, for the constant $C_{n,d}$ of Lemma 2.2, we have

$$\#\mathrm{TotRam}_d(k, f) \geq \#\mathrm{TotRamGen}_d(k, f) \geq (q - 1)^n - C_{n,d}q^{n-1}.$$

For a new constant $C_{n,d,r,s}$, we will have that for $q := \#k > C_{n,d,r,s}$, both

$$\#\mathrm{TotRam}_d(k, f) \geq q^n/2$$

and

$$C(r + s)C^{r+s} \leq q/2.$$

So for q larger than $(1+C)^2 + C_{n,d,r,s}$, we have the estimate

$$|\sum_{\chi \in \mathrm{TotRam}_d(k,f)} \mathrm{Trace}(\Xi(\theta_d(\chi,\rho)))| \le \frac{\#\mathrm{TotRam}_d(k,f)4(r+s+1)C^{r+s}}{\sqrt{\#k}}.$$

This concludes the deduction of Theorem 1.2 from Corollary 2.8. It remains only to prove Theorem 2.4.

3. Proof of Theorem 2.4

Let us recall the statement.

Theorem 3.1. (Restatement of Theorem 2.4) *Suppose $n \ge 2$, $d \ge 2$, and $p > d$. Suppose χ is a character of $B^\times$ which lies in* $\mathrm{TotRamGen}_d(k,f)$. *For each integer $1 \le a \le d$, denote by $[a]$ the a-th power map $x \mapsto x^a$ of $\mathbb{G}_m$ to itself. Then the direct sum object*

$$\mathcal{F}_d(\chi) := \bigoplus_{a=1}^{d} [a]_\star(N(\chi^a))$$

is pure of weight zero and lies in the Tannakian category $\mathcal{P}_{arith}$ in the sense of [Ka-CE]. *It has generic rank $nd(d+1)/2$, Tannakian "dimension" $(n-1)d$, it has at most $d(d+1)n$ bad characters, and it has*

$$G_{geom} = G_{arith} = (GL(n-1))^d.$$

Proof. We have a priori inclusions

$$G_{geom} \subset G_{arith} \subset (GL(n-1))^d,$$

so it suffices to show that $G_{geom} = (GL(n-1))^d$.

For each $1 \le a \le d$, χ^a lies in $\mathrm{TotRamGen}_1(k,f)$, so by Theorem 2.3 the object $N(\chi^a)$ has $G_{geom} = GL(n-1)$. By [Ka-CE, Thm. 8.1], the direct image object $[a]_\star(N(\chi^a))$ also has $G_{geom} = GL(n-1)$ (simply because $GL(n-1)$ is connected). As explained in [Ka-CE, Thm. 13.5 and Chapter 20], in order to show that $\bigoplus_{a=1}^{d}[a]_\star(N(\chi^a))$ has $G_{geom} = (GL(n-1))^d$, we must show two things.

(1) For the Tannakian determinants "det"$([a]_\star(N(\chi^a)))$, their direct sum

$$\bigoplus_{a=1}^{d} \text{“det”}([a]_\star(N(\chi^a)))$$

has $G_{geom} = (GL(1))^d$.

(2) For each $1 \le a < b \le d$, there is no geometric isomorphism from either $[a]_\star(N(\chi^a))$ or its Tannakian dual with $([b]_\star(N(\chi^b))) \star_{mid} \mathcal{H}$, the middle multiplicative convolution of $[b]_\star(N(\chi^b))$ with any one dimensional object $\mathcal{H}$ in the category $\mathcal{P}_{geom}$ of [Ka-CE].

The first of these will be shown in Theorem 3.5. The second will be shown in Theorem 3.6, the paragraph preceding it, and Theorem 3.7.

Our first task is to compute the Tannakian determinants. Recall that the one-dimensional objects in $\mathcal{P}_{geom}$ are the delta objects δ_α, $a \in \mathbb{G}_m(\overline{k})$, (these are the hypergeometrics of type $(0,0)$) and the multiplicative translates of the (shifts by [1] of the) irreducible hypergeometric sheaves

$$\mathcal{H}(!, \psi; \rho_1, ..., \rho_n; \Lambda_1, ..., \Lambda_m)$$

of type (n, m) with $n \geq 0, m \geq 0, n+m \geq 1$. Recall that irreducibility means that either n or m is zero or, if both $n, m > 0$ that no ρ_i is any Λ_j. Recall also that the middle convolution of two such hypergeometrics is obtained by first concatenating separately the upper parameters (the ρ's) and the lower parameters (the Λ's) of the two convolvees, and then canceling each character that appears both up and down, i.e., performing the **Cancel** operation of [Ka-ESDE, 9.3.1]. This description of middle convolution of irreducible hypergeometrics follows from [Ka-ESDE, 8.4.7] and the *Perv/Neg* point of view of Gabber-Loeser [G-L, Remark page 535].

Given a finite nonempty set S of distinct characters of $\pi_1^{tame}(\mathbb{G}_m/\overline{k})$, and an integer $m \geq 1$ we denote by $\mathrm{Roots}_m(S)$ the set

$$\mathrm{Roots}_m(\sigma) := \{\text{characters } \tau | \tau^m \in S\}.$$

If $m = m_0 p^r$ with m_0 prime to p, then $\#\mathrm{Roots}_m(S) = m_0 \# S$.

Given an integer a prime to p, we denote by Char_a the group of characters of $\pi_1^{tame}(\mathbb{G}_m/\overline{k})$ of order dividing a. For σ a character, we denote by $\sigma\mathrm{Char}_a$ the set of characters $\sigma\rho$ with $\rho \in \mathrm{Char}_a$.

Our next task is to describe the local monodromies at 0 and ∞ of each of the objects $[a]_\star(N(\chi^a))$. We do this in terms of the geometric components $\chi_i, i = 1, ..., n$, of $\chi \in \mathrm{TotRamGen}_d(k, f)$. We number the geometric components so that

$$\chi_1^{d!n} \neq \prod_i \chi_i^{d!}.$$

Lemma 3.2. *For $1 \leq a \leq d$, and $\chi \in \mathrm{TotRamGen}_d(k, f)$, the local monodromy of $[a]_\star(N(\chi^a))$ at 0 is tame. The na tame characters occurring in it are the characters $\chi_i\mathrm{Char}_a$, for $i = 1, ..., n$. These na characters are all distinct.*

Proof. The local monodromy of $N(\chi^a)$ at 0 is tame, with characters the χ_i^a. These are all distinct, because $\chi \in \mathrm{TotRamGen}_d(k, f)$. When we apply $[a]_\star$, the local monodromy remains tame, now with characters the $\mathrm{Roots}_a(\{\chi_i^a, i = 1, .., n\})$. These are precisely the characters $\chi_i\mathrm{Char}_a$, for $i = 1, ..., n$. They are all distinct because the $\chi_i^a, i = 1, .., n$, are all distinct. $\square$

