ON A QUESTION OF KEATING AND RUDNICK

NICHOLAS M. KATZ

Introduction

We work over a finite field $k = \mathbb{F}_q$ inside a fixed algebraic closure $\overline{k}$. We fix a squarefree monic polynomial $f(X) \in k[X]$ of degree $n \geq 2$. We form the k-algebra

$$B := k[X]/(f(X)),$$

which is finite étale over k of degree n. We denote by $u \in B$ the image of X in B under the "reduction mod f" homomorphism $k[X] \to B$. Thus we may write this homomorphism as

$$g(X) \in k[X] \mapsto g(u) \in B.$$

We denote by $B^\times$ the multiplicative group of B, and by χ a character

$$\chi : B^\times \to \mathbb{C}^\times.$$

We extend χ to all of B by decreeing that $\chi(b) := 0$ if $b \in B$ is not invertible.

The (possibly imprimitive) Dirichlet L-function $L(\chi, T)$ attached to this data is the power series in $\mathbb{C}[[T]]$ given by

$$L(\chi, T) := \sum_{\text{monic g(X)} \in \text{k[X]}} \chi(g(u))T^{\deg(g)} = \sum_{n \geq 0} A_n T^n,$$

$$A_n := \sum_{g(X) \in k[X] \text{ monic of deg. n, gcd(f,g)=1}} \chi(g(u)).$$

If χ is nontrivial, then $L(\chi, T)$ is a polynomial in T of degree $n - 1$.

Moreover, if χ is "as ramified as possible"[1], then this L-function is "pure of weight one", i.e., in its factored form $\prod_{i=1}^{n-1}(1 - \beta_i T)$, each reciprocal root β_i has complex absolute value

$$|\beta_i|_{\mathbb{C}} = \sqrt{q}.$$

[1]Factor f as a product of distinct monic irreducible polynomials, say $f = \prod_j f_j$. Then B is canonically the product of the algebras $B_j := k[X]/(f_j(X))$, and χ is the product of characters χ_j of these factors. The condition "as ramified as possible" is that each χ_j be nontrivial, and that the restriction of χ to $k^\times \subset B^\times$ be nontrivial.

For such a χ, its "unitarized" L-function $L(\chi, T/\sqrt{q})$ is the reversed characteristic polynomial $\det(1 - TA_\chi)$ of some element A_χ in the unitary group $U(n-1)$ (e.g., take $A_\chi := \mathrm{Diag}(\beta_1/\sqrt{\mathrm{q}}, ..., \beta_{\mathrm{n}-1}/\sqrt{\mathrm{q}})$). In $U(n-1)$, conjugacy classes are determined by their characteristic polynomials, so $L(\chi, T/\sqrt{q})$ is $\det(1-T\theta_\chi)$ for a well defined conjugacy class θ_χ in $U(n-1)$. In order to keep track of the input data (k, f, χ), we denote this conjugacy class

$$\theta_{k,f,\chi}.$$

Now suppose E/k is a finite extension field of k. Our polynomial f remains squarefree over E. We form the E-algebra $B_E := E[X]/(f(X))$, and for each character χ of $B_E^\times$ which is as ramified as possible, we get a conjugacy class $\theta_{E,f,\chi}$. The question posed by Keating and Rudnick was to show that for fixed f, the collections of conjugacy classes

$$\{\theta_{E,f,\chi}\}_{\chi \text{ char. of } \mathrm{B_E^\times} \text{ as ramified as possible}}$$

become equidistributed in the space $U(n-1)^\#$of conjugacy classes of $U(n-1)$ (for the measure induced by Haar measure on $U(n-1)$) as E runs over larger and larger finite extensions of k.

In fact, we will show something slightly stronger, where we fix the degree $n \geq 2$, but allow sequences of input data (k_i, f_i), with k_i a finite field (of possibly varying characteristic) and $f_i(X) \in k_i[X]$ squarefree of degree n. We will show that, in any such sequence in which $\#k_i$ is archimedeanly increasing to ∞, the collections of conjugacy classes

$$\{\theta_{k_i,f_i,\chi}\}_{\chi \text{ char. of } \mathrm{B_i^\times} \text{ as ramified as possible}}$$

become equidistributed in $U(n-1)^\#$.

In an appendix, we give an analogous result for"even" characters which are as ramified as possible, given that they are even, under the additional hypothesis that each $f_i(X) \in k_i[X]$ have a zero in k_i. Here the equidistribution is in the space of conjugacy classes of $U(n-2)$. Already the case when each $f_i(X) \in k_i[X]$ is an irreducible cubic seems to be open.

1. Preliminaries on the L-function

We return to our initial situation, a finite field k, an integer $n \geq 2$, a squarefree polynomial $f(X) \in k[X]$, and the finite étale k-algebra $B := k[X]/(f(X))$. We have the algebra-valued functor $\mathbb{B}$ on variable k-algebras R defined by

$$\mathbb{B}(R) := B_R := B \otimes_k R = R[X]/(f(X)),$$

and the group-valued functor $\mathbb{B}^\times$ on variable k-algebras R defined by

$$\mathbb{B}^\times(R) := B_R^\times = \mathbb{B}(R)^\times.$$

Because f is squarefree, $\mathbb{B}^\times$ is a smooth commutative groupscheme over k[2], which over any extension field E of k in which f factors completely becomes isomorphic to the n-fold product of $\mathbb{G}_m$ with itself. More precisely, if f factors completely over E, say $f(X) = \prod_{i=1}^n (X - a_i)$, then for any E-algebra R, we have an R-algebra isomorphism

$$\mathbb{B}(R) = R[X]/(\prod_{i=1}^n (X - a_i)) \cong \prod_{i=1}^n R$$

of $\mathbb{B}(R)$ with the n-fold product of R with itself, its algebra structure given by componentwise operations, under which the image u of X maps by

$$u \mapsto (a_1, ..., a_n).$$

So for any E-algebra R, we have $\mathbb{B}^\times(R) := \mathbb{B}(R)^\times \cong (R^\times)^n$.

For E/k a finite extension field, B_E is a finite etale B algebra which as a B-module is free of rank $\deg(E/k)$. Let us denote by $\mathbb{B}_E$ the functor on k-algebras $R \mapsto \mathbb{B}_E(R) := B_E \otimes_k R$. Then $\mathbb{B}_E(R)$ is a finite étale $\mathbb{B}(R)$-algebra, so we have the norm map

$$\mathrm{Norm}_{E/k} : \mathbb{B}_E \to \mathbb{B}.$$

Its restriction to unit groups gives a homomorphism of tori which is étale surjective,

$$\mathrm{Norm}_{E/k} : \mathbb{B}_E^\times \to \mathbb{B}^\times,$$

whose restriction to k-valued points gives a surjective[3] homomorphism

$$\mathrm{Norm}_{E/k} : \mathbb{B}^\times(E) \to \mathbb{B}^\times(k).$$

We will also have occasion to consider $\mathbb{B}(R)$ as a finite étale R-algebra which is free of rank n as an R-module, giving us **another** norm map

$$\mathrm{Norm}_{B/k} : \mathbb{B}(R) \to R,$$

which by restriction gives a homomorphism which is étale surjective, with geometrically connected kernel,

$$\mathrm{Norm}_{B/k} : \mathbb{B}^\times(R) \to R^\times.$$

For any finite extension E/k, this second norm map

$$\mathrm{Norm}_{B/k} : \mathbb{B}^\times(E) \to E^\times$$

[2] In fact $\mathbb{B}^\times$ is the generalized Jacobian of $\mathbb{P}^1/k$ with respect to the modulus $\infty \cup \{f = 0 \text{ in } \mathbb{A}^1\}$.

[3] By Lang's theorem [La, Thm. 2], because its kernel is smooth and geometrically connected

is surjective.

How is all this related to our L-function? For each integer $r \geq 1$, denote by k_r/k the unique extension field of k of degree r (inside our fixed algebraic closure of k). Recall that $f(X) \in k[X]$ is squarefree of degree $n \geq 1$, and that u denotes the image of X in $B = k[X]/(f(X))$.

Lemma 1.1. *For χ a character of B, we have the identity*

$$L(\chi, T) = \exp(\sum_{r \geq 1} S_r T^r/r), \quad S_r = \sum_{t \in \mathbb{A}^1[1/f](k_r)} \chi(\mathrm{Norm}_{k_r/k}(u-t)).$$

Proof. The key observation is that if $\alpha \in \mathbb{A}^1[1/f](k_d)$ generates the extension k_d/k, and has monic irreducible polynomial $P(X)$ over k, then $\gcd(f, P) = 1$ and $P(X) = \mathrm{Norm}_{k_d/k}(X - \alpha)$ in $k[X]$. Hence $P(u) = \mathrm{Norm}_{k_d/k}(u - \alpha)$ in B. We apply this as follows.

