ON A QUESTION OF KEATING AND RUDNICK

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INTRODUCTION

We work over a finite field \( k = \mathbb{F}_q \) inside a fixed algebraic closure \( \overline{k} \). We fix a squarefree monic polynomial \( f(X) \in k[X] \) of degree \( n \geq 2 \). We form the \( k \)-algebra \( B := k[X]/(f(X)) \), which is finite étale over \( k \) of degree \( n \). We denote by \( u \in B \) the image of \( X \) in \( B \) under the “reduction mod \( f \)” homomorphism \( k[X] \to B \).

Thus we may write this homomorphism as \( g(X) \in k[X] \mapsto g(u) \in B \).

We denote by \( B^\times \) the multiplicative group of \( B \), and by \( \chi \) a character \( \chi : B^\times \to \mathbb{C}^\times \).

We extend \( \chi \) to all of \( B \) by decreeing that \( \chi(b) := 0 \) if \( b \in B \) is not invertible.

The (possibly imprimitive) Dirichlet \( L \)-function \( L(\chi,T) \) attached to this data is the power series in \( \mathbb{C}[[T]] \) given by

\[
L(\chi,T) := \sum_{\text{monic } g(X) \in k[X]} \chi(g(u))T^{\deg(g)} = \sum_{n \geq 0} A_n T^n,
\]

\[
A_n := \sum_{g(X) \in k[X] \text{ monic of } \deg n, \gcd(f,g) = 1} \chi(g(u)).
\]

If \( \chi \) is nontrivial, then \( L(\chi,T) \) is a polynomial in \( T \) of degree \( n-1 \).

Moreover, if \( \chi \) is “as ramified as possible”\(^1\), then this \( L \)-function is “pure of weight one”, i.e., in its factored form \( \prod_{i=1}^{n-1} (1 - \beta_i T) \), each reciprocal root \( \beta_i \) has complex absolute value \( |\beta_i|_C = \sqrt{q} \).

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\(^1\)Factor \( f \) as a product of distinct monic irreducible polynomials, say \( f = \prod_j f_j \). Then \( B \) is canonically the product of the algebras \( B_j := k[X]/(f_j(X)) \), and \( \chi \) is the product of characters \( \chi_j \) of these factors. The condition “as ramified as possible” is that each \( \chi_j \) be nontrivial, and that the restriction of \( \chi \) to \( k^\times \subset B^\times \) be nontrivial.
For such a $\chi$, its “unitarized” $L$-function $L(\chi, T/\sqrt{q})$ is the reversed characteristic polynomial $\det(1 - TA_\chi)$ of some element $A_\chi$ in the unitary group $U(n-1)$ (e.g., take $A_\chi := \text{Diag}(\beta_1/\sqrt{q}, \ldots, \beta_{n-1}/\sqrt{q})$). In $U(n-1)$, conjugacy classes are determined by their characteristic polynomials, so $L(\chi, T/\sqrt{q})$ is $\det(1 - T\theta_\chi)$ for a well defined conjugacy class $\theta_\chi$ in $U(n-1)$. In order to keep track of the input data $(k, f, \chi)$, we denote this conjugacy class $\theta_{k,f,\chi}$.

Now suppose $E/k$ is a finite extension field of $k$. Our polynomial $f$ remains squarefree over $E$. We form the $E$-algebra $B_E := E[X]/(f(X))$, and for each character $\chi$ of $B_E$ which is as ramified as possible, we get a conjugacy class $\theta_{E,f,\chi}$. The question posed by Keating and Rudnick was to show that for fixed $f$, the collections of conjugacy classes $\{\theta_{E,f,\chi}\}_\chi$ of $B_E^*\times E$ as ramified as possible become equidistributed in the space $U(n-1)^\#$ of conjugacy classes of $U(n-1)$ (for the measure induced by Haar measure on $U(n-1)$) as $E$ runs over larger and larger finite extensions of $k$.

In fact, we will show something slightly stronger, where we fix the degree $n \geq 2$, but allow sequences of input data $(k_i, f_i)$, with $k_i$ a finite field (of possibly varying characteristic) and $f_i(X) \in k_i[X]$ squarefree of degree $n$. We will show that, in any such sequence in which $#k_i$ is archimedeanly increasing to $\infty$, the collections of conjugacy classes $\{\theta_{k_i,f_i,\chi}\}_\chi$ of $B_i^*\times k_i$ as ramified as possible become equidistributed in $U(n-1)^\#$.

In an appendix, we give an analogous result for “even” characters which are as ramified as possible, given that they are even, under the additional hypothesis that each $f_i(X) \in k_i[X]$ have a zero in $k_i$. Here the equidistribution is in the space of conjugacy classes of $U(n-2)$. Already the case when each $f_i(X) \in k_i[X]$ is an irreducible cubic seems to be open.

1. Preliminaries on the $L$-function

We return to our initial situation, a finite field $k$, an integer $n \geq 2$, a squarefree polynomial $f(X) \in k[X]$, and the finite étale $k$-algebra $B := k[X]/(f(X))$. We have the algebra-valued functor $\mathbb{B}$ on variable $k$-algebras $R$ defined by

$$\mathbb{B}(R) := B_R := B \otimes_k R = R[X]/(f(X)),$$
and the group-valued functor $\mathbb{B}^\times$ on variable $k$-algebras $R$ defined by

$$\mathbb{B}^\times(R) := B_R^\times = \mathbb{B}(R)^\times.$$  

Because $f$ is squarefree, $\mathbb{B}^\times$ is a smooth commutative groupscheme over $k^2$, which over any extension field $E$ of $k$ in which $f$ factors completely becomes isomorphic to the $n$-fold product of $\mathbb{G}_m$ with itself. More precisely, if $f$ factors completely over $E$, say $f(X) = \prod_{i=1}^n (X - a_i)$, then for any $E$-algebra $R$, we have an $R$-algebra isomorphism

$$\mathbb{B}(R) = R[X]/\left(\prod_{i=1}^n (X - a_i)\right) \cong \prod_{i=1}^n R$$

of $\mathbb{B}(R)$ with the $n$-fold product of $R$ with itself, its algebra structure given by componentwise operations, under which the image $u$ of $X$ maps by

$$u \mapsto (a_1, \ldots, a_n).$$

So for any $E$-algebra $R$, we have $\mathbb{B}^\times(R) := \mathbb{B}(R)^\times \cong (R^\times)^n$.

For $E/k$ a finite extension field, $B_E$ is a finite étale $B$-algebra which as a $B$-module is free of rank $\deg(E/k)$. Let us denote by $\mathbb{B}_E$ the functor on $k$-algebras $R \mapsto \mathbb{B}_E(R) := B_E \otimes_k R$. Then $\mathbb{B}_E(R)$ is a finite étale $\mathbb{B}(R)$-algebra, so we have the norm map

$$\text{Norm}_{E/k} : \mathbb{B}_E \to \mathbb{B}.$$  

Its restriction to unit groups gives a homomorphism of tori which is étale surjective,

$$\text{Norm}_{E/k} : \mathbb{B}_E^\times \to \mathbb{B}^\times,$$

whose restriction to $k$-valued points gives a surjective$^3$ homomorphism

$$\text{Norm}_{E/k} : \mathbb{B}^\times(E) \to \mathbb{B}^\times(k).$$

We will also have occasion to consider $\mathbb{B}(R)$ as a finite étale $R$-algebra which is free of rank $n$ as an $R$-module, giving us another norm map

$$\text{Norm}_{B/k} : \mathbb{B}(R) \to R,$$

which by restriction gives a homomorphism which is étale surjective, with geometrically connected kernel,

$$\text{Norm}_{B/k} : \mathbb{B}^\times(R) \to R^\times.$$  

For any finite extension $E/k$, this second norm map

$$\text{Norm}_{B/k} : \mathbb{B}^\times(E) \to E^\times$$

$^2$In fact $\mathbb{B}^\times$ is the generalized Jacobian of $\mathbb{P}^1/k$ with respect to the modulus $\infty \cup \{ f = 0 \text{ in } \mathbb{A}^1 \}$.

$^3$By Lang’s theorem [La, Thm. 2], because its kernel is smooth and geometrically connected
is surjective.

How is all this related to our $L$-function? For each integer $r \geq 1$, denote by $k_r/k$ the unique extension field of $k$ of degree $r$ (inside our fixed algebraic closure of $k$). Recall that $f(X) \in k[X]$ is squarefree of degree $n \geq 1$, and that $u$ denotes the image of $X$ in $B = k[X]/(f(X))$.

**Lemma 1.1.** For $\chi$ a character of $B$, we have the identity

$$L(\chi, T) = \exp(\sum_{r \geq 1} S_r T^r/r), \quad S_r = \sum_{t \in A^1[1/f](k_r)} \chi(\text{Norm}_{k_r/k}(u - t)).$$

**Proof.** The key observation is that if $\alpha \in A^1[1/f](k_d)$ generates the extension $k_d/k$, and has monic irreducible polynomial $P(X)$ over $k$, then $\gcd(f,P) = 1$ and $P(X) = \text{Norm}_{k_d/k}(X - \alpha)$ in $k[X]$. Hence $P(u) = \text{Norm}_{k_d/k}(u - \alpha)$ in $B$. We apply this as follows.

Write the $L$-function as the Euler product

$$L(\chi, T) = \prod_{\text{monic irred. } P(X), \, \gcd(f,P)=1} \frac{1}{1 - \chi(P(u))T^{\deg(P)}}.$$

Taking log’s, we must check that for each $r \geq 1$ we have the identity

$$\sum_{t \in A^1[1/f](k_r)} \chi(\text{Norm}_{k_r/k}(u-t)) = \sum_{d|r} \sum_{\text{irred } P, \deg(P)=d, \, \gcd(f,P) = 1} d \chi(P(u))^{r/d}.$$

To see this, partition the points $t \in A^1[1/f](k_r)$ according to their monic irreducible polynomials over $k$. For each divisor $d$ of $r$, and each monic irreducible $P(X)$ of degree $d$ with $\gcd(f,P) = 1$ and roots $\tau_1, \ldots, \tau_d$ in $A^1[1/f](k_d)$, each of the $d$ terms $\chi(\text{Norm}_{k_r/k}(u-\tau_i))$ is equal to $\chi(P(u))^{r/d}$ (simply because $\text{Norm}_{k_d/k}(u-\tau_i) = P(u)$, and, as $\tau_i \in k_d$, $\text{Norm}_{k_r/k}(u - \tau_i) = (\text{Norm}_{k_d/k}(u - \tau_i))^{r/d}$).


