corrected version of 6.16.6 (pp. 186−189 in katz−sarnak)

Proposition 6.16.6 Fix integers $r \geq 1$ and $N \geq 2$, and denote by

$$\mathbb{1} := \mathbb{1}_r := (1, 1, \ldots, 1) \text{ in } \mathbb{R}^r.$$

For any non−negative Borel measurable function function $g \geq 0$ on $\mathbb{R}^r$, denote by G the non−negative Borel measurable function function $G \geq 0$ defined by the Lebesgue integral

$$G(x) := \int_{[0,\, x(1)]} g(x - t\mathbb{1})dt := |x(1)|\int_{[0,\, 1]} g(x - tx(1)\mathbb{1})dt.$$

Fix an offset vector c in $\mathbb{Z}^r$:

$$1 \leq c(1) < c(2) < \ldots < c(r).$$

For each integer k with $0 \leq k \leq c(1)-1$, $c - k\mathbb{1}$ is again an offset vector, and we have the identity

$$\int_{\mathbb{R}^r} G d\mathrm{Off}\mu(U(N), \text{offsets } c) = \sum_{0 \leq k \leq c(1)-1} \int_{\mathbb{R}^r} g d\nu(c-k\mathbb{1}, U(N)).$$

proof The idea of the proof is that already used in proving 6.12.4, 6.12.6, and 6.14.12, namely to express the integrals involved as integrals over U(N) against Haar measure, and then to show that the integrands coincide on the set $U(N)^{\mathrm{reg}}$ of elements with N distinct eigenvalues.

The definition of $\nu(c, U(N))$ as a direct image from $U(N)\times U(1)$ gives

$$\int_{\mathbb{R}^r} g d\nu(c, U(N)) = \int_{U(N)}\int_{[0,\, 2\pi)} g(\theta(c(1))(e^{-i\varphi}A), \ldots, \theta(c(r))(e^{-i\varphi}A))d(\varphi/2\pi)dA$$

for any offset vector c. The definition of $\mathrm{Off}\mu(U(N), \text{offsets } c)$ as the expected value over U(N) of the measures $\mathrm{Off}\mu(A, U(N), \text{offsets } c)$ gives

$$\int_{\mathbb{R}^r} G d\mathrm{Off}\mu(U(N), \text{offsets } c) = \int_{U(N)} (\int_{\mathbb{R}^r} G d\mathrm{Off}\mu(A, U(N), \text{offsets } c)dA.$$

We will show that for each A in U(N) with N distinct eigenvalues, we have

$$\int_{\mathbb{R}^r} G d\mathrm{Off}\mu(A, U(N), \text{offsets } c)$$

$$= \sum_{0 \leq k \leq c(1)-1} \int_{[0,\, 2\pi)} g(\theta(c(1)-k)(e^{-i\varphi}A), \ldots, \theta(c(r)-k)(e^{-i\varphi}A))d(\varphi/2\pi).$$

To show this, we proceed as follows. Denote by $\varphi(i) := \varphi(i)(A)$ the (non−normalized) angles of A, defined for all i in $\mathbb{Z}$. For each i, let

$$s_i := (N/2\pi)(\varphi(i+1) - \varphi(i))$$

be the i'th normalized spacing of A, and let

$$S_i := (\varphi(i), \varphi(i+1)] \subset U(1)$$

be the half open interval between $\varphi(i)$ and $\varphi(i+1)$. By definition of $\mathrm{Off}\mu(A, U(N), \text{offsets } c)$, we have

$$N\int_{\mathbb{R}^r} G d\mathrm{Off}\mu(A, U(N), \text{offsets } c)$$

$$= \sum_{\ell \bmod N} G(s_{\ell+1} + s_{\ell+2} + \ldots + s_{\ell+c(1)}, \ldots, s_{\ell+1} + s_{\ell+2} + \ldots + s_{\ell+c(r)}).$$

Let us introduce the scalars

$$s_{\ell,a,b} := \sum_{a \leq i \leq b} s_{\ell+i}, \text{ if } a \leq b,$$

$$:= 0 \text{ if } a > b.$$

Then

$$N\int_{\mathbb{R}^r} G d\mathrm{Off}\mu(A, U(N), \text{offsets } c) = \sum_{\ell \bmod N} G(s_{\ell,1,c(1)}, s_{\ell,1,c(2)}, \ldots, s_{\ell,1,c(r)})$$

$$= \sum_{\ell \bmod N} G(s_{\ell,\mathbb{1},c}),$$

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where we denote by $s_{\ell,\mathbb{1},c}$ the vector $(s_{\ell,1,c(1)}, s_{\ell,1,c(2)},\dots, s_{\ell,1,c(r)})$.

