FROM CLAUSEN TO CARLITZ: LOW-DIMENSIONAL SPIN GROUPS AND IDENTITIES AMONG CHARACTER SUMS

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Dedicated to Pierre Deligne, with the utmost admiration

1. Introduction

Let k be a finite field, p its characteristic, q its cardinality, and

$$\psi : (k,+) \to \mathbb{Z}[\zeta_p]^\times \subset \mathbb{C}^\times$$

a nontrivial additive character of k. Given an integer $n \geq 1$ and an element $a \in \mathbb{G}_m(k) = k^\times$, the n-variable Kloosterman sum $Kl_n(\psi, k, a)$ is the complex number defined as

$$Kl_n(\psi, k, a) := \sum_{x_1 \ldots x_n = a, \ all \ \ x_i \in k} \psi(\sum_{i=1}^{n} x_i).$$

It was proven by Weil [Weil] in 1948, as a consequence of the Riemann Hypothesis for curves over finite fields, along lines foreseen by Davenport and Hasse [Dav-Ha] in 1934, that, at least if p is odd, one had the estimate

$$|Kl_2(\psi, k, a)| \leq 2q^{1/2}.$$

It was expected that in general one should have

$$|Kl_n(\psi, k, a)| \leq nq^{(n-1)/2},$$

but this was only proven nearly thirty years later, by Deligne [De-ST, 7.4].

In between, there appeared a 1969 result of Carlitz [Car], which did not then seem to fit into any known paradigm. Carlitz proved the expected estimate in the special case when $p = 2$ and $n = 3$. He did this by proving the following identity. Denote by $\psi_{\mathbb{F}_2}$ the unique nontrivial additive character of the field $\mathbb{F}_2$, and take for ψ the additive character

$$\psi := \psi_{\mathbb{F}_2} \circ Trace_{k/\mathbb{F}_2}$$

of k obtained by composition with the trace. With this *particular* choice of ψ, Carlitz proved the striking identity

$$Kl_2(\psi, k, a)^2 = q + Kl_3(\psi, k, a).$$

We will see that there is an analogous identity for $Kl_2(\psi, k, a)^2$ in any characteristic, but in odd characteristic it involves the so-called hypergeometric sums of type $(3, 1)$ (as opposed to 3-variable Kloosterman sums, which are hypergeometric sums of type $(3, 0)$). These identities are reminiscent of Clausen's famous 1828 identity [Clau] for the square of a hypergeometric function, and of Schläfli's (surprisingly) later 1871 identity [Schl, pages 141-142] for the square of a Bessel function. [Indeed, one might speculate that Carlitz was led to his identity by looking for a finite field analogue of Schläfli's.]

To understand these identities, we invoke the theory of hypergeometric sheaves (of which Kloosterman sheaves are a special case), and their properties of rigidity. In terms of these sheaves, what these identities amount to is roughtly the statement that if $\mathcal{H}$ is a *known* hypergeometric sheaf of rank 2, then $Sym^2(\mathcal{H})$ is a *known* hypergeomtric sheaf of rank 3.

In fact, there are some other identities for Kloosterman and hypergeometric sums in (a few) more variables, which come from the "accident" that for $3 \leq n \leq 6$, the spin double covering group of the special orthogonal group $SO(n)$

$$Spin(n) \to SO(n)$$

is a classical group. Here is a table. For $3 \leq n \leq 6$, we list below $Spin(n)$ as a classical group, and we tell how the tautological representation V_n of $SO(n)$ is built out of the "standard" representation std_n of $Spin(n)$ as classical group.

n	$Spin(n)$	V_n
3	$SL(2)$	$Sym^2(std_2)$
4	$SL(2) \times SL(2)$	$std_2 \otimes std_2$
5	$Sp(4)$	$\Lambda^2(std_4)/(triv)$
6	$SL(4)$	$\Lambda^2(std_4)$

The $n = 3$ case leads to formulas of the Carlitz type discussed above. The $n = 4$ case leads to identities for products of two suitable Kloosterman or hypergeometric sums, which are a finite field analogue of Schläfli's formula [Schl, pages 141-142] for the product of Bessel functions. The $n = 4$ case also leads, by the method of tensor induction, to finite field analogues of formulas [Bai2, 2.04, 2.07, and 2.09, on pages 245-246] or [EMOT, 4.3 (3), (4), (5)] for products of the form

$f(x)f(-x)$, with f a suitable Bessel or hypergeometric function. The product formula of Cayley and its later generalizations by Orr and Bailey, cf. [Cay], [Orr], and [BC], do *not* seem to fit into this framework, however.

The $n = 5$ and $n = 6$ cases lead to formulas for "Λ^2" of certain 4-variable sums, a sort of finite field analogue of an investigation begun by Appell, cf. [App, pp. 415-417].

In the cases of $n = 4, 5, 6$, a consideration of the Lie algebra aspects of these accidents leads to some other identities, which do not seem to occur in the classical hypergeometric literature.

Some of this material was presented at Beijing University in July, 2005, at the University of Wisconsin in November, 2005, and at the University of Minnesota in September, 2007. It is a pleasure to thank Dennis Stanton for pointing me to Bailey's 1928 paper [Bai2] with its plethora of hypergeometric identities; examination of [Bai2, 2.04,2.07,2.09] led to section 6 of this paper, and examination of [Bai2, 2.11] led to section 7.

2. Hypergeometric sums and hypergeometric sheaves, cf. [Ka-ESDE, Chapter 8] and [Ka-G_2, section 2]

2.1. **Hypergeometric sums.** Denote by $\mathbb{Q}_{ab}$ the field $\mathbb{Q}$(all roots of unity), say inside $\mathbb{C}$. Let k be a finite field of characteristic p and cardinality q, inside a fixed $\overline{\mathbb{F}_p}$, and ψ a nontrivial $\mathbb{Q}_{ab}^\times$-valued additive character of k. Fix two non-negative integers n and m, at least one of which is nonzero. Let $\chi_1, ..., \chi_n$ be an unordered list of n $\mathbb{Q}_{ab}^\times$-valued multiplicative characters of $k^\times$, some possibly trivial, and not necessarily distinct. We denote by $\mathbb{I}$ the trivial multiplicative character. Let $\rho_1, ..., \rho_m$ be another such list, but of length m. For E/k a finite extension field (inside the fixed $\overline{\mathbb{F}_p}$), denote by ψ_E the nontrivial additive character of E obtained from ψ by composition with the trace map $Trace_{E/k}$, and denote by $\chi_{i,E}$ (resp. $\rho_{j,E}$) the multiplicative character of E obtained from χ_i (resp. ρ_j) by composition with the norm map $Norm_{E/k}$. For $a \in E^\times$, the hypergeometric sum $Hyp(\psi; \chi_i's; \rho_j's)(a, E)$ is the cyclotomic integer defined as follows. Denote by $V(n, m, a)$ the hypersurface in $(\mathbb{G}_m)^n \times (\mathbb{G}_m)^m/E$, with coordinates $x_1, ..., x_n, y_1, ..., y_m$, defined by the equation

$$\prod_i x_i = a \prod_j y_j.$$

Then

$$Hyp(\psi; \chi_i\ 's; \rho_j\ 's)(a, E) :=$$

$$\sum_{V(n,m,a)(E)} \psi_E(\sum_i x_i - \sum_j y_j) \prod_i \chi_{i,E}(x_i) \prod_j \overline{\rho}_{j,E}(y_j).$$

These hypergeometric sums are closely related to monomials in Gauss sums, by multiplicative Fourier transform. Recall that for Λ a multiplicative character of $E^\times$, the Gauss sum $g(\psi_E, \Lambda)$ is defined by

$$g(\psi_E, \Lambda) := \sum_{x \in E^\times} \psi_E(x)\Lambda(x).$$

Thus for Λ the trivial character $\mathbb{I}$, we have

$$-g(\psi_E, \mathbb{I}) = 1.$$

For E/k a finite extension field, and for any multiplicative character Λ of $E^\times$ we have the formula

$$\sum_{a \in E^\times} \Lambda(a) Hyp(\psi; \chi_i\,'s; \rho_j\,'s)(a, E) = \prod_i g(\psi_E, \chi_{i,E}\Lambda) \prod_j g(\overline{\psi_E}, \overline{\rho}_{j,E}\overline{\Lambda}).$$

By Fourier inversion, for each $a \in E^\times$ we have the formula

$$\sum_{\Lambda} \overline{\Lambda}(a) \prod_i g(\psi_E, \chi_{i,E}\Lambda) \prod_j g(\overline{\psi_E}, \overline{\rho}_{j,E}\overline{\Lambda}) = (\#E-1) Hyp(\psi; \chi_i\,'s; \rho_j\,'s)(a, E).$$

2.2. **Hypergeometric sheaves and their trace functions.** Now make the disjointness assumption that no χ_i is a ρ_j. Then for every prime $\ell \neq p$, and every embedding of $\mathbb{Q}_{ab}$ into $\overline{\mathbb{Q}}_\ell$, there exists a geometrically irreducible middle extension $\overline{\mathbb{Q}}_\ell$-sheaf

$$\mathcal{H}(\psi; \chi_i\,'s; \rho_j\,'s)$$

on $\mathbb{G}_m/k$ of rank $Max(n, m)$ and pure of weight $n + m - 1$. It is lisse on $\mathbb{G}_m$ if $n \neq m$, and it is lisse on $\mathbb{G}_m - 1$ if $n = m$. Its trace function on $\mathbb{G}_m$ incarnates the above hypergeometric sums, as follows. For any finite extension E/k and any $a \in E^\times - 1$, (or, if $n \neq m$, for any $a \in E^\times$), we denote by $Frob_{E,a}$ the Frobenius conjugacy class in $\pi_1(\mathbb{G}_m - 1)$ (or, if $n \neq m$, in $\pi_1(\mathbb{G}_m)$) attached to a as an E-valued point of $\mathbb{G}_m - 1$ (or, if $n \neq m$, of $\mathbb{G}_m$).Then we have

$$Trace(Frob_{E,a}|\mathcal{H}(\psi; \chi_i\,'s; \rho_j\,'s)) = (-1)^{n+m-1} Hyp(\psi; \chi_i\,'s; \rho_j\,'s)(a, E).$$

In the "missing" case when $n = m$ and $a = 1$, this formula remains valid if we interpret the left hand side to mean the trace of $Frob_E$ on the stalk of $\mathcal{H}$ at 1, i.e., on the space $\mathcal{H}^{I_1}$ of inertial invariants at the point 1. A hypergeometric sheaf of type $(n, 0)$, $\mathcal{H}(\psi; \chi_i\,'s; \emptyset)$, is called a Kloosterman sheaf of rank n, denoted $\mathcal{K}\ell(\psi; \chi_i\,'s)$. Thus the Kloosterman sum $Kl_n(\psi, k, a)$ is, up to the sign $(-1)^{n-1}$, the trace of $Frob_{k,t}$ on $\mathcal{K}\ell(\psi; \mathbb{I}, ..., \mathbb{I}$ n times).

Because $\mathcal{H}(\psi;\chi_i\,'s;\rho_j\,'s)$ is geometrically irreducible, it is a fortiori arithmetically irreducible. Therefore any auxiliary sheaf $\mathcal{F}$, lisse on $\mathbb{G}_m - 1$ (or on $\mathbb{G}_m$, if $n \neq m$), built out of $\mathcal{H}$ by "an opertion of linear algebra", i.e., by pushing out by a representation of $GL(rank(\mathcal{H}))$ as algebraic group over $\mathbb{Q}$, and any subquotient $\mathcal{G}$ of such an $\mathcal{F}$, is itself arithmetically semisimple, and hence, by Chebotarev, is determined by its traces $Trace(Frob_{E,a}|\mathcal{G})$, as E ranges over all finite extensions of k, and as a runs over $E^\times - 1$ (or over $E^\times$, if $n \neq m$).

Given a hypergeometric sheaf $\mathcal{H}(\psi;\chi_i\,'s;\rho_j\,'s)$ on $\mathbb{G}_m/k$ and a Kummer sheaf $\mathcal{L}_\Lambda$ attached to a multiplicative character Λ of k, we have an isomorphism

$$\mathcal{L}_\Lambda \otimes \mathcal{H}(\psi;\chi_i\,'s;\rho_j\,'s) \cong \mathcal{H}(\psi;\Lambda\chi_i\,'s;\Lambda\rho_j\,'s),$$

because the two sides have, by inspection, the same traces.

Under multiplicative inversion, we have

$$inv^\star\mathcal{H}(\psi;\chi_i\,'s;\rho_j\,'s)) \cong \mathcal{H}(\overline{\psi};\overline{\rho}_j\,'s;\overline{\chi}_i\,'s)).$$

The linear dual of $\mathcal{H} := \mathcal{H}(\psi;\chi_i\,'s;\rho_j\,'s)$ is given by

$$\mathcal{H}(\psi;\chi_i\,'s;\rho_j\,'s)^\vee \cong \mathcal{H}(\overline{\psi};\overline{\chi}_i\,'s;\overline{\rho}_j\,'s)(n+m-1).$$

If $((\prod_j \rho_j)/(\prod_i \chi_i))(-1) = 1$, a condition which always holds over the quadratic extension of k, then multiplicative translation $t \mapsto (-1)^{n-m}t$ carries $\mathcal{H}(\psi;\chi_i\,'s;\rho_j\,'s)$ into $\mathcal{H}(\overline{\psi};\chi_i\,'s;\rho_j\,'s)$. More generally, for A the constant

$$A := ((\prod_j \rho_j)/(\prod_i \chi_i))(-1) = \pm 1,$$

and A^{deg} the corresponding geometrically constant lisse sheaf of rank one, we have

$$[t \mapsto (-1)^{n-m}t]^\star\mathcal{H}(\psi;\chi_i\,'s;\rho_j\,'s) \cong \mathcal{H}(\overline{\psi};\chi_i\,'s;\rho_j\,'s)\bigotimes A^{deg}.$$

2.3. **Local monodromy at 1, if n=m.** If $n = m$, the local monodromy of $\mathcal{H}$ at 1, i.e., its restriction to the inertia group I_1 at the point 1, is a tame pseudoreflection, whose determinant is $(\prod_j \rho_j)/(\prod_i \chi_i)$, viewed as a tame character of I_1. Here we view multiplicative characters of $k^\times$ as characters of I_1^{tame} as follows. First we use additive translation to identify I_1 with I_0. Then we view multiplicative characters of $k^\times$ as characters of I_0^{tame} in two steps, as follows. First use the inclusion (which is in fact an isomorphism)

$$I_0^{tame} \subset \pi_1^{tame}(\mathbb{G}_m \otimes \overline{k}),$$

and then use the isomorphism (given by the Lang torsor)

$$\pi_1^{tame}(\mathbb{G}_m \otimes \overline{k}) \cong \mathrm{liminv}_{\mathrm{E/k,Norm_{E/k}}}\mathrm{E}^\times.$$