Lemma 3.3. *Write n as n_0p^r with n_0 prime to p. For $1 \le a \le d$, and $\chi \in \mathrm{TotRamGen}_d(k, f)$, the local monodromy of $[a]_\star(N(\chi^a))$ at ∞ is the direct sum of a tame part and a totally wild part. The tame part has rank n_0a. The n_0a tame characters occurring in it are the characters $\mathrm{Roots}_n(\prod_i \chi_i)\mathrm{Char}_a$. These n_0a characters are all distinct. None of them is χ_1, and moreover none of them lies in $\chi_1\mathrm{Char}_b$ for any $1 \le b \le d$.*

Proof. By Frobenius reciprocity, a tame character ρ occurs in the local monodromy at ∞ of $N(\chi^a) = [(-1)^n f]_\star \mathcal{L}_{\prod_i \chi_i^a(a_i - t)}$ if and only if $f^\star(\rho)$, which is ρ^n as a character of the inertia (local monodromy) group $I(\infty)$ at ∞ , occurs in the local monodromy at ∞ of $\mathcal{L}_{\prod_i \chi_i^a(a_i - t)}$, i.e., if and only if $\rho^n = \prod_i \chi_i^a$. Applying $[a]_\star$, the characters in the tame part of the local monodromy of $[a]_\star(N(\chi^a))$ at ∞ are the a'th roots of these, i.e. they are the set $\mathrm{Roots}_{na}(\prod_i \chi_i^a)$, or equivalently the characters $\mathrm{Roots}_n(\prod_i \chi_i)\mathrm{Char}_a$. We claim that none of these is $\chi_1\sigma$ for any $\sigma \in \mathrm{Char}_b$ for any $1 \le b \le d$. Indeed, suppose

$$(\chi_1\sigma)^{an} = \prod_i \chi_i^a.$$

If $b = a$, then $\chi_1^{an} = \prod_i \chi_i^a$, which is impossible as $\chi_1^{d!n} \ne \prod_i \chi_i^{d!}$. If $b \ne a$, then ab divides $d!$, and taking the b'th power of our putative relation we again get a contradiction. $\square$

Lemma 3.4. *The Tannakian determinant of $[a]_\star(N(\chi^a))$ is a multiplicative translate of the $[1]$ shift of the hypergeometric sheaf*

$$\mathbf{Cancel}\mathcal{H}(!, \psi; \chi_1\mathrm{Char}_a, .., \chi_n\mathrm{Char}_a; \mathrm{Roots}_n(\prod_i \chi_i)\mathrm{Char}_a).$$

Proof. Given the local monodromies at 0 and ∞ of $[a]_\star(N(\chi^a))$, this is immediate from [Ka-CE, 16.1 and 16.3]. $\square$

Theorem 3.5. *The direct sum object $\bigoplus_{a=1}^d$ "det"$([a]_\star(N(\chi^a)))$ has $G_{geom} = (GL(1))^d$.*

Proof. Let us denote

$$\mathcal{H}_a := \mathbf{Cancel}\mathcal{H}(!, \psi; \chi_1\mathrm{Char}_a, .., \chi_n\mathrm{Char}_a; \mathrm{Roots}_n(\prod_i \chi_i)\mathrm{Char}_a).$$

We must show that for any nonzero integer vector $(v_1, ..., v_d)$, the Tannakian tensor product

$$\otimes_{\mathrm{Tan},a} \mathcal{H}_a^{\otimes_{\mathrm{Tan}} v_i}$$

is not geometrically a delta object. Eliminating those v_i which are 0, we distinguish two cases.

If all the nonzero v_i have the same sign, it suffices to treat the case when all nonzero v_i are positive. In this case, all of the "downstairs" characters $\mathrm{Roots}_n(\prod_i \chi_i)\mathrm{Char}_a)$ in any factor are disjoint from any of the $\chi_1\mathrm{Char}_b$ "upstairs" characters in that or any other of the factors. In particular, when we compute the Tannakian tensor product in question, the characters $\chi_1\mathrm{Char}_b$ survive upstairs, in fact with multiplicity v_r, from any $\mathcal{H}_r^{\otimes_{\mathrm{Tan}} v_r}$ factor with $v_r > 0$.

If the signs of the v_i are mixed, say some $v_i > 0$ and some disjoint $v_j = -w_j$ with $w_j > 0$. Then we must show there is no geometric isomorphism between

$$\otimes_{\mathrm{Tan},i}\mathcal{H}_i^{\otimes_{\mathrm{Tan}} v_i}$$

and any multiplicative translate of

$$\otimes_{\mathrm{Tan},j}\mathcal{H}_j^{\otimes_{\mathrm{Tan}} w_j}.$$

For this, we argue as follows. Choose the largest a such that $\mathcal{H}_a$ is one of the Tannakian tensor factors in our putative relation. To fix ideas, say this occurs on the left hand side, i.e., that $v_a > 0$ and $v_c = 0$ for $c > a$.

On the left side, the upstairs characters which survive include, for each b with $v_b > 0$, the characters $\chi_1\mathrm{Char}_b$, each taken with multiplicity v_b. And all upstairs surviving characters lie in one of the sets $\chi_i\mathrm{Char}_b$ for some i and for some b with $v_b > 0$.

On the right side, the upstairs characters which survive include, for each b with $v_b < 0$, the characters $\chi_1\mathrm{Char}_b$, each taken with multiplicity $w_b = -v_b$. And all upstairs surviving characters lie in one of the sets $\chi_i\mathrm{Char}_b$ for some i and for some b with $v_b < 0$. The key point is that the b's which occur on the right hand side are all $< a$

To see that there is no geometric isomorphism, it suffices to show that the two sides have nonisomorphic local monodromies at 0. For this, it suffices to exhibit one character which occurs upstairs on the left side but is absent from the the list of upstairs characters on the right side. For this, we choose a character Λ_a of order a. So $\chi_1\Lambda_a$ occurs on the left hand side. We claim it is not an upstairs character on the right hand side. It is visibly not in any of the sets $\chi_1\mathrm{Char}_b$ for any $b < a$. If $i > 1$, it is not in any of the sets $\chi_1\mathrm{Char}_b$ for any $b < a$, simply because ab divides $d!$, and already $\chi_1^{d!} \neq \chi_i^{d!}$ for any $i > 1$, because χ lies in $\mathrm{TotRamGen}_d(k, f)$. □

Notice that for the case $n = 2$, each object $[a]_\star(N(\chi^a)$ is one-dimensional, so is its own determinant. So Theorem 3.1 (i.e. Theorem

2.4) is proven for $n = 2$. We next do the case $n \geq 4$. The case $n = 3$ will require a slightly different argument.