Write the L-function as the Euler product

$$L(\chi, T) = \prod_{\text{monic irred. P(X), gcd(f,P)=1}} \frac{1}{1 - \chi(P(u))T^{\deg(P)}}.$$

Taking log's, we must check that for each $r \geq 1$ we have the identity

$$\sum_{t \in \mathbb{A}^1[1/f](k_r)} \chi(\mathrm{Norm}_{k_r/k}(u-t)) = \sum_{d|r} \sum_{\text{irred P,deg(P)=d, gcd(f,P)=1}} d\chi(P(u))^{r/d}.$$

To see this, partition the points $t \in \mathbb{A}^1[1/f](k_r)$ according to their monic irreducible polynomials over k. For each divisor d of r, and each monic irreducible $P(X)$ of degree d with $\gcd(f, P) = 1$ and roots $\tau_1, ..., \tau_d$ in $\mathbb{A}^1[1/f](k_d)$, each of the d terms $\chi(\mathrm{Norm}_{k_r/k}(u - \tau_i))$ is equal to $\chi(P(u))^{r/d}$ (simply because $\mathrm{Norm}_{k_d/k}(u - \tau_i) = P(u)$, and, as $\tau_i \in k_d$, $\mathrm{Norm}_{k_r/k}(u - \tau_i) = (\mathrm{Norm}_{k_d/k}(u - \tau_i))^{r/d}$). $\square$

2. Cohomological genesis

We now choose a prime number ℓ invertible in k, and an embedding of $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, into $\overline{\mathbb{Q}_\ell}$. In this way, we view χ as a $\overline{\mathbb{Q}_\ell}^\times$-valued character of $B^\times$. Attached to χ, we have the "Kummer sheaf" $\mathcal{L}_\chi$ on $\mathcal{B}^\times$. Recall that $\mathcal{L}_\chi$ is obtained as follows. We have the $q = \#k$'th power Frobenius endomorphism F_k of $\mathbb{B}$. The Lang torsor, i.e., the finite étale galois covering $1 - F_k : \mathbb{B}^\times \to \mathbb{B}^\times$, has structural group $B^\times = \mathbb{B}^\times(k)$. We then push out this $B^\times$-torsor on $\mathbb{B}^\times$ by $\overline{\chi}$, to obtain the $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{L}_\chi$ on $\mathbb{B}^\times$. It is lisse of rank one and pure of weight zero.

We have a k-morphism (in fact an embedding)

$$\mathbb{A}^1[1/f] \subset \mathbb{B}^\times,$$

given on R-valued points, R any k-algebra, by

$$t \in \mathbb{A}^1[1/f](R) \mapsto u - t \in \mathbb{B}(R).$$

Lemma 2.1. *For any k-algebra R, and any $t \in \mathbb{A}^1(R) = R$, we have the identity*

$$\mathrm{Norm}_{B/k}(u - t) = (-1)^n f(t) \in R.$$

Proof. In the k-algebra $B = k[X]/(f(X))$, multiplication by u (the class of X in B) has characteristic polynomial f (theory of the "companion matrix"), i.e., taking for R the polynomial ring $k[T]$, we have $\mathrm{Norm}_{B/k}(T - u) = f(T) \in R = k[T]$, hence $\mathrm{Norm}_{B/k}(u - T) = (-1)^n f(T) \in k[T]$, and this is the universal case of the asserted identity. □

We denote by $\mathcal{L}_{\chi(u-t)}$ the lisse $\overline{\mathbb{Q}_\ell}$-sheaf of rank one on $\mathbb{A}^1[1/f]$ obtained as the pullback of $\mathcal{L}_\chi$ on $\mathbb{B}^\times$ by the embedding $t \mapsto u - t$ of $\mathbb{A}^1[1/f]$ into $\mathbb{B}^\times$. In view of Lemma 1.1, the L-function $L(\chi, T)$ is, via the chosen embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}_\ell}$, the L-function of $\mathbb{A}^1[1/f]/k$ with coefficients in $\mathcal{L}_{\chi(u-t)}$. This sheaf on $\mathbb{A}^1[1/f]$ is lisse of rank one and pure of weight zero. The compact cohomology groups

$$H^i_c := H^i_c(\mathbb{A}^1[1/f] \otimes_k \overline{k}, \mathcal{L}_{\chi(u-t)})$$

vanish for $i \neq 1, 2$, and by the Lefschetz trace formula we have the formula

$$L(\chi, T) = \det(1 - TFrob_k | H^1_c) / \det(1 - TFrob_k | H^2_c).$$

We now turn to a closer examination of these cohomology groups. For this, we first examine the sheaf $\mathcal{L}_{\chi(u-t)}$ geometrically, i.e., pulled back to $\mathbb{A}^1[1/f]/\overline{k}$, and describe in terms of translations of Kummer sheaves $\mathcal{L}_\rho$ on $\mathbb{G}_m$. Recall that the tame fundamental group $\pi_1^{tame}(\mathbb{G}_m/\overline{k})$ is the inverse limit over prime-to p integers N, growing multiplicatively, of the groups $\mu_N(\overline{k})$, via the N'th power Kummer coverings of $\mathbb{G}_m/\overline{k}$ by itself. It is also the inverse limit, over finite extension fields E/k growing by inclusion, of the multiplicative groups $E^\times$, with transition maps the Norm, via the Lang torsor coverings $1 - F_E$ of of $\mathbb{G}_m/\overline{k}$ by itself. For any continuous $\overline{\mathbb{Q}_\ell}^\times$-valued character ρ of $\pi_1^{tame}(\mathbb{G}_m/\overline{k})$, we have the corresponding Kummer sheaf $\mathcal{L}_\rho$ on $\mathbb{G}_m/\overline{k}$. The characters of finite order of $\pi_1^{tame}(\mathbb{G}_m/\overline{k})$ are precisely those which arise from characters ρ of $E^\times$ for some finite extension E/k. More precisely, a character ρ of finite order of $\pi_1^{tame}(\mathbb{G}_m/\overline{k})$ comes from a character of $E^\times$ if and only if $\rho = \rho^{\#E}$(equality as characters of $\pi_1^{tame}(\mathbb{G}_m/\overline{k})$). For such a character ρ, the Kummer sheaf $\mathcal{L}_\rho$ on $\mathbb{G}_m/\overline{k}$ begins life on $\mathbb{G}_m/E$.

To analyze the sheaf $\mathcal{L}_{\chi(u-t)}$ geometrically, first choose a finite extension field E/k in which f factors completely, say $f(X) = \prod_{i=1}^{n}(X - a_i)$. Then $\mathbb{B}(E)^{\times} \cong (E^{\times})^n$, and $\chi_E := \chi \circ \mathrm{Norm}_{E/k}$ as character of $(E^{\times})^n$ is of the form $(x_1, ..., x_n) \mapsto \prod \chi_i(x_i)$, for characters $\chi_1, ..., \chi_n$ of $E^{\times}$. Then $\mathbb{B}^{\times}$, pulled back to $\overline{k}$, becomes $\mathbb{G}_m^n$, and $\mathcal{L}_{\chi}$ on it becomes the external tensor product $\boxtimes_{i=1}^{n}\mathcal{L}_{\chi_i}$ of usual Kummer sheaves $\mathcal{L}_{\chi_i}$ on the factors. Over $\overline{k}$, the embedding of $\mathbb{A}^1[1/f]$ into $\mathbb{B}^{\times}$ given by $t \mapsto u - t$ becomes the embedding of $\mathbb{A}^1[1/f] \otimes_k \overline{k}$ into $\mathbb{G}_m^n$ given by

$$t \mapsto (a_1 - t, ..., a_n - t).$$

Thus the sheaf $\mathcal{L}_{\chi(u-t)}$ is geometrically isomorphic to the tensor product $\otimes_{i=1}^{n}\mathcal{L}_{\chi_i(a_i-t)}$ on $\mathbb{A}^1[1/f] \otimes_k \overline{k} = Spec(\overline{k}[t][1/f(t)])$.

Lemma 2.2. *With the notations of the previous paragraph, we have the following results.*

1. *We have $H_c^2 = 0$ if and only if some χ_i is nontrivial, in which case H_c^1 has dimension $n - 1$.*
2. *The group H_c^1 is pure of weight one if and only if every χ_i is nontrivial and the product $\prod_{i=1}^{n}\chi_i$ is nontrivial.*

Proof. Both assertions are invariant under finite extension of the ground field, so it suffices to treat universally the case in which f factors completely over k. The character χ_i is the local monodromy of $\mathcal{L}_{\chi(u-t)}$ at the point a_i, and the product $\prod_{i=1}^{n}\chi_i$ is its local monodromy at ∞. For assertion (1), we note that the group H_c^2 is either zero or one-dimensional. It is nonzero if and only if the lisse rank one sheaf $\mathcal{L}_{\chi(u-t)}$ is geometrically constant, i.e., if and only if its local monodromy at each of the points $\infty, a_1, ..., a_n$ is trivial. The dimension assertion results from the Euler-Poincaré formula: because $\mathcal{L}_{\chi(u-t)}$ is lisse of rank one and at worst tamely ramified at the missing points, it gives

$$\chi_c(\mathbb{A}^1[1/f] \otimes_k \overline{k}, \mathcal{L}_{\chi(u-t)}) = \chi_c(\mathbb{A}^1[1/f] \otimes_k \overline{k}, \overline{\mathbb{Q}_\ell}) = 1 - n.$$

For assertion (2), we argue as follows. If all the χ_i are trivial, i.e., if χ is trivial, then $\mathcal{L}_{\chi}$ on $\mathbb{B}^{\times}$ is trivial, $\mathcal{L}_{\chi(u-t)}$ on $\mathbb{A}^1[1/f]$ is trivial, and its H_c^1 has dimension n and is pure of weight zero.