2.4. **Local monodromy at 0, when $\mathbf{n \geq m}$.** Let us now suppose in addition that $n \geq m$, a situation we can always achieve by multiplicative inversion. Local monodromy at 0 is tame, and the action of a generator γ_0 of I_0^{tame} is the action of T on the $\overline{\mathbb{Q}}_\ell[T]$-module $\overline{\mathbb{Q}}_\ell[T]/(P(T))$, for $P(T)$ the polynomial

$$P(T) := \prod_i (T - \chi_i(\gamma_0)).$$

Here we use view multiplicative characters χ_i of $k^\times$ as characters of I_0^{tame} via the inclusion, in fact an isomorphism,

$$I_0^{tame} \subset \pi_1^{tame}(\mathbb{G}_m \otimes \overline{k}) \cong \mathrm{liminv}_{\mathrm{E/k,Norm_{E/k}}} \mathrm{E}^\times.$$

2.5. **Local monodromy at $\boldsymbol{\infty}$, when $\mathbf{n \geq m}$.** Local monodromy at ∞ is the direct sum of an m-dimensional tame summand, say $Tame_m$, and, if $n > m$, a totally wild summand,

$$Wild(\psi; \chi_i\ 's; \rho_j\ 's),$$

of dimension $n-m$, Swan conductor 1, and all upper numbering breaks equal to $1/(n-m)$. On $Tame_m$, the action of a generator γ_∞ of I_∞^{tame} is the action of T on the $\overline{\mathbb{Q}}_\ell[T]$-module $\overline{\mathbb{Q}}_\ell[T]/(Q(T))$, for $Q(T)$ the polynomial

$$Q(T) := \prod_j (T - \rho_j(\gamma_\infty)).$$

Here we use view multiplicative characters ρ_j of $k^\times$ as characters of I_∞^{tame} via the inclusion, again an isomorphism,

$$I_\infty^{tame} \subset \pi_1^{tame}(\mathbb{G}_m \otimes \overline{k}) \cong \mathrm{liminv}_{\mathrm{E/k,Norm_{E/k}}} \mathrm{E}^\times.$$

The isomorphism class of any totally wild representation of I_∞ with Swan conductor 1, is determined, up to *unique* multiplicative translation on $\mathbb{G}_m \otimes \overline{k}$, by its rank $n-m$ and its determinant. More precisely, the isomorphism class of

$$Wild(\psi; \chi_i\ 's; \rho_j\ 's)$$

depends only on the triple (the character ψ, the rank $n-m$, the character $(\prod_i \chi_i)/(\prod_j \rho_j)$), cf. [Ka-ESDE, 8.6.4].] We denote by

$$Wild_{n-m}(\psi, \alpha),\ \alpha := (\prod_i \chi_i)/(\prod_j \rho_j),$$

one representation in this isomorphism class. When $n-m \geq 2$, then $det(Wild_{n-m}(\psi, \alpha)) = \alpha$.

For any $d \geq 1$, we obtain a choice of $Wild_d(\psi, \alpha)$ as the I_∞-representation attached to any Kloosterman sheaf $\mathcal{K}\ell(\psi; \chi_1, ..., \chi_d)$ of type $(d, 0)$ with $\prod_i \chi_i = \alpha$.

If d is prime to the characteristic p, then denoting by $[d] : \mathbb{G}_m \to \mathbb{G}_m$ the d'th power map, and by $\mathcal{L}_{\psi_d}$ the Artin Schreier sheaf attached to the character $\psi_d(x) := \psi(dx)$, one knows [Ka-GKM, 5.6.2] that the direct image $[d]_\star\mathcal{L}_{\psi_d}$ is geometrically isomorphic to the Kloosterman sheaf of rank d

$$\mathcal{K}\ell(\psi; \text{all characters of order dividing } d).$$

Its $Wild_d$ thus has trivial determinant if d is odd, and has determinant the quadratic character χ_2 if d is even. But as d is prime to the characteristic p, any multiplicative character α has, over a possibly bigger finite field, a d'th root. So $Wild_d(\psi, \alpha)$ is the tensor product of a tame character with (the restriction to I_∞ of) $[d]_\star\mathcal{L}_{\psi_d}$. More precisely, we have

$$Wild_d(\psi, \alpha) \cong [d]_\star(\mathcal{L}_\alpha \otimes \mathcal{L}_{\psi_d} \text{ if d is odd},$$

$$Wild_d(\psi, \alpha) \cong [d]_\star(\mathcal{L}_{\alpha\chi_2} \otimes \mathcal{L}_{\psi_d} \text{ if d is even},$$

where we write χ_2 for the quadratic character. Therefore the pullback $[d]^\star Wild_d(\psi, \alpha)$ has a known and explicit structure as a representation of the inertia group I_∞: it is the restriction to I_∞ of

$$\mathcal{L}_{\alpha(x)} \otimes \bigoplus_{\zeta \in \mu_d(\overline{k})} \mathcal{L}_{\psi(\zeta dx)}, \text{ if d is odd},$$

$$\mathcal{L}_{(\alpha\chi_2)(x)} \otimes \bigoplus_{\zeta \in \mu_d(\overline{k})} \mathcal{L}_{\psi(\zeta dx)}, \text{ if d is even}.$$

What is the structure of $Wild_d(\psi, \alpha)$ when d is not prime to p, e.g., when d is a power q of p? In this case, one knows [Ka-GKM, 1.14, 1.14.2, 1.15] that the restriction of $Wild_q(\psi, \alpha)$ to the wild inertia group P_∞ is absolutely irreducible. In fact, this *restriction* $Wild_q(\psi, \alpha)|P_\infty$ detects multiplicative translations, as the following (slightly more general) lemma (combined with [Ka-ESDE, 8.6.4]) shows, applied to two multiplicative translates M and N of $Wild_q(\psi, \alpha)$.

Lemma 2.5.1. *Let k be a finite field, $\ell \neq p$ a prime, q a power of p, M and N two absolutely irreducible $\overline{\mathbb{Q}}_\ell$-representations of I_∞, both of dimension q. Suppose that $det(M) = det(N)$ as characters of I_∞. Suppose also that there exists an isomorphism of P_∞-representations*

$$M|P_\infty \cong N|P_\infty.$$

Then there exists an isomorphism of I_∞-representations

$$M \cong N.$$

Proof. If $q = 1$, there is nothing to prove, since $M = det(M)$ and $N = det(N)$. Suppose now that $q > 1$. By [Ka-GKM, 1.15], both $M|P_\infty$ and $N|P_\infty$ are absolutely irreducible. Hence the internal hom $\mathcal{H}om_{P_\infty}(M, N)$ is a one-dimensional representation of I_∞ on which P_∞ acts trivially, so is a Kummer sheaf $\mathcal{L}_\chi$ for some tame character χ. The canonical I_∞-equivariant composition map

$$\mathcal{H}om_{P_\infty}(M, N) \otimes M \to N$$

is an isomorphism, so we have

$$\mathcal{L}_\chi \otimes M \cong N.$$

Comparing determinants, we conclude that $\chi^q = \mathbb{I}$ is the trivial character. As q is a power of p, χ itself is trivial. □

2.6. **Frobenius action on inertial invariants.**

Lemma 2.6.1. *Let k be a finite field, $\ell \neq p$ a prime, and $\mathcal{H} := \mathcal{H}(\psi; \chi_i\ 's; \rho_j\ 's)$ on $\mathbb{G}_m/k$ a hypergeometric sheaf of type (n, m). Suppose that the trivial character $\mathbb{I}$ is among the χ_i (respectively among the ρ_j). Then the space $\mathcal{H}^{I_0}$ of I_0-invariants (respectively the space $\mathcal{H}^{I_\infty}$ of I_∞-invariants) is one-dimensional, and $Frob_k$ acts on it as the monomial in gauss sums*

$$Trace(Frob|\mathcal{H}^{I_0}) = \prod_{i=1}^{n}(-g(\psi, \chi_i)) \prod_{j=1}^{m}(-g(\overline{\psi}, \overline{\rho_j})).$$

Proof. By multiplicative inversion, it suffices to treat the I_0 case. That $\mathcal{H}^{I_0}$ is one-dimensional follows from the structure of the I_0-representation recalled above, that each character that occurs in it occurs in a single Jordan block. As $\mathbb{I}$ occurs at 0, it does not occur at ∞, by disjointness, and hence $\mathcal{H}^{I_\infty} = 0$. So if we denote by

$$j : \mathbb{G}_m \to \mathbb{P}^1$$

the inclusion, we have a short exact sequence of sheaves on $\mathbb{P}^1$,

$$0 \to j_!\mathcal{H} \to j_\star\mathcal{H} \to \mathcal{H}^{I_0} \text{ conc. at } 0 \to 0.$$

We claim that all the cohomology groups $H^i(\mathbb{P}^1 \otimes \overline{k}, j_\star\mathcal{H})$ vanish. To see this, notice that $\mathcal{H}$ is not only geometrically irreducible, but when it has rank one it is geometrically nonconstant (because it is ramified at ∞, for type $(1, 0)$,or at 0 , for type $(0, 1)$or at 1, for type $(1, 1)$. Hence both the H^0 and the H^2 vanish, so it suffices to see that the Euler characteristic vanishes. But this is obvious from the above short exact sequence, since $\mathcal{H}$ has Euler characteristic -1. So from the long exact sequence of cohomology, we get a coboundary isomorphism

$$\mathcal{H}^{I_0} \cong H^1_c(\mathbb{G}_m \otimes \overline{k}, \mathcal{H}),$$

while all the other groups $H^i_c(\mathbb{G}_m \otimes \overline{k}, \mathcal{H})$, $i \neq 1$, vanish. By the Lefschetz Trace Formula applied to $\mathcal{H}$, we have

$$Trace(Frob_k | H^1_c(\mathbb{G}_m \otimes \overline{k}, \mathcal{H})) = -\sum_{t \in k^\times} Trace(Frob_{k,t}|\mathcal{H}) =$$

$$-\sum_{t \in k^\times} (-1)^{n+m-1} Hyp(\psi; \chi_i' s; \rho_j' s)(t,k) = \prod_{i=1}^{n} (-g(\psi, \chi_i)) \prod_{j=1}^{m} (-g(\overline{\psi}, \overline{\rho_j})),$$

this last equality by the relation, recalled above, of hypergeometric sums to monomials in Gauss sums. □

2.7. **Rigidity properties.** We next recall from [Ka-ESDE, 8.5.3.1] the fundamental rigidity result for hypergeometric sheaves of type (n,m). Consider first the case when $n > m$. Suppose we are given a lisse, geometrically irreducible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on $\mathbb{G}_m/k$ of rank $n \geq 1$ which is tame at 0 and has $Swan_\infty(\mathcal{F}) = 1$. Suppose in addition that all the tame characters which occur in the local monodromy at both 0 and ∞ are characters of $k^\times$, i.e., have order dividing $\#k^\times$. Then $\mathcal{F}$ is geometrically isomorphic to a unique multiplicative translate, by an element of $\mathbb{G}_m(k) = k^\times$, of a unique hypergeometric sheaf $\mathcal{H}(\psi; \chi_i\,'s; \rho_j\,'s)$ on $\mathbb{G}_m/k$ of type (n,m) for some $m < n$.

What about the case $n = m$? Suppose we are given a geometrically irreducible middle extension sheaf on $\mathbb{G}_m$ of rank $n \geq 1$ which is lisse on $\mathbb{G}_m - 1$, tame at both 0 and ∞, and whose local monodromy at 1 is a tame pseudoreflection. Suppose in addition that all the tame characters which occur in the local monodromy at both 0 and ∞ are characters of $k^\times$, i.e., have order dividing $\#k^\times$. Then $\mathcal{F}$ is geometrically isomorphic to a unique hypergeometric sheaf $\mathcal{H}(\psi; \chi_i\,'s; \rho_j\,'s)$ on $\mathbb{G}_m/k$ of type (n,n). [In the case $n = m$, the geometric isomorphism class is independent of the choice of ψ.]

Suppose now we are given a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on $\mathbb{G}_m/k$ of rank $n \geq 1$ which is is tame at 0 and has $Swan_\infty(\mathcal{F}) = 1$, but which is not assumed to be geometrically irreducible. Suppose that all the tame characters $\chi_i, i = 1, ..., n$ occuring in the local monodromy at 0 , respectively all the tame characters $\rho_j, j = 1, ..., m$ with $m < n$ at ∞ are characters of $k^\times$, and make the disjointness assumption that no χ_i is a ρ_j. Then in fact $\mathcal{F}$ is geometrically irreducible, hence is geometrically isomorphic to a unique multiplicative translate, by an element of $\mathbb{G}_m(k) = k^\times$, of the hypergeometric sheaf $\mathcal{H}(\psi; \chi_i\,'s; \rho_j\,'s)$ of type (n,m) on $\mathbb{G}_m/k$. Indeed, denote by $Wild(\mathcal{F})$ the wild part of the I_∞-representation of $\mathcal{F}$. Work over $\overline{k}$, and form the Jordan-Holder constituents, say $\mathcal{F}_i$, of $\mathcal{F}$ as lisse sheaf on $\mathbb{G}_m/\overline{k}$. Since $Wild(\mathcal{F})$ has Swan conductor 1, it is irreducible, and so is contained in precisely one of the Jordan-Holder constituents, say $\mathcal{F}_0$. The other Jordan-Holder constituents are then tame at both 0 and ∞, so are necessarily Kummer sheaves. But the existence of such a Kummer sheaf constituent contradicts the disjointness assumption that no χ_i is a ρ_j. Therefore $\mathcal{F}$ is geometrically irreducible, as asserted.

Similarly, suppose we are given a middle extension sheaf $\mathcal{F}$ on $\mathbb{G}_m/k$ which is lisse on $\mathbb{G}_m - 1$ and whose local monodromy at 1 is a tame pseudoreflection, but which is not assumed to be geometrically irreducible. Suppose that all the tame characters $\chi_i, i = 1, ..., n$ occuring in the local monodromy at 0 , respectively all the tame characters $\rho_j, j = 1, ..., n$ at ∞ are characters of $k^\times$, and make the disjointness assumption that no χ_i is a ρ_j. Then $\mathcal{F}$ is geometrically irreducible. The key point is that $\mathcal{F}$ is a middle extension sheaf on $\mathbb{G}_m$ whose Euler characteristic is -1. One knows the Euler characteristic of any middle extension sheaf on $\mathbb{G}_m$ is a negative integer, and that it is zero if and only if the sheaf is a successive extension of Kummer sheaves. Then conclude as above that among the Jordan-Holder constituents, there is precisely one whose Euler characteristic is -1, and that there can be no others, since these would be Kummer sheaves, not allowed by the disjointness.

2.8. **A twisting principle.** To end this section, we recall the following general principle, valid on any smooth, geometrically connected k-scheme U/k. Suppose $\mathcal{F}$ and $\mathcal{G}$ are geometrically irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaves on U which are geometrically isomorphic. The there exists a unit $A \in \overline{\mathbb{Q}_\ell}^\times$ such that after tensoring with the geometrically constant lisse rank one sheaf A^{deg}, we have an arithmetic isomorphism $\mathcal{F} \cong \mathcal{G} \otimes A^{deg}$.