Theorem 3.6. *Suppose $n \geq 4$. For each $1 \leq a < b \leq d$, there is no geometric isomorphism from either $[a]_\star(N(\chi^a))$ or its Tannakian dual with $([b]_\star(N(\chi^b))) \star_{mid} \mathcal{H}$, the middle multiplicative convolution of $[b]_\star(N(\chi^b))$ with any one dimensional object $\mathcal{H}$ in the category $\mathcal{P}_{geom}$ of* [Ka-CE].

Proof. Let us denote by $\mathcal{H}_a$, respectively $\mathcal{H}_b$, the Tannakian determinant of $([a]_\star(N(\chi^a)))$, respectively of $([b]_\star(N(\chi^b)))$. Suppose there is a geometric isomorphism of either $[a]_\star(N(\chi^a))$ or its Tannakian dual with $([b]_\star(N(\chi^b))) \star_{mid} \mathcal{H}$ for some $\mathcal{H}$. Because $\mathbb{V}_b := ([b]_\star(N(\chi^b)))$ has Tannakian dimension $n-1$, for any one-dimensional $\mathcal{H}$ we have

$$\text{“det”}(\mathbb{V}_b \otimes_{\mathrm{Tan}} \mathcal{H}) \cong \text{“det”}(\mathbb{V}_b) \otimes_{\mathrm{Tan}} \mathcal{H}^{\otimes_{\mathrm{Tan}}, n-1}.$$

So we find a geometric isomorphism from either $\mathcal{H}_a$ or its Tannakian dual with $\mathcal{H}_b \otimes_{\mathrm{Tan}} \mathcal{H}^{\otimes_{\mathrm{Tan}}, n-1}$. Here we are dealing with one-dimensional and hence invertible objects. So either the “ratio” $\mathcal{H}_a \otimes_{\mathrm{Tan}} \mathcal{H}_b^{\otimes_{\mathrm{Tan}}, -1}$ or the Tannakian dual of $\mathcal{H}_a \otimes_{\mathrm{Tan}} \mathcal{H}_b$ is of the form $\mathcal{H}^{\otimes_{\mathrm{Tan}}, n-1}$ for some one-dimensional $\mathcal{H}$. The key point here is that both $\mathcal{H}_a$ and $\mathcal{H}_b$ have the following property:

(Mult_1): every character that occurs either upstairs or downstairs does so with multiplicity one.

Forming the Tannakian dual of a one-dimensional object simply interchanges upstairs and downstairs characters. When we do middle convolution of two one-dimensional objects which each have the Mult_1 property, we get a one-dimensional object in which every character that occurs, either upstairs or downstairs, does so with multiplicity at most two. Therefore in both the “ratio” $\mathcal{H}_a \otimes_{\mathrm{Tan}} \mathcal{H}_b^{\otimes_{\mathrm{Tan}}, -1}$ and the Tannakian dual of $\mathcal{H}_a \otimes_{\mathrm{Tan}} \mathcal{H}_b$, every character that occurs, either upstairs or downstairs, does so with multiplicity at most two. But in an object of the form $\mathcal{H}^{\otimes_{\mathrm{Tan}}, n-1}$, every character that occurs, either upstairs or down, does so with multiplicity divisible by $n-1$. As $n \geq 4$, the conclusion is that there are **no** characters either upstairs or down in $\mathcal{H}$. In other words, $\mathcal{H}$ is a delta object, and we have an isomorphism of either $[a]_\star(N(\chi^a))$ or its Tannakian dual with a multiplicative translate of $([b]_\star(N(\chi^b)))$. But there can be no such geometric isomorphism, as the two sides have different generic ranks, namely na and nb. □

Theorem 3.7. *Suppose $n = 3$. For each $1 \leq a < b \leq d$, there is no geometric isomorphism from either $[a]_\star(N(\chi^a))$ or its Tannakian dual with $([b]_\star(N(\chi^b))) \star_{mid} \mathcal{H}$, the middle multiplicative convolution of*

$[b]_\star(N(\chi^b))$ *with any one dimensional object* $\mathcal{H}$ *in the category* $\mathcal{P}_{geom}$ *of* [Ka-CE].

Proof. The key observation here is that for a two dimensional object V its dual is $V \otimes (\det(V)^\vee$. So it suffices to show there is no geometric isomorphism of $[a]_\star(N(\chi^a))$ with (the Tannakian dual of $([b]_\star(N(\chi^b)))) \star_{mid} \mathcal{H}$. If there were, then passing to determinants we find that $\mathcal{H}_a \otimes_{\mathrm{Tan}} \mathcal{H}_b$ is of the form $\mathcal{H}^{\otimes_{\mathrm{Tan}},2}$. But in $\mathcal{H}_a \otimes_{\mathrm{Tan}} \mathcal{H}_b = \mathcal{H}_a \star_{md} \mathcal{H}_b$, there is no cancellation of characters in either $\chi_1\mathrm{Char}_a$ or in $\chi_1\mathrm{Char}_b$. In particular, for each character Λ_b of order $b > a$, the character $\chi_1\Lambda_b$ occurs upstairs with multiplicity one. But in an object of the form $\mathcal{H}^{\otimes_{\mathrm{Tan}},2}$, every character that occurs, either upstairs or down, does so with even multiplicity. Therefore $\mathcal{H}$ is a delta object, i.e. we get a geometric isomorphism of $[a]_\star(N(\chi^a))$ with a multiplicative translate of the Tannakian dual of $([b]_\star(N(\chi^b)))$. Just as above, this is impossible, as the two sides have different generic ranks $3a$ and $3b$. □

This concludes the proof of Theorem 3.1 (i.e. of Theorem 2.4), and with it the proof of Theorem 1.2 □

4. Proof of Theorem 1.3; first steps

Suppose we are given a pair of characters $(\chi, \Lambda) \in \mathrm{TotRamPairs}(k, f)$. Their geometric components are

$$(\chi_1, ..., \chi_n), \quad (\Lambda_1, ..., \Lambda_n), \quad (\chi_1\Lambda_1, ..., \chi_n\Lambda_n).$$

We say that this totally ramified pair is very generic if the following fifteen conditions all hold.