Suppose now that χ is nontrivial, i.e., that at least one χ_i is nontrivial. Denote by $j : \mathbb{A}^1[1/f] \subset \mathbb{P}^1$ the inclusion. Then we have a short exact sequence of sheaves on $\mathbb{P}^1$

$$0 \to j_!\mathcal{L}_{\chi(u-t)} \to j_\star\mathcal{L}_{\chi(u-t)} \to Pct \to 0,$$

in which Pct is a skyscraper sheaf, supported at those of the points $\infty, a_1, ..., a_n$ where the local monodromy is trivial, and is punctually pure of weight zero with one-dimensional stalk at each of these points. The long exact cohomology sequence then gives a short exact sequence

$$0 \to H^0(\mathbb{P}^1/\overline{k}, Pct) \to H^1(\mathbb{P}^1/\overline{k}, j_!\mathcal{L}_{\chi(u-t)}) \to H^1(\mathbb{P}^1/\overline{k}, j_\star\mathcal{L}_{\chi(u-t)}) \to 0$$

in which the middle term $H^1(\mathbb{P}^1/\overline{k}, j_!\mathcal{L}_{\chi(u-t)})$ is the cohomology group H^1_c, the third term $H^1(\mathbb{P}^1/\overline{k}, j_\star\mathcal{L}_{\chi(u-t)})$ is pure of weight one [De-Weil II, 3.2.3], and the first term, $H^0(\mathbb{P}^1/\overline{k}, Pct)$ is pure of weight zero and of dimension the number of points among $\infty, a_1, ..., a_n$ where the local monodromy is trivial. □

Given a character χ of $B^\times$, how do we determine what $\mathcal{L}_{\chi(u-t)}$ looks like, geometrically? We know that, in terms of the factorization of f, say $f(X) = \prod_{i=1}^n (X - a_i)$ over some finite extension field E/k, $\mathcal{L}_{\chi(u-t)}$ is geometrically isomorphic to the tensor product $\otimes_{i=1}^n \mathcal{L}_{\chi_i(a_i-t)}$ on $\mathbb{A}^1[1/f] \otimes_k \overline{k} = Spec(\overline{k}[t][1/f(t)])$. We have an easy interpretation of the product $\prod_i \chi_i$ of all the χ_i.

Lemma 2.3. *For $\rho :=$ the restriction of χ to $k^\times$ ($k^\times$ seen as a subgroup of $B^\times$), the composition $\rho \circ \mathrm{Norm}_{E/k}$ is the character of $E^\times$ given by the product $\prod_i \chi_i$ of all the χ_i.*

Proof. Under the E-linear isomorphism of $B_E = E[X]/(f)$ with the n-fold self product of E, E viewed as the constant polynomials is diagonally embedded. Thus $\prod_i \chi_i$ is the effect of $\chi \circ \mathrm{Norm}_{B_E/B}$ on $E^\times$ (viewed as a subgroup of $B_E^\times$). The restriction to $E^\times$ of this norm map $\mathrm{Norm}_{B_E/B} : B_E^\times \to B^\times$ is the norm map $\mathrm{Norm}_{E/k} : E^\times \to k^\times$. □

To further analyze this question, in a "k-rational" way, we first factor our squarefree monic f as a product of distinct monic k-irreducible polynomials, say

$$f = \prod P_i, \ \deg(P_i) := d_i.$$

Then with

$$B_{P_i} := k[X]/(P_i),$$

we have an isomorphism of k-algebras

$$B := k[X]/(f) \cong \prod_i B_{P_i}, \ g \mapsto (g \mod P_i)_i,$$

and a character χ of $B^\times$ is uniquely of the form

$$\chi(g) = \prod_i \chi_{P_i}(g \mod P_i),$$

for characters χ_{P_i} of $B_{P_i}^\times$.

So it suffices treat the case when f is a single irreducible polynomial P of some degree $d \geq 1$. Choose a root a of P in our chosen $\overline{k}$. This choice gives an isomorphism of B_P with the unique extension field k_d/k of degree d_i inside $\overline{k}$, namely $g \mapsto g(a)$. Via this isomorphism, the character χ_P becomes a character χ of $k_d^\times$. After extension of scalars from k to k_d, we have a k_d-linear isomorphism

$$B_P \otimes_k k_d = k_d[X]/(P) \cong \prod_{\sigma \in Gal(k_d/k)} k_d,\ g(X) \mapsto (g(\sigma(a))_\sigma.$$

Then for $g(X) \in k_d[X]/(P_i)$, its k_d/k-Norm down to B_P is

$$\mathrm{Norm}_{k_d/k}(g(X)) = \prod_{\tau \in Gal(k_d/k)} g^\tau(X) \mod P = \prod_{\tau \in Gal(k_d/k)} g^\tau(a) \in k_d.$$

So we have the identity

$$(\chi \circ \mathrm{Norm}_{k_d/k})(g(X)) = \prod_{\tau \in Gal(k_d/k)} \chi(g^\tau(a)) = \prod_{\tau \in Gal(k_d/k)} (\chi \circ \tau)(g(\tau^{-1}(a)).$$

The arguments $g(\tau^{-1}(a))$ of the characters $\chi \circ \tau$ are just the components, in another order, of g in the isomorphism $k_d[X]/(P) \cong \prod_{\sigma \in Gal(k_d/k)} k_d$. In other words, the pullback of χ by the k_d/k-Norm from $B_P \otimes_k k_d$ down to B_P has components $(\chi, \chi^q, ..., \chi^{q^{d-1}})$. Thus we have the following lemma.

Lemma 2.4. *For P an irreducible monic k-polynomial of degree $d \geq 1$, and χ a character of $B_P^\times \cong k_d^\times$ (via $u \mapsto a$, a a chosen root of P in k_d), the sheaf $\mathcal{L}_\chi(u-t)$ on $\mathbb{A}^1[1/P]$ is geometrically isomorphic to the tensor product $\otimes_{i=0}^{d-1} \mathcal{L}_{\chi^{q^i}(a^{q^{-i}}-t)}$.*

Combining these last two lemmas with Lemma 2.2, we get the following result.

Lemma 2.5. *Let f be a squarefree monic k-polynomial of degree $n \geq 2$, $f = \prod_i P_i$ its factorization into monic k-irreducibles, χ a character of $B^\times$, and, for each P_i, χ_{P_i} the P_i-component of χ. The group $H_c^1(\mathbb{A}^1[1/f] \otimes_k \overline{k}, \mathcal{L}_{\chi(u-t)})$ is pure of weight one if and only if χ is nontrivial on $k^\times$ and each χ_{P_i} is nontrivial, in which case H_c^1 has dimension $n-1$ and $H_c^2 = 0$.*

3. The direct image theorem

In this section, we work over $\overline{k}$.

Theorem 3.1. *Suppose that $f(X) = \prod_{i=1}^n (X - a_i)$ is a squarefree polynomial of degree $n \geq 2$ over $\overline{k}$. Let $\chi_1, ..., \chi_n$ be characters of $\pi_1^{tame}(\mathbb{G}_m/\overline{k})$ of finite order, and form the lisse sheaf*

$$\mathcal{F} := \otimes_{i=1}^n \mathcal{L}_{\chi_i(a_i - t)}$$

on $\mathbb{A}^1[1/f] \otimes_k \overline{k}$. Then we have the following results.

(1) *For any scalar $\lambda \in \overline{k}^\times$, the direct image $[\lambda f]_\star \mathcal{F}$ of $\mathcal{F}$ by the polynomial map $\lambda f : \mathbb{A}^1[1/f]/\overline{k} \to \mathbb{G}_m/\overline{k}$ is a middle extension sheaf on $\mathbb{G}_m/\overline{k}$, of generic rank n, and the perverse sheaf $[\lambda f]_\star \mathcal{F}[1]$ is geometrically semisimple.*
(2) *If one of the χ_i, say χ_1, is a singleton among the χ's, in the sense that $\chi_1 \neq \chi_j$ for every $j \neq 1$, then the perverse sheaf $[\lambda f]_\star \mathcal{F}[1]$ on on $\mathbb{G}_m/\overline{k}$ is irreducible.*
(3) *If two of the χ_i, say χ_1 and χ_2 are each singletons among the χ's, then the irreducible perverse sheaf $[\lambda f]_\star \mathcal{F}[1]$ on on $\mathbb{G}_m/\overline{k}$ is not isomorphic to any nontrivial multiplicative translate of itself.*

Proof. To prove (1), we argue as follows. The map $\lambda f : \mathbb{A}^1[1/f]/\overline{k} \to \mathbb{G}_m/\overline{k}$ is finite and flat of degree n. As f has all distinct roots, its derivative f' is not identically zero, so over the dense open set U of $\mathbb{G}_m/\overline{k}$ obtained by deleting the images under λf of the zeroes of f', the map λf is finite étale of degree n. Thus $[\lambda f]_\star \mathcal{F}[1]$ has generic rank n. It is a middle extension because $\mathcal{F}$ is a middle extension on the source (being lisse), and λf is finite, flat, and generically étale, cf. [Ka-TLFM, first paragraph of the proof of 3.3.1]. On the dense open set U, $[\lambda f]_\star \mathcal{F}$ is (the pullback from some finite subfield E of $\overline{k}$ of) a lisse sheaf which is pure of weight zero, hence is geometrically semisimple [De-Weil II, 3.4.1 (iii)]. Therefore [BBD, 4.3.1 (ii)] the perverse sheaf $[\lambda f]_\star \mathcal{F}[1]|U$ is semisimple, and this property is preserved by middle extension from U to $\mathbb{G}_m/\overline{k}$.

Suppose now that χ_1 is a singleton among the χ's. We claim that $[\lambda f]_\star \mathcal{F}[1]$ is irreducible. Since $[\lambda f]_\star \mathcal{F}[1]$ is just a multiplicative translate of $f_\star \mathcal{F}[1]$, it suffices to show that $f_\star \mathcal{F}[1]$ is irreducible. Since $f_\star \mathcal{F}[1]$ is semisimple, we must show that the inner product

$$<f_\star \mathcal{F}[1], f_\star \mathcal{F}[1]> = 1.$$

By Frobenius reciprocity, we have

$$<f_\star \mathcal{F}[1], f_\star \mathcal{F}[1]> = <\mathcal{F}[1], f^\star f_\star \mathcal{F}[1]>.$$

So we must show that $\mathcal{F}[1]$ occurs at most once in $f^\star f_\star \mathcal{F}[1]$. We will show the stronger statement, that denoting by $I(a_1)$ the inertia group at the point $a_1 \in \mathbb{A}^1(\overline{k})$, the $I(a_1)$-representation of $\mathcal{F}[1]$ occurs at most once in the $I(a_1)$-representation of $f^\star f_\star \mathcal{F}[1]$. As a finite flat map of $\mathbb{A}^1$ to itself, f is finite étale over a neighborhood of 0 in the target (because f has n distinct roots $a_1, ..., a_n$, the preimages of 0). We first infer that the $I(0)$-representation of $f_\star \mathcal{F}[1]$ is the direct sum of the χ_i, and then that for each j the $I(a_j)$-representation of $f^\star f_\star \mathcal{F}[1]$ is the direct sum of the χ_i. At the point a_1, the $I(a_1)$-representation of $\mathcal{F}[1]$ is χ_1, and by the singleton hypothesis χ_1 occurs only once in the direct sum of the χ_i, so only once in the $I(a_1)$-representation of $f^\star f_\star \mathcal{F}[1]$.