3. The Carlitz identity

Theorem 3.1 (Carlitz)**.** *Denote by ψ the unique nontrivial additive character of the field $\mathbb{F}_2$. Then we have the following two equivalent results.*

(1) *For every odd prime ℓ, we have an isomorphism of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/\mathbb{F}_2$,*

$$Sym^2(\mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I})) \cong \mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I}).$$

(2) *For every finite extension field $E/\mathbb{F}_2$, and for every $t \in E^\times$, we have the Carlitz identity of Kloosterman sums*

$$Kl_2(\psi_E, E, t)^2 = \#E + Kl_3(\psi_E, E, t).$$

Proof. The first assertion implies the second, by passing to traces, and remembering [Ka-GKM, 7.4.1.2] that $det(\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I})) \cong \overline{\mathbb{Q}}_\ell(-1)$.The second implies the first, by Chebotarev, which guarantees that both sides have isomorphic semisimplifications. But both sides are arithmetically semisimple. Indeed, both $\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I}))$ and $\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I},\mathbb{I})$ are geometrically irreducible, hence arithmetically irreducible. The arithmetic semisimplicity of $\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I})$ implies that of anything built out of it by linear algebra, in particular $Sym^2(\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I}))$.

To prove the first assertion, we argue as follows. We know [Ka-GKM, 11.1] that the geometric monodromy group of $\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I}))$ is $SL(2)$. Therefore $Sym^2(\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I}))$ is geometrically irreducible, with geometric monodromy group $SO(3)$. Its local monodromy at 0 is $Unip(3)$:= a single unipotent Jordan block of size 3, simply because the local monodromy at 0 of $\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I}))$ is $Unip(2)$.

What about the local monodromy at ∞ of $Sym^2(\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I}))$? The local monodromy at ∞ of $\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I})$ has both slopes $1/2$, so

$$Swan_\infty(Sym^2(\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I}))) \leq 3/2.$$

But Swan conductors are nonnegative integers, so $Swan_\infty(Sym^2)$ is 0 or 1. It cannot be 0, otherwise $Sym^2(\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I}))$ is a lisse sheaf on $\mathbb{G}_m/\overline{k}$ which is tame at both 0 and ∞, so is a successive extension of rank one Kummer sheaves, so cannot be geometrically irreducible. Therefore $Swan_\infty(Sym^2(\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I}))) = 1$, and all the ∞-slopes are $\leq \mathbb{I}/2$. So either $Sym^2(\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I}))$ is totally wild at ∞, with all ∞-slopes $1/3$, or its I_∞-representation is the direct sum of a tame character and a totally wild part of rank 2 with both ∞-slopes $1/2$. We now show this second possibility cannot arise. Indeed, in this second case, rigidity would show that, geometrically, $Sym^2(\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I}))$ is a multiplicative translate of a hypergeometric sheaf of type $(3,1)$ of the shape $\mathcal{H}(\psi;\mathbb{I},\mathbb{I},\mathbb{I};\chi)$ for some nontrivial (by disjointness) χ. But this sheaf has geometric monodromy group SO(3), in particular is self dual, and this is possible only if $\chi = \overline{\chi}$. But in characteristic 2, the only such χ is the trivial character.

[Here is a second argument to show that $Sym^2(Wild_2(\psi,\mathbb{I}))$ is totally wild in characteristic 2. Since $\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I}))$ is geometrically self dual, $Wild_2(\psi,\mathbb{I})|P_\infty$ is self dual. So it is the same to show that $\mathcal{E}nd^0(Wild_2(\psi,\mathbb{I}))$ is totally wild, i.e., to show that the space of P_∞-invariants in $\mathcal{E}nd(Wild_2(\psi,\mathbb{I}))$ is one-dimensional. And this is just

Schur's lemma, because in characteristic2, $Wild_2(\psi, \mathbb{I})|P_\infty$ is absolutely irreducible.]

Therefore the local monodromy at ∞ of $Sym^2(\mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I}))$ is totally wild, with all ∞-slopes 1/3. So by rigidity, it is geometrically isomorphic to a $\mathbb{G}_m(\mathbb{F}_2)$-translate of $\mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I})$. Since $\mathbb{G}_m(\mathbb{F}_2)$ consists only of the identity element, we conclude that $Sym^2(\mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I}))$ and $\mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I})$ are geometrically isomorphic. As both are geometrically irreducible, there exists a unit $A \in \overline{\mathbb{Q}_\ell}^\times$ such that we have an arithmetic isomorphism

$$Sym^2(\mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I})) \cong \mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I}) \otimes A^{deg}.$$

It remains to show that $A = 1$. For this, we argue as follows. The "half" Tate-twisted sheaf $\mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I})(1/2)$ has $G_{arith} = G_{geom} = SL(2)$, and the Tate-twisted sheaf $\mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I})(1)$ has $G_{arith} = G_{geom} = SO(3)$, cf. [Ka-GKM, 11.1 and 11.3]. Hence both $Sym^2(\mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I}))(1)$ and $\mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I})(1)$ have $G_{arith} = G_{geom} = SO(3)$. Therefore A is a scalar which lies in$SO(3)$, so $A = 1$. ☐

Corollary 3.2. *For ψ the nontrivial additive character of $\mathbb{F}_2$, we have isomorphisms of I_∞-representations*

$$Sym^2(Wild_2(\psi; \mathbb{I})) \cong Wild_3(\psi, \mathbb{I}),$$

$$Wild_2(\psi; \mathbb{I}) \otimes Wild_2(\psi; \mathbb{I}) \cong Wild_3(\psi, \mathbb{I}) \oplus \overline{\mathbb{Q}}_\ell.$$

Proof. Restrict the isomorphism of part (1) of the theorem to the inertia group at ∞ to obtain the first assertion, and remember that $det(Wild_2(\psi; \mathbb{I})) = \mathbb{I}$ to deduce from it the second. ☐

We can now make use of the above corollary to prove a slight generalization of the theorem. Henceforth, we will not spell out the trace identities which result from, and are indeed equivalent to (by Chebotarev, cf. 2.2), arithmetic isomorphisms of arithmetically semisimple lisse sheaves.

Theorem 3.3. *Let k be a finite field of characteristic 2, and ψ_k the nontrivial additive character of k obtained from the ψ mof the previous theorem by composition with the trace. Let χ be any multiplicative character of $k^\times$, $\overline{\chi}$ the inverse character. Then for every odd prime ℓ, we have an isomorphism of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,*

$$Sym^2(\mathcal{K}\ell(\psi_k; \chi, \overline{\chi})) \cong \mathcal{K}\ell(\psi_k; \mathbb{I}, \chi^2, \overline{\chi}^2).$$

Proof. By the above corollary, and the fact that the I_∞-representation of $\mathcal{K}\ell(\psi_k; \chi, \overline{\chi})$ is independent of the choice of χ, we know we know that local monodromy at ∞ of $Sym^2(\mathcal{K}\ell(\psi_k; \chi, \overline{\chi}))$ is $Wild_3(\psi, \mathbb{I})$. Since

$Wild_3(\psi, \mathbb{I})$ is irreducible, we see that $Sym^2(\mathcal{K}\ell(\psi_k; \chi, \overline{\chi}))$ must be geometrically irreducible. Then by rigidity, we conclude that it is geometrically isomorphic to some $k^\times$-translate of $\mathcal{K}\ell(\psi_k; \mathbb{I}, \chi^2, \overline{\chi}^2)$. Looking again at the I_∞-representations, which detect multiplicative translations, we conclude that the two sheaves in question are geometrically isomorphic. As both are geometrically irreducible, there exists a unit $A \in \overline{\mathbb{Q}_\ell}^\times$ such that we have an arithmetic isomorphism

$$Sym^2(\mathcal{K}\ell(\psi_k; \chi, \overline{\chi})) \cong \mathcal{K}\ell(\psi_k; \mathbb{I}, \chi^2, \overline{\chi}^2) \otimes A^{deg}.$$

We then show that $A = 1$ as follows. Whatever the choice of χ, the half Tate-twisted sheaf $\mathcal{K}\ell(\psi_k; \chi, \overline{\chi})(1/2)$ has both its G_{geom} and its G_{arith} inside $SL(2)$, cf. [Ka-GKM, 7.4.1.2]. Therefore the Tate-twisted sheaf $Sym^2(\mathcal{K}\ell(\psi_k; \chi, \overline{\chi}))(1)$ has both its G_{geom} and its G_{arith} inside $S)(3)$. So the Tate-twisted sheaf $\mathcal{K}\ell(\psi_k; \mathbb{I}, \chi^2, \overline{\chi}^2)(1)$ has its G_{geom} inside $SO(3)$. This sheaf is orthogonally self dual, so its G_{arith} lies in $O(3)$. But again by [Ka-GKM, 7.4.1.2], its G_{arith} lies in $SL(3)$. So its G_{arith} lies in $SO(3)$. Thus again A is a scalar which lies in $SO(3)$, so $A = 1$. □

Here is a slightly more general restatement of this last theorem.

Corollary 3.4. *Let k be a finite field of characteristic 2, and ψ_k the nontrivial additive character of k obtained from the ψ mof the previous theorem by composition with the trace. Let Λ and ρ be any multiplicative characters of $k^\times$. Then for every odd prime ℓ, we have an isomorphism of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,*

$$Sym^2(\mathcal{K}\ell(\psi_k; \Lambda, \rho)) \cong \mathcal{K}\ell(\psi_k; \Lambda\rho, \Lambda^2, \rho^2).$$

Proof. Since we are in characteristic 2, every multiplicative character has a unique square root. So we can find characters η and χ of $k^\times$ such that $(\Lambda, \rho) = (\eta\chi, \eta\overline{\chi})$. Then we have $\mathcal{K}\ell(\psi_k; \Lambda, \rho) \cong \mathcal{L}_\eta \otimes \mathcal{K}\ell(\psi_k; \chi, \overline{\chi})$, and the result now follows by applying the previous corollary to $\mathcal{K}\ell(\psi_k; \chi, \overline{\chi})$. □

What happens in odd characteristic? Here $\mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I})$ is replaced by a hypergeometric sheaf of type $(3, 1)$, which involves the quadratic character (and hence "makes no sense" in characteristic 2).

Theorem 3.5. *Let k be a finite field of odd characteristic p, and ψ a nontrivial additive character of k. Denote by χ_2 the quadratic character, i.e., the unique character of $k^\times$ of order 2. Then for every prime $\ell \neq p$, we have an isomorphism of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$*

$$Sym^2(\mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I})) \cong [x \mapsto 4x]^\star \mathcal{H}(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I}; \chi_2) \otimes A^{deg},$$

for A (minus) the reciprocal Gauss sum

$$A := 1/(-g(\overline{\psi},\chi_2)),\ \ g(\overline{\psi},\chi_2) := \sum_{x\in k^\times} \overline{\psi(x)}\chi_2(x).$$

Proof. We apply rigidity to the sheaf $Sym^2(\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I}))$. Since G_{geom} for $\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I}))$ is $SL(2)$, it follows that $Sym^2(\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I}))$ is geometrically irreducible, with G_{geom} the group $SO(3)$. Its local monodromy at 0 is $Unip(3)$. To analyze its local monodromy at ∞, we pull back by the squaring map [2], to find, as representation of I_∞,

$$[2]^\star Sym^2(\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I})) \cong [2]^\star Sym^2([2]_\star(\mathcal{L}_{\chi_2(x)}\otimes\mathcal{L}_{\psi(2x)})$$

$$\cong Sym^2([2]^\star[2]_\star(\mathcal{L}_{\chi_2(x)}\otimes\mathcal{L}_{\psi(2x)})) \cong Sym^2(\mathcal{L}_{\chi_2(x)}\otimes(\mathcal{L}_{\psi(2x)}\oplus\mathcal{L}_{\psi(-2x)}))$$

$$\cong \mathcal{L}_{\psi(4x)}\oplus\mathcal{L}_{\psi(-4x)}\oplus\overline{\mathbb{Q}_\ell}.$$

Thus we see that $Sym^2(\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I}))$ is a hypergeometric of type $(3,1)$, geometrically isomorphic to a multiplicative translate of $\mathcal{H}(\psi;\mathbb{I},\mathbb{I},\mathbb{I};\rho)$ for some nontrivial ρ. But as this Sym^2 is geometrically self dual, we must have $\rho=\chi_2$. By rigidity, we then know that for some $t\in\overline{k}^\times$, there exists a geometric isomorphism

$$Sym^2(\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I})) \cong [x\mapsto tx]^\star\mathcal{H}(\psi;\mathbb{I},\mathbb{I},\mathbb{I};\chi_2).$$

We must show that $t=4$. For this, write $t=s^2$, for some $s\in\overline{k}^\times$. Comparing the wild parts of the pullbacks by [2] of the I_∞-representations, we find

$$\mathcal{L}_{\psi(4x)}\oplus\mathcal{L}_{\psi(-4x)} \cong [2]^\star[x\mapsto s^2x]^\star Wild_2(\psi,\chi_2)$$

$$\cong [x\mapsto sx]^\star[2]^\star[2]_\star\mathcal{L}_{\psi(2x)} \cong [x\mapsto sx]^\star(\mathcal{L}_{\psi(2x)}\oplus\mathcal{L}_{\psi(-2x)})$$

$$\cong \mathcal{L}_{\psi(2sx)}\oplus\mathcal{L}_{\psi(-2sx)}.$$

Thus $s=\pm 2$, and so $t=4$.

So there exists a geometric isomorphism

$$Sym^2(\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I})) \cong [x\mapsto 4x]^\star\mathcal{H}(\psi;\mathbb{I},\mathbb{I},\mathbb{I};\chi_2).$$

As both sides are geometrically isomorphic, there exists a a unit $A\in\overline{\mathbb{Q}_\ell}^\times$ such that we have an arithmetic isomorphism

$$Sym^2(\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I})) \cong [x\mapsto 4x]^\star\mathcal{H}(\psi;\mathbb{I},\mathbb{I},\mathbb{I};\chi_2)\otimes A^{deg}.$$

We must show that $A=1/(-g(\overline{\psi},\chi_2))$. Tate-twisting, we have an arithmetic isomorphism

$$Sym^2(\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I}))(1) \cong [x\mapsto 4x]^\star\mathcal{H}(\psi;\mathbb{I},\mathbb{I},\mathbb{I};\chi_2)(1)\otimes A^{deg}.$$

But now the source $Sym^2(\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I}))(1)$ has $G_{geom}=G_{arith}=SO(3)$ (because $\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I})(1/2)$ has $G_{geom}=G_{arith}=SL(2)$). On the other hand, the Tate-twisted target $[x \mapsto 4x]^\star \mathcal{H}(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I}; \chi_2)(3/2)$ is orthogonally self dual, so also

$$[x \mapsto 4x]^\star \mathcal{H}(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I}; \chi_2)(1) \otimes (1/(-g(\overline{\psi}, \chi_2)))^{deg}$$

is orthogonally self dual. But by [Ka-ESDE, 8.12.2], this last sheaf has arithmetically trivial determinant, so has its $G_{arith} \subset SO(3)$. So the ratio of A to $1/(-g(\overline{\psi}, \chi_2))$ is a scalar in $SO(3)$, so is 1. □

Corollary 3.6. *Let k be a finite field of odd characteristic p, and ψ a nontrivial additive character of k. Denote by χ_2 the quadratic character. We have isomorphisms of representations of I_∞*

$$Sym^2(Wild_2(\psi, \mathbb{I})) \cong \mathcal{L}_{\chi_2} \oplus [x \mapsto 4x]^\star Wild_2(\psi, \chi_2),$$

$$Wild_2(\psi, \mathbb{I}) \otimes Wild_2(\psi, \mathbb{I}) \cong \mathbb{I} \oplus \mathcal{L}_{\chi_2} \oplus [x \mapsto 4x]^\star Wild_2(\psi, \chi_2),$$

Proof. Restrict the isomorphism of the previous theorem to the inertia group at ∞ to obtain the first assertion, and use the fact that $det(Wild_2(\psi, \mathbb{I})) = \mathbb{I}$ to infer from it the second. □

Using this corollary, we now give a slight generalization of the theorem.