(1) For every $1 \le i \le n$, $\chi_i^n \neq \prod_{j=1}^n \chi_j$.
(2) For every $1 \le i \le n$, $\chi_i^n \neq \prod_{j=1}^n \Lambda_j$.
(3) For every $1 \le i \le n$, $\chi_i^{2n} \neq (\prod_{j=1}^n \chi_j)(\prod_{j=1}^n \Lambda_j)$.
(4) For every $1 \le i \le n$, $\Lambda_i^n \neq \prod_{j=1}^n \chi_j$.
(5) For every $1 \le i \le n$, $\Lambda_i^n \neq \prod_{j=1}^n \Lambda_j$.
(6) For every $1 \le i \le n$, $\Lambda_i^{2n} \neq (\prod_{j=1}^n \chi_j)(\prod_{j=1}^n \Lambda_j)$.
(7) For every $1 \le i \le n$, $(\chi_i\Lambda_i)^n \neq (\prod_{j=1}^n \chi_j)^2$.
(8) For every $1 \le i \le n$, $(\chi_i\Lambda_i)^n \neq (\prod_{j=1}^n \Lambda_j)^2$.
(9) For every $1 \le i \le n$, $(\chi_i\Lambda_i)^n \neq (\prod_{j=1}^n \chi_j)(\prod_{j=1}^n \Lambda_j)$.
(10) $\prod_{j=1}^n \chi_j \neq \prod_{j=1}^n \Lambda_j$.
(11) $(\prod_{j=1}^n \chi_j)^2 \neq (\prod_{j=1}^n \chi_j)(\prod_{j=1}^n \Lambda_j)$.
(12) $(\prod_{j=1}^n \Lambda_j)^2 \neq (\prod_{j=1}^n \chi_j)(\prod_{j=1}^n \Lambda_j)$.
(13) For $1 \le i < j \le n$, $\chi_i \neq \chi_j$.
(14) For $1 \le i < j \le n$, $\Lambda_i \neq \Lambda_j$.

(15) For $1 \le i < j \le n$, $\chi_i \Lambda_i \neq \chi_j \Lambda_j$.

We denote by $\mathrm{TotRamVeryGenPairs}(k, f)$ the set of totally ramified pair which are very generic. Let us denote by $\mathrm{Pairs}(k, f)$ the set of all pairs of characters. Thus

$$\mathrm{TotRamVeryGenPairs}(k, f) \subset \mathrm{TotRamPairs}(k, f) \subset \mathrm{Pairs}(k, f).$$

This notion of $\mathrm{TotRamVeryGenPairs}(k, f)$ is somewhat ad hoc. It's *raison d'être* is that the fraction of pairs not in it is $O(1/q)$, proven below in Lemma 4.1, and it enjoys the stability property of Lemma 4.3.

We first give a crude estimate for the fraction of elements in $\mathrm{Pairs}(k, f)$ which fail to lie in $\mathrm{TotRamVeryGenPairs}(k, f)$. When no confusion is likely, we write $q := \#k$.

Lemma 4.1. *The fraction of elements in* $\mathrm{Pairs}(k, f)$ *which fail to lie in* $\mathrm{TotRamVeryGenPairs}(k, f)$ *is at most* $(3n^3 + 17n + 4)/(q - 1)$.

Proof. A pair of characters (χ, Λ) fails to lie in $\mathrm{TotRamVeryGenPairs}(k, f)$ if any of a long list of identities holds. For each of these identities, we will estimate the fraction of pairs (χ, Λ) on which it holds. The sum of these estimated fractions will be the asserted upper bound.

If χ fails to be totally ramified, it is because one of the $n+1$ identities $\chi_i = \mathbb{1}$, for $i = 1, ..., n$ or $\prod_{j=1}^{n} \chi_j = \mathbb{1}$. Each χ_i is a geometric component of a character $\chi_{[j]}$ of one of the factors $B_j^\times$, itself a finite field which is a finite extension of $k = \mathbb{F}_q$. If any geometric component of $\chi_{[j]}$ is trivial, then all geometric components of $\chi_{[j]}$ are trivial, and $\chi_{[j]}$ itself is trivial. [Indeed, if we identify $B_j^\times$ with an $\mathbb{F}_{q^d}^\times$, then the geometric components of $\chi_{[j]}$ are all the Galois conjugates of $\chi_{[j]}$, namely $\chi_{[j]}^{q^i}$ for $i = 0, ..., d-1$.] Thus a given $\chi_{[j]}$ is trivial for $1/(q^d - 1) \le 1/(q - 1)$ of all pairs, and there are at most n factors of B. The product $\prod_{j=1}^{n} \chi_j$ is (the composition with the norm of B/k of) the character $\chi|k^\times$ of $k^\times$. The norm map being surjective, the condition that $\chi|k = \mathbb{1}$ occurs for $1/(q-1)$ of all pairs. So the condition that χ fail to be totally ramified occurs for at most $(n + 1)/(q - 1)$ of all pairs.

Similarly, the condition that Λ fail to be totally ramified occurs for at most $(n+1)/(q-1)$ of all pairs. By means of the shearing automorphism $(\chi, \Lambda) \mapsto (\chi, \chi\Lambda)$ of the space of pairs, we see that the condition that $\chi\Lambda$ fail to be totally ramified occurs for at most $(n + 1)/(q - 1)$ of all pairs. So all in all, the failure to lie in $\mathrm{TotRamPairs}(k, f)$ occurs for at most $3(n + 1)/(q - 1)$.

We next turn to condition (1). Suppose $\chi_1^n = \prod_{j=1}^{n} \chi_j$. We must distinguish two cases. If B has two or more factors B_j, we may assume that χ_1 is a component of $\chi_{[1]}$. a character of $B_1^\times$, and that for some $r \geq 2$, $\chi_r, \chi_{r+1}, ..., \chi_n$ are the geometric components of $\chi_{[2]}$, a character of $B_2^\times$. We use the equation to "solve" for $\prod_{j=r}^n \chi_j$. This product is the compostion of $\chi_{[2]}$ with the surjective $\mathrm{Norm}_{B_2/k} : B_2^\times \to k^\times$. Thus we solve for a character of $k^\times$, so we get a fraction at most $1/(q-1)$. Repeating this for each index i, we get a fraction at most $n/(q-1)$.

The second case of condition (1) is when there is only one factor, i.e., when B is itself a field, so the field of q^n elements. In this case, the product $\prod_{j=1}^n \chi_j$ is a character of $k^\times$, so the component χ_1 is one of at most $n(q-1)$ characters. So the fraction is at most $n(q-1)/(q^n-1)$, which for $n \geq 2$ is $\leq 2/(q-1)$. So in the second case, repeating for each index i, we get a fraction at most $2n/(q-1)$.

So from condition (1) we get a fraction of at most $2n/(q-1)$, whatever the case.

We turn to condition (2). For each i, solve for the character $\prod_j \Lambda_j$, which is the composition with the norm of $\Lambda|k$. Repeating for each index i, we get a fraction of at most $n/(q-1)$ where (2) fails.

We turn to condition (3). Again solve for $\prod_j \Lambda_j$, so again a fraction at most $n/(q-1)$ where (3) fails.