Suppose now that both χ_1 and χ_2 are singletons. We must show that for any scalar $\lambda \neq 1$ in $\overline{k}^\times$, the perverse irreducible sheaves $[\lambda f]_\star \mathcal{F}[1]$ and $f_\star \mathcal{F}[1]$ on $\mathbb{G}_m/\overline{k}$ are not isomorphic. We argue by contradiction, and thus suppose the two are isomorphic. Choose a finite subfield E of $\overline{k}$ over which the scalar λ, the points a_i, the characters χ_i and the open set U are all defined, so that we may speak of the geometrically irreducible perverse sheaves $[\lambda f]_\star \mathcal{F}(1/2)[1]$ and $f_\star \mathcal{F}(1/2)[1]$ on $\mathbb{G}_m/E$. Each of these is pure of weight zero. On the dense open set $U \subset \mathbb{G}_m/E$, the sheaves $[\lambda f]_\star \mathcal{F}$ and $f_\star \mathcal{F}$ are lisse and geometrically isomorphic, so one is a constant field twist of the other, say $[\lambda f]_\star \mathcal{F}|U \cong f_\star \mathcal{F} \otimes \alpha^{\deg}|U$, for some scalar $\alpha \in \overline{\mathbb{Q}_\ell}^\times$. Taking middle extensions, we find an arithmetic isomorphism

$$[\lambda f]_\star \mathcal{F}(1/2)[1] \cong f_\star \mathcal{F}(1/2)[1] \otimes \alpha^{\deg}$$

on $\mathbb{G}_m/E$. Because both $[\lambda f]_\star \mathcal{F}(1/2)[1]$ and $f_\star \mathcal{F}(1/2)[1]$ are pure of weight zero, the scalar α must be pure of weigh zero. This arithmetic isomorphism implies that (and, given the geometric irreducibility, is in fact equivalent to the fact that) for any finite extension L/E, and any point $t \in L^\times$, we have an equality of traces

$$\mathrm{Trace}(Frob_{L,t}|[\lambda f]_\star \mathcal{F}(1/2)) = \alpha^{\deg(L/E)} \mathrm{Trace}(Frob_{L,t}|f_\star \mathcal{F}(1/2)).$$

Because $[\lambda f]_\star \mathcal{F}(1/2)[1])$ is a geometrically irreducible perverse sheaf on $\mathbb{G}_m/E$ which is pure of weight zero, we have the estimate, as L/E runs over larger and larger finite extensions,

$$\sum_{t \in \mathbb{G}_m(L)} |\mathrm{Trace}(Frob_{L,t}|[\lambda f]_\star \mathcal{F}(1/2)|^2 = 1 + O(1/\sqrt{\#L}),$$

or equivalently the estimate

$$\sum_{t \in \mathbb{G}_m(L)} |\mathrm{Trace}(Frob_{L,t}|[\lambda f]_\star \mathcal{F}|^2 = \#L + O(\sqrt{\#L}).$$

Indeed, it suffices to check that this second estimate holds instead for the sum over points $t \in U(L)$, as this sum omits at most $\#(\mathbb{G}_m \setminus U)(\overline{k})$ terms, each of which is itself $O(1)$. Because $[\lambda f]_\star \mathcal{F}$ is lisse on U and pure of weight zero, the sum over U is given, by the Lefschetz trace formula, in terms of the sheaf $End := End([\lambda f]_\star \mathcal{F})$ as

$$\mathrm{Trace}(Frob_L|H^2_c(U/\overline{k}, End) - \mathrm{Trace}(Frob_L|H^1_c(U/\overline{k}, End).$$

The sheaf End is pure of weight zero. By the geometric irreducibility of $([\lambda f]_\star\mathcal{F})|U$, the $\pi_1^{geom}(U)$-coinvariants of End are just the constants $\overline{\mathbb{Q}_\ell}$, so the group H^2_c above is just $\overline{\mathbb{Q}_\ell}(-1)$, on which $Frob_L$ acts as $\#L$. The H^1_c group is mixed of weight ≤ 1, so we get the asserted estimate.

We now rewrite the sum of squares as follows.The sheaves $\mathcal{F}$ and

$$\overline{\mathcal{F}} := \otimes_{i=1}^n \mathcal{L}_{\overline{\chi_i}(a_i-t)}$$

have complex conjugate trace functions, as do the their direct images by any λf. As α is pure of weight zero, we have $\overline{\alpha} = 1/\alpha$. So we have

$$\alpha^{\deg(L/E)} \sum_{t\in\mathbb{G}_m(L)} |\mathrm{Trace}(Frob_{L,t}|[\lambda f]_\star\mathcal{F}|^2 =$$

$$= \sum_{t\in\mathbb{G}_m(L)} (\mathrm{Trace}(Frob_{L,t}|[\lambda f]_\star\mathcal{F}))(\mathrm{Trace}(Frob_{L,t} f_\star\overline{\mathcal{F}})) =$$

$$= \alpha^{\deg(L/E)}\#L + O(\sqrt{\#L}).$$

We now rewrite this penultimate sum as

$$\sum_{t\in\mathbb{G}_m(L)} (\sum_{x\in L,\ \lambda f(x)=t} \mathrm{Trace}(Frob_{L,x}|\mathcal{F}))(\sum_{y\in L,\ f(y)=t} Trace(Frob_{L,y}|\overline{\mathcal{F}})) =$$

$$\sum_{(x,y)\in\mathbb{A}^2(L),\ \lambda f(x)=f(y)\neq 0} \mathrm{Trace}(Frob_{L,x}|\mathcal{F})\mathrm{Trace}(Frob_{L,y}|\overline{\mathcal{F}}).$$

For $j : \mathbb{A}^1[1/f] \subset \mathbb{A}^1$,if we add the n^2 terms

$$\mathrm{Trace}(Frob_{L,x}|j_\star\mathcal{F})\mathrm{Trace}(Frob_{L,y}|j_\star\overline{\mathcal{F}})$$

for the points $(x, y) \in \mathbb{A}^2(L)$ with $f(x) = f(y) = 0$, i.e., for the n^2 points (a_i, a_j), we only change our sum by $O(1)$ (and we don't change it at all if all the χ_i are nontrivial). So we end up with the estimate

$$\sum_{(x,y)\in\mathbb{A}^2(L),\ \lambda f(x)=f(y)} \mathrm{Trace}(Frob_{L,x}|j_\star\mathcal{F})\mathrm{Trace}(Frob_{L,y}|j_\star\overline{\mathcal{F}}) =$$

$$= \alpha^{\deg(L/E)}\#L + O(\sqrt{\#L}).$$

We now explain how this estimate leads to a contradiction. Consider the affine curve of equation $\lambda f(x) = f(y)$ in $\mathbb{A}^2$. It is singular at the finitely many points (a, b) which are pairs of critical points of f, i.e., $f'(a) = f'(b) = 0$, such that $\lambda f(a) = f(b)$. It is nonsingular at each pair of zeroes (a_i, a_j) of f. Replacing E by a finite extension if necessary, we may further assume that each irreducible component of the curve $\lambda f(x) = f(y)$ over E is geometrically irreducible (i.e., that each irreducible factor of $\lambda f(x) - f(y)$ in $E[x, y]$ remains irreducible in $\overline{k}[x,y]$). The penultimate sum is, up to an $O(1)$ term, the sum over the irreducible components C_j of the curve $\lambda f(x) = f(y)$, of the sums

$$\sum_{(x,y)\in\mathbb{C}_j(L)} \mathrm{Trace}(Frob_{L,x}|j_\star\mathcal{F})\mathrm{Trace}(Frob_{L,y}|j_\star\overline{\mathcal{F}}).$$

By the estimate for the sum, over the various C_j, of these sums, there is at least one irreducible component, call it C for which this sum is **not** $O(\sqrt{\#L})$. The equation of any C_j divides the polynomial $\lambda f(x)-f(y)$, whose highest degree term is $\lambda x^n - y^n$. Therefore the highest degree term any divisor is a product of linear terms $\mu y - x$, with the various possible μ's the n'th roots of λ. So an irreducible component C_i, given by a degree d_i divisor of $\lambda f(x) - f(y)$, is finite flat of degree d_i over the y-line (and over the x line as well).

On the original curve $\lambda f(x) = f(y)$, for each a_j there are n points (a_j, y) on the curve, namely $y = a_i$ for $i = 1, ..., n$. On an irreducible component C_j, given by a degree d_j divisor of $\lambda f(x) - f(y)$, there are at most d_j values of y such that (a_1, y) lies on C_j. Each of these points is a smooth point of the original curve, so it lies only on the irreducible component C_j. As there are $n = \sum d_j$ points (a_1, y) on the original curve, we must have exactly d_j points on C_j of the form (a_1, y).