## 2. Cohomological genesis

We now choose a prime number $\ell$ invertible in $k$, and an embedding of $\overline{Q}$, the algebraic closure of $Q$ in $C$, into $\overline{Q}_\ell$. In this way, we view $\chi$ as a $\overline{Q}_\ell^\times$-valued character of $B^\times$. Attached to $\chi$, we have the “Kummer sheaf” $L_\chi$ on $B^\times$. Recall that $L_\chi$ is obtained as follows. We have the $q = \#k^\times$th power Frobenius endomorphism $F_k$ of $B$. The Lang torser, i.e., the finite étale galois covering $1 - F_k : B^\times \to B^\times$, has structural group $B^\times = B^\times(k)$. We then push out this $B^\times$-torser on $B^\times$ by $\overline{\chi}$, to obtain the $\overline{Q}_\ell$-sheaf $L_\chi$ on $B^\times$. It is lisse of rank one and pure of weight zero.

We have a $k$-morphism (in fact an embedding)

$$A^1[1/f] \subset B^\times,$$
given on $R$-valued points, $R$ any $k$-algebra, by
\[ t \in A^1[1/f](R) \mapsto u - t \in B(R). \]

**Lemma 2.1.** For any $k$-algebra $R$, and any $t \in A^1(R) = R$, we have the identity
\[ \text{Norm}_{B/k}(u - t) = (-1)^nf(t) \in R. \]

**Proof.** In the $k$-algebra $B = k[X]/(f(X))$, multiplication by $u$ (the class of $X$ in $B$) has characteristic polynomial $f$ (theory of the “companion matrix”), i.e., taking for $R$ the polynomial ring $k[T]$, we have $\text{Norm}_{B/k}(T - u) = f(T) \in R = k[T]$, hence $\text{Norm}_{B/k}(u - T) = (-1)^nf(T) \in k[T]$, and this is the universal case of the asserted identity. \hfill \square

We denote by $L_{\chi(u-t)}$ the lisse $\overline{Q}_l$-sheaf of rank one on $A^1[1/f]$ obtained as the pullback of $L_{\chi}$ on $B^\times$ by the embedding $t \mapsto u - t$ of $A^1[1/f]$ into $B^\times$. In view of Lemma 1.1, the $L$-function $L(\chi, T)$ is, via the chosen embedding of $\overline{Q}$ into $\overline{Q}_l$, the $L$-function of $A^1[1/f]/k$ with coefficients in $L_{\chi(u-t)}$. This sheaf on $A^1[1/f]$ is lisse of rank one and pure of weight zero. The compact cohomology groups
\[ H^1_c := H^1_c(A^1[1/f] \otimes_k \overline{k}, L_{\chi(u-t)}) \]
vanish for $i \neq 1, 2$, and by the Lefschetz trace formula we have the formula
\[ L(\chi, T) = \det(1 - TFrob_k|H^1_c)/\det(1 - TFrob_k|H^2_c). \]

We now turn to a closer examination of these cohomology groups. For this, we first examine the sheaf $L_{\chi(u-t)}$ geometrically, i.e., pulled back to $A^1[1/f]/\overline{k}$, and describe in terms of translations of Kummer sheaves $L_\rho$ on $\mathbb{G}_m$. Recall that the tame fundamental group $\pi^\text{tame}_1(\mathbb{G}_m/\overline{k})$ is the inverse limit over prime-to-$p$ integers $N$, growing multiplicatively, of the groups $\mu_N(\overline{k})$, via the $N$’th power Kummer coverings of $\mathbb{G}_m/\overline{k}$ by itself. It is also the inverse limit, over finite extension fields $E/k$ growing by inclusion, of the multiplicative groups $E^\times$, with transition maps the Norm, via the Lang torsor coverings $1 - F_E$ of of $\mathbb{G}_m/\overline{k}$ by itself. For any continuous $\overline{Q}_l^\times$-valued character $\rho$ of $\pi^\text{tame}_1(\mathbb{G}_m/\overline{k})$, we have the corresponding Kummer sheaf $L_\rho$ on $\mathbb{G}_m/\overline{k}$. The characters of finite order of $\pi^\text{tame}_1(\mathbb{G}_m/\overline{k})$ are precisely those which arise from characters $\rho$ of $E^\times$ for some finite extension $E/k$. More precisely, a character $\rho$ of finite order of $\pi^\text{tame}_1(\mathbb{G}_m/\overline{k})$ comes from a character of $E^\times$ if and only if $\rho = \rho^F_E$ (equality as characters of $\pi^\text{tame}_1(\mathbb{G}_m/\overline{k})$). For such a character $\rho$, the Kummer sheaf $L_\rho$ on $\mathbb{G}_m/\overline{k}$ begins life on $\mathbb{G}_m/E$. 

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To analyze the sheaf $\mathcal{L}_{\chi_{(u-t)}}$ geometrically, first choose a finite extension field $E/k$ in which $f$ factors completely, say $f(X) = \prod_{i=1}^{n}(X - a_i)$. Then $\mathbb{B}(E)^\times \cong (E^\times)^n$, and $\chi_E := \chi \circ \text{Norm}_{E/k}$ as character of $(E^\times)^n$ is of the form $(x_1, \ldots, x_n) \mapsto \prod \chi_i(x_i)$, for characters $\chi_1, \ldots, \chi_n$ of $E^\times$. Then $\mathbb{B}^\times$, pulled back to $\overline{k}$, becomes $\mathbb{G}_m^n$, and $\mathcal{L}_\chi$ on it becomes the external tensor product $\bigotimes_{i=1}^{n} \mathcal{L}_{\chi_i}$ of usual Kummer sheaves $\mathcal{L}_{\chi_i}$ on the factors. Over $\overline{k}$, the embedding of $\mathbb{A}^1[1/f]$ into $\mathbb{B}^\times$ given by $t \mapsto u - t$ becomes the embedding of $\mathbb{A}^1[1/f] \otimes_k \overline{k}$ into $\mathbb{G}_m^n$ given by

$$t \mapsto (a_1 - t, \ldots, a_n - t).$$

Thus the sheaf $\mathcal{L}_{\chi_{(u-t)}}$ is geometrically isomorphic to the tensor product $\bigotimes_{i=1}^{n} \mathcal{L}_{\chi_i(a_i-t)}$ on $\mathbb{A}^1[1/f] \otimes_k \overline{k} = \text{Spec}(\overline{k}[t][1/f(t)])$.

**Lemma 2.2.** With the notations of the previous paragraph, we have the following results.

1. We have $H^2_c = 0$ if and only if some $\chi_i$ is nontrivial, in which case $H^1_c$ has dimension $n - 1$.
2. The group $H^1_c$ is pure of weight one if and only if every $\chi_i$ is nontrivial and the product $\prod_{i=1}^{n} \chi_i$ is nontrivial.

**Proof.** Both assertions are invariant under finite extension of the ground field, so it suffices to treat universally the case in which $f$ factors completely over $k$. The character $\chi_i$ is the local monodromy of $\mathcal{L}_{\chi_{(u-t)}}$ at the point $a_i$, and the product $\prod_{i=1}^{n} \chi_i$ is its local monodromy at $\infty$. For assertion (1), we note that the group $H^2_c$ is either zero or one-dimensional. It is nonzero if and only if the lisse rank one sheaf $\mathcal{L}_{\chi_{(u-t)}}$ is geometrically constant, i.e., if and only if its local monodromy at each of the points $\infty$, $a_1$, ..., $a_n$ is trivial. The dimension assertion results from the Euler-Poincaré formula: because $\mathcal{L}_{\chi_{(u-t)}}$ is lisse of rank one and at worst tamely ramified at the missing points, it gives

$$\chi_c(\mathbb{A}^1[1/f] \otimes_k \overline{k}, \mathcal{L}_{\chi_{(u-t)}}) = \chi_c(\mathbb{A}^1[1/f] \otimes_k \overline{k}, \overline{\mathbb{Q}}_l) = 1 - n.$$

For assertion (2), we argue as follows. If all the $\chi_i$ are trivial, i.e., if $\chi$ is trivial, then $\mathcal{L}_\chi$ on $\mathbb{B}^\times$ is trivial, $\mathcal{L}_{\chi_{(u-t)}}$ on $\mathbb{A}^1[1/f]$ is trivial, and its $H^1_c$ has dimension $n$ and is pure of weight zero.

Suppose now that $\chi$ is nontrivial, i.e., that at least one $\chi_i$ is nontrivial. Denote by $j : \mathbb{A}^1[1/f] \subset \mathbb{P}^1$ the inclusion. Then we have a short exact sequence of sheaves on $\mathbb{P}^1$

$$0 \to j_* \mathcal{L}_{\chi_{(u-t)}} \to j_* \mathcal{L}_{\chi_{(u-t)}} \to Pct \to 0,$$

in which $Pct$ is a skyscraper sheaf, supported at those of the points $\infty$, $a_1$, ..., $a_n$ where the local monodromy is trivial, and is punctually
pure of weight zero with one-dimensional stalk at each of these points. The long exact cohomology sequence then gives a short exact sequence 
\[ 0 \rightarrow H^0(\mathbb{P}^1/\overline{k}, Pct) \rightarrow H^1(\mathbb{P}^1/\overline{k}, j_! L_{\chi(u-t)}) \rightarrow H^1(\mathbb{P}^1/\overline{k}, j_! L_{\chi(u-t)}) \rightarrow 0 \]
in which the middle term \( H^1(\mathbb{P}^1/\overline{k}, j_! L_{\chi(u-t)}) \) is the cohomology group \( H^1_{c} \), the third term \( H^1(\mathbb{P}^1/\overline{k}, j_\ast L_{\chi(u-t)}) \) is pure of weight one [De-Weil II, 3.2.3], and the first term, \( H^0(\mathbb{P}^1/\overline{k}, Pct) \) is pure of weight zero and of dimension the number of points among \( \infty, a_1, ..., a_n \) where the local monodromy is trivial.