Conditions (4), (5), (6) are renamings of conditions (2), (1), (3), with χ's and Λ's interchanged. So these all together give a fraction at most $4n/(q-1)$ where any of (4), (5), (6) fails.

We turn to condition (7). Just as in discussing condition (1), we distinguish two cases. If there are two or more factors B_j, in the notations of the discussion of (1) we use the equality $(\chi_1\Lambda_1)^n = (\prod_{j=1}^n \chi_j)^2$ to "solve" for $(\prod_{j=r}^n \chi_j)^2$, which is the square of a character of $k^\times$, so we get a fraction of $2/(q-1)$. Repeating for each i, we get a fraction at most $2n/(q-1)$.

The second case of condition (7) is when there is only one factor. In the equality $(\chi_1\Lambda_1)^n = (\prod_{j=1}^n \chi_j)^2$, the right hand side is a character of $k^\times$, so $\chi_1\Lambda_1$ is one of at most $n(q-1)$ characters, and just as in the discussion of (1) above, this gives a fraction of at most $2/(q-1)$. Repeating for each index i, we get a fraction at most $2n/(q-1)$.

Condition (8) is the renaming of (7), with χ's and Λ's interchanged. So again a fraction at most $2n/(q-1)$.

Condition (9) is the renaming of condition (5) after the automorphism $(\chi, \Lambda) \mapsto (\chi, \chi\Lambda)$ of the space $\mathrm{Pairs}(k, f)$, so again a fraction at most $2n/(q-1)$.

For condition (10), failure is the condition that χ/Λ be trivial on $k^\times$, so a fraction $1/(q-1)$. Conditions (11) and (12) are each equivalent to (10), and are only listed for symmetry in later use.

For condition (13), we factor B as the product of fields B_j, and χ as the corresponding product of characters $\chi_{[j]}$ of the $B_j^\times$. If χ_1 is a geometric component of $\chi_{[1]}$, the other geometric components of $\chi_{[1]}$ are its (not necessarily distinct) conjugates $\chi_1^{q^a}$ for $0 \le a < \deg(B_1/k)$. Let us denote $d_j := \deg(B_j/k)$. We again distinguish cases.

If χ_2 is also a geometric component of $\chi_{[1]}$, possible only if $d_1 > 1$, then $\chi_{[1]}$ is (the composition with the norm of) a character of a proper subfield of B_1. So in this case $\chi_{[1]}$ is one of at most $\sum_{r|d_1, r \le d_1/2}(q^r - 1) \le (d_1/2)(q^{d_1/2} - 1)$ characters out of $q^{d_1} - 1$ possible. So here the fraction is at most

$$\frac{(d_1/2)(q^{d_1/2} - 1)}{q^{d_1} - 1} = (d_1/2)/(q^{d_1/2} + 1) \le n/(q-1).$$

If χ_2 is a geometric component of a different factor, say of the character $\chi_{[2]}$ of $B_2^\times$, we argue as follows. If $d_1 = d_2$, then $\chi_{[1]}$ and $\chi_{[2]}$ have the same restrictions to $k^\times$ ($k^\times$ seen inside $B_1^\times$ and inside $B_2^\times$). So in this case the fraction is at most $1/(q-1)$.

If $d_1 \ne d_2$, say $d_1 > d_2$. Then for some $e \le d_2 - 1 \le d_1 - 2$, $\chi_1^{q^e} = \chi_1$ (because this holds for χ_2). Thus χ_1 is a character of a proper subfield of $B_1^\times$, and hence $\chi_{[1]}$ is a character of a proper subfield, and the fraction is at most $n/(q-1)$. So in all cases the $\chi_1 = \chi_2$ equality gives a fraction of at most $n/(q-1)$. Summing over all $1 \le i < j \le n$, condition (13) gives a fraction of at most $n^3/(q-1)$.

Conditions (14) and (15) are each renamings of condition (13), so each gives separately a fraction at most $n^3/(q-1)$.

So all in all we get a fraction of at most $(3n^3 + 17n + 4)/(q-1)$. □

Notice that if (χ, Λ) lies in $\mathrm{TotRamVeryGenPairs}(k, f)$, then each of the characters $\chi, \Lambda, \chi\Lambda$ is totally ramified and generic (in the sense of Section 2).

The key technical result is this.

Theorem 4.2. *Suppose $n \ge 2$ and $p > 2$. Suppose (χ, Λ) lies in* $\mathrm{TotRamVeryGenPairs}(k, f)$. *Denote by* [2] *the squaring map $x \mapsto x^2$ of $\mathbb{G}_m$ to itself. Then the direct sum object*

$$N(\chi, \Lambda) := N(\chi) \oplus N(\Lambda) \oplus [2]_\star(N(\chi\Lambda))$$

is pure of weight zero and lies in the Tannakian category $\mathcal{P}_{arith}$ in the sense of [Ka-CE]. *It has generic rank $4n$, Tannakian "dimension" $3(n-1)$, it has at most $8n$ bad characters, and it has*

$$G_{geom} = G_{arith} = (GL(n-1))^3.$$

We first explain how to apply this result.

Let ρ be a character of $k^\times$ which is good for $N(\chi,\Lambda)$, i.e., a character such that for the inclusion $j : \mathbb{G}_m \subset \mathbb{P}^1$, the canonical "forget supports" map is an isomorphism $j_!(N(\chi,\Lambda)\otimes\mathcal{L}_\rho) \cong Rj_\star(N(\chi,\Lambda)\otimes\mathcal{L}_\rho)$. Thus ρ is good for $N(\chi,\Lambda)$ if and only if the following conditions hold: ρ is good for both $N(\chi)$ and $N(\Lambda)$, and ρ^2 is good for $N(\chi\Lambda)$.

Denote by ρ_{Norm} the character of $B^\times$ given by $\rho\circ\mathrm{Norm}_{B/k}$. Then ρ is good for $N(\chi,\Lambda)$ if and only if the three characters $\chi\rho_{\mathrm{Norm}}, \Lambda\rho_{\mathrm{Norm}}, \chi\Lambda\rho^2_{\mathrm{Norm}}$ are all totally ramified. In this case, the conjugacy classes $\theta_{k,f,\chi\rho_{\mathrm{Norm}}}$, $\theta_{k,f,\Lambda\rho_{\mathrm{Norm}}}$, and $\theta_{k,f,\chi\Lambda\rho^2_{\mathrm{Norm}}}$ are given respectively by

$$\det(1-T\theta_{k,f,\chi\rho_{\mathrm{Norm}}}) = \det(1-TFrob_k|H^0_c(\mathbb{G}_m/\overline{k}, N(\chi)\otimes\mathcal{L}_\rho),$$

$$\det(1-T\theta_{k,f,\Lambda\rho_{\mathrm{Norm}}}) = \det(1-TFrob_k|H^0_c(\mathbb{G}_m/\overline{k}, N(\Lambda)\otimes\mathcal{L}_\rho),$$

$$\det(1-T\theta_{k,f,\chi\Lambda\rho^2_{\mathrm{Norm}}}) = \det(1-TFrob_k|H^0_c(\mathbb{G}_m/\overline{k}, [2]_\star(N(\chi\Lambda))\otimes\mathcal{L}_\rho),$$

cf. [Ka-QKRPD, end of §4].