Now consider an irreducible component C on which our sum is not $O(\sqrt{\#L})$. Let us denote by $\mathcal{C}$ the dense open set of the smooth locus of C which, via f, lies over $\mathbb{G}_m$. The sum

$$\sum_{(x,y)\in\mathcal{C}(L)} \mathrm{Trace}(Frob_{L,x}|\mathcal{F})\mathrm{Trace}(Frob_{L,y}|\overline{\mathcal{F}})$$

differs only by $O(1)$ from the sum over C, so it too is not $O(\sqrt{\#L})$. In terms of the (restriction to $\mathcal{C}$ of the) lisse, pure of weight zero, lisse of rank one sheaf

$$\mathcal{G} := \otimes_{i=1}^n \mathcal{L}_{\chi_i(a_i - x)} \otimes_{i=1}^n \mathcal{L}_{\overline{\chi_i}(a_i - y)}$$

on $\mathbb{A}^2[1/(f(x)f(y))]$, this last sum is

$$\sum_{(x,y)\in\mathcal{C}(L)} \mathrm{Trace}(Frob_{L,(x,y)}|\mathcal{G}).$$

By the Lefschetz trace formula, this sum is

$$\mathrm{Trace}(Frob_L|H^2_c(\mathcal{C}/\overline{k}, \mathcal{G})) - \mathrm{Trace}(Frob_L|H^1_c(\mathcal{C}/\overline{k}, \mathcal{G})).$$

Because $\mathcal{G}$ is pure of weight zero and lisse of rank one, the H^2_c is either zero or is is one-dimensional and pure of weight two, and this second case only occurs when $\mathcal{G}$ is geometrically constant on $\mathcal{C}$. The H^1_c is mixed of weight ≤ 1. So the failure of an $O(\sqrt{\#L})$ estimate means that the H^2_c is nonzero, and hence that $\mathcal{G}$ is geometrically constant on $\mathcal{C}$.

Suppose first that the equation of C is of degre $d \geq 2$. Then there are d points (a_1, a_i) on C, at least one of which is of the form (a_1, a_i) with $a_i \neq a_1$. The curve C is finite etale over both the x-line and the y-line at the point (a_1, a_i). So the functions $x - a_1$ **and** $y - a_i$ are each uniformizing parameters at this point. From the expression for $\mathcal{G}$, at the point (a_1, a_i) on C its inertia group representation is that of $\mathcal{L}_{\chi_1(x-a_1)} \otimes \mathcal{L}_{\overline{\chi_i}(y-a_i)}$. In other words, its inertia group representation at (a_1, a_i) is the character χ_1/χ_i. But this character is nontrivial (because χ_1 is a singleton), contradicting the geometric constance of $\mathcal{G}$ on $\mathcal{C}$.

It remains to treat the case in which the equation for C is of degree one. In this case, the above argument still works unless the unique point on C of the form (a_1, y) has $y = a_1$. In this case, we use the fact that we have a second singleton, χ_2. Using this singleton, we could still use the above argument unless the unique point on C of the form (a_2, y) has $y = a_2$. So we only need treat the case when both the points (a_1, a_1) and (a_2, a_2) lie on C. But in this case, the equation for C, being of degree one, must be $y = x$. But if $y - x$ divides $\lambda f(x) - f(y)$, we reduce mod $y - x$ to find that $(\lambda - 1)f(x) = 0$, and hence $\lambda = 1$, contradiction. $\square$

4. A preliminary estimate

In this section, we continue with a squarefree monic k-polynomial f of degree $n \geq 2$, $B := k[X]/(f)$, and a character χ of $B^\times$. Over a finite extension E/k where f factors completely, say $f(X) = \prod_i (X - a_i)$, the lisse rank one sheaf $\mathcal{L}_{\chi(u-t)}$ on $\mathbb{A}^1[1/f]/k$ becomes isomorphic to the sheaf $\otimes_{i=1}^n \mathcal{L}_{\chi_i(a_i - t)}$ on $\mathbb{A}^1[1/f]/E$.

Theorem 4.1. *Let χ be a character of $B^\times$ whose constituent characters χ_i satisfy the following three conditions.*

1. *The χ_i are pairwise distinct.*
2. *The product $\prod_i \chi_i$ is nontrivial, (i.e., χ is nontrivial on $k^\times$).*
3. *For at least one index i, $\chi_i^n \neq \prod_i \chi_i$.*

Fix $\lambda \in k^\times$, and form the perverse sheaf

$$N(\lambda, \chi) := [\lambda f]_\star(\mathcal{L}_{\chi(u-t)})(1/2)[1]$$

on $\mathbb{G}_m/k$. Then we have the following results.

1. *$N(\lambda, \chi)$ is geometrically irreducible, pure of weight zero, and lies in the Tannakian category $\mathcal{P}_{arith}$ in the sense of* [Ka-CE]. *It has generic rank n, Tannakian "dimension" $n - 1$, and it has at most $2n$ bad characters.*

(2) $N(\lambda,\chi)$ *is geometrically Lie-irreducible in* $\mathcal{P}$.
(3) $N(\lambda,\chi)$ *has* $G_{geom} = G_{arith} = GL(n-1)$.

Proof. By Theorem 3.1 and the disjointness of the χ_i, $N(\lambda,\chi)$ is geometrically irreducible. It visibly has generic rank n. As $n \geq 2$, it is not a Kummer sheaf, so, being geometrically irreducible, it lies in $\mathcal{P}$. Its Tannakian dimension is

$$\chi_c(\mathbb{G}_m/\overline{k}, N(\lambda,\chi)) = -\chi_c(\mathbb{G}_m/\overline{k}, [\lambda f]_\star(\mathcal{L}_{\chi(u-t)})) =$$
$$= -\chi_c(\mathbb{A}^1[1/f]/\overline{k}, \mathcal{L}_{\chi(u-t)}) = -\chi_c(\mathbb{A}^1[1/f]/\overline{k}, \otimes_{i=1}^n \mathcal{L}_{\chi_i(a_i-t)}) = n-1.$$

Because $N(\lambda,\chi)$ has generic rank n, it has at most $2n$ bad characters, namely those whose inverses occur in either its $I(0)$-representation or in its $I(\infty)$-representation.

On some dense open set $j : U \subset \mathbb{G}_m$, $[\lambda f]_\star(\mathcal{L}_{\chi(u-t)})$ is a lisse sheaf of rank n, which is pure of weight zero, hence $j^\star N(\lambda,\chi)$ is pure of weight zero (the Tate twist $(1/2)$ offsets the shift $[1]$). By irreducibility $N(\lambda,\chi)$ must be the middle extension of $j^\star N(\lambda,\chi)$, cf.[BBD, 5.3.8], so remains pure of weight zero [BBD, 5.3.2]. Again by the disjointness of the χ_i, part (3) of Theorem 3.1, together with [Ka-CE, Cor. 8.3], we get that $N(\lambda,\chi)$ is geometrically Lie-irreducible in $\mathcal{P}$.

It remains to explain why $N(\lambda,\chi)$ has $G_{geom} = G_{arith} = GL(n-1)$. Since we have a priori inclusions $G_{geom} \subset G_{arith} \subset GL(n-1)$, it suffices to prove that $G_{geom} = GL(n-1)$. The idea is to apply [Ka-CE, Thm. 17.1]. We may compute G_{geom} after extension of scalars to E. Suppose that $\chi_1^n \neq \prod_i \chi_i$. The construction $M \mapsto M \otimes \mathcal{L}_{\overline{\chi_1}}$ induces a Tannakian isomorphism of $<N(\lambda,\chi)>_{arith}$ with $<N(\lambda,\chi) \otimes \mathcal{L}_{\overline{\chi_1}}>_{arith}$. So it suffices to prove that $N(\lambda,\chi) \otimes \mathcal{L}_{\overline{\chi_1}}$ has $G_{geom} = GL(n-1)$. By the disjointness assumption on the χ_i, the trivial character $\mathbb{1}$ occurs exactly once in the $I(0)$-representation of $N(\lambda,\chi) \otimes \mathcal{L}_{\overline{\chi_1}}$. So by [Ka-CE, Thm. 17.1], it suffices to show that the trivial character does not occur in its $I(\infty)$-representation, or equivalently that χ_1 does not occur in the $I(\infty)$-representation of $N(\lambda,\chi)$. This $I(\infty)$-representation is $[\lambda f]_\star \mathcal{L}_{\prod_i \chi_i}$, and $\mathcal{L}_{\chi_1}$ occurs in it if and only if $[\lambda f]^\star(\mathcal{L}_{\chi_1})$ occurs in $\mathcal{L}_{\prod_i \chi_i}$. Because λf has degree n, the pullback $[\lambda f]^\star(\mathcal{L}_{\chi_1})$ is geometrically isomorphic to $\mathcal{L}_{\chi_1^n}$ as $I(\infty)$-representation. So if $\chi_1^n \neq \prod_i \chi_i$, then $\mathcal{L}_{\chi_1}$ does not occur in the $I(\infty)$-representation $[\lambda f]_\star \mathcal{L}_{\prod_i \chi_i}$, and we conclude by applying [Ka-CE, Thm. 17.1] to $N(\lambda,\chi) \otimes \mathcal{L}_{\overline{\chi_1}}$. □

Corollary 4.2. *Let* χ *be a character of* $B^\times$ *whose constituent characters* χ_i *satisfy the three conditions of the previous theorem. Suppose that* $q := \#k$ *satisfies the inequality* $\sqrt{q} \geq 1+2n$. *For each character* ρ *of* $k^\times$ *which is good for* $N(\lambda,\chi)$ *(i.e., such that for* $j : \mathbb{G}_m \subset \mathbb{P}^1$ *the inclusion, the "'forget supports" map gives an isomorphism* $j_!(N(\lambda,\chi) \otimes \mathcal{L}_\rho) \cong$

$Rj_\star(N(\lambda,\chi)\otimes\mathcal{L}_\rho)$, or equivalently, $\overline{\rho}$ does not occur in the local monodromy at either 0 or ∞ of $N(\lambda f,\chi)$), denote by $\theta_{k,\lambda f,\chi,\rho}$ the conjugacy class in $U(n-1)$ whose reversed characteristic polynomial is given by

$$\det(1-T\theta_{k,\lambda f,\chi,\rho}) = \det(1-TFrob_k|H^0_c(\mathbb{G}_m/\overline{k}, N(\lambda,\chi)\otimes\mathcal{L}_\rho).$$