Given a character \( \chi \) of \( B^\times \), how do we determine what \( L_{\chi(u-t)} \) looks like, geometrically? We know that, in terms of the factorization of \( f \), say \( f(X) = \prod_{i=1}^{n} (X - a_i) \) over some finite extension field \( E/k \), \( L_{\chi(u-t)} \) is geometrically isomorphic to the tensor product \( \otimes_{i=1}^{n} L_{\chi_i(a_i-t)} \) on \( A^1[1/f] \otimes_k \overline{k} = \text{Spec}([\overline{k}[t][1/f(t)]) \). We have an easy interpretation of the product \( \prod_i \chi_i \) of all the \( \chi_i \).

**Lemma 2.3.** For \( \rho := \text{the restriction of } \chi \text{ to } k^\times (k^\times \text{ seen as a subgroup of } B^\times) \), the composition \( \rho \circ \text{Norm}_{E/k} \) is the character of \( E^\times \) given by the product \( \prod_i \chi_i \) of all the \( \chi_i \).

**Proof.** Under the \( E \)-linear isomorphism of \( B_E = E[X]/(f) \) with the \( n \)-fold self product of \( E \), \( E \) viewed as the constant polynomials is diagonally embedded. Thus \( \prod_i \chi_i \) is the effect of \( (\chi \circ \text{Norm}_{B_E/B}) \) on \( E^\times \) (viewed as a subgroup of \( B_E^\times \)). The restriction to \( E^\times \) of this norm map \( \text{Norm}_{B_E/B} : B^\times_E \rightarrow B^\times \) is the norm map \( \text{Norm}_{E/k} : E^\times \rightarrow k^\times \).

To further analyze this question, in a “\( k \)-rational” way, we first factor our squarefree monic \( f \) as a product of distinct monic \( k \)-irreducible polynomials, say 
\[ f = \prod P_i, \quad \deg(P_i) := d_i. \]
Then with 
\[ B_{P_i} := k[X]/(P_i), \]
we have an isomorphism of \( k \)-algebras 
\[ B := k[X]/(f) \cong \prod_i B_{P_i}, \quad g \mapsto (g \mod P_i)_i, \]
and a character \( \chi \) of \( B^\times \) is uniquely of the form 
\[ \chi(g) = \prod_i \chi_{P_i}(g \mod P_i), \]
for characters \( \chi_{P_i} \) of \( B^\times_{P_i} \).

So it suffices treat the case when \( f \) is a single irreducible polynomial \( P \) of some degree \( d \geq 1 \). Choose a root \( a \) of \( P \) in our chosen \( \overline{k} \). This
choice gives an isomorphism of $B_P$ with the unique extension field $k_d/k$ of degree $d_i$ inside $\overline{k}$, namely $g \mapsto g(a)$. Via this isomorphism, the character $\chi_P$ becomes a character $\chi$ of $k_d^{\times}$. After extension of scalars from $k$ to $k_d$, we have a $k_d$-linear isomorphism

$$B_P \otimes_k k_d = k_d[X]/(P) \cong \prod_{\sigma \in \text{Gal}(k_d/k)} k_d, \quad g(X) \mapsto (g(\sigma(a))_\sigma.$$ 

Then for $g(X) \in k_d[X]/(P)$, its $k_d/k$-Norm down to $B_P$ is

$$\text{Norm}_{k_d/k}(g(X)) = \prod_{\tau \in \text{Gal}(k_d/k)} g^\tau(X) \mod P = \prod_{\tau \in \text{Gal}(k_d/k)} g^\tau(a) \in k_d.$$

So we have the identity

$$(\chi \circ \text{Norm}_{k_d/k})(g(X)) = \prod_{\tau \in \text{Gal}(k_d/k)} \chi(g^\tau(a)) = \prod_{\tau \in \text{Gal}(k_d/k)} (\chi \circ \tau)(g(\tau^{-1}(a))).$$

The arguments $g(\tau^{-1}(a))$ of the characters $\chi \circ \tau$ are just the components, in another order, of $g$ in the isomorphism $k_d[X]/(P) \cong \prod_{\sigma \in \text{Gal}(k_d/k)} k_d$.

In other words, the pullback of $\chi$ by the $k_d/k$-Norm from $B_P \otimes_k k_d$ down to $B_P$ has components $(\chi, \chi^q, \ldots, \chi^{d-1})$. Thus we have the following lemma.

**Lemma 2.4.** For $P$ an irreducible monic $k$-polynomial of degree $d \geq 1$, and $\chi$ a character of $B_P^\times \cong k_d^{\times}$ (via $u \mapsto a$, a a chosen root of $P$ in $k_d$), the sheaf $\mathcal{L}_\chi(u - t)$ on $\mathbb{A}^1[1/P]$ is geometrically isomorphic to the tensor product $\otimes_{i=0}^{d-1} \mathcal{L}_{\chi^i(aq^i - 1 - t)}$.

Combining these last two lemmas with Lemma 2.2, we get the following result.

**Lemma 2.5.** Let $f$ be a squarefree monic $k$-polynomial of degree $n \geq 2$, $f = \prod_i P_i$ its factorization into monic $k$-irreducibles, $\chi$ a character of $B^\times$, and, for each $P_i$, $\chi_{P_i}$ the $P_i$-component of $\chi$. The group $H^1_\text{c}(\mathbb{A}^1[1/f] \otimes_k \overline{k}, \mathcal{L}_{\chi(u - t)})$ is pure of weight one if and only if $\chi$ is nontrivial on $k^\times$ and each $\chi_{P_i}$ is nontrivial, in which case $H^1_\text{c}$ has dimension $n - 1$ and $H^2_\text{c} = 0$.

3. **The direct image theorem**

In this section, we work over $\overline{k}$.

**Theorem 3.1.** Suppose that $f(X) = \prod_{i=1}^n (X - a_i)$ is a squarefree polynomial of degree $n \geq 2$ over $\overline{k}$. Let $\chi_1, \ldots, \chi_n$ be characters of $\pi_1^{\text{tame}}(\mathbb{G}_m/\overline{k})$ of finite order, and form the lisse sheaf

$$\mathcal{F} := \otimes_{i=1}^n \mathcal{L}_{\chi_i(a_i - t)}$$
on \( \mathbb{A}^1[1/f] \otimes_k \overline{k} \). Then we have the following results.

1. For any scalar \( \lambda \in \overline{k}^\times \), the direct image \([\lambda f]_*F\) of \( F\) by the polynomial map \( \lambda f : \mathbb{A}^1[1/f]/\overline{k} \to \mathbb{G}_m/\overline{k} \) is a middle extension sheaf on \( \mathbb{G}_m/\overline{k} \), of generic rank \( n \), and the perverse sheaf \([\lambda f]_*F[1]\) is geometrically semisimple.

2. If one of the \( \chi_i\), say \( \chi_1\), is a singleton among the \( \chi\)'s, in the sense that \( \chi_1 \neq \chi_j \) for every \( j \neq 1 \), then the perverse sheaf \([\lambda f]_*F[1]\) on \( \mathbb{G}_m/\overline{k} \) is irreducible.

3. If two of the \( \chi_i\), say \( \chi_1 \) and \( \chi_2\) are each singletons among the \( \chi\)'s, then the irreducible perverse sheaf \([\lambda f]_*F[1]\) on \( \mathbb{G}_m/\overline{k} \) is not isomorphic to any nontrivial multiplicative translate of itself.

Proof. To prove (1), we argue as follows. The map \( \lambda f : \mathbb{A}^1[1/f]/\overline{k} \to \mathbb{G}_m/\overline{k} \) is finite and flat of degree \( n \). As \( f\) has all distinct roots, its derivative \( f'\) is not identically zero, so over the dense open set \( U \) of \( \mathbb{G}_m/\overline{k} \) obtained by deleting the images under \( \lambda f\) of the zeroes of \( f'\), the map \( \lambda f\) is finite étale of degree \( n \). Thus \([\lambda f]_*F[1]\) has generic rank \( n \).

It is a middle extension because \( F\) is a middle extension on the source (being lisse), and \( \lambda f\) is finite, flat, and generically étale, cf. [Ka-TLFM, first paragraph of the proof of 3.3.1]. On the dense open set \( U \), \([\lambda f]_*F\) is (the pullback from some finite subfield \( E \) of \( \overline{k} \) of) a lisse sheaf which is pure of weight zero, hence is geometrically semisimple [De-Weil II, 3.4.1 (iii)]. Therefore [BBD, 4.3.1 (ii)] the perverse sheaf \([\lambda f]_*F[1]\)|\( U\) is semisimple, and this property is preserved by middle extension from \( U \) to \( \mathbb{G}_m/\overline{k} \).

Suppose now that \( \chi_1 \) is a singleton among the \( \chi\)'s. We claim that \([\lambda f]_*F[1]\) is irreducible. Since \([\lambda f]_*F[1]\) is just a multiplicative translate of \( f_*F[1]\), it suffices to show that \( f_*F[1]\) is irreducible. Since \( f_*F[1]\) is semisimple, we must show that the inner product

\[ <f_*F[1], f_*F[1]> = 1.\]

By Frobenius reciprocity, we have


So we must show that \( F[1]\) occurs at most once in \( f^* f_*F[1]\). We will show the stronger statement, that denoting by \( I(a_1)\) the inertia group at the point \( a_1 \in \mathbb{A}^1(\overline{k})\), the \( I(a_1)\)-representation of \( F[1]\) occurs at most once in the \( I(a_1)\)-representation of \( f^* f_*F[1]\). As a finite flat map of \( \mathbb{A}^1\) to itself, \( f\) is finite étale over a neighborhood of 0 in the target (because \( f\) has \( n \) distinct roots \( a_1, ..., a_n\), the preimages of 0). We first infer that the \( I(0)\)-representation of \( f_*F[1]\) is the direct sum of the \( \chi_i\), and then
that for each $j$ the $I(a_j)$-representation of $f^*f_*\mathcal{F}[1]$ is the direct sum of the $\chi_i$. At the point $a_1$, the $I(a_1)$-representation of $\mathcal{F}[1]$ is $\chi_1$, and by the singleton hypothesis $\chi_1$ occurs only once in the direct sum of the $\chi_i$, so only once in the $I(a_1)$-representation of $f^*f_*\mathcal{F}[1]$.