Now recall the definition of G in terms of g, to see that

$$G(s_{\ell,\mathbb{1},c}) = \int_{[0,\, s_{\ell,1,c(1)}]} g(s_{\ell,\mathbb{1},c} - t\mathbb{1})dt = \int_{(0,\, s_{\ell,1,c(1)}]} g(s_{\ell,\mathbb{1},c} - t\mathbb{1})dt.$$

We break the interval $(0, s_{\ell,1,c(1)}]$ into c(1) disjoint intervals

$$(0, s_{\ell,1,c(1)}] = (0, \sum_{1 \le i \le c(1)} s_{\ell+i}) = \coprod_{0 \le k \le c(1)-1} (s_{\ell,1,k}, s_{\ell,1,k+1}].$$

Thus we get

$$G(s_{\ell,\mathbb{1},c}) = \sum_{0 \le k \le c(1)-1} \int_{(s_{\ell,1,k},\, s_{\ell,1,k+1}]} g(s_{\ell,\mathbb{1},c} - t\mathbb{1})dt$$

$$= \sum_{0 \le k \le c(1)-1} \int_{(0,\, s_{\ell+k+1}]} g(s_{\ell,\mathbb{1},c} - s_{\ell,1,k}\mathbb{1} - t\mathbb{1})dt.$$

At this point, we observe that we have the relation

$$s_{\ell,\mathbb{1},c} - s_{\ell,1,k}\mathbb{1} = s_{\ell+k,\mathbb{1},c-k\mathbb{1}}.$$

So the previous identity becomes

$$G(s_{\ell,\mathbb{1},c}) = \sum_{0 \le k \le c(1)-1} \int_{[0,\, s_{\ell+k+1}]} g(s_{\ell+k,\mathbb{1},c-k\mathbb{1}} - t\mathbb{1})dt.$$

Summing over ℓ and shifting ℓ by k+1, we obtain

$$N\int_{\mathbb{R}^r} G d\mathrm{Off}\mu(A, U(N), \text{offsets } c)$$

$$= \sum_{0 \le k \le c(1)-1} \sum_{\ell \bmod N} \int_{[0,\, s_\ell]} g(s_{\ell-1,\mathbb{1},c-k\mathbb{1}} - t\mathbb{1})dt.$$

So we are reduced to showing that for each k with $0 \le k \le c(1) - 1$, we have

$$(1/N)\sum_{\ell \bmod N} \int_{[0,\, s_\ell]} g(s_{\ell-1,\mathbb{1},c-k\mathbb{1}} - t\mathbb{1})dt$$

$$= \int_{[0,\, 2\pi)} g(\theta(c(1)-k)(e^{-i\varphi}A),\dots, \theta(c(r)-k)(e^{-i\varphi}A))d(\varphi/2\pi).$$

This is a statement about the offset vector $c-k\mathbb{1}$, so it suffices to treat universally the case when k=0, i.e., to show that for any offset vector c in $\mathbb{Z}^r$ we have

$$(1/N)\sum_{\ell \bmod N} \int_{[0,\, s_\ell]} g(s_{\ell-1,\mathbb{1},c} - t\mathbb{1})dt$$

$$= \int_{[0,\, 2\pi)} g(\theta(c(1))(e^{-i\varphi}A),\dots, \theta(c(r))(e^{-i\varphi}A))d(\varphi/2\pi).$$

To show this, it suffices to show that for each ℓ we have

$$\int_{S_\ell} g(\theta(c(1))(e^{-i\varphi}A),\dots, \theta(c(r))(e^{-i\varphi}A))d(\varphi/2\pi)$$

$$= (1/N)\int_{[0,\, s_\ell]} g(s_{\ell-1,\mathbb{1},c} - t\mathbb{1})dt.$$

But this is a tautology: as φ runs in $(\varphi(\ell), \varphi(\ell+1)]$, $\theta(c)(e^{-i\varphi}A)$ runs from $s_{\ell-1,\mathbb{1},c}$ to $s_{\ell-1,\mathbb{1},c} - s_\ell\mathbb{1}$. QED