Let Λ be a nontrivial irreducible representation of $U(n-1)$ which occurs in $std^{\otimes a}\otimes(std^\vee)^{\otimes b}$. Then we have the estimate

$$|\sum_{\rho\in Good(k,\lambda f,\chi)} Trace(\Lambda(\theta_{k,\lambda f,\chi,\rho}))|$$

$$\leq (\#Good(k,\lambda f,\chi))2(a+b+1)(2n)^{a+b}/\sqrt{q}.$$

Proof. By Theorem 4.1, $N(\lambda,\chi)$ has $G_{geom} = G_{arith} = GL(n-1)$. So this is [Ka-CE, Remark 7.5 and the proof of Theorem 28.1], applied to $N := N(\lambda,\chi)$ with the constant C there, an upper bound for each of the generic rank, the number of bad characters, and the Tannakian dimension of N, taken to be $2n$. □

The interest of this Corollary is that the (trivial) Leray spectral sequence for $[\lambda f]_!$ gives a $Frob_k$-isomorphism of cohomology groups

$$H^0_c(\mathbb{G}_m/\overline{k}, N(\lambda,\chi)\otimes\mathcal{L}_\rho) \cong H^0_c(\mathbb{A}^1[1/f]/\overline{k}, \mathcal{L}_{\chi(u-t)}\otimes\mathcal{L}_{\rho(\lambda f(t))}(1/2)[1]) =$$

$$= H^1_c(\mathbb{A}^1[1/f]/\overline{k}, \mathcal{L}_{\chi(u-t)}\otimes\mathcal{L}_{\rho(\lambda f(t))})(1/2).$$

By Lemma 2.1, $\mathrm{Norm}_{B/k}(u-t) = (-1)^n f(t)$. So if we denote by ρ_{Norm} the character of $B^\times$ given by

$$\rho_{\mathrm{Norm}} := \rho\circ \mathrm{Norm}B/k,$$

then $\mathcal{L}_{\rho((-1)^n f(t))}$ is $\mathcal{L}_{\rho_{\mathrm{Norm}}(u-t)}$, and the conjugacy class $\theta_{k,(-1)^n f,\chi,\rho}$ is none other than the conjugacy class $\theta_{k,f,\chi\rho_{\mathrm{Norm}}}$ of the Introduction.

5. The equidistribution theorem

We continue with a squarefree monic k-polynomial f of degree $n\geq 2$, $B := k[X]/(f)$, and a character χ of $B^\times$. Over a finite extension E/k where f factors completely, say $f(X) = \prod_i (X-a_i)$, the lisse rank one sheaf $\mathcal{L}_{\chi(u-t)}$ on $\mathbb{A}^1[1/f]/k$ becomes isomorphic to the sheaf $\otimes_{i=1}^n \mathcal{L}_{\chi_i(a_i-t)}$ on $\mathbb{A}^1[1/f]/E$.

Let us say that χ is "totally ramified" (what we called "as ramified as possible" in the Introduction) if each χ_i and the product $\prod_i \chi_i$ are all nontrivial. In view of Lemma 2.2, χ is totally ramified if and only if the group $H^1_c(\mathbb{A}^1[1/f]/\overline{k}, \mathcal{L}_{\chi(u-t)})$ is pure of weight one, or equivalently if and only if the group $H^0_c(\mathbb{A}^1[1/f]/\overline{k}, \mathcal{L}_{\chi(u-t)}(1/2)[1])$ is pure of weight zero, in which case it has dimension $n-1$.

Let us say that a totally ramified χ is "generic" if, **in addition** to being totally ramified, its constituent characters χ_i satisfy the three conditions of Theorem 4.1. We denote by

$$TotRam(k,f), \text{resp. } TotRamGen(k,f)$$

the sets of totally ramified (respectively totally ramified and generic) characters of $B^\times$.

Lemma 5.1. *Let χ be a totally ramified character of $B^\times$. Let ρ be a character of $k^\times$ which is good for $N((-1)^n f,\chi)$. Then the product character $\chi\rho_{\mathrm{Norm}}$ is totally ramified. Moreover, χ is generic if and only if $\chi\rho_{\mathrm{Norm}}$ is generic.*

Proof. Indeed, if geometrically we have $\mathcal{L}_{\chi(u-t)} \cong \otimes_{i=1}^n \mathcal{L}_{\chi_i(a_i-t)}$, then $\mathcal{L}_{\chi\rho_{\mathrm{Norm}}(u-t)} \cong \otimes_{i=1}^n \mathcal{L}_{\chi_i\rho(a_i-t)}$; we view ρ and the χ_i as characters of $\pi_1^{tame}(\mathbb{G}_m/\overline{k})$, to make sense of the products $\chi_i\rho$. Alternatively, if f splits over E, think of ρ as the character $x \mapsto \rho(\mathrm{Norm}_{E/k}(x))$ of $E^\times$. Thus the constituent characters of $\chi\rho_{\mathrm{Norm}}$ are the $\chi_i\rho$. That ρ is good for $N((-1)^n f,\chi)$ means precisely $\overline{\rho}$ does not occur in the local monodromy of $N((-1)^n f,\chi)$ at either 0 or $infty$. Its absence at 0 is the nontriviality of each $\chi_i\rho$. Its absence at ∞ is that $\rho^n\prod_i \chi_i$ is nontrivial, i.e., that $\prod_i(\chi_i\rho)$ is nontrivial. Thus $\chi\rho_{\mathrm{Norm}}$ is totally ramified. If in addition χ is generic, say $\chi_1^n \neq \prod_i \chi_i$, then $(\chi_1\rho)^n \neq \prod_i(\chi_i\rho)$, and hence $\chi\rho_{\mathrm{Norm}}$ is generic as well. Conversely, if χ is totally ramified and $\chi\rho_{\mathrm{Norm}}$ is totally ramified and generic, then $\overline{\rho}$ is good for $N((-1)^n f,\chi\rho_{\mathrm{Norm}})$, and so by the previous argument χ is totally ramified and generic. $\square$

We now combine this lemma with Corollary 4.2 to get a result concerning those conjugacy classes $\theta_{k,f,\chi}$ of the Introduction whose χ is totally ramified and generic.

Corollary 5.2. *Suppose $\sqrt{q} \geq 1+2n$. Let Λ be a nontrivial irreducible representation of $U(n-1)$ which occurs in $std^{\otimes a} \otimes (std^\vee)^{\otimes b}$. Then we have the estimate*

$$\Big|\sum_{\chi \in TotRamGen(k,f)} \mathrm{Trace}(\Lambda(\theta_{k,f,\chi}))\Big|$$

$$\leq (\#TotRamGen(k,f))2(a+b+1)(2n)^{a+b}/\sqrt{q}.$$

Proof. Let us say that two totally ramified generic characters χ and χ' of $B^\times$ are equivalent if $\chi' = \chi\rho_{\mathrm{Norm}}$ for some (necessarily unique) character ρ of $k^\times$. Break the terms of the sum into equivalence classes. The sum over the equivalence class of χ is precisely the sum bounded by Corollary 4.2, with λ there taken to be $(-1)^n$. $\square$

Our final task is to infer from this estimate an estimate for the sum over **all** χ in $TotRam(k, f)$. For this, we now turn to giving upper and lower bounds for $\#TotRam(k, f)$ and for $\#TotRamGen(k, f)$. We define three monic integer polynomials of degree n,

$$P_{\mathrm{all},n}(X) := X^n - 1,$$

$$P_{TR,n}(X) := (X-2)^n - \sum_{0 \le i \le n-1} X^i,$$

and

$$P_{TRG,n}(X) := (X-1-n)^n + (X-2)^n - X^n + 1 - n \sum_{0 \le i \le n-1} X^i.$$

Lemma 5.3. *For $q := \#k$, we have the (trivial) estimate*

$$\#TotRam(k, f) \le P_{\mathrm{all},n}(q) = q^n - 1.$$

Proof. Indeed, $B^\times$ is a subset of $B \setminus \{0\}$, whose cardinality is $q^n - 1$, so $q^n - 1$ is an upper bound for the total number of characters of $B^\times$. $\square$

Lemma 5.4.

$$\#TotRam(k, f) \ge P_{TR,n}(q) = (q-2)^n - \sum_{0 \le i \le n-1} q^i,$$

and

$$\#(\{\mathrm{char's\ of\ B^\times}\} \setminus \mathrm{TotRam(k,f)}) \le \mathrm{q^n - 1 - (q-2)^n + \sum_{0 \le i \le n-1} q^i}.$$

Proof. Factor f as the product of k-irreducible monic polynomials P_i of degree d_i. Thus $n = \sum_i d_i$, and $\#B^\times = \prod_i (q^{d_i} - 1) \ge (q-1)^n$. So there are at least $(q-1)^n$ characters χ of $B^\times$. We now count the characters which violate or satisfy the two conditions of being totally ramified.

Since $k^\times \subset B^\times$, the restriction map on characters is surjective. So the condition that $\chi | k^\times$ be nontrivial disqualifies $\#B^\times/(q-1) = (\prod_i (q^{d_i} - 1))/(q-1)$ of them.