Suppose now that both $\chi_1$ and $\chi_2$ are singletons. We must show that for any scalar $\lambda \neq 1$ in $k^\times$, the perverse irreducible sheaves $[\lambda f]^*\mathcal{F}[1]$ and $f^*\mathcal{F}[1]$ on $\mathbb{G}_m/k$ are not isomorphic. We argue by contradiction, and thus suppose the two are isomorphic. Choose a finite subfield $E$ of $k$ over which the scalar $\lambda$, the points $a_i$, the characters $\chi_i$ and the open set $U$ are all defined, so that we may speak of the geometrically irreducible perverse sheaves $[\lambda f]^*\mathcal{F}(1/2)[1]$ and $f^*\mathcal{F}(1/2)[1]$ on $\mathbb{G}_m/E$. Each of these is pure of weight zero. On the dense open set $U \subset \mathbb{G}_m/E$, the sheaves $[\lambda f]^*\mathcal{F}$ and $f^*\mathcal{F}$ are lisse and geometrically isomorphic, so one is a constant field twist of the other, say $[\lambda f]^*\mathcal{F}|_U \sim f^*\mathcal{F} \otimes \alpha_{\deg}|_U$, for some scalar $\alpha \in \overline{\mathbb{Q}}^\times$. Taking middle extensions, we find an arithmetic isomorphism $[\lambda f]^*\mathcal{F}(1/2)[1] \sim f^*\mathcal{F}(1/2)[1] \otimes \alpha_{\deg}$ on $\mathbb{G}_m/E$. Because both $[\lambda f]^*\mathcal{F}(1/2)[1]$ and $f^*\mathcal{F}(1/2)[1]$ are pure of weight zero, the scalar $\alpha$ must be pure of weight zero. This arithmetic isomorphism implies that (and, given the geometric irreducibility, is in fact equivalent to the fact that) for any finite extension $L/E$, and any point $t \in L^\times$, we have an equality of traces

$$\text{Trace}(\text{Frob}_{L,t}[\lambda f]^*\mathcal{F}(1/2)) = \alpha_{\deg(L/E)} \text{Trace}(\text{Frob}_{L,t}[f^*\mathcal{F}(1/2)].$$

Because $[\lambda f]^*\mathcal{F}(1/2)[1]$ is a geometrically irreducible perverse sheaf on $\mathbb{G}_m/E$ which is pure of weight zero, we have the estimate, as $L/E$ runs over larger and larger finite extensions,

$$\sum_{t \in \mathbb{G}_m(L)} |\text{Trace}(\text{Frob}_{L,t}[\lambda f]^*\mathcal{F}(1/2))|^2 = 1 + O(1/\sqrt{\#L}),$$

or equivalently the estimate

$$\sum_{t \in \mathbb{G}_m(L)} |\text{Trace}(\text{Frob}_{L,t}[\lambda f]^*\mathcal{F})|^2 = \#L + O(\sqrt{\#L}).$$

Indeed, it suffices to check that this second estimate holds instead for the sum over points $t \in U(L)$, as this sum omits at most $\#(\mathbb{G}_m \setminus U(\overline{k}))$ terms, each of which is itself $O(1)$. Because $[\lambda f]^*\mathcal{F}$ is lisse on $U$ and pure of weight zero, the sum over $U$ is given, by the Lefschetz trace formula, in terms of the sheaf $\text{End} := \text{End}([\lambda f]^*\mathcal{F})$ as

$$\text{Trace}(\text{Frob}_L[H^2_c(U/\overline{k}, \text{End})] = \text{Trace}(\text{Frob}_L[H^1_c(U/\overline{k}, \text{End})].$$
The sheaf $\text{End}$ is pure of weight zero. By the geometric irreducibility of $([\lambda f], \mathcal{F})|U$, the $\pi_1^{\text{geom}}(U)$-coinvariants of $\text{End}$ are just the constants $\mathbb{Q}_\ell$, so the group $H^2_c$ above is just $\mathbb{Q}_\ell(-1)$, on which $\text{Frob}_L$ acts as $#L$. The $H^1_c$ group is mixed of weight $\leq 1$, so we get the asserted estimate.

We now rewrite the sum of squares as follows. The sheaves $\mathcal{F}$ and $\mathcal{F} := \bigotimes_{i=1}^n \mathcal{L}_{\chi_i(a_i - t)}$ have complex conjugate trace functions, as do their direct images by any $\lambda f$. As $\alpha$ is pure of weight zero, we have $\alpha = 1/\alpha$. So we have

$$\alpha^{\deg(L/E)} \sum_{t \in G_m(L)} |\text{Trace}(\text{Frob}_{L,t}[\lambda f], \mathcal{F})|^2 = \sum_{t \in G_m(L)} (\text{Trace}(\text{Frob}_{L,t}[\lambda f], \mathcal{F}))(\text{Trace}(\text{Frob}_{L,t} f, \mathcal{F})) = \alpha^{\deg(L/E)} #L + O(\sqrt{#L}).$$

We now rewrite this penultimate sum as

$$\sum_{t \in G_m(L)} \sum_{x \in L, \lambda f(x) = t} \text{Trace}(\text{Frob}_{L,x} f, \mathcal{F})(\sum_{y \in L, f(y) = t} \text{Trace}(\text{Frob}_{L,y}[\mathcal{F}] = \sum_{(x,y) \in A^2(L), \lambda f(x) = f(y) \neq 0} \text{Trace}(\text{Frob}_{L,x} f, \mathcal{F}) \text{Trace}(\text{Frob}_{L,y}[\mathcal{F}]).$$

For $j : A^1[1/f] \subset A^1$, if we add the $n^2$ terms

$$\text{Trace}(\text{Frob}_{L,x} j, \mathcal{F}) \text{Trace}(\text{Frob}_{L,y} j, \mathcal{F})$$

for the points $(x, y) \in A^2(L)$ with $f(x) = f(y) = 0$, i.e., for the $n^2$ points $(a_i, a_j)$, we only change our sum by $O(1)$ (and we don’t change it at all if all the $\chi_i$ are nontrivial). So we end up with the estimate

$$\sum_{(x,y) \in A^2(L), \lambda f(x) = f(y)} \text{Trace}(\text{Frob}_{L,x} f, \mathcal{F}) \text{Trace}(\text{Frob}_{L,y}[\mathcal{F}] = \alpha^{\deg(L/E)} #L + O(\sqrt{#L}).$$

We now explain how this estimate leads to a contradiction. Consider the affine curve of equation $\lambda f(x) = f(y)$ in $A^2$. It is singular at the finitely many points $(a, b)$ which are pairs of critical points of $f$, i.e., $f'(a) = f'(b) = 0$, such that $\lambda f(a) = f(b)$. It is nonsingular at each pair of zeroes $(a_i, a_j)$ of $f$. Replacing $E$ by a finite extension if necessary, we may further assume that each irreducible component of the curve $\lambda f(x) = f(y)$ over $E$ is geometrically irreducible (i.e., that each irreducible factor of $\lambda f(x) - f(y)$ in $E[x, y]$ remains irreducible
in \( \overline{F}[x, y] \). The penultimate sum is, up to an \( O(1) \) term, the sum over the irreducible components \( C_j \) of the curve \( \lambda f(x) = f(y) \), of the sums

\[
\sum_{(x, y) \in C_j(L)} \text{Trace}(\text{Frob}_{L, x}|j_* \mathcal{F}) \text{Trace}(\text{Frob}_{L, y}|j_* \mathcal{F}).
\]

By the estimate for the sum, over the various \( C_j \), of these sums, there is at least one irreducible component, call it \( C_j \) for which this sum is not \( O(\sqrt{\# L}) \). The equation of any \( C_j \) divides the polynomial \( \lambda f(x) - f(y) \), whose highest degree term is \( \lambda x^n - y^n \). Therefore the highest degree term any divisor is a product of linear terms \( \mu y - x \), with the various \( \mu \)'s the \( n \)'th roots of \( \lambda \). So an irreducible component \( C_j \), given by a degree \( d_j \) divisor of \( \lambda f(x) - f(y) \), is finite flat of degree \( d_j \) over the \( y \)-line (and over the \( x \) line as well).

On the original curve \( \lambda f(x) = f(y) \), for each \( a_j \) there are \( n \) points \((a_j, y)\) on the curve, namely \( y = a_i \) for \( i = 1, \ldots, n \). On an irreducible component \( C_j \), given by a degree \( d_j \) divisor of \( \lambda f(x) - f(y) \), there are at most \( d_j \) values of \( y \) such that \((a_1, y)\) lies on \( C_j \). Each of these points is a smooth point of the original curve, so it lies only on the irreducible component \( C_j \). As there are \( n = \sum d_j \) points \((a_1, y)\) on the original curve, we must have exactly \( d_j \) points on \( C_j \) of the form \((a_1, y)\).

Now consider an irreducible component \( C \) on which our sum is not \( O(\sqrt{\# L}) \). Let us denote by \( \mathcal{C} \) the dense open set of the smooth locus of \( C \) which, via \( f \), lies over \( \mathbb{G}_m \). The sum

\[
\sum_{(x, y) \in \mathcal{C}(L)} \text{Trace}(\text{Frob}_{L, x}|\mathcal{F}) \text{Trace}(\text{Frob}_{L, y}|\overline{\mathcal{F}})
\]

differs only by \( O(1) \) from the sum over \( C \), so it too is not \( O(\sqrt{\# L}) \). In terms of the (restriction to \( \mathcal{C} \) of the) lisse, pure of weight zero, lisse of rank one sheaf

\[
\mathcal{G} := \otimes_{i=1}^n \mathcal{L}_{X_i(a_i-x)} \otimes_{i=1}^n \mathcal{L}_{X_i(a_i-y)}
\]
on \( \mathbb{A}^2[1/(f(x)f(y))] \), this last sum is

\[
\sum_{(x, y) \in \mathcal{C}(L)} \text{Trace}(\text{Frob}_{L, (x, y)}|\mathcal{G}).
\]

By the Lefschetz trace formula, this sum is

\[
\text{Trace}(\text{Frob}_L|H^2_c(\mathcal{C}/\overline{k}, \mathcal{G})) - \text{Trace}(\text{Frob}_L|H^1_c(\mathcal{C}/\overline{k}, \mathcal{G})).
\]

Because \( \mathcal{G} \) is pure of weight zero and lisse of rank one, the \( H^2_c \) is either zero or is is one-dimensional and pure of weight two, and this second case only occurs when \( \mathcal{G} \) is geometrically constant on \( \mathcal{C} \). The \( H^1_c \) is mixed of weight \( \leq 1 \). So the failure of an \( O(\sqrt{\# L}) \) estimate means
that the $H^2_c$ is nonzero, and hence that $\mathcal{G}$ is geometrically constant on $\mathcal{C}$.