Theorem 3.7. *Let k be a finite field of odd characteristic p, and ψ a nontrivial additive character of k. Denote by χ_2 the quadratic character. Let ρ be any multiplicative character of $k^\times$, $\overline{\rho}$ the inverse character. Suppose that*

$$\rho^2 \neq \chi_2.$$

Then for every prime $\ell \neq p$, we have an isomorphism of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$

$$Sym^2(\mathcal{K}\ell(\psi; \rho, \overline{\rho})) \cong [x \mapsto 4x]^\star \mathcal{H}(\psi; \mathbb{I}, \rho^2, \overline{\rho}^2; \chi_2) \otimes A^{deg},$$

for A (minus) the reciprocal Gauss sum

$$A := 1/(-g(\overline{\psi}, \chi_2)), \;\; g(\overline{\psi}, \chi_2) := \sum_{x \in k^\times} \overline{\psi(x)} \chi_2(x).$$

Proof. The previous corollary identifies the I_∞ representation attached to $Sym^2(\mathcal{K}\ell(\psi; \rho, \overline{\rho}))$, and shows that the only tame character in it is χ_2. By the assumption that $\rho^2 \neq \chi_2$, we may form the geometrically irreducbile hypergeometric sheaf $\mathcal{H}(\psi; \mathbb{I}, \rho^2, \overline{\rho}^2; \chi_2)$, and conclude that $Sym^2(\mathcal{K}\ell(\psi; \rho, \overline{\rho}))$ is geometrically isomorphic to a multiplicative translate of it. Exactly as above we compute the multiplicative translate. We compute the constant A as above, again using the fact that

$$[x \mapsto 4x]^\star \mathcal{H}(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I}; \chi_2)(1) \otimes (1/(-g(\overline{\psi}, \chi_2)))^{deg}$$

is orthogonally self dual, and, by [Ka-ESDE, 8.12.2], it has arithmetically trivial determinant, so has its $G_{arith} \subset SO(3)$. ☐

Here is a slightly more general restatement of this last theorem.

Corollary 3.8. *Let k be a finite field of odd characteristic p, and ψ a nontrivial additive character of k. Denote by χ_2 the quadratic character. Let Λ and ρ be any multiplicative characters of $k^\times$. Suppose that*

$$\rho \neq \chi_2\Lambda.$$

Then for every prime $\ell \neq p$, we have an isomorphism of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,

$$Sym^2(\mathcal{K}\ell(\psi;\Lambda,\rho)) \cong [x \mapsto 4x]^\star\mathcal{H}(\psi;\Lambda\rho,\Lambda^2,\rho^2;\rho\Lambda\chi_2) \otimes A^{deg},$$

for

$$A := 1/(\rho(4)\Lambda(4)(-g(\overline{\psi},\chi_2))).$$

Proof. If the character $\Lambda\rho$ had a square root (as a character of $k^\times$), as it automatically did in characteristic 2, then just as in characteristic 2 this would result from the previous $(\rho,\overline{\rho})$ theorem by the same twisting argument. In any case, such a square root exists over at most a quadratic extension of k, so we do get a geometric isomorphism between $Sym^2(\mathcal{K}\ell(\psi;\Lambda,\rho))$ and $[x \mapsto 4x]^\star\mathcal{H}(\psi;\Lambda\rho,\Lambda^2,\rho^2;\rho\Lambda\chi_2)$. Hence for some unit A we get an isomorphism

$$Sym^2(\mathcal{K}\ell(\psi;\Lambda,\rho)) \cong [x \mapsto 4x]^\star\mathcal{H}(\psi;\Lambda\rho,\Lambda^2,\rho^2;\rho\Lambda\chi_2) \otimes A^{deg},$$

and the problem remains to evaluate A. To do this, we may, by tame twisting, reduce to the case when at least one of the two characters Λ and ρ is trivial, say $\Lambda = \mathbb{I}$, and other,ρ, is not χ_2.

To treat this case, we argue as follows. Since $\rho \neq \chi_2$, we see that in the I_0 representation of $Sym^2(\mathcal{K}\ell(\psi_k;\mathbb{I},\rho))$, the unipotent part is a single Jordan block (of dimension 2 if ρ is trivial, and otherwise of dimension 1), and hence the space of I_0-invariants is one-dimensional. Since we have an a priori inclusion of invariants,

$$Sym^2((\mathcal{K}\ell(\psi;\mathbb{I},\rho))^{I_0}) \subset (Sym^2(\mathcal{K}\ell(\psi;\mathbb{I},\rho)))^{I_0},$$

and both sides are one-dimensional, this inclusion is an isomorphism, and we get the identity

$$Trace(Frob_k|Sym^2(\mathcal{K}\ell(\psi;\mathbb{I},\rho))^{I_0}) = (Trace(Frob_k|\mathcal{K}\ell(\psi;\mathbb{I},\rho)))^{I_0})^2$$
$$= (-g(\psi,\rho))^2.$$

On the other hand, the I_0-invariants in

$$[x \mapsto 4x]^\star\mathcal{H}(\psi;\mathbb{I},\rho,\rho^2;\rho\chi_2) \otimes A^{deg}$$

are one-dimension, with $Frob_k$-eigenvalue

$$(-g(\psi,\rho))(-g(\psi,\rho^2))(-g(\overline{\psi},\overline{\rho\chi_2}))A.$$

Equate these two expressions for the $Frob_k$-eigenvalue, rewrite $g(\psi,\rho^2) = \rho^2(2)g(\psi_2,\rho^2)$, and use the Hasse-Davenport identity [Dav-Ha, 0.9I]

$$g(\psi_2,\rho^2)g(\psi,\chi_2) = g(\psi,\rho)g(\psi,\rho\chi_2)$$

to obtain the asserted value of A. □

4. The original Clausen isomorphism

The hypergeometric sheaf analogue of a generalized hypergeometric function

$${}_nF_{m-1}(a_1,...,a_n;b_1,...,b_{m-1};x)$$

in which no a_i is either an integer or differs from any b_j by an integer is a (geometrically irreducible) hypergeometric sheaf

$$\mathcal{H}(\psi;\mathbb{I},\beta_1,...,\beta_{m-1};\alpha_1,...,\alpha_n)$$

of type (m,n), with the disjointness condition that no α_i is either $\mathbb{I}$ or any β_j. In 1828, Clausen [Clau] proved the identity for hypergeometric functions

$${}_2F_1(a,b;a+b+1/2;x)^2 = {}_3F_2(2a,a+b,2b;a+b+1/2,2a+2b;x).$$

The corresponding statement for hypergeometric sheaves in odd characteristic is this.

Theorem 4.1. *Let k be a finite field of odd characteristic p, ψ a nontrivial additive character of k, and α,β two multiplicative characters of $k^\times$, neither of which is $\mathbb{I}$ or χ_2, and such that $\alpha\beta \neq \mathbb{I}$ and $\alpha \neq \beta\chi_2$. Put*

$$\gamma := \alpha\beta\chi_2$$

. Then for any $\ell \neq p$ we have an isomorphism of middle extension sheaves on $\mathbb{G}_m/k$

$$Sym^2(\mathcal{H}(\psi;\mathbb{I},\gamma;\alpha,\beta)) \cong \mathcal{H}(\psi;\mathbb{I},\gamma,\gamma^2;\alpha^2,\beta^2,\alpha\beta)\otimes A^{deg},$$

for A the constant

$$A = -\#kg(\psi,\gamma)g(\psi,\alpha^2)g(\psi,\beta^2)g(\psi,\alpha\beta)/g(\psi,\gamma^2)g(\psi,\alpha)^2g(\psi,\beta)^2.$$

Proof. The hypotheses on α and β insure that both $\mathcal{H}(\psi;\mathbb{I},\gamma;\alpha,\beta)$ and $\mathcal{H}(\psi;\mathbb{I},\gamma,\gamma^2;\alpha^2,\beta^2,\alpha\beta)$ are irreducible hypergeometric sheaves. The ratio $\gamma/\alpha\beta$ is χ_2, hence the local monodromy around 1 of $\mathcal{H}(\psi;\mathbb{I},\gamma;\alpha,\beta)$ is a true reflection. Hence the Sym^2 has its local monodromy around 1 *also* a true reflection. Also the ratio $\gamma^3/\alpha^3\beta^3$ is χ_2, hence the local monodromy around 1 of $\mathcal{H}(\psi;\mathbb{I},\gamma,\gamma^2;\alpha^2,\beta^2,\alpha\beta)$ is also a true reflection. Now both sides of the asserted isomorphism have isomorphic local monodromies around each of 0, ∞, and 1, so the two sides are, by rigidity, geometrically isomorphic. We then compute the constant A by comparing the expressions of the $Frob_k$-eigenvalue on the one-dimensional space of I_0-invariants on the two sides. □

In characteristic 2, we get a degenerate analogue of the Clausen isomorphism, by erasing the term $\gamma := \alpha\beta\chi_2$ that "doesn't make sense". For ease of exposition, we interchange 0 and ∞ as well.

Theorem 4.2. *Let k be a finite field of characteristic 2, ψ the nontrivial additive character of k obtained by composition with the trace from the nontrivial additive character of the prime field $\mathbb{F}_2$. Let α,β be two multiplicative characters of $k^\times$, such that α,β, and $\alpha\beta$ are all nontrivial. Then for any odd ℓ we have an isomorphism of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$*

$$Sym^2(\mathcal{H}(\psi;\alpha,\beta;\mathbb{I})) \cong \mathcal{H}(\psi;\alpha^2,\beta^2,\alpha\beta;\mathbb{I},\alpha^2\beta^2)\otimes A^{deg},$$

for A the constant

$$A = g(\psi,\alpha)g(\psi,\beta)/\#k.$$

Proof. The I_∞-representation of $\mathcal{H}(\psi;\alpha,\beta;\mathbb{I})$ is

$$\mathbb{I}\bigoplus Wild_1(\psi,\alpha\beta) \cong \mathbb{I}\bigoplus \mathcal{L}_{\alpha\beta}\otimes\mathcal{L}_\psi.$$

Because we are in characteristic 2, $\mathcal{L}_\psi\otimes\mathcal{L}_\psi$ is trivial, so the I_∞-representation of $Sym^2(\mathcal{H}(\psi;\alpha,\beta;\mathbb{I}))$ is

$$\mathbb{I}\bigoplus\mathcal{L}_{\alpha^2\beta^2}\bigoplus\mathcal{L}_{\alpha\beta}\otimes\mathcal{L}_\psi.$$

This is also the I_∞-representation of $\mathcal{H}(\psi;\alpha^2,\beta^2,\alpha\beta;\mathbb{I},\alpha^2\beta^2)$, so by rigidity the two sides are geometrically isomorphic. We then compute the constant A by comparing the expressions of the $Frob_k$-eigenvalue on the one-dimensional space of I_0-invariants on the two sides, remembering that for our choice of ψ, we have $g(\psi,\chi) = g(\psi,\chi^2)$ for any multiplicative character χ. □

If we twist both sides by the square of a Kummer sheaf $\mathcal{L}_\gamma$, and rescale α and β, we get the following slight generalization.

Theorem 4.3. *Let k be a finite field of characteristic 2, ψ the nontrivial additive character of k obtained by composition with the trace from the nontrivial additive character of the prime field $\mathbb{F}_2$. Let α,β,γ be multiplicative characters of $k^\times$, such that $\alpha\neq\gamma,\beta\neq\gamma$, and $\alpha\beta\neq\gamma^2$.*

Then for any odd ℓ *we have an isomorphism of lisse* $\overline{\mathbb{Q}}_\ell$*-sheaves on* $\mathbb{G}_m/k$

$$Sym^2(\mathcal{H}(\psi;\alpha,\beta;\gamma)) \cong \mathcal{H}(\psi;\alpha^2,\beta^2,\alpha\beta;\gamma^2,\alpha^2\beta^2/\gamma^2)\otimes A^{deg},$$

for A *the constant*

$$A = g(\psi,\alpha/\gamma)g(\psi,\beta/\gamma)/\#k.$$

5. The $SL(2)\times SL(2)$ case

We begin with characteristic 2, where, as always, the results are a bit easier.

Theorem 5.1. *Let* k *be a finite field of characteristic* 2*, and* ψ_k *the nontrivial additive character of* k *obtained from the unique nontrivial* ψ *of the the prime field* $\mathbb{F}_2$ *by composition with the trace. Let* χ *and* ρ *be any multiplicative characters of* $k^\times$*,* $\overline{\chi}$ *and* $\overline{\rho}$*the inverse characters. Suppose that*

$$\chi \neq \rho, \chi \neq \overline{\rho}.$$

Then we have the following two equivalent results.

