I am indebted to Douglas Ulmer for pointing out, in great and careful detail, a number of errors and omissions in Chapter 5 of *Twisted L-Functions and Monodromy*. For the convenience of the reader, what follows is a corrected version of Chapter 5. The corrections and additions are typed in this font rather than in the font of the book, to make them relatively easy to locate. Because of the additions, the page numbers match less and less well as the chapter goes on, but should still help the reader find his place.

Chapter 5: Twist Sheaves and Their Monodromy

5.0 Families of twists: basic definitions and constructions

(5.0.1) In this section, we make explicit the "families of twists" we will be concerned with. We fix an algebraically closed field k, a proper smooth connected curve C/k whose genus is denoted g, and a prime number ℓ invertible in k. We also fix an integer $r \geq 1$, and an irreducible middle extension $\overline{\mathbb{Q}}_\ell$–sheaf $\mathcal{F}$ on C of generic rank r. This means that for some dense open set U in C, with $j : U \to C$ the inclusion, $\mathcal{F}|U$ is a lisse sheaf of rank r on U which is irreducible in the sense that the corresponding r–dimensional $\overline{\mathbb{Q}}_\ell$–representation of $\pi_1(U)$ is irreducible, and $\mathcal{F}$ on C is obtained from the lisse irreducible sheaf $\mathcal{F}|U$ on U by direct image: $\mathcal{F} \cong j_*(\mathcal{F}|U) := j_*j^*\mathcal{F}$.

(5.0.2) We say that $\mathcal{F}$ is self–dual if for every dense open set U on which it is lisse, $\mathcal{F}|U$ is self–dual as lisse sheaf, i.e., isomorphic to its contragredient. It is equivalent to say that the perverse sheaf $\mathcal{F}[1]$ on C is self–dual, but we will not need this more sophisticated point of view.

(5.0.3) The finite set of points of C at which $\mathcal{F}$ fails to be lisse, i.e., the set of points x for which the inertia group I(x) acts nontrivially on $\mathcal{F}$, will be denoted $\mathrm{Sing}(\mathcal{F})$, the set of "singularities" of $\mathcal{F}$. Thus $\mathcal{F}$ is lisse on $C - \mathrm{Sing}(\mathcal{F})$, and $\mathrm{Sing}(\mathcal{F})$ is minimal with this property.

(5.0.4) We fix an effective divisor $D = \sum a_i P_i$ on C, whose degree $d := \sum a_i$ satisfies $d \geq 2g+1$. Some or all or none of the points P_i may lie in $\mathrm{Sing}(\mathcal{F})$. We denote by L(D) the Riemann–Roch space $H^0(C, I^{-1}(D))$, and we view L(D) as a space of functions (maps to $\mathbb{A}^1$) on the open curve C – D.

(5.0.5) Corresponding to the choice of D as the "points at ∞" of C, we break up the set $\mathrm{Sing}(\mathcal{F})$ as the disjoint union

(5.0.5.1) $$\mathrm{Sing}(\mathcal{F}) := \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}} \amalg \mathrm{Sing}(\mathcal{F})_\infty$$

where

(5.0.5.2) $$\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}} := \mathrm{Sing}(\mathcal{F}) \cap (C-D),$$

(5.0.5.3) $$\mathrm{Sing}(\mathcal{F})_\infty := \mathrm{Sing}(\mathcal{F}) \cap D.$$

Lemma 5.0.6 Given a finite subset S of C–D, denote by

$$\mathrm{Fct}(C, d, D, S) \subset L(D)$$

the set of nonzero functions f in L(D) with the following property:

the divisor of zeroes of f, $f^{-1}(0)$, consists of d = degree(D) distinct points, none of which lies in $S \cup D$. Then Fct(C, d, D, S) is (the set of k–points of) a dense open set *Fct*(C, d, D, S) in L(D) (viewed as the set of k–points of an affine space $\mathbb{A}^{d+1-g}$ over k).

proof The projective space $\mathbb{P}(L(D)^\vee)$ of lines in L(D) is the space of effective divisors of degree d which are linearly equivalent to D. In the space $\mathrm{Sym}^d(C)$ of all effective divisors of degree D, those consisting of d distinct points, none of which lies in S∪D, form an open set, say U_1. When we map $\mathrm{Sym}^d(C)$ to $\mathrm{Jac}^d(C)$, the fibre over the class of D is $\mathbb{P}(L(D)^\vee)$. The intersection of this fibre with U_1 is an open set U_2 in $\mathbb{P}(L(D)^\vee)$. The inverse image U_3 of this set in L(D) – {0} is the set Fct(C, d, D, S) in L(D), which is thus open.