Suppose first that the equation of $\mathcal{C}$ is of degree $d \geq 2$. Then there are $d$ points $(a_1, a_i)$ on $\mathcal{C}$, at least one of which is of the form $(a_1, a_i)$ with $a_i \neq a_1$. The curve $\mathcal{C}$ is finite etale over both the $x$-line and the $y$-line at the point $(a_1, a_i)$. So the functions $x - a_1$ and $y - a_i$ are each uniformizing parameters at this point. From the expression for $\mathcal{G}$, at the point $(a_1, a_i)$ on $\mathcal{C}$ its inertia group representation is that of $L_{\chi_1}(x - a_1) \otimes L_{\chi_i}(y - a_i)$. In other words, its inertia group representation at $(a_1, a_i)$ is the character $\chi_1/\chi_i$. But this character is nontrivial (because $\chi_1$ is a singleton), contradicting the geometric constance of $\mathcal{G}$ on $\mathcal{C}$.

It remains to treat the case in which the equation for $\mathcal{C}$ is of degree one. In this case, the above argument still works unless the unique point on $\mathcal{C}$ of the form $(a_1, y)$ has $y = a_1$. In this case, we use the fact that we have a second singleton, $\chi_2$. Using this singleton, we could still use the above argument unless the unique point on $\mathcal{C}$ of the form $(a_2, y)$ has $y = a_2$. So we only need treat the case when both the points $(a_1, a_1)$ and $(a_2, a_2)$ lie on $\mathcal{C}$. But in this case, the equation for $\mathcal{C}$, being of degree one, must be $y = x$. But if $y - x$ divides $\lambda f(x) - f(y)$, we reduce mod $y - x$ to find that $(\lambda - 1)f(x) = 0$, and hence $\lambda = 1$, contradiction.

□

4. A PRELIMINARY ESTIMATE

In this section, we continue with a squarefree monic $k$-polynomial $f$ of degree $n \geq 2$, $B := k[X]/(f)$, and a character $\chi$ of $B^\times$. Over a finite extension $E/k$ where $f$ factors completely, say $f(X) = \prod_i (X - a_i)$, the lisse rank one sheaf $L_{\chi(u-t)}$ on $\mathbb{A}^1[1/f]/k$ becomes isomorphic to the sheaf $\otimes_{i=1}^n L_{\chi_i(a_i-t)}$ on $\mathbb{A}^1[1/f]/E$.

**Theorem 4.1.** Let $\chi$ be a character of $B^\times$ whose constituent characters $\chi_i$ satisfy the following three conditions.

(1) The $\chi_i$ are pairwise distinct.

(2) The product $\prod_i \chi_i$ is nontrivial, (i.e., $\chi$ is nontrivial on $k^\times$).

(3) For at least one index $i$, $\chi_i^n \neq \prod_i \chi_i$.

Fix $\lambda \in k^\times$, and form the perverse sheaf

$$N(\lambda, \chi) := [\lambda f]_* (L_{\chi(u-t)}(1/2))[1]$$

on $\mathbb{G}_m/k$. Then we have the following results.

(1) $N(\lambda, \chi)$ is geometrically irreducible, pure of weight zero, and lies in the Tannakian category $\mathcal{P}_{\text{arith}}$ in the sense of [Ka-CE]. It has generic rank $n$, Tannakian “dimension” $n - 1$, and it has at most $2n$ bad characters.
(2) $N(\lambda, \chi)$ is geometrically Lie-irreducible in $\mathcal{P}$.
(3) $N(\lambda, \chi)$ has $G_{\text{geom}} = G_{\text{arith}} = GL(n-1)$.

Proof. By Theorem 3.1 and the disjointness of the $\chi_i$, $N(\lambda, \chi)$ is geometrically irreducible. It visibly has generic rank $n$. As $n \geq 2$, it is not a Kummer sheaf, so, being geometrically irreducible, it lies in $\mathcal{P}$. Its Tannakian dimension is

$$\chi_c(\mathbb{G}_m/k, N(\lambda, \chi)) = -\chi_c(\mathbb{G}_m/k, [\lambda f]_*(\mathcal{L}_{\chi(u-t)})) =$$

$$= -\chi_c(\mathbb{A}^1[1/f]/k, \mathcal{L}_{\chi(u-t)}) = -\chi_c(\mathbb{A}^1[1/f]/k, \otimes_{i=1}^n \mathcal{L}_{\chi_i(a_i-t)}) = n - 1.$$ 

Because $N(\lambda, \chi)$ has generic rank $n$, it has at most $2n$ bad characters, namely those whose inverses occur in either its $I(0)$-representation or in its $I(\infty)$-representation.

On some dense open set $j : U \subset \mathbb{G}_m$, $[\lambda f]_*(\mathcal{L}_{\chi(u-t)})$ is a lisse sheaf of rank $n$, which is pure of weight zero, hence $j^*N(\lambda, \chi)$ is pure of weight zero (the Tate twist $(1/2)$ offsets the shift $[1]$). By irreducibility $N(\lambda, \chi)$ must be the middle extension of $j^*N(\lambda, \chi)$, cf. [BBD, 5.3.8], so remains pure of weight zero [BBD, 5.3.2]. Again by the disjointness of the $\chi_i$, part (3) of Theorem 3.1, together with [Ka-CE, Cor. 8.3], we get that $N(\lambda, \chi)$ is geometrically Lie-irreducible in $\mathcal{P}$.

It remains to explain why $N(\lambda, \chi)$ has $G_{\text{geom}} = G_{\text{arith}} = GL(n-1)$. Since we have a priori inclusions $G_{\text{geom}} \subset G_{\text{arith}} \subset GL(n-1)$, it suffices to prove that $G_{\text{geom}} = GL(n-1)$. The idea is to apply [Ka-CE, Thm. 17.1]. We may compute $G_{\text{geom}}$ after extension of scalars to $E$. Suppose that $\chi^1_i \neq \prod_i \chi_i$. The construction $M \mapsto M \otimes \mathcal{L}_{\chi^1}$ induces a Tannakian isomorphism of $<N(\lambda, \chi)>_{\text{arith}}$ with $<N(\lambda, \chi) \otimes \mathcal{L}_{\chi^1}>_{\text{arith}}$. So it suffices to prove that $N(\lambda, \chi) \otimes \mathcal{L}_{\chi^1}$ has $G_{\text{geom}} = GL(n-1)$. By the disjointness assumption on the $\chi_i$, the trivial character $1$ occurs exactly once in the $I(0)$-representation of $N(\lambda, \chi) \otimes \mathcal{L}_{\chi^1}$. So by [Ka-CE, Thm. 17.1], it suffices to show that the trivial character does not occur in its $I(\infty)$-representation, or equivalently that $\chi_1$ does not occur in the $I(\infty)$-representation of $N(\lambda, \chi)$. This $I(\infty)$-representation is $[\lambda f]_* \mathcal{L}_{\prod_i \chi_i}$, and $\mathcal{L}_{\chi_1}$ occurs in it if and only if $[\lambda f]^*(\mathcal{L}_{\chi_1})$ occurs in $\mathcal{L}_{\prod_i \chi_i}$. Because $\lambda f$ has degree $n$, the pullback $[\lambda f]^*(\mathcal{L}_{\chi_1})$ is geometrically isomorphic to $\mathcal{L}_{\chi_1^*}$ as $I(\infty)$-representation. So if $\chi^1_i \neq \prod_i \chi_i$, then $\mathcal{L}_{\chi_1}$ does not occur in the $I(\infty)$-representation $[\lambda f]_* \mathcal{L}_{\prod_i \chi_i}$, and we conclude by applying [Ka-CE, Thm. 17.1] to $N(\lambda, \chi) \otimes \mathcal{L}_{\chi^1}$. \hfill $\Box$

**Corollary 4.2.** Let $\chi$ be a character of $B^\times$ whose constituent characters $\chi_i$ satisfy the three conditions of the previous theorem. Suppose that $q := \# k$ satisfies the inequality $\sqrt{q} \geq 1 + 2n$. For each character $\rho$ of $k^\times$ which is good for $N(\lambda, \chi)$ (i.e., such that for $j : \mathbb{G}_m \subset \mathbb{P}^1$ the inclusion, the “forget supports” map gives an isomorphism $j!(N(\lambda, \chi) \otimes \mathcal{L}_\rho) \cong$
$R_{\chi}(\mathcal{N}(\lambda, \chi) \otimes \mathcal{L}_p)$, or equivalently, $\overline{\rho}$ does not occur in the local monodromy at either $0$ or $\infty$ of $\mathcal{N}(\lambda f, \chi)$, denote by $\theta_{k,\lambda f,\chi,\rho}$ the conjugacy class in $U(n-1)$ whose reversed characteristic polynomial is given by

$$\det(1 - T \theta_{k,\lambda f,\chi,\rho}) = \det(1 - TF_{\text{Frob}}k|H^0_c(\mathbb{G}_m/\mathbb{F}, \mathcal{N}(\lambda, \chi) \otimes \mathcal{L}_p)).$$