(1) *For every odd prime* ℓ*, we have an isomorphism of lisse* $\overline{\mathbb{Q}}_\ell$*-sheaves on* $\mathbb{G}_m/k$*,*

$$\mathcal{K}\ell(\psi_k;\chi,\overline{\chi})\otimes\mathcal{K}\ell(\psi_k;\rho,\overline{\rho}) \cong \mathcal{H}(\psi_k;\chi\rho,\overline{\chi}\rho,\chi\overline{\rho},\overline{\chi\rho};\mathbb{I})(1).$$

(1bis) *For every odd prime* ℓ*, we have an isomorphism of lisse* $\overline{\mathbb{Q}}_\ell$*-sheaves on* $\mathbb{G}_m/k$*,*

$$\mathcal{K}\ell(\psi_k;\mathbb{I},\chi^2)\otimes\mathcal{K}\ell(\psi_k;\mathbb{I},\rho^2) \cong \mathcal{H}(\psi_k;\mathbb{I},\chi^2,\rho^2,\chi^2\rho^2;\chi\rho)\otimes A^{deg}.$$

Proof. The assertions (1) and (1bis) are equivalent; we pass from (1) to (1bis) by tensoring with the Kummer sheaf $\mathcal{L}_\chi\otimes\mathcal{L}_\rho \cong \mathcal{L}_{\chi\rho}$. To prove (1), we remark that under the hypotheses on χ and ρ, none of the four characters $(\chi\rho,\overline{\chi}\rho,\chi\overline{\rho},\overline{\chi\rho})$ is trivial, so the geometrically irreducible hypergeometric sheaf on the right exists. By the corollary in section 3 on $Sym^2(Wild_2(\psi,\mathbb{I}))$, we see that both sides of the alleged isomorphism are lisse sheaves with isomorphic I_0 and I_∞-representations. Then by the lemma in section 2, the left hand side is itself geometrically irreducible. Hence one side is an A^{deg} twist of some multiplicative translate of the other. But the wild parts of the I_∞-representations of the two sides are already isomorphic, and each has $Swan_\infty = 1$, so no multiplicative translation is needed, both sides are already geometrically isomorphic. So we have an isomorphism

$$\mathcal{K}\ell(\psi_k;\chi,\overline{\chi})\otimes\mathcal{K}\ell(\psi_k;\rho,\overline{\rho}) \cong \mathcal{H}(\psi_k;\chi\rho,\overline{\chi}\rho,\chi\overline{\rho},\overline{\chi\rho};\mathbb{I})\otimes A^{deg}.$$

It remains to show that $A = 1/\#k$. For this, we pass to form (1bis). Twisting both sides of the above isomorphism by the Kummer sheaf $\mathcal{L}_\chi \otimes \mathcal{L}_\rho \cong \mathcal{L}_{\chi\rho}$, we obtain an isomorphism

$$\mathcal{K}\ell(\psi_k; \mathbb{I}, \chi^2) \otimes \mathcal{K}\ell(\psi_k; \mathbb{I}, \rho^2) \cong \mathcal{H}(\psi_k; \mathbb{I}, \chi^2, \rho^2, \chi^2\rho^2; \chi\rho) \otimes A^{deg}.$$

We obtain the asserted value of A by computing the $Frob_k$-eigenvalue on the one-dimensional space of I_0-invariants on the two sides. We obtain the equality

$$(-g(\psi_k, \chi^2))((-g(\psi_k, \rho^2))$$
$$= (-g(\psi_k, \chi^2))((-g(\psi_k, \rho^2))((-g(\psi_k, \chi^2\rho^2))(-\overline{g(\psi_k, \chi\rho)})A.$$

Now use the fact that for any multiplicative character Λ, here $\chi\rho$, our choice of ψ_k as coming from the prime field, leads, for $p = 2$ to the identity

$$g(\psi_k, \Lambda) = g(\psi_k, \Lambda^2).$$

□

Because in characteristic 2 every multiplicative character has a unique square root, Kummer twisting gives the following slight generalization of the previous result.

Theorem 5.2. *Let k be a finite field of characteristic 2, and ψ_k the nontrivial additive character of k obtained from the unique nontrivial ψ of the the prime field $\mathbb{F}_2$ by composition with the trace. Let $\chi, \rho, \Lambda, \beta$ be any multiplicative characters of $k^\times$, and let η be the unique character with*

$$\eta^2 = \chi\rho\Lambda\beta.$$

Suppose that none of $\chi\Lambda, \chi\beta, \rho\Lambda, \rho\beta$ is η. Then for every odd prime ℓ, we have an isomorphism of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,

$$\mathcal{K}\ell(\psi_k; \chi, \rho) \otimes \mathcal{K}\ell(\psi_k; \Lambda, \beta) \cong \mathcal{H}(\psi_k; \chi\Lambda, \chi\beta, \rho\Lambda, \rho\beta; \eta)(1).$$

What happens in odd characteristic? Here we obtain a finite field analogue of Schläfli's identity, cf. [Schl, pages 141-142] and [Bai2, 2.03 on page 245].

Theorem 5.3. *Let k be a finite field of odd characteristic p, ψ a nontrivial additive character of k. Let $\alpha, \rho, \Lambda, \beta, \eta$ be multiplicative characters of $k^\times$. Suppose that*

$$\eta^2 = \alpha\rho\Lambda\beta,$$

and that none of $\alpha\Lambda, \alpha\beta, \rho\Lambda, \rho\beta$ is either η or $\eta\chi_2$. Then for each prime $\ell \neq p$, we have an isomorphism of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,

$$\mathcal{K}\ell(\psi_k; \alpha, \rho) \otimes \mathcal{K}\ell(\psi_k; \Lambda, \beta)$$

$$\cong [x \mapsto 4x]^{\star}\mathcal{H}(\psi_k; \alpha\Lambda, \alpha\beta, \rho\Lambda, \rho\beta; \eta, \eta\chi_2)(1) \otimes A^{deg},$$

for A the constant

$$A = 1/(\eta^2(2)(-g(\overline{\psi}, \chi_2)).$$

Proof. To obtain a geometrical isomorphism between the two sides, we may extend scalars and assume in addition that both $\alpha\rho$ and $\Lambda\beta$ are squares. Then tensoring with a suitable Kummer sheaf reduces us to the case where

$$\alpha\rho = \Lambda\beta = \mathbb{I}.$$

In this case the assertion is that there exists a geometric isomorphism

$$\mathcal{K}\ell(\psi_k; \alpha, \overline{\alpha}) \otimes \mathcal{K}\ell(\psi_k; \beta, \overline{\beta})$$
$$\cong [x \mapsto 4x]^{\star}\mathcal{H}(\psi_k; \alpha\beta, \alpha/\beta, \beta/\alpha, 1/\alpha\beta; \mathbb{I}, \chi_2).$$

This is immediate from rigidity and the odd p corollary in section 3.

Return now to the general case. Now reduce to the case when

$$\alpha = \beta = \mathbb{I},$$

by tensoring both sides with the Kummer sheaf $\mathcal{L}_{\overline{\alpha}} \otimes \mathcal{L}_{\overline{\beta}}$ and replacing η by $\eta\overline{\alpha}\overline{\beta}$. So we have an isomorphism

$$\mathcal{K}\ell(\psi_k; \mathbb{I}, \rho) \otimes \mathcal{K}\ell(\psi_k; \mathbb{I}, \Lambda)$$
$$\cong [x \mapsto 4x]^{\star}\mathcal{H}(\psi_k; \mathbb{I}, \Lambda, \rho, \rho\Lambda; \eta, \eta\chi_2)(1) \otimes A^{deg},$$

with $\eta^2 = \rho\Lambda$. To evaluate A, we again compute the $Frob_k$-eigenvalue on the one-dimensional space of I_0-invariants on the two sides, and equate the expressions. We get

$$g(\psi, \rho)g(\psi, \Lambda) = (A/\#k)g(\psi, \rho)g(\psi, \Lambda)g(\psi, \rho\Lambda)g(\overline{\psi}, \overline{\eta})g(\overline{\psi}, \overline{\eta\chi_2}).$$

Using the Hasse-Davenport relation

$$g(\overline{\psi}, \overline{\eta})g(\overline{\psi}, \overline{\eta\chi_2}) = g(\overline{\psi}, \chi_2)g(\overline{\psi_2}, \overline{\eta^2})$$
$$= \eta^2(2)g(\overline{\psi}, \chi_2)g(\overline{\psi}, \overline{\eta^2}),$$

we find the asserted value of A. □

6. Another $SL(2) \times SL(2)$ case, via tensor induction

We refer to [C-R-MRT, 13] and to [Ev] for background on tensor induction, and to [Ka-ESDE, 10.3-10.6] for instances of its application to hypergeometric sheaves (although the case we will be considering in this section was not discussed there).

We work over a finite field k of odd characteristic p. Then the squaring map

$$[2] : \mathbb{G}_m/k \to \mathbb{G}_m/k$$

makes the source into a finite etale ± 1-torsor over the target. Given a lisse sheaf $\mathcal{F}$ on the source $\mathbb{G}_m/k$, the tensor product $\mathcal{F} \otimes [x \mapsto -x]^\star \mathcal{F}$ has a preferred descent, called the tensor induction,

$$[2]_{\otimes\star}\mathcal{F}$$

in the notation of [Ka-ESDE, 10.5.1]. It is a lisse sheaf on the target that with the property under Kummer pullback $[2]^\star$, we have

$$[2]^\star [2]_{\otimes\star}\mathcal{F} \cong \mathcal{F} \otimes [x \mapsto -x]^\star \mathcal{F}.$$

The trace function of the tensor induction is given by the following recipe.

Proposition 6.1. *Let E/k be a finite extension, $t \in E^\times$. Then we have the following formulas.*

(1) *If t is a square in E, say $t = s^2$ with $s \in E$, then we have the product formula*

$$Trace(Frob_{E,s^2}|[2]_{\otimes\star}\mathcal{F}) = Trace(Frob_{E,s}|\mathcal{F})Trace(Frob_{E,-s}|\mathcal{F}).$$

(2) *If t is not a square in E, then the square roots $\pm s$ of t generate the quadratic extension E_2/E, and we have the formula*

$$Trace(Frob_{E,t}|[2]_{\otimes\star}\mathcal{F}) = Trace(Frob_{E_2,s}|\mathcal{F}) = Trace(Frob_{E_2,-s}|\mathcal{F}).$$

Proof. The first formula is just the spelling out of the isomorphism

$$[2]^\star [2]_{\otimes\star}\mathcal{F} \cong \mathcal{F} \otimes [x \mapsto -x]^\star \mathcal{F}$$

at the E-valued point s of the source. In the second formula, the last equality results from the fact that $\mathcal{F}$ is defined, by pullback from $\mathbb{G}_m/k$, on $\mathbb{G}_m/E$, so that E-conjugate points (here s and $-s$) in a finite extension (here E_2) have the same trace. To prove the first formula, first use the base change property [Ka-ESDE, 10.5.3] of tensor induction, here by the map $t : Spec(E) \to \mathbb{G}_m$ toward the target, to reduce to a question about tensor induction $\otimes Ind_{E_2}^E$ from $Gal(\overline{k}/E_2)$ to $Gal(\overline{k}/E)$. The key point then is that if V is a representation of $Gal(\overline{k}/E_2$, whose $Frob_{E_2}$-eigenvalues are $\alpha_1, ..., \alpha_{n=dim(V)}$, then the n^2 eigenvalues of $Frob_E$ on the tensor induction $\otimes Ind_{E_2}^E(V)$ are the same $\alpha_1, ..., \alpha_n$, together with, for each $i < j$, both square roots of $\alpha_i\alpha_j$, cf. [Ka-ESDE, 10.4.5 with n=2]. Consequently, we have the equality of traces

$$Trace(Frob_{E_2}|V) = Trace(Frob_E|Ind_{E_2}^E(V)),$$

which is the asserted equality. $\square$

Remark 6.2. If instead of 2 we consider the ℓ_0-power map $[\ell_0]$ for any prime ℓ_0, then so long as we work over a finite field k of characteristic $p \neq \ell_0$ which contains all the ℓ_0'th roots of unity, we have the two analogous formulas for $Trace(Frob_{E,t}|[\ell_0]_{\otimes\star}\mathcal{F})$, now depending on whether or not t is an ℓ_0'th power in E. We leave their elaboration to the reader.

The following result is the finite field analogue of a classical identity, cf. [Bai2, 2.04 on page 245].

Theorem 6.3. *Let k be a finite field of odd characteristic p, ψ a nontrivial additive character of k. Let α, β, ρ be multiplicative characters of $k^\times$ such that*

$$\rho^2 = \alpha\beta.$$

Then for each prime $\ell \neq p$, we have an isomorphism of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,

$$[2]_{\otimes\star}\mathcal{K}\ell(\psi;\alpha,\beta) \cong [x \mapsto -x/4]^\star\mathcal{K}\ell(\psi;\alpha,\beta,\rho,\chi_2\rho) \otimes A^{deg},$$

for A the constant

$$A = (\alpha\beta)(2)/(-g(\psi,\chi_2)).$$

Proof. Let us consider the lisse, rank four sheaf

$$\mathcal{F} := [2]_{\otimes\star}\mathcal{K}\ell(\psi;\alpha,\beta).$$

Our first task is to show that it is a Kloosterman sheaf. We know that its Kummer pullback by [2] is the tensor product

$$[2]^\star\mathcal{F} \cong \mathcal{K}\ell(\psi;\alpha,\beta) \otimes [x \mapsto -x]^\star\mathcal{K}\ell(\psi;\alpha,\beta).$$

Pulling back again by [2], and comparing I_∞-representations on both sides, we see that $[4]^\star\mathcal{F}$ is the sum of four characters, each of Swan conductor 1. Hence $\mathcal{F}$ as I_∞-representation is totally wild, with Swan conductor 1. Its I_0-representation is tame, and the tame characters occurring in it are precisely $\alpha, \beta, \rho, \chi_2\rho$; this is a special case of [Ka-ESDE, 10.6.3 and its elaboration 10.6.5(1)]. Hence by rigidity [Ka-GKM, 8.7.1], $\mathcal{F}$ is a Kloosterman sheaf, and there exists a geometric isomorphism of $\mathcal{F}$ with some multiplicative translate of $\mathcal{K}\ell(\psi;\alpha,\beta,\rho,\chi_2\rho)$. Looking again at the I_∞-representation of $[4]^\star\mathcal{F}$, we see that the multiplicative translate is as asserted.

To compute the constant A, we argue as follows. We first twist by a Kummer sheaf to reduce to the case when $\beta = \mathbb{I}$. More precisely, for $\Lambda := \alpha/\beta$, we have

$$\mathcal{K}\ell(\psi;\alpha,\beta) \cong \mathcal{L}_\beta \otimes \mathcal{K}\ell(\psi;\Lambda,\mathbb{I}).$$

By the "additivity" of tensor induction [Ka-ESDE, 10.3.2(i)], we have

$$[2]_{\otimes\star}\mathcal{K}\ell(\psi;\alpha,\beta) \cong ([2]_{\otimes\star}\mathcal{L}_\beta)\otimes([2]_{\otimes\star}\mathcal{K}\ell(\psi;\Lambda,\mathbb{I})).$$

Using the trace formula for a tensor induction, we see that

$$[2]_{\otimes\star}\mathcal{L}_\beta \cong \mathcal{L}_\beta\otimes(\beta(-1))^{deg}.$$

Let us denote $\gamma := \rho/\beta$, so that

$$\gamma^2 = \Lambda.$$

Thus we obtain, for our unknown A, an isomorphism

$$[2]_{\otimes\star}\mathcal{K}\ell(\psi;\alpha,\beta) \cong \mathcal{L}_\beta\otimes(\beta(-1))^{deg}\otimes[x\mapsto -x/4]^\star\mathcal{K}\ell(\psi;\Lambda,\mathbb{I},\gamma,\chi_2\gamma)\otimes A^{deg}$$
$$\cong (\beta(4))^{deg}\otimes[x\mapsto -x/4]^\star(\mathcal{L}_\beta\otimes\mathcal{K}\ell(\psi;\Lambda,\mathbb{I},\gamma,\chi_2\gamma)\otimes A^{deg})$$
$$\cong (\beta(4))^{deg}\otimes[x\mapsto -x/4]^\star\mathcal{K}\ell(\psi;\alpha,\beta,\rho,\chi_2\rho)\otimes A^{deg}.$$

So it suffices to treat the case when $\beta=\mathbb{I}$ (and $\gamma^2=\alpha$). In this case, we willl compute the $Frob_k$-eigenvalue of the space of inertial invariants at 0. We know that

$$[2]_{\otimes\star}\mathcal{K}\ell(\psi;\alpha,\mathbb{I}) \cong [x\mapsto -x/4]^\star\mathcal{K}\ell(\psi;\alpha,\mathbb{I},\gamma,\chi_2\gamma)\otimes A^{deg},$$

and that

$$[2]^\star[2]_{\otimes\star}\mathcal{K}\ell(\psi;\alpha,\mathbb{I}) \cong \mathcal{K}\ell(\psi;\alpha,\mathbb{I})\otimes[x\mapsto -x]^\star\mathcal{K}\ell(\psi;\alpha,\mathbb{I}).$$

Suppose first that $\alpha\neq\chi_2$ and $\alpha\neq\mathbb{I}$. Then in the tensor product

$$\mathcal{K}\ell(\psi;\alpha,\mathbb{I})\otimes[x\mapsto -x]^\star\mathcal{K}\ell(\psi;\alpha,\mathbb{I}),$$

the space of I_0-invariants is one-dimensional, and the $Frob_k$-eigenvalue on it is $(-g(\psi,\alpha))^2$.The space of I_0-invariants in

$$[2]_{\otimes\star}\mathcal{K}\ell(\psi;\alpha,\mathbb{I}) \cong [x\mapsto -x/4]^\star\mathcal{K}\ell(\psi;\alpha,\mathbb{I},\gamma,\chi_2\gamma)\otimes A^{deg}$$

is one dimensional, and, as we have just noted, remains so after pullback by [2]. So we can read its $Frob_k$-eigenvalue after pullback by [2]. Thus this eigenvalue is $(-g(\psi,\alpha))^2$. On the other hand, this eigenvalue is

$$A(-g(\psi,\alpha))(-g(\psi,\gamma))(-g(\psi,\chi_2\gamma)).$$

Remembering that $\alpha=\gamma^2$, we obtain the asserted value of A using the Hasse-Davenport formula.