Lemma 4.3. *Suppose* (χ,Λ) *lies in* $\mathrm{TotRamVeryGenPairs}(k,f)$. *A character* ρ *of* $k^\times$ *is good for* $N(\chi,\Lambda)$ *if and only if the pair* $(\chi\rho_{\mathrm{Norm}},\Lambda\rho_{\mathrm{Norm}})$ *lies in* $\mathrm{TotRamVeryGenPairs}(k,f)$.

Proof. A character ρ is good for $N(\chi,\Lambda)$ if and only if the three characters $\chi\rho_{\mathrm{Norm}}, \Lambda\rho_{\mathrm{Norm}}, \chi\Lambda\rho^2_{\mathrm{Norm}}$ are all totally ramified. To check the fifteen auxiliary conditions defining $\mathrm{TotRamVeryGenPairs}(k,f)$ inside $\mathrm{TotRamPairs}(k,f)$, look at geometric components. Let the geometric components of χ,Λ, and $\chi\Lambda$ respectively be $\chi_1,...,\chi_n$, $\Lambda_1,...,\Lambda_n$, and $\chi_1\Lambda_1,...,\chi_n\Lambda_n$. Then the geometric components of $\chi\rho_{\mathrm{Norm}}, \Lambda\rho_{\mathrm{Norm}}$, and $\chi\Lambda\rho^2_{\mathrm{Norm}}$ respectively are $\chi_1\rho,...,\chi_n\rho$, $\Lambda_1\rho,...,\Lambda_n\rho$, and $\chi_1\Lambda_1\rho^2,...,\chi_n\Lambda_n\rho^2$. The fifteen conditions are each homogenous in ρ, so the assertion is obvious. ☐

The general equidistribution theorem [Ka-CE, Remark 7.5 and the proof of Theorem 28.1] gives the following corollary of the theorem.

Corollary 4.4. *Let* $n\geq 2$, $p>2$. *Suppose* (χ,Λ) *lies in*

$$\mathrm{TotRamVeryGenPairs}(k,f).$$

Let Ξ *be an irreducible nontrivial representation of the triple product* $(U(n-1))^3$ *which occurs in* $\mathbb{V}^{\otimes r}\otimes(\mathbb{V}^\vee)^{\otimes s}$ *for* $\mathbb{V} := \oplus_{i=1}^3 std_{n-1,i}$, *for* $std_{n-1,i}$ *the standard representation of the i'th factor* $U(n-1)$ *in the triple product* $(U(n-1))^3$. *Define the constant*

$$C := 8n.$$

Denote by Good$(N(\chi,\Lambda),k)$ *the set of characters* ρ *which are good for* $N(\chi,\Lambda)$. *For each good* ρ, *denote by* $\theta(k,\chi,\Lambda,\rho)$ *the conjugacy class in* $(U(n-1))^3$ *given by*

$$\theta(k,\chi,\Lambda,\rho) := (\theta_{k,f,\chi\rho_{\mathrm{Norm}}}, \theta_{k,f,\Lambda\rho_{\mathrm{Norm}}}, \theta_{k,f,\chi\Lambda\rho^2_{\mathrm{Norm}}}).$$

Then for $\#k > (1+C)^2$, *we have the estimate*

$$|\sum_{\rho\in \mathrm{Good}(N(\chi,\Lambda),k)} \mathrm{Trace}(\Xi(\theta_d(k,\chi,\rho)))| \le \frac{\#\mathrm{Good}(N(\chi,\Lambda,k))2(r+s+1)C^{r+s}}{\sqrt{\#k}}.$$

This corollary itself has the following corollary.

Corollary 4.5. *Let* $n\ge 2$, $p>2$. *Let* Ξ *be an irreducible nontrivial representation of the triple product* $(U(n-1))^3$ *which occurs in* $\mathbb{V}^{\otimes r}\otimes(\mathbb{V}^\vee)^{\otimes s}$ *for* $\mathbb{V} := \oplus_{i=1}^3 std_{n-1,i}$, *for* $std_{n-1,i}$ *the standard representation of the i'th factor* $U(n-1)$ *in the triple product* $(U(n-1))^3$. *Define the constant*

$$C := 8n.$$

Denote by $\theta(k,\chi,\Lambda)$ *the conjugacy class in* $(U(n-1))^3$ *given by*

$$\theta(k,\chi,\Lambda) := (\theta_{k,f,\chi}, \theta_{k,f,\Lambda}, \theta_{k,f,\chi\Lambda}).$$

Then for $\#k > (1+C)^2$, *we have the estimate*

$$|\sum_{(\chi,\Lambda)\in \mathrm{TotRamVeryGenPairs}(k,f)} \mathrm{Trace}(\Xi(\theta_d(k,\chi,\rho)))| \le$$

$$\le \frac{\#\mathrm{TotRamVeryGenPairs}(k,f)2(r+s+1)C^{r+s}}{\sqrt{\#k}}.$$

Proof. On the set TotRamVeryGenPairs(k,f), we have an equivalence relation $(\chi,\Lambda)\cong(\chi',\Lambda')$ if, for some ρ, $(\chi',\Lambda') = (\chi\rho_{\mathrm{Norm}}, \Lambda\rho_{\mathrm{Norm}})$. The sums in the corollary above are the sums over equivalence classes. Added them up, we get the asserted estimate. ☐

Using this, we conclude the proof of the target theorem as follows. We have inequalities

$$q^{2n} = (\#B)^2 > (\#B^\times)^2 = \#\mathrm{Pairs}(k,f) \ge \#\mathrm{TotRamPairs}(k,f) \ge$$

$$\ge \#\mathrm{TotRamVeryGenPairs}(k,f) \ge \#\mathrm{Pairs}(k,f)(1-(3n^3+17n+4)/(q-1)),$$

the last inequality by Lemma 2.1. On the other hand, $\#B^\times \ge (q-1)^n$, so for an explicit constant C_n we will have

$$q^{2n}\#\mathrm{TotRamPairs}(k,f) \ge \#\mathrm{TotRamVeryGenPairs}(k,f) \ge q^{2n} - C_n q^{2n-1}.$$