The condition that each constituent character χ_i is nontrivial is equivalent to the condition that when we write χ as the product of characters χ_{P_i} of the factors $(k[X]/(P_i))^\times$, each χ_{P_i} is nontrivial. So there are $\prod_i (q^{d_i} - 2)$ choices of χ which satisfy this condition. If we now omit the ones which are trivial on $k^\times$, we are left with at least

$$\prod_i (q^{d_i} - 2) - (\prod_i (q^{d_i} - 1))/(q-1)$$

characters which are totally ramified. From the inequalities $q^d - 2 \geq (q-2)^d$ and $\prod_i (q^{d_i} - 1) \leq q^n - 1$ we get

$$\#TotRam(k, f) \geq \prod_i (q^{d_i} - 2) - (\prod_i (q^{d_i} - 1))/(q-1) \geq$$

$$\geq (q-2)^n - \sum_{0 \leq i \leq n-1} q^i.$$

Combining this with the previous lemma, we get the asserted upper bound for the number of characters of $B^\times$ which are not totally ramified. $\square$

Lemma 5.5. *For $q \geq n+1$, we have the estimate*

$$\#TotRamGen(k, f) \geq P_{TRG,n}(q) =$$

$$= (q-1-n)^n + (q-2)^n - q^n + 1 - n \sum_{0 \leq i \leq n-1} q^i.$$

Proof. We now count the characters which violate or satisfy the two additional condtions which make a totally ramified character generic

We first turn to the condition that for at least one of the χ_i, $\chi_i^n \neq \prod_i \chi_i$. Suppose first that f is itself irreducible. Then χ is a character of the field $B^\times \cong \mathbb{F}_{q^n}^\times$, and its constituent characters $\chi_1, ..., \chi_n$ are the characters $\chi, \chi^q, ..., \chi^{q^{n-1}}$. The condition that $\chi^n \neq \prod_i \chi_i$ is the condition that $\chi^n \neq \chi^{1+q+...+q^{n-1}}$, which disqualifies at most $1 + q + ... + q^{n-1} - n$ possible χ.

If f is not irreducible, let P be an irreducible factor of some degree $d < n$, and χ_P the P-constituent of χ. The the constituents of χ_P as character of $(k[X]/(P))^\times$ are $\chi_P, \chi_P^q, ..., \chi_P^{q^{d-1}}$. Think of these as the first d constituents of χ. We can be sure that there is some choice of index $j \in [1, d]$ such that $\chi_j^n \neq \prod_i \chi_i$ if we have

$$\prod_{1 \leq j \leq d} \chi_j^n \neq (\prod_i \chi_i)^d.$$

This is the condition that

$$\chi_P^{(n-d)(1+q+...+q^{d-1})} \neq (\prod_{d+1 \leq i \leq n} \chi_i)^d.$$

So for any given choice of the P_i-components of χ for all the **other** irreducible factors P_i of f, at most $(n-d)(1+q+...+q^{d-1})$ characters χ_P are disqualified. So the total number of characters χ which fail this second condition is at most $(n-d)(1+q+...+q^{d-1})\prod_{P_i\neq P}(q^{d_i}-1)$. From the inequality

$$(n-d)(1+q+...+q^{d-1})\prod_{P_i\neq P}(q^{d_i}-1)=(n-d)(\prod_{\mathrm{all\ P_i}}(q^{d_i}-1))(q-1)$$
$$\leq (n-1)(q^n-1)/(q-1)$$

we see that the in either case, f irreducible or not, there are at most

$$(n-1)(\sum_{0\leq i\leq n-1}q^i)$$

characters χ of $B^\times$ which violate this first condition.

We now turn to the condition that the constituents χ_i be all distinct. Again we factor f, and this time collect the factors according to their degrees. Suppose that there are e_i factors $P_{d_i,j}, j=1,..,e_i$ whose degrees are d_i. The first condition for distinctness is that for each $P_{d_i,j}$-component $\chi_{P_{d_i,j}}$ the d_i characters $\chi_{P_{d_i,j}}^{q^i}$ for $0\leq i\leq d_i-1$ are all distinct, or in other words that the orbit of $\chi_{P_{d_i,j}}$ under the q'th power map has full length d_i, rather than some proper divisor of d_i. The characters of $\mathbb{F}_{q^{d_i}}^\times$ whose orbit length is a proper divisor of d_i are those which come from (by composition with the relative norm) characters of subfields $\mathbb{F}_{q^r}$ for some proper divisor r of d_i. So the number of such short-orbit characters is at most $\sum_{r|d_i,r<d_i}(q^r-1)$, and this is trivially bounded by

$$\sum_{r|d_i,r<d_i}(q^r-1)\leq -1+\sum_{r|d_i,r<d_i}q^r\leq -1+\sum_{1\leq r\leq d_i/2}q^r\leq -1+[d_i/2]q^{[d_i/2]}.$$

So the number of full-orbit characters of $\mathbb{F}_{q^{d_i}}^\times$ is at least

$$q^{d_i}-[d_i/2]q^{[d_i/2]}\geq q^{d_i}-q^{d_i-1}.$$

Suppose now that for each irreducible factor $P_{d_i,j}$ of f, we have chosen a full-orbit (i.e., orbit length d_i) character. For irreducibles of different degrees, there can be no equality of their constituent characters, because the orbit-lengths are different. If there are $e_i\geq 2$ irreducible factors of the same degree d_i, say $P_{d_i,1},...,P_{d_i,e_i}$, then we may choose $\chi_{P_{d_i,1}}$ to be any of the at least $q^{d_i}-q^{d_i-1}$ full-orbit characters of $\mathbb{F}_{q^{d_i}}^\times$. Then we must choose $\chi_{P_{d_i,2}}$ to be a full-orbit character of $\mathbb{F}_{q^{d_i}}^\times$ which does lie in the orbit of $\chi_{P_{d_i,1}}$, thus excluding d_i possible full-orbit characters. Continuing in this way, we see that there are at least

$$\prod_{d_i\ \mathrm{which\ occur}}(\prod_{j=0}^{e_i-1}(q^{d_i}-q^{d_i-1}-jd_i))$$

characters χ of $B^\times$ all of whose constituents are distinct.

Because $q \geq n+1$, each factor $(q^{d_i} - q^{d_i - 1} - jd_i)$ satisfies

$$(q^{d_i} - q^{d_i-1} - jd_i) \geq (q^{d_i} - q^{d_i-1} - n) \geq (q-1-n)^{d_i}.$$

[For the last inequality, write $q = X + n + 1$; then we are saying that $(X+n+1)^{d-1}(X+n) \geq X^d + n$, which obviously holds for $X \geq 0$ and $d \geq 1$.] Thus for $q \geq n+1$, there are at least

$$(q-1-n)^n$$

characters χ of $B^\times$ all of whose constituents are distinct.

Removing from these those which violate the first condition, we are left with at least

$$(q-1-n)^n - (n-1)(\sum_{0 \leq i \leq n-1} q^i)$$

characters which, if totally ramified, are also generic. We have already seen that at most

$$q^n - 1 - (q-2)^n + \sum_{0 \leq i \leq n-1} q^i$$

characters fail to be totally ramified. Taking (some of) these away, we end up with at least

$$(q-1-n)^n - (n-1)(\sum_{0 \leq i \leq n-1} q^i) - (q^n - 1 - (q-2)^n + \sum_{0 \leq i \leq n-1} q^i) =$$

$$= (q-1-n)^n + (q-2)^n - q^n + 1 - n \sum_{0 \leq i \leq n-1} q^i$$

characters which are totally ramified and generic. □

Lemma 5.6. *We have the estimate*

$$\#(TotRam(k,f) \setminus TotRamGen(k,f) \leq P_{\mathrm{all},n}(q) - P_{TRG,n}(q) =$$

$$= (q-1-n)^n + (q-2)^n - 2q^n + 2 - n \sum_{0 \leq i \leq n-1} q^i.$$

Proof. Combine Lemmas 5.3 and 5.5. □

Lemma 5.7. *There exists a real constant C_n such that for $q \geq C_n$, we have*

$$P_{\mathrm{all},n}(q) - P_{TRG,n}(q) \leq P_{TRG,n}(q)/\sqrt{q}.$$

Proof. The difference $P_{\mathrm{all},n}(X) - P_{TRG,n}(X)$ is a real polynomial of degree $n-1$, while $P_{TRG,n}(X)$ is a real polynomial which is monic of degree n. □

Theorem 5.8. *Suppose $q \geq C_n$ and $\sqrt{q} \geq 1+2n$. Let Λ be a nontrivial irreducible representation of $U(n-1)$ which occurs in $std^{\otimes a} \otimes (std^\vee)^{\otimes b}$. Then we have the estimate*

$$|\sum_{\chi \in TotRam(k,f)} \mathrm{Trace}(\Lambda(\theta_{k,f,\chi}))|$$

$$\leq (\#TotRamGen(k,f))4(a+b+1)(2n)^{a+b}/\sqrt{q}.$$

Proof. We break the sum into two pieces, the sum over $\chi \in TotRamGen(k,f)$, and the sum over $\chi \in TotRam(k,f) \setminus TotRamGen(k,f)$. By Corollary 5.2, the absolute value of the first sum is bounded by

$$(\#TotRamGen(k,f))2(a+b+1)(2n)^{a+b}/\sqrt{q}.$$

The second sum has at most

$$P_{\mathrm{all},n}(q) - P_{TRG,n}(q) \leq P_{TRG,n}(q)/\sqrt{q} \leq (\#TotRamGen(k,f))/\sqrt{q}$$

terms, each of which, being the trace of a unitary conjugacy class in a representation of dimension at most $(n-1)^{a+b}$, is bounded in absolute value by $(n-1)^{a+b}$. So the absolute value of the second sum is bounded by

$$(\#TotRamGen(k,f))(n-1)^{a+b}/\sqrt{q},$$

which is less than the upper bound for the first sum. So doubling the upper bound for the first sum is safe. □

Corollary 5.9. *Suppose $q \geq C_n$ and $\sqrt{q} \geq 1+2n$. Let Λ be a nontrivial irreducible representation of $U(n-1)$ which occurs in $std^{\otimes a} \otimes (std^\vee)^{\otimes b}$. Then we have the estimate*

$$|(1/\#TotRam(k,f)) \sum_{\chi \in TotRam(k,f)} \mathrm{Trace}(\Lambda(\theta_{k,f,\chi}))|$$

$$\leq 4(a+b+1)(2n)^{a+b}/\sqrt{q}.$$

Proof. Indeed, $\#TotRamGen(k,f) \leq \#TotRam(k,f)$. □

Thus we obtain our target result.