Let $\Lambda$ be a nontrivial irreducible representation of $U(n-1)$ which occurs in $\text{std}^{a} \otimes (\text{std}^{b})^{\otimes b}$. Then we have the estimate

$$| \sum_{\rho \in \text{Good}(k,\lambda f,\chi)} \text{Trace}(\Lambda(\theta_{k,\lambda f,\chi,\rho})) | \leq (#\text{Good}(k,\lambda f,\chi))2(a+b+1)(2n)^{a+b}/\sqrt{q}.$$

**Proof.** By Theorem 4.1, $\mathcal{N}(\lambda, \chi)$ has $G_{\text{geom}} = G_{\text{arith}} = GL(n-1)$. So this is [Ka-CE, Remark 7.5 and the proof of Theorem 28.1], applied to $\mathcal{N} := \mathcal{N}(\lambda, \chi)$ with the constant $C$ there, an upper bound for each of the generic rank, the number of bad characters, and the Tannakian dimension of $\mathcal{N}$, taken to be $2n$. \qed

The interest of this Corollary is that the (trivial) Leray spectral sequence for $[\lambda f]$! gives a $F_{\text{Frob}}k$-isomorphism of cohomology groups

$$H^0_c(\mathbb{G}_m/\overline{k}, \mathcal{N}(\lambda, \chi) \otimes \mathcal{L}_p) \cong H^0_c(\mathbb{A}^1[1/f]/\overline{k}, \mathcal{L}_{\chi(u-t)} \otimes \mathcal{L}_{\rho(\lambda f(t))}(1/2)[1]) =$$

$$= H^1_c(\mathbb{A}^1[1/f]/\overline{k}, \mathcal{L}_{\chi(u-t)} \otimes \mathcal{L}_{\rho(\lambda f(t))})(1/2).$$

By Lemma 2.1, $\text{Norm}_{B/k}(u-t) = (-1)^n f(t)$. So if we denote by $\rho_{\text{Norm}}$ the character of $B^\times$ given by

$$\rho_{\text{Norm}} := \rho \circ \text{Norm}B/k,$$

then $\mathcal{L}_{\rho((-1)^n f(t))}$ is $\mathcal{L}_{\rho_{\text{Norm}}(u-t)}$, and the conjugacy class $\theta_{k,(-1)^n f,\chi,\rho}$ is none other than the conjugacy class $\theta_{k,f,\chi,\rho_{\text{Norm}}}$ of the Introduction.

5. THE EQUIDISTRIBUTION THEOREM

We continue with a squarefree monic $k$-polynomial $f$ of degree $n \geq 2$, $B := k[X]/(f)$, and a character $\chi$ of $B^\times$. Over a finite extension $E/k$ where $f$ factors completely, say $f(X) = \prod_i(X - a_i)$, the lisse rank one sheaf $\mathcal{L}_{\chi(u-t)}$ on $\mathbb{A}^1[1/f]/k$ becomes isomorphic to the sheaf $\otimes_{i=1}^n \mathcal{L}_{\chi_i(a_i-t)}$ on $\mathbb{A}^1[1/f]/E$.

Let us say that $\chi$ is “totally ramified” (what we called “as ramified as possible” in the Introduction) if each $\chi_i$ and the product $\prod i \chi_i$ are all nontrivial. In view of Lemma 2.2, $\chi$ is totally ramified if and only if the group $H^1_c(\mathbb{A}^1[1/f]/\overline{k}, \mathcal{L}_{\chi(u-t)})$ is pure of weight one, or equivalently if and only if the group $H^0_c(\mathbb{A}^1[1/f]/\overline{k}, \mathcal{L}_{\chi(u-t)}(1/2)[1])$ is pure of weight zero, in which case it has dimension $n-1$. 
Let us say that a totally ramified $\chi$ is “generic” if, in addition to being totally ramified, its constituent characters $\chi_i$ satisfy the three conditions of Theorem 4.1. We denote by

$$\text{TotRam}(k, f), \text{resp. TotRamGen}(k, f)$$

the sets of totally ramified (respectively totally ramified and generic) characters of $B^\times$.

**Lemma 5.1.** Let $\chi$ be a totally ramified character of $B^\times$. Let $\rho$ be a character of $k^\times$ which is good for $N((-1)^nf, \chi)$. Then the product character $\chi\rho_{\text{Norm}}$ is totally ramified. Moreover, $\chi$ is generic if and only if $\chi\rho_{\text{Norm}}$ is generic.

**Proof.** Indeed, if geometrically we have $L_{\chi(u-t)} \cong \otimes_{i=1}^n L_{\chi_i(a_i-t)}$, then $L_{\chi\rho_{\text{Norm}}(u-t)} \cong \otimes_{i=1}^n L_{\chi_i\rho(a_i-t)}$; we view $\rho$ and the $\chi_i$ as characters of $\pi_{1,\text{tame}}(\mathcal{G}_m/k)$, to make sense of the products $\chi_i\rho$. Alternatively, if $f$ splits over $E$, think of $\rho$ as the character $x \mapsto \rho(\text{Norm}_{E/k}(x))$ of $E^\times$. Thus the constituent characters of $\chi\rho_{\text{Norm}}$ are the $\chi_i\rho$. That $\rho$ is good for $N((-1)^nf, \chi)$ means precisely $\bar{\rho}$ does not occur in the local monodromy of $N((-1)^nf, \chi)$ at either 0 or $\infty$. Its absence at 0 is the nontriviality of each $\chi_i\rho$. Its absence at $\infty$ is that $\rho \prod_i \chi_i$ is nontrivial, i.e., that $\prod_i (\chi_i\rho)$ is nontrivial. Thus $\chi\rho_{\text{Norm}}$ is totally ramified. If in addition $\chi$ is generic, say $\chi^n \neq \prod_i \chi_i$, then $(\chi_1\rho)^n \neq \prod_i (\chi_i\rho)$, and hence $\chi\rho_{\text{Norm}}$ is generic as well. Conversely, if $\chi$ is totally ramified and $\chi\rho_{\text{Norm}}$ is totally ramified and generic, then $\bar{\rho}$ is good for $N((-1)^nf, \chi\rho_{\text{Norm}})$, and so by the previous argument $\chi$ is totally ramified and generic. □

We now combine this lemma with Corollary 4.2 to get a result concerning those conjugacy classes $\theta_{k, f, \chi}$ of the Introduction whose $\chi$ is totally ramified and generic.

**Corollary 5.2.** Suppose $\sqrt{q} \geq 1 + 2n$. Let $\Lambda$ be a nontrivial irreducible representation of $U(n-1)$ which occurs in $\text{std}^a \otimes (\text{std}^\vee)^b$. Then we have the estimate

$$|\sum_{\chi \in \text{TotRamGen}(k, f)} \text{Trace}(\Lambda(\theta_{k, f, \chi}))| \leq \#\text{TotRamGen}(k, f)2(a + b + 1)(2n)^{a+b}/\sqrt{q}.$$

**Proof.** Let us say that two totally ramified generic characters $\chi$ and $\chi'$ of $B^\times$ are equivalent if $\chi' = \chi\rho_{\text{Norm}}$ for some (necessarily unique) character $\rho$ of $k^\times$. Break the terms of the sum into equivalence classes. The sum over the equivalence class of $\chi$ is precisely the sum bounded by Corollary 4.2, with $\lambda$ there taken to be $(-1)^n$. □
Our final task is to infer from this estimate an estimate for the sum over all $\chi$ in $\text{TotRam}(k, f)$. For this, we now turn to giving upper and lower bounds for $\#\text{TotRam}(k, f)$ and for $\#\text{TotRamGen}(k, f)$. We define three monic integer polynomials of degree $n$,

\[ P_{\text{all}, n}(X) := X^n - 1, \]

\[ P_{\text{TR}, n}(X) := (X - 2)^n - \sum_{0 \leq i \leq n-1} X^i, \]

and

\[ P_{\text{TRG}, n}(X) := (X - 1 - n)^n + (X - 2)^n - X^n + 1 - n \sum_{0 \leq i \leq n-1} X^i. \]

**Lemma 5.3.** For $q := \#k$, we have the (trivial) estimate

\[ \#\text{TotRam}(k, f) \leq P_{\text{all}, n}(q) = q^n - 1. \]

**Proof.** Indeed, $B^\times$ is a subset of $B \setminus \{0\}$, whose cardinality is $q^n - 1$, so $q^n - 1$ is an upper bound for the total number of characters of $B^\times$. $\Box$

**Lemma 5.4.**

\[ \#\text{TotRam}(k, f) \geq P_{\text{TR}, n}(q) = (q - 2)^n - \sum_{0 \leq i \leq n-1} q^i, \]

and

\[ \#(\{\text{chars of } B^\times \} \setminus \text{TotRam}(k, f)) \leq q^n - 1 - (q - 2)^n + \sum_{0 \leq i \leq n-1} q^i. \]

**Proof.** Factor $f$ as the product of $k$-irreducible monic polynomials $P_i$ of degree $d_i$. Thus $n = \sum_i d_i$, and $\#B^\times = \prod_i (q^{d_i} - 1) \geq (q - 1)^n$. So there are at least $(q - 1)^n$ characters $\chi$ of $B^\times$. We now count the characters which violate or satisfy the two conditions of being totally ramified.

Since $k^\times \subset B^\times$, the restriction map on characters is surjective. So the condition that $\chi|k^\times$ be nontrivial disqualifies $\#B^\times/(q - 1) = (\prod_i (q^{d_i} - 1))/(q - 1)$ of them.

The condition that each constituent character $\chi_i$ is nontrivial is equivalent to the condition that when we write $\chi$ as the product of characters $\chi_{P_i}$ of the factors $(k[X]/(P_i))^\times$, each $\chi_{P_i}$ is nontrivial. So there are $\prod_i (q^{d_i} - 2)$ choices of $\chi$ which satisfy this condition. If we now omit the ones which are trivial on $k^\times$, we are left with at least

\[ \prod_i (q^{d_i} - 2) - (\prod_i (q^{d_i} - 1))/(q - 1) \]
characters which are totally ramified. From the inequalities $q^d - 2 \geq (q - 2)^d$ and $\prod_i (q^{d_i} - 1) \leq q^n - 1$ we get

$$#\text{TotRam}(k, f) \geq \prod_i (q^{d_i} - 2) - (\prod_i (q^{d_i} - 1))/(q - 1) \geq (q - 2)^n - \sum_{0 \leq i \leq n-1} q^i.$$  

Combining this with the previous lemma, we get the asserted upper bound for the number of characters of $B^\times$ which are not totally ramified. □

**Lemma 5.5.** For $q \geq n + 1$, we have the estimate

$$#\text{TotRamGen}(k, f) \geq P_{TRG,n}(q) =$$

$$= (q - 1 - n)^n + (q - 2)^n - q^n + 1 - n \sum_{0 \leq i \leq n-1} q^i.$$  

*Proof.* We now count the characters which violate or satisfy the two additional conditions which make a totally ramified character generic.