We need to estimate the sum

$$\sum_{(\chi,\Lambda)\in \mathrm{TotRamPairs}(k,f)} \mathrm{Trace}(\Xi(\theta_d(k,\chi,\rho))),$$

which differs from the sum

$$\sum_{(\chi,\Lambda)\in \mathrm{TotRamVeryGenPairs}(k,f)} \mathrm{Trace}(\Xi(\theta_d(k,\chi,\rho)))$$

by at most $C_n q^{2n-1}$ terms, each of which is bounded in absolute value by $\deg(\Xi) \le C^{r+s}$. For $q \ge (1+C_n)^2$, we will have

$$C_n q^{2n-1} \le (q^{2n} - C_n q^{2n-1})/\sqrt{q} \le \#\mathrm{TotRamPairs}(k,f)/\sqrt{q}.$$

So for $q \ge (1+C_n)^2$ we get the estimate

$$|\sum_{(\chi,\Lambda)\in \mathrm{TotRamPairs}(k,f)} \mathrm{Trace}(\Xi(\theta_d(k,\chi,\rho)))| \le$$

$$\le \frac{\#\mathrm{TotRamVeryGenPairs}(k,f)2(r+s+1)C^{r+s}}{\sqrt{\#k}} + \frac{\#\mathrm{TotRamPairs}(k,f)C^{r+s}}{\sqrt{\#k}}$$

$$\le \frac{\#\mathrm{TotRamPairs}(k,f)(2(r+s+1)C^{r+s} + C^{r+s})}{\sqrt{\#k}}.$$

5. Proof of Theorem 4.2

We recall the statement.

Theorem 5.1. (Restatement of Theorem 4.2) *Suppose $n \ge 2$ and $p > 2$. Suppose (χ,Λ) lies in* TotRamVeryGenPairs(k,f). *Denote by* [2] *the squaring map $x \mapsto x^2$ of of $\mathbb{G}_m$ to itself. Then the direct sum object*

$$N(\chi,\Lambda) := N(\chi) \oplus N(\Lambda) \oplus [2]_\star(N(\chi\Lambda))$$

is pure of weight zero and lies in the Tannakian category $\mathcal{P}_{arith}$ in the sense of [Ka-CE]. *It has generic rank $4n$, Tannakian "dimension" $3(n-1)$, it has at most $8n$ bad characters, and it has*

$$G_{geom} = G_{arith} = (GL(n-1))^3.$$

Proof. We have a priori inclusions $G_{geom} \subset G_{arith} \subset (GL(n-1))^3$, so it suffices to show that $G_{geom} = (GL(n-1))^3$. Each of $\chi, \Lambda, \chi\Lambda$ is totally ramified and generic, so by Theorem 2.2, the objects $N(\chi), N(\Lambda)$, and $N(\chi\Lambda)$ each have $G_{geom} = GL(n-1)$. By [Ka-CE, Thm. 8.1], the direct image object $[2]_\star(N(\chi\Lambda))$ has $G_{geom} = GL(n-1)$.

As explained in [Ka-CE, Thm. 13.5 and Chapter 20], In order to show that $N(\chi,\Lambda)$ has $G_{geom} = (GL(n-1))^3$, we must show four things, cf. [Ka-CE, Thm. 13.5 and Chapter 20] and the proof of Theorem 3.5 above.

(1) The direct sum of the Tannakian determinants

$$\text{“}\det\text{”}N(\chi) \oplus \text{“}\det\text{”}N(\Lambda) \oplus \text{“}\det\text{”}[2]_\star(N(\chi\Lambda))$$

has $G_{geom} = (GL(1))^3$.

(2) If $n \geq 3$, there is no geometric isomorphism from either $N(\chi)$ or its Tannakian dual with $N(\Lambda) \star_{mid} \mathcal{H}$ for any one dimensional object $\mathcal{H}$ in the category $\mathcal{P}_{geom}$ of [Ka-CE].

(3) If $n \geq 3$, there is no geometric isomorphism from either $N(\chi))$ or its Tannakian dual with $([2]_\star(N(\chi\Lambda))) \star_{mid} \mathcal{H}$, for any one dimensional object $\mathcal{H}$ in the category $\mathcal{P}_{geom}$ of [Ka-CE].

(4) If $n \geq 3$, there is no geometric isomorphism from either $N(\Lambda))$ or its Tannakian dual with $([2]_\star(N(\chi\Lambda))) \star_{mid} \mathcal{H}$ for any one dimensional object $\mathcal{H}$ in the category $\mathcal{P}_{geom}$ of [Ka-CE].

We begin with an analysis of the Tannakian determinants. Let us denote

$$\mathcal{H}_\chi := \text{“}\det\text{”}N(\chi),\ \mathcal{H}_\Lambda := \det\text{”}N(\Lambda),\ \mathcal{H}_{[2],\chi\Lambda} := \text{“}\det\text{”}[2]_\star(N(\chi\Lambda)).$$

Each of these is, geometrically, either a delta function or a multiplicative translate of the (shift by [1] of) an irreducible hypergeometric sheaf

$$\mathcal{H}(!, \psi; \rho_1, ..., \rho_n; \sigma_1, ..., \sigma_m)$$

of type (n, m) with $n \geq 0, m \geq 0, n+m \geq 1$. We refer to the ρ_i as the upstairs characters, the σ_j as the downstairs characters.

Given a character ρ of $\pi_1^{tame}(\mathbb{G}_m/\overline{k})$, and an integer $m \geq 1$, recall from Section 3 that we denote by $\mathrm{Roots}_m(\rho)$ the set of characters σ with $\sigma^m = \rho$. [If $m = m_0 p^r$ with m_0 prime to p, this is a set of cardinality m_0.]

Lemma 5.2. *The Tannakian determinants $\mathcal{H}_\chi, \mathcal{H}_\Lambda, \mathcal{H}_{[2],\chi\Lambda}$ are, geometrically, multiplicative translates of (the shifts by* [1] *of) the irreducible hypergeometric sheaves*

$$\mathcal{H}(!, \psi; \chi_1, ..., \chi_n; \mathrm{Roots}_n(\prod_j \chi_j)),$$

$$\mathcal{H}(!, \psi; \Lambda_1, ..., \Lambda_n; \mathrm{Roots}_n(\prod_j \Lambda_j)),$$

$$\mathcal{H}(!, \psi; \mathrm{Roots}_2(\chi_1\Lambda_1), ..., \mathrm{Roots}_2(\chi_n\Lambda_n); \mathrm{Roots}_{2n}(\prod_j(\chi_j\Lambda_j))).$$

Proof. This follows almost entirely from Lemma 3.4 above. The only point that needs to be observed is that no **Cancel** takes place, thanks to conditions (1), (5), and (9) defining TotRamVeryGenPairs. □

The following lemma will be crucial.