Theorem 5.10. *Fix an integer $n \geq 2$ and a sequence of data (k_i, f_i) with k_i a finite field (of possibly varying characteristic) and $f_i(X) \in k_i[X]$ squarefree of degree n. If $\#k_i$ is archimedeanly increasing to ∞, the collections of conjugacy classes*

$$\{\theta_{k_i,f_i,\chi}\}_{\chi \in TotRam(k_i,f_i)}$$

become equidistributed in $U(n-1)^\#$ as $\#k_i \to \infty$.

6. Appendix: The case of "even" characters

We continue to work with a squarefree monic polynomial $f(X) \in k[X]$ of degree $n \geq 2$, and the k-algebra $B := k[X]/(f(X)X)$. We say that a character χ of $B^\times$ is even if it is trivial on $k^\times$ (viewed as a subgroup of $B^\times$).

Lemma 6.1. *The chararcter χ is even if and only if $\mathcal{L}_{\chi(u-t)}$ is lisse at ∞ (more precisely, if and only if, denoting by $j : \mathbb{A}^1[1/f] \subset \mathbb{P}^1$ the inclusion, the middle extension sheaf $j_\star\mathcal{L}_{\chi(u-t)}$ on $\mathbb{P}^1$ is lisse at ∞). Moreover, for even χ we have the formula*

$$\mathrm{Trace}(Frob_{k,\infty}|j_\star\mathcal{L}_{\chi(u-t)}) = 1.$$

Proof. The first assertion is immediate from the geometric isomorphism of $\mathcal{L}_{\chi(u-t)}$ with the tensor product $\otimes_{i=1}^n \mathcal{L}_{\chi_i(a_i-t)}$, together with Lemma 2.3. For the second assertion, we argue as follows. We have a morphism $\mathbb{G}_m \to \mathbb{B}^\times$ given by $t \mapsto 1/t$. The corresponding pullback sheaf $\mathcal{L}_{\chi(1/t)}$ on $\mathbb{G}_m$ is trivial, i.e., isomorphic to the constant sheaf $\overline{\mathbb{Q}_\ell}$, precisely because χ is trivial on $k^\times$. So on $\mathbb{G}_m[1/f]$, we have arithmetic isomorphisms

$$\mathcal{L}_{\chi(u-t)} \cong \mathcal{L}_{\chi(u-t)} \otimes \mathcal{L}_{\chi(1/t)} \cong \mathcal{L}_{\chi(u/t-1)}.$$

In terms of the uniformizing parameter $s := 1/t$ at ∞, we have $\mathcal{L}_{\chi(u-t)} \cong \mathcal{L}_{\chi(su-1)}$. Extending $\mathcal{L}_{\chi(su-1)}$ across ∞, i.e., across $s = 0$, by direct image, we get

$$\mathrm{Trace}(Frob_{k,\infty}|j_\star\mathcal{L}_{\chi(u-t)}) = \mathrm{Trace}(Frob_{k,0}|j_\star\mathcal{L}_{\chi(su-1)}) = \chi(-1) = 1,$$

the last equality because, once again, χ is trivial on $k^\times$. ☐

Let us say that an even character χ is totally ramified if, in the geometric isomorphism

$$\mathcal{L}_{\chi(u-t)} \cong \otimes_{i=1}^n \mathcal{L}_{\chi_i(a_i-t)},$$

each χ_i is nontrivial. The we have the following lemma, analogous to Lemma 2.5

Lemma 6.2. *The even character χ is totally ramified if and only if the group $H^1_c(\mathbb{P}^1[1/f] \otimes_k \overline{k}, j_\star\mathcal{L}_{\chi(u-t)})$ is pure of weight one, in which case H^1_c has dimension $n-2$, and $H^2_c = 0$.*

Let us denote by $TotRamEven(k,f)$ the set of even characters of $B^\times$ which are totally ramified. Attached to each $\chi \in TotRamEven(k,f)$, we have a conjugacy class $\theta_{k,f,\chi} \in U(n-2)^\#$, defined by its reversed characteristic polynomial via the equation

$$\det(1 - T\sqrt{\#k}\theta_{k,f,\chi}) = \det(1 - TFrob_k|H^1_c(\mathbb{P}^1[1/f] \otimes_k \overline{k}, j_\star\mathcal{L}_{\chi(u-t)})).$$

Keating and Rudnick make the following conjecture, the "even" version of Theorem 5.10.

Conjecture 6.3. *Fix an integer $n \geq 3$ and a sequence of data (k_i, f_i) with k_i a finite field (of possibly varying characteristic) and $f_i(X) \in k_i[X]$ squarefree of degree n. If $\#k_i$ is archimedeanly increasing to ∞, the collections of conjugacy classes*

$$\{\theta_{k_i,f_i,\chi}\}_{\chi \in TotRamEven(k_i,f_i)}$$

become equidistributed in $U(n-2)^{\#}$ as $\#k_i \to \infty$.

At present, we can prove this only under the additional (and highly artificial) hypothesis that each $f_i(X) \in k_i[X]$ has a zero in k_i.

Theorem 6.4. *Fix an integer $n \geq 3$ and a sequence of data (k_i, f_i) with k_i a finite field (of possibly varying characteristic) and $f_i(X) \in k_i[X]$ squarefree of degree n. Suppose each f_i has a zero in k_i. If $\#k_i$ is archimedeanly increasing to ∞, the collections of conjugacy classes*

$$\{\theta_{k_i,f_i,\chi}\}_{\chi \in TotRamEven(k_i,f_i)}$$

become equidistributed in $U(n-2)^{\#}$ as $\#k_i \to \infty$.

Proof. Replacing each f_i by an additive translate $X \mapsto X + a_i$ of itself, we reduce to the case when each f_i is of the form $f_i(X) = Xg_i(X)$, with $g_i \in k_i[X]$ squarefree and having $g_i(0) \neq 0$.

The idea is that the theorem is a consequence of a (slight variant of) Theorem 5.10, applied to the g_i. To explain this, let us fix a finite field k, a squarefree monic $g(X) \in k[X]$ of degree $n-1$ with $g(0) \neq 0$, and put $f(X) := Xg(X)$. Let us write $B_f := k[X]/(f(X))$, $B_g := k[X]/(g(X))$, $B_X := k[X]/(X) \cong k$. Then

$$B_f \cong k \times B_g.$$

For $P(X)$ a monic irreducible in $k[X]$ which is prime to f, the image of $P(X)$ in $B_f^{\times}$ is, via this isomorphism, the pair

$$(P(0), P \bmod g) = (\text{the scalar } P(0) \in k^{\times}) \times (1, P/P(0) \bmod g).$$

For an even character χ_f of $B_f^{\times}$, with components χ_X, χ_g, we therefore have

$$\chi_f(P \bmod f) = \chi_f(1, P/P(0) \bmod g) = \chi_g(P/P(0)).$$

If χ_f lies in $TotRamEven(k, f)$ then χ_X is nontrivial, each constituent chararcter χ_i of χ_g is nontrivial, and, by the evenness of χ_f, the restriction of χ_g to $k^{\times}$ is the inverse of the nontrivial character χ_X. In other words, $\chi_g \in TotRam(k, g)$. Conversely, given

$\chi_g \in TotRam(k,g)$, define χ_X to be the restriction to $k^\times$ of $1\chi_g$; then the pair (χ_X, χ_g) taken to be χ_f lies in $TotRamEven(k,f)$.

For $P(X) = X - t$ a linear irreducible, and χ_f even, we have

$$\chi_f(X-t) = \chi_g((X-t)/(-t)) = \chi_g(1 - X/t).$$

Exactly as in section 2 of this paper, we find an arithmetic isomorphism on $\mathbb{A}^1[1/f] = \mathbb{G}_m[1/g]$,

$$\mathcal{L}_{\chi_f(u-t)} \cong \mathcal{L}_{\chi_g(1-u/t)}.$$

In terms of the parameter $s := 1/t$ on $\mathbb{G}_m$, and the palindrome $g^{pal}(s) := s^{deg(g)}g(t)$ of g, our sheaf becomes $\mathcal{L}_{\chi_g(1-us)}$ on $\mathbb{G}_m[1/g^{pal}]$, and has an obvious lisse extension across $s = 0$ to the sheaf $\mathcal{L}_{\chi_g(1-us)}$ on $\mathbb{A}^1[1/g^{pal}]$. [N.B. Here the u is still the image of X in B_g, and χ_g is our character of $B_g^\times$. But it is the zeroes of $g^{pal}(s)$ we must avoid.]

We now define conjugacy classes $\Theta_{k,g,\chi_g} \in U(n-2)^\#$, for each $\chi_g \in TotRam(k,g)$, through their reversed characteristic polynomials

$$\det(1 - T\sqrt{\#k}\Theta_{k,g,\chi_g}) = \det(1 - TFrob_k | H^1_c(\mathbb{A}^1[1/g^{pal}] \otimes_k \overline{k}, \mathcal{L}_{\chi_g(1-us)}).$$

With these preliminaries out of the way, we see that we have reduced Theorem 6.4 to the variant of Theorem 5.10 for the conjugacy classes $\{\Theta_{k,g,\chi_g}\}_{\chi_g \in TotRam(k,g)}$. To prove this variant, we repeat the proof of Theorem 5.10, but looking at the direct image by g^{pal} of $\mathcal{L}_{\chi_g(1-us)}$ (rather than looking at the direct image by $(-1)^{deg(g)}g$ of $\mathcal{L}_{\chi_g(u-t)}$, as we did in proving Theorem 5.10). $\square$

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Princeton University, Mathematics, Fine Hall, NJ 08544-1000, USA
E-mail address: `nmk@math.princeton.edu`