In the case when $\alpha=\chi_2$, the space of I_0-invariants in the tensor product

$$\mathcal{K}\ell(\psi;\alpha,\mathbb{I})\otimes[x\mapsto -x]^\star\mathcal{K}\ell(\psi;\alpha,\mathbb{I}),$$

is two dimensional, but $Frob_k$ nonetheless acts on it by the scalar $(-g(\psi,\alpha))^2$, and again we conclude by using the Hasse-Davenport formula.

In the case when $\alpha = \mathbb{I}$, the space of I_0-invariants in

$$[x \mapsto -x/4]^\star \mathcal{K}\ell(\psi; \alpha, \mathbb{I}, \gamma, \chi_2\gamma) = [x \mapsto -x/4]^\star \mathcal{K}\ell(\psi; I, \mathbb{I}, \mathbb{I}, \chi_2)$$

is one-dimensional, with $Frob_k$-eigenvalue of weight 1, namely $-g(\psi, \chi_2)$. After pullback by [2], we get a second I_0-invariant, whose weight is 3 (because in $\mathcal{K}\ell(\psi; I, \mathbb{I}, \mathbb{I}, \chi_2)$, χ_2 occurs just once, cf. [Ka-GKM, 7.0.7 (2) and 7.0.8 (2)]). Looking at

$$\mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I}) \otimes [x \mapsto -x]^\star \mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I}),$$

we see that the I_0-representation is unipotent, with Jordan blocks of sizes 3 and 1. On the 2-dimensional space of its I_0-invariants, we have $Frob_k$-eigenvalues of weights 0 and 2, namely 1 and $\#k$. Since the 3-dimensional unipotent block here must be the [2] pullback of the 3-dimensional unipotent block in $[x \mapsto -x/4]^\star \mathcal{K}\ell(\psi; I, \mathbb{I}, \mathbb{I}, \chi_2)$, we conclude that it is on both sides the $Frob_k$-eigenvalues of lowest weights which must match, i.e., $1 = (-g(\psi, \chi_2))A$. □

Let us spell out the trace formulas which result, the first of which is the finite field analogue of a classical identity, cf. [Bai2, 2.04 on page 245] or [EMOT, 4.3 (3)].

Corollary 6.4. *Let k be a finite field of odd characteristic p, ψ a nontrivial additive character of k, and $\ell \neq p$ a prime. Let α, β, ρ be multiplicative characters of $k^\times$ such that*

$$\rho^2 = \alpha\beta.$$

Then for each each finite extension field E/k, and for each $t \in E^\times$, we have the following formulas.

(1) *If $t = s^2$ with $s \in E$, then*

$$Trace(Frob_{E,-s^2/4} | \mathcal{K}\ell(\psi; \alpha, \beta, \rho, \chi_2\rho))/(-g(\psi_E, \chi_{2,E})$$
$$= Trace(Frob_{E,s} | \mathcal{K}\ell(\psi; \alpha, \beta)) Trace(Frob_{E,-s} | \mathcal{K}\ell(\psi; \alpha, \beta)).$$

(2) *If t is a nonsquare in E, denote by E_2/E the quadratic extension, and let $s \in E_2$ have $s^2 = t$. Then*

$$Trace(Frob_{E,-t/4} | \mathcal{K}\ell(\psi; \alpha, \beta, \rho, \chi_2\rho))/(-g(\psi_E, \chi_{2,E})$$
$$= Trace(Frob_{E_2,s} | \mathcal{K}\ell(\psi; \alpha, \beta)).$$

There is another theorem of the same flavor, but now for hypergeometrics of type $(2, 1)$ instead of $(2, 0)$, which is the finite field analogue of the classical identities [Bai2, 2.07 and 2.09 on pages 245-246], or [EMOT, 4.3 (4) and (5)]. Bailey attributes the second of these to Ramanujan, cf. [Har, page 503, line 6] where the statement is slightly different.

Theorem 6.5. *Let k be a finite field of odd characteristic p, ψ a nontrivial additive character of k. Let $\alpha, \beta, \gamma, \rho$ be multiplicative characters of $k^\times$ such that*

$$\rho^2 = \alpha\beta$$

and such that

$$\alpha \neq \gamma, \beta \neq \gamma, \alpha\beta \neq \gamma^2.$$

Then for each prime $\ell \neq p$, we have an isomorphism of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,

$$[2]_{\otimes\star}\mathcal{H}(\psi; \alpha, \beta; \gamma) \cong [x \mapsto x/4]^\star\mathcal{H}(\psi; \alpha, \beta, \rho, \chi_2\rho; \gamma, \alpha\beta/\gamma) \otimes A^{deg},$$

for A the constant

$$A = (\alpha\beta)(2)\gamma(-1)(-g(\psi, \alpha/\gamma))(-g(\psi, \beta/\gamma))/(\#k)(-g(\psi, \chi_2)).$$

Proof. Again we consider the lisse, rank four sheaf

$$\mathcal{F} := [2]_{\otimes\star}\mathcal{H}(\psi; \alpha, \beta; \gamma).$$

Our first task is to show that it is a hypergeometric sheaf of type $(4, 2)$. The I_∞-representation of $\mathcal{H}(\psi; \alpha, \beta; \gamma)$ is the direct sum

$$\mathcal{L}_\gamma \oplus (\mathcal{L}_{\alpha\beta/\gamma} \otimes \mathcal{L}_\psi).$$

So the I_∞-representation of $[2]^\star\mathcal{F}$ is

$$(\mathcal{L}_\gamma \oplus (\mathcal{L}_{\alpha\beta/\gamma} \otimes \mathcal{L}_\psi)) \otimes (\mathcal{L}_\gamma \oplus (\mathcal{L}_{\alpha\beta/\gamma} \otimes \mathcal{L}_{\overline{\psi}}))$$

$$\cong \mathcal{L}_{\gamma^2} \oplus \mathcal{L}_{(\alpha\beta/\gamma)^2} \oplus (\mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_\psi) \oplus (\mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\overline{\psi}}).$$

Thus this I_∞-representation has $Swan_\infty = 2$. Hence $Swan_\infty(\mathcal{F}) = 1$, and its wild part is of dimension 2.

We next compute the tame part of the I_∞-representation of $\mathcal{F}$. In virtue of [Ka-GKM, 10.6.2], we can do this by a global argument: the I_∞-representation of $\mathcal{F}$ is the same as that of

$$[2]_{\otimes\star}(\mathcal{L}_\gamma \oplus (\mathcal{L}_{\alpha\beta/\gamma} \otimes \mathcal{L}_\psi)).$$

By a basic property of tensor induction [Ka-ESDE, 10.3.2 (2ter)], this last object contains the direct sum of the tensor inductions of the individual summands

$$[2]_{\otimes\star}(\mathcal{L}_\gamma) \bigoplus [2]_{\otimes\star}(\mathcal{L}_{\alpha\beta/\gamma} \otimes \mathcal{L}_\psi)).$$

As already noted above, the trace property of tensor induction shows that

$$[2]_{\otimes\star}(\mathcal{L}_\gamma) \cong \mathcal{L}_\gamma \otimes (\gamma(-1))^{deg}.$$

As for the second factor, by "additivity" we have

$$[2]_{\otimes\star}(\mathcal{L}_{\alpha\beta/\gamma} \otimes \mathcal{L}_\psi)) \cong ([2]_{\otimes\star}(\mathcal{L}_{\alpha\beta/\gamma})) \otimes ([2]_{\otimes\star}(\mathcal{L}_\psi)).$$

The trace property of tensor induction shows that

$$[2]_{\otimes\star}(\mathcal{L}_\psi) \cong \overline{\mathbb{Q}}_\ell.$$

So we find that the I_∞-representation of $\mathcal{F}$ contains the 2-dimensional tame subrepresentation

$$\mathcal{L}_\gamma \oplus \mathcal{L}_{\alpha\beta/\gamma},$$

which for dimension reasons must be the entire tame part.

On the other hand, the tame characters occurring in the I_0-representation of $\mathcal{F}$ are precisely $\alpha, \beta, \rho, \chi_2\rho$, and by the hypotheses none of these is either γ or $\alpha\beta/\gamma$. So by rigidity, $\mathcal{F}$ is, geometrically, a multiplicative translate of $\mathcal{H}(\psi; \alpha, \beta, \rho, \chi_2\rho; \gamma, \alpha\beta/\gamma)$. Comparing the wild parts of the pullbacks by [2], we see that the multiplicative translate is as asserted.

To compute the constant A, a bit of care is needed. If $\alpha \neq \chi_2\beta$,we proceed as in the proof of Theorem 6.3, first doing a Kummer twist to reduce to the case when $\alpha = \mathbb{I}$, then comparing the $Frob_k$ eigenvalues on the one-dimensional spaces of I_0-invariants; the case $\alpha = \mathbb{I} = \beta$ requires a separate argument. If $\gamma^2 \neq \chi_2\alpha\beta$, we first do a Kummer twist to reduce to the case when $\gamma = \mathbb{I}$, then we compare eigenvalues of $Frob_k$ eigenvalues on the one-dimensional spaces of I_∞-invariants. Fortunately, we are always in one of these two cases. Indeed, if we have both $\alpha = \chi_2\beta$ and $\gamma^2 = \chi_2\alpha\beta$, then the first relation gives

$$\alpha\beta = \chi_2\beta^2.$$

By the second relation, we have

$$\alpha\beta = \chi_2\gamma^2.$$

Thus $\beta^2 = \gamma^2$, so either $\beta = \gamma$ or $\chi_2\beta = \gamma$. The first possibility is ruled out by hypothesis. But as $\alpha = \chi_2\beta$, the second possibility forces $\alpha = \gamma$, again ruled out. □

7. Yet another $SL(2) \times SL(2)$ case, this time a false alarm

In Bailey's 1928 paper [Bai2, 2.11], we find the following hypergeometric identity:

$$\begin{aligned} &{}_1F_1(a; 2a; x)\,{}_1F_1(b; 2b; -x) \\ &= {}_2F_3((a+b)/2, (a+b+1)/2; a+1/2, b+1/2, a+b; x^2/4), \end{aligned}$$

which looks like some strange hybrid of the identities we have considered in the last two sections. In fact, it has a rather more benign explanation. The key point, as Dennis Stanton explained to me, is the following identity, the finite field analogue of [Bai2, 2.02].

Theorem 7.1. *Let k be a finite field of odd characteristic p, ψ a nontrivial additive character of k. For $t \in k^\times$, denote by ψ_t the nontrivial additive character $x \mapsto \psi(tx)$. Let $\alpha \neq \mathbb{I}$ be a multiplicative character of $k^\times$. Then for each prime $\ell \neq p$, we have an isomorphism of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,*

$$[2]^\star \mathcal{K}\ell(\alpha\chi_2, \mathbb{I}) \cong [x \mapsto -4x]^\star(\mathcal{L}_{\psi_{-1/2}} \otimes \mathcal{H}(\psi; \alpha^2, \mathbb{I}; \alpha)) \otimes A^{deg},$$

for A the constant

$$A = \alpha(1/4)(-g(\psi, \chi_2))/\#k.$$

Equivalently, we have an isomorphism

$$\mathcal{L}_{\psi_{-2}} \otimes [2]^\star \mathcal{K}\ell(\alpha\chi_2, \mathbb{I})$$
$$\cong [x \mapsto -4x]^\star(\mathcal{H}(\psi; \alpha^2, \mathbb{I}; \alpha)) \otimes A^{deg}.$$

Proof. We first look at the I_0-representations. Because $\alpha \neq \mathbb{I}$, the I_0-representation of $[2]^\star \mathcal{K}\ell(\alpha\chi_2, \mathbb{I})$ is either a single unipotent block, if $\alpha = \chi_2$, or is the sum of two distinct tame characters, $\mathbb{I}$ and α^2. Hence the two sides of our putative isomorphism have isomorphic I_0-representations.

The $I_\infty 0$-representation of $\mathcal{K}\ell(\alpha\chi_2, \mathbb{I})$ is $[2]_\star(\mathcal{L}_\alpha \otimes \mathcal{L}_{\psi_2})$, hence that of $[2]^\star \mathcal{K}\ell(\alpha\chi_2, \mathbb{I})$ is

$$\mathcal{L}_\alpha \otimes \mathcal{L}_{\psi_2} \bigoplus \mathcal{L}_\alpha \otimes \mathcal{L}_{\psi_{-2}}.$$

The I_∞-representation of $\mathcal{H}(\psi; \alpha^2, \mathbb{I}; \alpha))$ is

$$\mathcal{L}_\alpha \bigoplus \mathcal{L}_\alpha \otimes \mathcal{L}_\psi,$$

as follows from [Ka-ESDE, 8.12.2(2)]. Thus both sides of our putative isomorphism have isomorphic I_∞-representations. Looking at the second formulation, we see by rigidity that the left hand side $\mathcal{L}_{\psi_{-2}} \otimes [2]^\star \mathcal{K}\ell(\alpha\chi_2, \mathbb{I})$ is in fact a hypergeometric sheaf, and that we have the asserted geometric isomorphism. As always, we compute the constant A by looking at the eigenvalue of $Frob_k$ on the one-dimensional spaces of I_0-invariants on the two sides. ☐

Theorem 7.2. *Let k be a finite field of odd characteristic p, ψ a nontrivial additive character of k. For $t \in k^\times$, denote by ψ_t the nontrivial additive character $x \mapsto \psi(tx)$. Let α, β, η be multiplicative characters of $k^\times$. Suppose that $\alpha \neq \mathbb{I}, \beta \neq \mathbb{I}, \alpha \neq \beta, \alpha\beta \neq \mathbb{I}$, and $\eta^2 = \alpha\beta$. Then for each prime $\ell \neq p$, we have an isomorphism of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,*

$$\mathcal{H}(\psi; \alpha^2, \mathbb{I}; \alpha) \otimes [x \mapsto -x]^\star(\mathcal{H}(\psi; \beta^2, \mathbb{I}; \beta)$$
$$\cong [x \mapsto x^2/4]^\star \mathcal{H}(\psi; \mathbb{I}, \chi_2\alpha, \chi_2\beta, \alpha\beta; \eta, \chi_2\eta) \otimes A^{deg},$$

for A the constant

$$A = (\alpha\beta)(2)/(-g(\psi, \chi_2)).$$

Proof. This is a formal consequence of (the [2] pullback of) Theorem 5.3, using the theorem above to rewrite its two tensor factors. ☐

8. The $Sp(4)$ case

In this section, we exploit the fact that the spin group of $SO(5)$ is $Sp(4)$, via the representation $\Lambda^4(std_4)/(triv)$ of $Sp(4)$.