We first turn to the condition that for at least one of the $\chi_i$, $\chi_i^n \neq \prod_i \chi_i$. Suppose first that $f$ is itself irreducible. Then $\chi$ is a character of the field $B^\times \cong \mathbb{F}_q^\times$, and its constituent characters $\chi_1, \ldots, \chi_n$ are the characters $\chi, \chi q, \ldots, \chi q^{n-1}$. The condition that $\chi^n \neq \prod_i \chi_i$ is the condition that $\chi^n \neq \chi^{1+q+\ldots+q^{n-1}}$, which disqualifies at most $1 + q + \ldots + q^{n-1} - n$ possible $\chi$.

If $f$ is not irreducible, let $P$ be an irreducible factor of some degree $d < n$, and $\chi_P$ the $P$-constituent of $\chi$. The constituents of $\chi_P$ as character of $(k[X]/(P))^\times$ are $\chi_P, \chi_P q, \ldots, \chi_P q^{d-1}$. Think of these as the first $d$ constituents of $\chi$. We can be sure that there is some choice of index $j \in [1, d]$ such that $\chi_j^n \neq \prod_i \chi_i$ if we have

$$\prod_{1 \leq j \leq d} \chi_j^n \neq (\prod_i \chi_i)^d.$$  

This is the condition that

$$\chi_P^{(n-d)(1+q+\ldots+q^{d-1})} \neq (\prod_{d+1 \leq i \leq n} \chi_i)^d.$$  

So for any given choice of the $P_i$-components of $\chi$ for all the other irreducible factors $P_i$ of $f$, at most $(n-d)(1+q+\ldots+q^{d-1})$ characters $\chi_P$ are disqualified. So the total number of characters $\chi$ which fail this
second condition is at most \((n - d)(1 + q + \ldots + q^{d-1}) \prod_{P_i \neq P}(q^{d_i} - 1)\). From the inequality
\[
(n - d)(1 + q + \ldots + q^{d-1}) \prod_{P \neq P_i} (q^{d_i} - 1) = (n - d)( \prod_{\text{all } P_i} (q^{d_i} - 1))(q - 1)
\]
\[
\leq (n - 1)(q^n - 1)/(q - 1)
\]
we see that in either case, \(f\) irreducible or not, there are at most
\[
(n - 1)\left( \sum_{0 \leq i \leq n - 1} q^i \right)
\]
characters \(\chi\) of \(B^x\) which violate this first condition.

We now turn to the condition that the constituents \(\chi_i\) be all distinct. Again we factor \(f\), and this time collect the factors according to their degrees. Suppose that there are \(e_i\) factors \(P_{d_i,j}, j = 1, \ldots, e_i\) whose degrees are \(d_i\). The first condition for distinctness is that for each \(P_{d_i,j}\)-component \(\chi_{P_{d_i,j}}\) the \(d_i\) characters \(\chi_{q d_i}^{q^i}\) for \(0 \leq i \leq d_i - 1\) are all distinct, or in other words that the orbit of \(\chi_{P_{d_i,j}}\) under the \(q\)'th power map has full length \(d_i\), rather than some proper divisor of \(d_i\).

The characters of \(\mathbb{F}_{q^{d_i}}\) whose orbit length is a proper divisor of \(d_i\) are those which come from (by composition with the relative norm) characters of subfields \(\mathbb{F}_{q^r}\) for some proper divisor \(r\) of \(d_i\). So the number of such short-orbit characters is at most
\[
\sum_{r|d_i, r<d_i} (q^r - q^r - 1)
\]
So the number of full-orbit characters of \(\mathbb{F}_{q^{d_i}}\) is at least
\[
q^{d_i} - [d_i/2]q^{[d_i/2]}\geq q^{d_i} - q^{d_i-1}.
\]

Suppose now that for each irreducible factor \(P_{d_i,j}\) of \(f\), we have chosen a full-orbit (i.e., orbit length \(d_i\)) character. For irreducibles of different degrees, there can be no equality of their constituent characters, because the orbit-lengths are different. If there are \(e_i \geq 2\) irreducible factors of the same degree \(d_i\), say \(P_{d_i,1}, \ldots, P_{d_i,e_i}\), then we may choose \(\chi_{P_{d_i,1}}\) to be any of the at least \(q^{d_i} - q^{d_i-1}\) full-orbit characters of \(\mathbb{F}_{q^{d_i}}\). Then we must choose \(\chi_{P_{d_i,2}}\) to be a full-orbit character of \(\mathbb{F}_{q^{d_i}}\) which does lie in the orbit of \(\chi_{P_{d_i,1}}\), thus excluding \(d_i\) possible full-orbit characters. Continuing in this way, we see that there are at least
\[
\prod_{d_i \text{ which occur } j=0}^{e_i-1} (\prod_{j=0} (q^{d_i} - q^{d_i-1} - j d_i))
\]
characters $\chi$ of $B^\times$ all of whose constituents are distinct.

Because $q \geq n + 1$, each factor $(q^{d_i} - q^{d_i-1} - jd_i)$ satisfies
\[
(q^{d_i} - q^{d_i-1} - jd_i) \geq (q^{d_i} - q^{d_i-1} - n) \geq (q - 1 - n)^{d_i}.
\]

[For the last inequality, write $q = X + n + 1$; then we are saying that $(X + n + 1)^{d-1}(X + n) \geq X^d + n$, which obviously holds for $X \geq 0$ and $d \geq 1$.] Thus for $q \geq n + 1$, there are at least
\[
(q - 1 - n)^n
\]
characters $\chi$ of $B^\times$ all of whose constituents are distinct.

Removing from these those which violate the first condition, we are
left with at least
\[
(q - 1 - n)^n - (n - 1)\left(\sum_{0 \leq i \leq n-1} q^i\right)
\]
characters which, if totally ramified, are also generic. We have already
seen that at most
\[
q^n - 1 - (q - 2)^n + \sum_{0 \leq i \leq n-1} q^i
\]
characters fail to be totally ramified. Taking (some of) these away, we
end up with at least
\[
(q - 1 - n)^n - (n - 1)\left(\sum_{0 \leq i \leq n-1} q^i\right) - (q^n - 1 - (q - 2)^n + \sum_{0 \leq i \leq n-1} q^i) = 
= (q - 1 - n)^n + (q - 2)^n - q^n + 1 - n \sum_{0 \leq i \leq n-1} q^i
\]
characters which are totally ramified and generic. $\square$

**Lemma 5.6.** We have the estimate
\[
\#(\text{TotRam}(k, f) \setminus \text{TotRamGen}(k, f) \leq P_{\text{all},n}(q) - P_{\text{TRG},n}(q) = 
= (q - 1 - n)^n + (q - 2)^n - 2q^n + 2 - n \sum_{0 \leq i \leq n-1} q^i.
\]

**Proof.** Combine Lemmas 5.3 and 5.5. $\square$

**Lemma 5.7.** There exists a real constant $C_n$ such that for $q \geq C_n$, we have
\[
P_{\text{all},n}(q) - P_{\text{TRG},n}(q) \leq P_{\text{TRG},n}(q) / \sqrt{q}.
\]

**Proof.** The difference $P_{\text{all},n}(X) - P_{\text{TRG},n}(X)$ is a real polynomial of
degree $n - 1$, while $P_{\text{TRG},n}(X)$ is a real polynomial which is monic of
degree $n$. $\square$
Theorem 5.8. Suppose \( q \geq C_n \) and \( \sqrt{q} \geq 1+2n \). Let \( \Lambda \) be a nontrivial irreducible representation of \( U(n-1) \) which occurs in \( \text{std}^a \otimes (\text{std}^d)^b \). Then we have the estimate
\[
\left| \sum_{\chi \in \text{TotRam}(k,f)} \text{Trace}(\Lambda(\theta_{k,f,\chi})) \right| \leq (\#\text{TotRamGen}(k,f))4(a + b + 1)(2n)^{a+b}/\sqrt{q}.
\]

Proof. We break the sum into two pieces, the sum over \( \chi \in \text{TotRamGen}(k,f) \), and the sum over \( \chi \in \text{TotRam}(k,f) \setminus \text{TotRamGen}(k,f) \). By Corollary 5.2, the absolute value of the first sum is bounded by
\[
(\#\text{TotRamGen}(k,f))2(a + b + 1)(2n)^{a+b}/\sqrt{q}.
\]
The second sum has at most
\[
P_{\text{all},n}(q) - P_{\text{TRG},n}(q) \leq P_{\text{TRG},n}(q)/\sqrt{q} \leq (\#\text{TotRamGen}(k,f))/\sqrt{q}
\]
terms, each of which, being the trace of a unitary conjugacy class in a representation of dimension at most \((n-1)^{a+b}\), is bounded in absolute value by \((n-1)^{a+b}\). So the absolute value of the second sum is bounded by
\[
(\#\text{TotRamGen}(k,f))(n-1)^{a+b}/\sqrt{q},
\]
which is less than the upper bound for the first sum. So doubling the upper bound for the first sum is safe. \(\square\)

Corollary 5.9. Suppose \( q \geq C_n \) and \( \sqrt{q} \geq 1+2n \). Let \( \Lambda \) be a nontrivial irreducible representation of \( U(n-1) \) which occurs in \( \text{std}^a \otimes (\text{std}^d)^b \). Then we have the estimate
\[
\left| (1/\#\text{TotRam}(k,f)) \sum_{\chi \in \text{TotRam}(k,f)} \text{Trace}(\Lambda(\theta_{k,f,\chi})) \right| \leq 4(a + b + 1)(2n)^{a+b}/\sqrt{q}.
\]

Proof. Indeed, \( \#\text{TotRamGen}(k,f) \leq \#\text{TotRam}(k,f) \). \(\square\)

Thus we obtain our target result.