Lemma 5.3. *Write $n = n_0 p^r$ with n_0 prime to p. The downstairs parameters in the three objects $\mathcal{H}_\chi, \mathcal{H}_\Lambda, \mathcal{H}_{[2],\chi\Lambda}$ form a set of $4n_0$ distinct characters. This set is disjoint from any of the upstairs characters of any of the three objects.*

Proof. The first assertion results from conditions (10), (11), and (12) defining TotRamVeryGenPairs. The second assertion results from conditions (1) through (9). ☐

Lemma 5.4. *In any Tannakian tensor product*

$$\mathcal{H}_\chi^{\otimes a} \otimes \mathcal{H}_\Lambda^{\otimes b} \otimes \mathcal{H}_{[2],\chi\Lambda}^{\otimes c},$$

(1) *Each of the downstairs characters of $\mathcal{H}_\chi$ occurs with multiplicity $|a|$, as a downstairs character if $a > 0$ and as an upstairs character if $a < 0$, and not at all in the opposite position. It is absent if $a = 0$.*
(2) *Each of the downstairs characters of $\mathcal{H}_\Lambda$ occurs with multiplicity $|b|$, as a downstairs character if $b > 0$ and as an upstairs character if $b < 0$, and not at all in the opposite position. It is absent if $b = 0$.*
(3) *Each of the downstairs characters of $\mathcal{H}_{[2],\chi\Lambda}$ occurs with multiplicity $|c|$, as a downstairs character if $c > 0$ and as an upstairs character if $c < 0$, and not at all in the opposite position. It is absent if $c = 0$.*

Proof. This follows from the previous lemma, which shows there is never cancellation involving any of the $4n_0$ downstairs characters in forming a Tannakian tensor product; they cannot cancel among themselves, nor can they cancel any of the upstairs characters. ☐

Corollary 5.5. *The Tannakian tensor product*

$$\mathcal{H}_\chi^{\otimes a} \otimes \mathcal{H}_\Lambda^{\otimes b} \otimes \mathcal{H}_{[2],\chi\Lambda}^{\otimes c}$$

is geometrically a delta function if and only if $a = b = c = 0$.

It remains now to show for $n \geq 3$, we have the asserted nonisomorphisms. Here we use the multiplicity argument, already used above in the proof of Theorem 3.6, that if V and W are $n-1$-dimensional objects and L is a one dimensional object, then $V \cong W \otimes L$ implies $\det(V) \cong \det(W) \otimes L^{\otimes n-1}$, and hence that the ratio $\det(V) \otimes \det(W)^\vee \cong L^{\otimes n-1}$. Similarly, $V^\vee \cong W \otimes L$ implies $\det(V)^\vee \otimes \det(W)^\vee \cong L^{\otimes n-1}$. But for V and W two of our objects $N(\chi), N(\Lambda), [2]_\star N(\chi\Lambda)$, both ratios

$\det(V) \otimes \det(W)^\vee$ and $\det(V)^\vee \otimes \det(W)^\vee$ have the downstairs characters of their inputs appearing with multiplicity one. Therefore the putative $\mathcal{H}$ must be a delta function.

It remains to show there is no geometric isomorphism of either $N(\chi)$ or its Tannakian dual with a multiplicative translate of either $N(\Lambda)$ or of $[2]_\star(N(\chi\Lambda))$. We must also show there is no geometric isomorphism of either $N(\Lambda)$ or its Tannakian dual with a multiplicative translate of $[2]_\star(N(\chi\Lambda))$. The isomorphisms toward a multiplicative translate of $[2]_\star(N(\chi\Lambda))$ cannot exist, because the source has generic rank n and the target has generic rank $2n$.

To see that $N(\chi)$ is not geometrically isomorphic to a multiplicative translate of $N(\Lambda)$, it suffice to observe that their local monodromies at ∞ are not isomorphic. Indeed, by condition (10), $\prod_i \chi_i \neq \prod_i \Lambda_i$, and the tame parts of the local monodromies at ∞ are the direct sums of the n'th roots of these unequal products.

To see that the Tannakian dual of $N(\chi)$ is not geometrically isomorphic to a multiplicative translate of $N(\Lambda)$, it suffices to observe that their local monodromies at ∞ are not isomorphic. For the Tannakian dual of $N(\chi)$, this local monodromy is tame, the direct sum of the χ_i. For $N(\Lambda)$, the tame part of its local monodromy at ∞ is the sum of the n'th roots of $\prod_i \Lambda_i$. By condition (2), no χ_j^n is this product.

This concludes the proof of Theorem 5.1 (i.e. of Theorem 4.2), and with it of Theorem 1.3. □

References

[De-Weil II] Deligne, P., La conjecture de Weil II. Publ. Math. IHES 52 (1981), 313-428.

[G-L] Gabber, O., and Loeser, F., Faisceaux pervers ℓ-adiques sur un tore. Duke Math. J. 83 (1996), no. 3, 501-606

[Ka-ESDE] Katz, N., Exponential sums and differential equations. Annals of Mathematics Studies, 124. Princeton Univ. Press, Princeton, NJ, 1990. xii+430 pp.

[Ka-CE] Katz, N., Convolution and Equidistribution: Sato-Tate Theorems for Finite-Field Mellin Transforms. Annals of Mathematics Studies, 180. Princeton Univ. Press, Princeton, NJ, 2012.

[Ka-MG] Katz, N., On the monodromy groups attached to certain families of exponential sums. Duke Math. J. 54 (1987), no. 1, 4156.

[Ka-MMP] Katz, N., Moments, monodromy, and perversity: a Diophantine perspective. Annals of Mathematics Studies, 159. Princeton University Press, Princeton, NJ, 2005. viii+475 pp.

[Ka-QKRPD] Katz, N., On a question of Keating and Rudnick about primitive Dirichlet characters with squarefree conductor. Int. Math. Res. Not. IMRN 2013, no. 14, 3221-3249.

[Ka-Semi] Katz, N., A semicontinuity result for monodromy under degeneration. Forum Math. 15 (2003), no. 2, 191-200.

[Ka-WVQKR] Katz, N., Witt vectors and a question of Keating and Rudnick. Int. Math. Res. Not. IMRN 2013, no. 16, 36133638.

[Weil] Weil, A., Variétés abéliennes et courbes algébriques. Actualités Sci. Ind., no. 1064 = Publ. Inst. Math. Univ. Strasbourg 8 (1946). Hermann & Cie., Paris, 1948. 165 pp.

Princeton University, Mathematics, Fine Hall, NJ 08544-1000, USA
E-mail address: `nmk@math.princeton.edu`