Any rank 4 Kloosterman sheaf of the form $\mathcal{K}\ell(\psi; \alpha, \overline{\alpha}, \beta, \overline{\beta})(3/2)$ is symplectically self dual. So its Λ^2 is the direct sum of $\overline{\mathbb{Q}}_\ell$ and a rank 5 sheaf whose G_{arith} lies in $SO(5)$, namely

$$\Lambda^2(\mathcal{K}\ell(\psi; \alpha, \overline{\alpha}, \beta, \overline{\beta})(3/2)/\overline{\mathbb{Q}}_\ell.$$

We begin in characteristic 2, with a special case.

Theorem 8.1. *Let ψ be the nontrivial additive character of $\mathbb{F}_2$. Then for any odd prime ℓ we have an isomorphism of of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/\mathbb{F}_2$,*

$$\Lambda^2(\mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I})(3/2))/\overline{\mathbb{Q}}_\ell \cong \mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I})(2).$$

Proof. The sheaf $\mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I})(3/2)$ has $G_{geom} = G_{arith} = Sp(4)$, cf. [Ka-GKM, 11.1, 11.3]. So its $\Lambda^2/\overline{\mathbb{Q}}_\ell$ has $G_{geom} = G_{arith} = SO(5)$ and in particular is geometrically irreducible. Its I_0-representation is $Unip(5)$ (because the I_0-representation of $\mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I})3/2)$ is $Unip(4)$).

What about its I_∞-representation. It cannot be tame, otherwise $\Lambda^2/\overline{\mathbb{Q}}_\ell$ is geometrically a successive extension of Kummer sheaves, contradicting its geometric irreducibility. All the ∞-slopes of $\mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I})3/2)$ are $1/4$, so all the ∞-slopes of $\Lambda^2/\overline{\mathbb{Q}}_\ell$ are at most $1/4$. Thus $Swan_\infty(\Lambda^2/\overline{\mathbb{Q}}_\ell) \leq 5/4$. As Swan conductors are nonnegative integers, and our representation is not tame, we conclude that

$$Swan_\infty(\Lambda^2/\overline{\mathbb{Q}}_\ell) = 1,$$

and all its ∞-slopes are $\leq 1/4$. So the wild part of the I_∞-representation has dimension either 4 or 5. We will show that the dimension is 5. If not, our sheaf is, by rigidity and the fact that we are over $\mathbb{F}_2$, geometrically isomorphic to a selfdual hypergeometric sheaf of the form

$$\mathcal{H}(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I}; \chi)$$

for some nontrivial character χ. The autoduality implies $\chi = \overline{\chi}$, which in characteristic 2 forces χ to be trivial. [Alternatively, χ must be a nontrivial multiplicative character of $\mathbb{F}_2$, and there are none.]

[Here is another argument to show that $\Lambda^2(Wild_4(\psi,\mathbb{I})/\overline{\mathbb{Q}}_\ell$ is totally wild in characteristic 2. We know that $\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I},\mathbb{I},\mathbb{I})$ is, geometrically, symplectically self dual, and hence that $Wild_4(\psi,\mathbb{I})|P_\infty$ is symplectically self dual. So it suffices to show that in $\Lambda^2(Wild_4(\psi,\mathbb{I})/\overline{\mathbb{Q}}_\ell$, the space of P_∞-invariants vanishes, or equivalently that in $\Lambda^2(Wild_4(\psi,\mathbb{I}))$, the space of P_∞-invariants has dimension ≤ 1. But in the larger space

$$Wild_4(\psi,\mathbb{I})\otimes Wild_4(\psi,\mathbb{I})\cong \mathcal{E}nd(Wild_4(\psi,\mathbb{I})),$$

the space of P_∞-invariants has dimension 1, because $Wild_4(\psi,\mathbb{I})|P_\infty$ is absolutely irreducible.]

So by rigidity, we conclude that we have an isomorphism

$$\Lambda^2(\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I},\mathbb{I},\mathbb{I})(3/2))/\overline{\mathbb{Q}}_\ell\cong \mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I},\mathbb{I},\mathbb{I},\mathbb{I})(2)\otimes A^{deg},$$

for some A. To see that $A=1$, observe that both $\Lambda^2(\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I},\mathbb{I},\mathbb{I})(3/2))/\overline{\mathbb{Q}}_\ell$ and $\mathcal{K}\ell(\psi;\mathbb{I},\mathbb{I},\mathbb{I},\mathbb{I},\mathbb{I})(2)$ have $G_{arith}\subset SO(5)$ (in fact $G_{arith}=SO(5)$, but we don't need that here), and hence A is a scalar in $SO(5)$. □

Corollary 8.2. *Let ψ be the nontrivial additive character of $\mathbb{F}_2$. Then for any odd prime ℓ we have an isomorphisms of I_∞-representations*

$$\Lambda^2(Wild_4(\psi;\mathbb{I}))/\overline{\mathbb{Q}}_\ell\cong Wild_5(\psi;\mathbb{I}),$$

$$\Lambda^2(Wild_4(\psi;\mathbb{I}))\cong Wild_5(\psi;\mathbb{I})\oplus\overline{\mathbb{Q}}_\ell.$$

Proof. Restrict the isomorphism of the theorem to the I_∞-representations of the two sides to get the first, which immediately implies the second. □

Using this corollary, we get the following more general statement in characteristic 2.

Theorem 8.3. *Let k be a finite field of characteristic 2, and ψ the nontrivial additive character of k obtained from the unique nontrivial additive character of the the prime field $\mathbb{F}_2$ by composition with the trace. Fix multiplicative characters α,β of k. Then for every odd prime ℓ, we have an isomorphism of of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,*

$$\Lambda^2(\mathcal{K}\ell(\psi;\alpha,\overline{\alpha},\beta,\overline{\beta})(3/2))/\overline{\mathbb{Q}}_\ell\cong \mathcal{K}\ell(\psi;\mathbb{I},\alpha\beta,\overline{\alpha}\beta,\alpha\overline{\beta},\overline{\alpha}\overline{\beta})(2).$$

Proof. By the corollary above, both sides have isomorphic I_∞-representations. By inspection they have the same characters occuring at 0, so by rigidity the two sides are geometrically isomorphic. The $\Lambda^2/\overline{\mathbb{Q}}_\ell$ has $G_{arith}\subset SO(5)$ (because $\mathcal{K}\ell(\psi;\alpha,\overline{\alpha},\beta,\overline{\beta})(3/2)$ has $G_{arith}\subset Sp(4)$). The rank 5 Kloosterman sheaf in question is orthogonally self dual, and has trivial determinant [Ka-GKM, 7.4.1.3]. So just as in the previous theorem, there is no A^{deg} twisting. □

We now turn to the case of odd characteristic p. Here we have a direct description of $Wild_4(\psi, \mathbb{I})$, namely

$$Wild_4(\psi, \mathbb{I}) \cong [4]_\star(\mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi(4x)}).$$

Thus we have

$$[4]^\star Wild_4(\psi, \mathbb{I}) \cong \mathcal{L}_{\chi_2(x)} \otimes (\bigoplus_{\zeta \in \mu_4(\overline{k})} \mathcal{L}_{\psi(4\zeta x)}).$$

Passing to Λ^2, and using the fact that the nonzero sums of two distinct elements of $\mu_4(\overline{k})$, i.e., $\pm 1 \pm i$, are the fourth roots of -4, we see that

$$[4]^\star \Lambda^2(Wild_4(\psi, \mathbb{I})) \cong \overline{\mathbb{Q}}_\ell \oplus \overline{\mathbb{Q}}_\ell \oplus (\bigoplus_{\gamma^4=-4} \mathcal{L}_{\psi(4\gamma x)}).$$

We begin, as we did in characteristic 2 with a special case.

Theorem 8.4. *Let k be a finite field of odd characteristic p, ψ a nontrivial additive character of k. Then for every prime $\ell \neq p$ we have an isomorphism of of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,*

$$\Lambda^2(\mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I})(3/2))/\overline{\mathbb{Q}}_\ell \cong [x \mapsto -4x]^\star \mathcal{H}(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I}; \chi_2)(2) \otimes A^{deg},$$

for A the constant

$$A = 1/(-g(\overline{\psi}, \chi_2)).$$

Proof. The sheaf $\mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I})(3/2)$ has $G_{arith} = G_{geom} = Sp(4)$. Hence its $\Lambda^2/\overline{\mathbb{Q}}_\ell$ has $G_{arith} = G_{geom} = SO(5)$, and in particular is geometrically irreducible. From the discussion above of $[4]^\star \Lambda^2(Wild_4(\psi, \mathbb{I}))$, we see that the wild part of the I_∞-representation of $\Lambda^2/\overline{\mathbb{Q}}_\ell$ is of dimension 4 and has Swan conductor 1. So by rigidity, there is a geomtric isomorphism between $\Lambda^2/\overline{\mathbb{Q}}_\ell$ and some multiplicative translate of $\mathcal{H}(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I}; \rho)$, for some nontrivial ρ. But as this sheaf is self dual, we must have $\rho = \chi_2$. That the multiplicative translate is as asserted results immediately from the explicit shape of $[4]^\star \Lambda^2(Wild_4(\psi, \mathbb{I}))$ recalled above, cf. the proof of Theorem 3.5. So we find that for some A, we have an isomorphism

$$\Lambda^2(\mathcal{K}\ell(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I})(3/2))/\overline{\mathbb{Q}}_\ell \cong [x \mapsto -4x]^\star \mathcal{H}(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I}; \chi_2)(2) \otimes A^{deg},$$

and it remains to evaluate A. By [Ka-ESDE, 8.8.1, 8.8.2, 8.12.2(3)], for the asserted value of A the sheaf $[x \mapsto -4x]^\star \mathcal{H}(\psi; \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I}; \chi_2)(2) \otimes A^{deg}$ has $G_{arith} \subset SO(5)$, and we conclude as before. □

Corollary 8.5. *Let k be a finite field of odd characteristic p, ψ a nontrivial additive character of k. Then for every prime $\ell \neq p$ we have isomorphisms of I_∞-representations*

$$\Lambda^2(Wild_4(\psi, \mathbb{I}))/\overline{\mathbb{Q}}_\ell \cong \mathcal{L}_{\chi_2} \oplus [x \mapsto -4x]^\star Wild_4(\psi, \chi_2),$$

$$\Lambda^2(Wild_4(\psi, \mathbb{I})) \cong \overline{\mathbb{Q}}_\ell \oplus \mathcal{L}_{\chi_2} \oplus [x \mapsto -4x]^\star Wild_4(\psi, \chi_2).$$

Proof. Restrict the isomorphism of the theorem to the I_∞-representations of the two sides to get the first, which immediately implies the second. □

Almost exactly as in the case of characteristic 2, we can use this corollary to get the following more general result. The details are left to the reader.

Theorem 8.6. *Let k be a finite field of odd characteristic p, ψ a nontrivial additive character of k. Fix multiplicative characters α, β of k. Then for every prime $\ell \neq p$, we have an isomorphism of of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,*

$$\Lambda^2(\mathcal{K}\ell(\psi; \alpha, \overline{\alpha}, \beta, \overline{\beta})(3/2))/\overline{\mathbb{Q}}_\ell$$

$$\cong [x \mapsto -4x]^\star \mathcal{H}(\psi; \mathbb{I}, \alpha\beta, \overline{\alpha}\beta, \alpha\overline{\beta}, \overline{\alpha}\overline{\beta}; \chi_2)(2) \otimes A^{deg},$$

for A the constant

$$A = 1/(-g(\overline{\psi}, \chi_2)).$$

9. The $SL(4)$ case

In this section, we exploit the fact that the spin group of $SO(6)$ is $SL(4)$, via the representation $\Lambda^4(std_4)$ of $SL(4)$.

We begin with the situation in characteristic 2.

Theorem 9.1. *Let k be a finite field of characteristic 2, and ψ the nontrivial additive character of k obtained from the unique nontrivial additive character of the the prime field $\mathbb{F}_2$ by composition with the trace. Fix multiplicative characters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of k. Suppose that each product $\alpha_i\alpha_j$ with $i \neq j$ is nontrivial, but that $\alpha_1\alpha_2\alpha_3\alpha_4 = \mathbb{I}$. Then for any odd prime ℓ we have an isomorphism of of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,*

$$\Lambda^2(\mathcal{K}\ell(\psi; \alpha_1, \alpha_2, \alpha_3, \alpha_4)(3/2))$$

$$\cong \mathcal{H}(\alpha_1\alpha_2, \alpha_1\alpha_3, \alpha_1\alpha_4, \alpha_2\alpha_3, \alpha_2\alpha_4, \alpha_3\alpha_4; \mathbb{I})(3).$$

Proof. Using Corollary 6.2, we see that the two sides have isomorphic I_∞-representations. They visibly have the same characters occurring in their I_0-representations. So by rigidity, they are geometrically isomorphic. The sheaf $\mathcal{K}\ell(\psi; \alpha_1, \alpha_2, \alpha_3, \alpha_4)(3/2)$ has $G_{arith} \subset SL(4)$, cf.[7.4.1.3-4]Ka-GKM, so its Λ^2 has its $G_{arith} \subset SO(6)$. By [Ka-GKM, 8.8.1-2 and 8.12.2], the target

$$\mathcal{H}(\alpha_1\alpha_2, \alpha_1\alpha_3, \alpha_1\alpha_4, \alpha_2\alpha_3, \alpha_2\alpha_4, \alpha_3\alpha_4; \mathbb{I})(3).$$

also has its $G_{arith} \subset SO(6)$.Since the only scalars in $SO(6)$ are ± 1, for A some choice of ± 1, we have an isomorphism after twisting the target with A^{deg}. In order to determine A, we will prove the slightly more general theorem below. $\square$

Theorem 9.2. *Let k be a finite field of characteristic 2, and ψ the nontrivial additive character of k obtained from the unique nontrivial additive character of the the prime field $\mathbb{F}_2$ by composition with the trace. Fix multiplicative characters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of k. Denote by η the unique multiplicative character with*

$$\eta^2 = \alpha_1\alpha_2\alpha_3\alpha_4.$$

Suppose that none of $\alpha_i\alpha_j$ with $i \neq j$ is η. Then for any odd prime ℓ we have an isomorphism of of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,

$$\Lambda^2(\mathcal{K}\ell(\psi;\alpha_1,\alpha_2,\alpha_3,\alpha_4))$$
$$\cong \mathcal{H}(\alpha_1\alpha_2,\alpha_1\alpha_3,\alpha_1\alpha_4,\alpha_2\alpha_3,\alpha_2\alpha_4,\alpha_3\alpha_4;\eta).$$

Proof. We begin by remarking that the two sides are geometrically isomorphic; indeed if we twist $\mathcal{K}\ell(\psi;\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ by the unique square root of η, we reduce to the situation of the previous theorem, where we already established the geometric isomorphism.