Theorem 5.10. Fix an integer \( n \geq 2 \) and a sequence of data \((k_i, f_i)\) with \( k_i \) a finite field (of possibly varying characteristic) and \( f_i(X) \in k_i[X] \) squarefree of degree \( n \). If \#\(k_i\) is archimedeanly increasing to \( \infty \), the collections of conjugacy classes
\[
\{\theta_{k_i,f_i,\chi}\}_{\chi \in \text{TotRam}(k_i,f_i)}
\]
become equidistributed in \( U(n-1)^\# \) as \#\(k_i \to \infty \).
6. Appendix: The case of “even” characters

We continue to work with a squarefree monic polynomial $f(X) \in k[X]$ of degree $n \geq 2$, and the $k$-algebra $B := k[X]/(f(X)X)$. We say that a character $\chi$ of $B^\times$ is even if it is trivial on $k^\times$ (viewed as a subgroup of $B^\times$).

**Lemma 6.1.** The character $\chi$ is even if and only if $L_{\chi(u-t)}$ is lisse at $\infty$ (more precisely, if and only if, denoting by $j : \mathbb{A}^1[1/f] \subset \mathbb{P}^1$ the inclusion, the middle extension sheaf $j_*L_{\chi(u-t)}$ on $\mathbb{P}^1$ is lisse at $\infty$). Moreover, for even $\chi$ we have the formula

$$\text{Trace}(Frob_{k,\infty}|j_*L_{\chi(u-t)}) = 1.$$  

**Proof.** The first assertion is immediate from the geometric isomorphism of $L_{\chi(u-t)}$ with the tensor product $\otimes_{i=1}^n L_{\chi_i(a_i-t)}$, together with Lemma 2.3. For the second assertion, we argue as follows. We have a morphism $\mathbb{G}_m \rightarrow B^\times$ given by $t \mapsto 1/t$. The corresponding pullback sheaf $L_{\chi(1/t)}$ on $\mathbb{G}_m$ is trivial, i.e., isomorphic to the constant sheaf $\mathbb{Q}_\ell$, precisely because $\chi$ is trivial on $k^\times$. So on $\mathbb{G}_m[1/f]$, we have arithmetic isomorphisms

$$L_{\chi(u-t)} \cong L_{\chi(u-t)} \otimes L_{\chi(1/t)} \cong L_{\chi(u/t-1)}.$$  

In terms of the uniformizing parameter $s := 1/t$ at $\infty$, we have $L_{\chi(u-t)} \cong L_{\chi(su-1)}$. Extending $L_{\chi(su-1)}$ across $\infty$, i.e., across $s = 0$, by direct image, we get

$$\text{Trace}(Frob_{k,\infty}|j_*L_{\chi(u-t)}) = \text{Trace}(Frob_{k,0}|j_*L_{\chi(su-1)}) = \chi(-1) = 1,$$

the last equality because, once again, $\chi$ is trivial on $k^\times$.

Let us say that an even character $\chi$ is totally ramified if, in the geometric isomorphism

$$L_{\chi(u-t)} \cong \otimes_{i=1}^n L_{\chi_i(a_i-t)},$$

each $\chi_i$ is nontrivial. The we have the following lemma, analogous to Lemma 2.5

**Lemma 6.2.** The even character $\chi$ is totally ramified if and only if the group $H^1_c(\mathbb{P}^1[1/f] \otimes_k \overline{k}, j_*L_{\chi(u-t)})$ is pure of weight one, in which case $H^1_c$ has dimension $n - 2$, and $H^2_c = 0$.

Let us denote by $\text{TotRamEven}(k, f)$ the set of even characters of $B^\times$ which are totally ramified. Attached to each $\chi \in \text{TotRamEven}(k, f)$, we have a conjugacy class $\theta_{k,f,\chi} \in U(n - 2)^\#$, defined by its reversed characteristic polynomial via the equation

$$\det(1 - T \sqrt{\#k\theta_{k,f,\chi}}) = \det(1 - TFrob_k|H^1_c(\mathbb{P}^1[1/f] \otimes_k \overline{k}, j_*L_{\chi(u-t)})).$$
Keating and Rudnick make the following conjecture, the “even” version of Theorem 5.10.

**Conjecture 6.3.** Fix an integer \( n \geq 3 \) and a sequence of data \((k_i, f_i)\) with \( k_i \) a finite field (of possibly varying characteristic) and \( f_i(X) \in k_i[X] \) squarefree of degree \( n \). If \( \#k_i \) is archimedeanly increasing to \( \infty \), the collections of conjugacy classes

\[
\{\theta_{k_i, f_i, \chi}\}_{\chi \in \text{TotRamEven}(k_i, f_i)}
\]

become equidistributed in \( U(n-2)\# \) as \( \#k_i \to \infty \).

At present, we can prove this only under the additional (and highly artificial) hypothesis that each \( f_i(X) \in k_i[X] \) has a zero in \( k_i \).

**Theorem 6.4.** Fix an integer \( n \geq 3 \) and a sequence of data \((k_i, f_i)\) with \( k_i \) a finite field (of possibly varying characteristic) and \( f_i(X) \in k_i[X] \) squarefree of degree \( n \). Suppose each \( f_i \) has a zero in \( k_i \). If \( \#k_i \) is archimedeanly increasing to \( \infty \), the collections of conjugacy classes

\[
\{\theta_{k_i, f_i, \chi}\}_{\chi \in \text{TotRamEven}(k_i, f_i)}
\]

become equidistributed in \( U(n-2)\# \) as \( \#k_i \to \infty \).

**Proof.** Replacing each \( f_i \) by an additive translate \( X \mapsto X + a_i \) of itself, we reduce to the case when each \( f_i \) is of the form \( f_i(X) = Xg_i(X) \), with \( g_i \in k_i[X] \) squarefree and having \( g_i(0) \neq 0 \).

The idea is that the theorem is a consequence of a (slight variant of) Theorem 5.10, applied to the \( g_i \). To explain this, let us fix a finite field \( k \), a squarefree monic \( g(X) \in k[X] \) of degree \( n-1 \) with \( g(0) \neq 0 \), and put \( f(X) := Xg(X) \). Let us write \( B_f := k[X]/(f(X)) \), \( B_g := k[X]/(g(X)) \), \( B_X := k[X]/(X) \cong k \). Then

\[
B_f \cong k \times B_g.
\]

For \( P(X) \) a monic irreducible in \( k[X] \) which is prime to \( f \), the image of \( P(X) \) in \( B_f^\times \) is, via this isomorphism, the pair

\[
(P(0), P \mod g) = (\text{the scalar } P(0) \in k^\times) \times (1, P/P(0) \mod g).
\]

For an even character \( \chi_f \) of \( B_f^\times \), with components \( \chi_X, \chi_g \), we therefore have

\[
\chi_f(P \mod f) = \chi_f(1, P/P(0) \mod g) = \chi_g(P/P(0)).
\]

If \( \chi_f \) lies in \( \text{TotRamEven}(k, f) \) then \( \chi_X \) is nontrivial, each constituent character \( \chi_i \) of \( \chi_g \) is nontrivial, and, by the evenness of \( \chi_f \), the restriction of \( \chi_g \) to \( k^\times \) is the inverse of the nontrivial character \( \chi_X \). In other words, \( \chi_g \in \text{TotRam}(k, g) \). Conversely, given
\( \chi_g \in \text{TotRam}(k, g) \), define \( \chi_X \) to be the restriction to \( k^\times \) of \( 1 \chi_g \); then the pair \((\chi_X, \chi_g)\) taken to be \( \chi_f \) lies in \( \text{TotRamEven}(k, f) \).

For \( P(X) = X - t \) a linear irreducible, and \( \chi_f \) even, we have

\[
\chi_f(X - t) = \chi_g((X - t)/(-t)) = \chi_g(1 - X/t).
\]

Exactly as in section 2 of this paper, we find an arithmetic isomorphism on \( A^1[1/f] = \mathbb{G}_m[1/g] \),

\[
\mathcal{L}_{\chi_f(u-t)} \cong \mathcal{L}_{\chi_g(1-u/t)}.
\]

In terms of the parameter \( s := 1/t \) on \( \mathbb{G}_m \), and the palindrome \( g^{pal}(s) := s^{\deg(g)}g(t) \) of \( g \), our sheaf becomes \( \mathcal{L}_{\chi_g(1-us)} \) on \( \mathbb{G}_m[1/g^{pal}] \), and has an obvious lisse extension across \( s = 0 \) to the sheaf \( \mathcal{L}_{\chi_g(1-us)} \) on \( A^1[1/g^{pal}] \).

[N.B. Here the \( u \) is still the image of \( X \) in \( B_g \), and \( \chi_g \) is our character of \( B_g^\times \). But it is the zeroes of \( g^{pal}(s) \) we must avoid.]

We now define conjugacy classes \( \Theta_{k,g,\chi_g} \in U(n-2)\# \), for each \( \chi_g \in \text{TotRam}(k, g) \), through their reversed characteristic polynomials

\[
\det(1 - T \sqrt{\#k}\Theta_{k,g,\chi_g}) = \det(1 - TFrob_k|H^1_c(A^1[1/g^{pal}] \otimes_k \mathbb{F}, \mathcal{L}_{\chi_g(1-us)})).
\]

With these preliminaries out of the way, we see that we have reduced Theorem 6.4 to the variant of Theorem 5.10 for the conjugacy classes \( \{\Theta_{k,g,\chi_g}\}_{\chi_g \in \text{TotRam}(k, g)} \). To prove this variant, we repeat the proof of Theorem 5.10, but looking at the direct image by \( g^{pal} \) of \( \mathcal{L}_{\chi_g(1-us)} \) (rather than looking at the direct image by \( (-1)^{\deg(g)}g \) of \( \mathcal{L}_{\chi_g(u-t)} \), as we did in proving Theorem 5.10). \( \square \)

\section*{References}