To see that U_3 is nonempty, we argue as follows. Suppose there exists a function f in L(D) whose divisor of poles is D and whose differential df is nonzero. Then for any t in k which is not a value taken by f on either S or on the set of zeroes in C–D of df, the function f–t lies in U_3 (it is nonzero on S, and it has simple zeroes because it has no zeroes in common with df).

Why does such an f exist? By Riemann–Roch, for each point P_i in D, $L(D - P_i)$ is a hyperplane in L(D): as k is infinite, L(D) is not the union of finitely many hyperplanes. So we can find a function f in L(D) whose divisor of poles is D. If any of the coefficients a_i in $D = \Sigma a_i P_i$ is invertible in k, then df is nonzero, because at P_i it has a pole of order $1+a_i$. If all a_i vanish in k, then k has charactertistic p, all the a_i are divisible by p, say $a_i = pb_i$, and $D = pD_0$, for D_0 the divisor $D_0 := \Sigma_i b_i P_i$. If df vanishes, then $f = g^p$ for some g in $L(D_0)$. In this case, pick a function g in $L(D - P_1)$ whose divisor of poles is $D - P_1$ (still possible by Riemann–Roch). Then dg is nonzero (it has a pole of order a_1 at P_1). For all but finite many values of t in k, f – tg still has divisor of poles D. For any such t, f – tg is the desired function. QED

Remark 5.0.7 Perhaps the simplest example to keep in mind is this. Take C to be $\mathbb{P}^1$, and take D to be d∞. So here C–D is $\mathbb{A}^1$ = Spec(k[X]), and *Fct*(C, d, D, S) is all the polynomials of degree d in one variable X with d distinct zeroes, none of which lies in S.

Notations 5.0.7.1 It will be convenient in the rest of this chapter to adopt the following notations. For any finite subset S of C, not necessarily contained in C - D, we define Fct(C, d, D, S) by

Fct(C, d, D, S) := Fct(C, d, D, S∩(C-D)).

And for Z a finite subscheme of C, we define

Fct(C, d, D, Z) := Fct(C, d, D, Z^{red}).:= Fct(C, d, D, Z^{red}∩(C-D)).

Notation 5.0.7.2 For f any nonzero rational function on C, we denote by div(f) the minimal finite subset of C outside of which f is invertible. In other words, div(f) consists of the zeroes and poles of f, each taken with multiplicity one.

(5.0.8) We now turn to our final piece of data, a nontrivial $\overline{\mathbb{Q}}_\ell^\times$–valued character χ of finite order n

≥ 2 of the tame fundamental group of $\mathbb{G}_m/k$, corresponding to a lisse rank one $\bar{\mathbb{Q}}_\ell$–sheaf $\mathcal{L}_\chi$ on $\mathbb{G}_m$. The order n of χ is necessarily invertible in k, indeed $\pi_1^{tame}(\mathbb{G}_m/k)$ is the inverse limit of the groups $\mu_N(k)$ over those N invertible in k, corresponding to the various Kummer coverings $x \mapsto x^n$ of $\mathbb{G}_m$ by itself.

(5.0.9) When k has positive characteristic, the $\mathcal{L}_\chi$'s having given order n are obtained concretely as follows. Take any finite subfield $\mathbb{F}_q$ of k which contains the n'th roots of unity (i.e., $q \equiv 1 \bmod n$), and take a character $\chi : (\mathbb{F}_q)^\times \to \bar{\mathbb{Q}}_\ell^\times$ of order n. View $\mathbb{G}_m/\mathbb{F}_q$ as an $(\mathbb{F}_q)^\times$–torsor over itself by the map ("Lang isogeny")

(5.0.9.1) $$1 - \mathrm{Frob}_q : x \mapsto x^{1-q},$$

and push out this torsor by the character $\chi : (\mathbb{F}_q)^\times \to \bar{\mathbb{Q}}_\ell^\times$ to obtain a lisse rank one $\mathcal{L}_\chi$ on $\mathbb{G}_m/\mathbb{F}_q$. Its pullback to $\mathbb{G}_m/k$ is an $\mathcal{L}_\chi$ of the same order n on $\mathbb{G}_m/k$, and every $\mathcal{L}_\chi$ of order n on $\mathbb{G}_m/k$ is obtained this way.