We now twist $\mathcal{K}\ell(\psi;\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ by the unique square root of $\overline{\alpha_1\alpha_2}$, we reduce to the case in which we have the additional condition $\alpha_1\alpha_2 = \mathbb{I}$. The advantage of this case is that now the space of I_0-invariants on both sides is one-dimensional, and it suffices to show that the $Frob_k$-eigenvalues are equal. Lemma 2.1 gives an explicit formula for this eigenvalue, call it E, on the $\mathcal{H}$ side. How do we compute the eigenvalue on the Λ^2 side?

Suppose first that $\alpha_1 = \alpha_2$. Because we are in characteristic 2, and $\alpha_1\alpha_2 = \mathbb{I}$, we must have $\alpha_1 = \alpha_2 = \mathbb{I}$. In this case, the I_0-unipotent subspace $(\mathcal{K}\ell)^{I_0-unipotent}$ of $\mathcal{K}\ell := \mathcal{K}\ell(\psi;\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ is a single Jordan block of size either 2 or 3, and Lemma 2.1 gives an explicit formula for the $Frob_k$-eigenvalue, call it B, on $\mathcal{K}\ell^{I_0}$. Denote $q := \#k$. It then follows from [De-Weil II, 1.8.4, 1.6.14.2-3] (cf. also [Ka-GKM, 7.0.6-7]) that the eigenvalues of (any element in D_0 whose image in $D_0/|_0$ is) $Frob_k$ on $(\mathcal{K}\ell)^{I_0-unipotent}$ are B, qB (resp. are B, qB, q^2B) if this space has dimension 2 (resp. has dimension 3). So on

$$\Lambda^2((\mathcal{K}\ell)^{I_0-unip}) = (\Lambda^2((\mathcal{K}\ell))^{I_0-unip},$$

the eigenvalues are qB^2 (resp. qB^2, q^2B^2, q^3B^2). Of these, the $Frob_k$-eigenvalue on $\Lambda^2(\mathcal{K}\ell))^{I_0}$ is always the one of lowest weight [Ka-GKM, 7.0.6-7], here qB^2. It remains to see that the eigenvalues coincide, i.e., that $D = qB^2$. This is immediate from the explicit formulas, and the fact that for our particular choice of ψ in characteristic 2, we have $g(\psi,\chi) = g(\psi,\chi^2)$ for any χ.

Suppose now that $\alpha_1 \neq \alpha_2$. Then the D_0-representation of

$$\mathcal{K}\ell := \mathcal{K}\ell(\psi; \alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

contains the direct sum

$$(\mathcal{K}\ell)^{\alpha_1-unipotent} \oplus (\mathcal{K}\ell)^{\alpha_2-unipotent}.$$

So the the D_0-representation of Λ^2 contains the tensor product

$$(\mathcal{K}\ell)^{\alpha_1-unipotent} \otimes (\mathcal{K}\ell)^{\alpha_2-unipotent}.$$

Since $\alpha_1\alpha_2 = \mathbb{I}$, this last tensor product is

$$(\mathcal{L}_{\overline{\alpha_1}} \otimes \mathcal{K}\ell)^{I_0-unipotent} \otimes (\mathcal{L}_{\overline{\alpha_2}} \otimes \mathcal{K}\ell)^{I_0-unipotent}.$$

Passing to I_0-invariants, we get a D_0-isomorphism

$$(\Lambda^2(\mathcal{K}\ell)^{I_0} \cong (\mathcal{L}_{\overline{\alpha_1}} \otimes \mathcal{K}\ell)^{I_0} \otimes (\mathcal{L}_{\overline{\alpha_2}} \otimes \mathcal{K}\ell)^{I_0}.$$

Once again, Lemma 2.1 gives us explicit formulas for the $Frob_k$-eigenvalues, say C_1 and C_2, on the spaces $(\mathcal{L}_{\overline{\alpha_i}} \otimes \mathcal{K}\ell)^{I_0}$ for $i = 1, 2$. Again, the verification that indeed $D = C_1C_2$ is immediate. □

We now turn to the situation in odd characteristic p.

Theorem 9.3. *Let k be a finite field of odd characteristic p, ψ a nontrivial additive character of k. Fix multiplicative characters $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \eta$ of k. Suppose that*

$$\eta^2 = \alpha_1\alpha_2\alpha_3\alpha_4,$$

and that none of $\alpha_i\alpha_j$ with $i \neq j$ is η or $\eta\chi_2$. Then for any prime $\ell \neq p$ we have an isomorphism of of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,

$$\Lambda^2(\mathcal{K}\ell(\psi; \alpha_1, \alpha_2, \alpha_3, \alpha_4))$$

$$\cong [x \mapsto -4x]^\star \mathcal{H}(\alpha_1\alpha_2, \alpha_1\alpha_3, \alpha_1\alpha_4, \alpha_2\alpha_3, \alpha_2\alpha_4, \alpha_3\alpha_4; \eta, \eta\chi_2) \otimes A^{deg},$$

for A the constant

$$A := 1/((\alpha_1\alpha_2\alpha_3\alpha_4)(2))(-g(\overline{\psi}, \chi_2)).$$

Proof. We first show that the two sides are geometrically isomorphic. For this, we may work over a finite extension field of k in which there η is a square, say $\beta^2\eta = \mathbb{I}$. Then twisting $\mathcal{K}\ell := \mathcal{K}\ell(\psi; \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ by the Kummer sheaf $\mathcal{L}_\beta$, we reduce to the case when $\alpha_1\alpha_2\alpha_3\alpha_4 = \mathbb{I}$. Then by Corollary 6.5, both sides have isomorphic I_∞-representations, and we conclude by rigidity.

We now return to the original setting. We claim that at least one of the six products $\alpha_i\alpha_j$ with $i \neq j$ is a square (of some multiplicative character of k). This is obviously true if two of the α_i coincide. In the case where the α_i are four distinct characters, either at least two are squares, or at least two are nonsquares, so we are done here as well. Renumbering, we may assume that $\alpha_1\alpha_2$ is a square, say $\alpha_1\alpha_2\beta^2 = \mathbb{I}$. Kummer twisting $\mathcal{K}\ell$ by $\mathcal{L}_\beta$, we reduce to the case when $\alpha_1\alpha_2 = \mathbb{I}$. From this point on, the proof goes along the same reduction into two cases as in characteristic 2, now using the Hasse-Davenport identity [Dav-Ha, 0.9I]

$$g(\psi_2, \rho^2)g(\psi, \chi_2) = g(\psi, \rho)g(\psi, \rho\chi_2)$$

to see that, in both cases, the $Frob_k$ eigenvalues match. □

10. Lie algebra aspects: the $SL(2) \times SL(2)$ case

In terms of the standard representation std_4 of $SO(4)$, the Adjoint representation of $SO(4)$ is $\Lambda^2(std_4)$. But if we think of std_4 as being the representation $std_{2,first} \otimes std_{2,second}$ of its spin group $SL(2) \times SL(2)$, then the Adjoint representation is $Sym^2(std_{2,first}) \bigoplus Sym^2(std_{2,second})$. So for any two lisse $\overline{\mathbb{Q}}_\ell$-sheaves $\mathcal{F}$ and $\mathcal{G}$ on $\mathbb{G}_m/k$ both of which have their $G_{arith} \subset SL(2)$, we have

$$\Lambda^2(\mathcal{F} \otimes \mathcal{G}) \cong Sym^2(\mathcal{F}) \bigoplus Sym^2(\mathcal{G}).$$

Thus we obtain from Theorems 5.1 and 5.3 the following theorems.

Theorem 10.1. *Let k be a finite field of characteristic 2, and ψ the nontrivial additive character of k obtained from the unique nontrivial additive character of the the prime field $\mathbb{F}_2$ by composition with the trace. Let χ and ρ be any multiplicative characters of $k^\times$, $\overline{\chi}$ and $\overline{\rho}$ the inverse characters. Suppose that*

$$\chi \neq \rho, \chi \neq \overline{\rho}.$$

Then for every odd prime ℓ, we have an isomorphism of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,

$$\Lambda^2(\mathcal{H}(\psi; \chi\rho, \overline{\chi}\rho, \chi\overline{\rho}, \overline{\chi\rho}; \mathbb{I})(2)).$$

$$\cong Sym^2(\mathcal{K}\ell(\psi; \chi, \overline{\chi})(1/2)) \bigoplus Sym^2(\mathcal{K}\ell(\psi; \rho, \overline{\rho})(1/2))$$

Theorem 10.2. *Let k be a finite field of odd characteristic p, ψ a nontrivial additive character of k. Let α, β be multiplicative characters of $k^\times$, $\overline{\alpha}$ and $\overline{\beta}$ the inverse characters. Suppose that*

$$\alpha^2 \neq \beta^2, \alpha^2 \neq \overline{\beta}^2.$$

Then for each prime $\ell \neq p$, we have an isomorphism of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,

$$[x \mapsto 4x]^\star \Lambda^2(\mathcal{H}(\psi_k; \alpha\beta, \alpha\overline{\beta}, \overline{\alpha}\beta, \overline{\alpha\beta}; \mathbb{I}, \chi_2)(2) \otimes A^{deg})$$

$$\cong Sym^2(\mathcal{K}\ell(\psi; \alpha, \overline{\alpha})(1/2)) \bigoplus Sym^2(\mathcal{K}\ell(\psi; \beta, \overline{\beta})(1/2)),$$

for A the constant

$$A = 1/(-g(\overline{\psi}, \chi_2)).$$

11. Lie algebra aspects: the $Sp(4)$ case

In terms of the standard representation std_4 of $Sp(4)$, the Adjoint representation of $Sp(4)$ is $Symn^2(std_4)$. In terms of the standard representation std_5 of $SO(5)$, the Adjoint representation of $SO(5)$ is $\Lambda^2(std_5)$. But the standard representation std_5 of $SO(5)$ is the representation $\Lambda^2(std_4)/(triv)$ of its spin group $Sp(4)$, and hence we have an isomorphism of representations of $Sp(4)$,

$$Symn^2(std_4) \cong \Lambda^2(\Lambda^2(std_4)/(triv)).$$

So for any lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on $\mathbb{G}_m/k$ with $G_{arith} \subset Sp(4)$, we have

$$Symn^2(\mathcal{F}) \cong \Lambda^2(\Lambda^2(\mathcal{F})/(triv)).$$

Thus we obtain from Theorems 8.3 and 8.6 the following theorems.

Theorem 11.1. *Let k be a finite field of characteristic 2, and ψ the nontrivial additive character of k obtained from the unique nontrivial additive character of the the prime field $\mathbb{F}_2$ by composition with the trace. Fix multiplicative characters α, β of k. Then for every odd prime ℓ, we have an isomorphism of of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,*

$$Sym^2(\mathcal{K}\ell(\psi; \alpha, \overline{\alpha}, \beta, \overline{\beta})(3/2))$$

$$\cong \Lambda^2(\mathcal{K}\ell(\psi; \mathbb{I}, \alpha\beta, \overline{\alpha}\beta, \alpha\overline{\beta}, \overline{\alpha}\overline{\beta})(2)).$$

Theorem 11.2. *Let k be a finite field of odd characteristic p, ψ a nontrivial additive character of k. Fix multiplicative characters α, β of k. Then for every prime $\ell \neq p$, we have an isomorphism of of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,*

$$Sym^2(\mathcal{K}\ell(\psi; \alpha, \overline{\alpha}, \beta, \overline{\beta})(3/2))$$

$$\cong [x \mapsto -4x]^\star \Lambda^2(\mathcal{H}(\psi; \mathbb{I}, \alpha\beta, \overline{\alpha}\beta, \alpha\overline{\beta}, \overline{\alpha}\overline{\beta}; \chi_2)(2) \otimes A^{deg}),$$

for A the constant

$$A = 1/(-g(\overline{\psi}, \chi_2)).$$

12. Lie algebra aspects: the $SL(4)$ case

In terms of the standard representation std_4 of $SL(4)$, the Adjoint representation of $SL(4)$ is $End^0(std_4)$. In terms of the standard representation std_6 of $SO(6)$, the Adjoint representation of $SO(6)$ is $\Lambda^2(std_6)$. But the standard representation std_6 of $SO(6)$ is the representation $\Lambda^2(std_4)$ of its spin group $SL(4)$, and hence we have an isomorphism of representations of $SL(4)$,

$$End^0(std_4) \cong \Lambda^2(\Lambda^2(std_4)).$$

So for any lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on $\mathbb{G}_m/k$ with $G_{arith} \subset SL(4)$, we have

$$End^0(\mathcal{F}) \cong \Lambda^2(\Lambda^2(\mathcal{F})).$$

Thus we obtain from Theorems 9.1 and 9.3 the following theorems.

Theorem 12.1. *Let k be a finite field of characteristic 2, and ψ the nontrivial additive character of k obtained from the unique nontrivial additive character of the the prime field $\mathbb{F}_2$ by composition with the trace. Fix multiplicative characters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of k. Suppose that each product $\alpha_i\alpha_j$ with $i \neq j$ is nontrivial, but that $\alpha_1\alpha_2\alpha_3\alpha_4 = \mathbb{I}$. Then for any odd prime ℓ we have an isomorphism of of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,*

$$End^0(\mathcal{K}\ell(\psi; \alpha_1, \alpha_2, \alpha_3, \alpha_4)(3/2))$$
$$\cong \Lambda^2(\mathcal{H}(\alpha_1\alpha_2, \alpha_1\alpha_3, \alpha_1\alpha_4, \alpha_2\alpha_3, \alpha_2\alpha_4, \alpha_3\alpha_4; \mathbb{I})(3)).$$

Theorem 12.2. *Let k be a finite field of odd characteristic p, ψ a nontrivial additive character of k. Fix multiplicative characters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of k. Suppose that*

$$\alpha_1\alpha_2\alpha_3\alpha_4 = \mathbb{I},$$

and that none of the products $\alpha_i\alpha_j$ with $i \neq j$ is $\mathbb{I}$ or χ_2. Then for any prime $\ell \neq p$ we have an isomorphism of of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{G}_m/k$,

$$End^0((\mathcal{K}\ell(\psi; \alpha_1, \alpha_2, \alpha_3, \alpha_4)(3/2))$$
$$\cong [x \mapsto -4x]^\star\Lambda^2(\mathcal{H}(\alpha_1\alpha_2, \alpha_1\alpha_3, \alpha_1\alpha_4, \alpha_2\alpha_3, \alpha_2\alpha_4, \alpha_3\alpha_4; \mathbb{I}, \chi_2)(3)\otimes A^{deg}),$$

for A the constant

$$A := 1/(-g(\overline{\psi}, \chi_2)).$$

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