(5.0.10) Given f in Fct(C, d, D, $\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}$), we may view f as mapping the open curve $C - D - f^{-1}(0)$ to $\mathbb{G}_m$, and we form the lisse rank one $\bar{\mathbb{Q}}_\ell$–sheaf $\mathcal{L}_{\chi(f)} := f^*\mathcal{L}_\chi$ on $C - D - f^{-1}(0)$. When no ambiguity is likely, we will also denote by $\mathcal{L}_{\chi(f)}$ the extension by direct image of this sheaf to all of C. We then "twist" $\mathcal{F}$ by $\mathcal{L}_{\chi(f)}$. This means that we pass to the open set

$$j : C - D - f^{-1}(0) - \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}} \subset C,$$

on which both $\mathcal{F}$ and $\mathcal{L}_{\chi(f)}$ are lisse, on that open set we form $\mathcal{F}\otimes\mathcal{L}_{\chi(f)}$, and then we take the direct image $j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})$ to C. Notice that this twisted sheaf $j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})$ on C is itself an irreducible middle extension.

(5.0.11) Since at each point of $f^{-1}(0)$ and at each point of $\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}$ one of the factors $\mathcal{F}$ or $\mathcal{L}_{\chi(f)}$ is lisse, the sheaf $j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}) \mid C-D$ is the literal tensor product $\mathcal{F}\otimes\mathcal{L}_{\chi(f)} \mid C-D$. Thus if we denote by $j_\infty : C - D \to C$ the inclusion, $j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})$ as defined above is obtained from the literal tensor product $\mathcal{F}\otimes\mathcal{L}_{\chi(f)} \mid C-D$ by taking direct image across D^{red}:

$$j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}) = j_{\infty*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}).$$

This alternate interpretation will be used later, in 5.2.4 and 5.2.5.

(5.0.12) We then form the cohomology groups $H^i(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}))$ with coefficients in the twist $j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})$. Our eventual goal is to study the variation of these cohomology groups as f varies. But first we must establish some basic properties of these groups for a fixed f.

5.1 Basic facts about the groups $H^i(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}))$

Lemma 5.1.1 Hypotheses and notations as in 5.0.1, 5.0.4, 5.0.8, and 5.0.10 above, the cohomology groups $H^i(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}))$ vanish for $i\neq 1$.

proof The H^i vanish for cohomological dimension reasons for i not in [0, 2]. For i=0, we have

$$H^0(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})) := H^0(C - D - f^{-1}(0) - \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}, \mathcal{F}\otimes\mathcal{L}_{\chi(f)}).$$

This group vanishes because $\mathcal{F}\otimes\mathcal{L}_{\chi(f)}$ is lisse on the open curve, it is irreducible ($\mathcal{F}$ is irreducible, and $\mathcal{L}_{\chi(f)}$ has rank one) and nontrivial (because $\mathcal{F}\otimes\mathcal{L}_{\chi(f)}$ is nontrivially ramified at each of the d points of $f^{-1}(0)$). So the H^0 is the invariants in a nontrivial irreducible representation, so vanishes. Similarly, the birational invariance of H^2_c gives

$$H^2(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})) := H^2_c(C - D - f^{-1}(0) - \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}, \mathcal{F}\otimes\mathcal{L}_{\chi(f)}),$$

which is the Tate–twisted coinvariants in the same representation, so also vanishes. QED

(5.1.2) We next compute the dimension of $H^1(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}))$, for f in $\mathrm{Fct}(C, d, D, \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}})$.

Given a point x in C(k), and a lisse sheaf $\mathcal{H}$ on some dense open set of C, we denote by $\mathcal{H}(x)$ the representation of I(x) given by $\mathcal{H}$ (strictly speaking, given by the pullback of $\mathcal{H}$ to the spectrum of the x–adic completion of the function field of C), and by $\mathcal{H}(x)^{I(x)}$, or simply $\mathcal{H}^{I(x)}$, the invariants in this representation. We will write $\mathcal{H}/\mathcal{H}^{I(x)}$ for $\mathcal{H}(x)/\mathcal{H}(x)^{I(x)}$. We will write

(5.1.2.1) $$\mathrm{drop}_x(\mathcal{H}) := \mathrm{drop}_x(\mathcal{H}(x)) := \dim(\mathcal{H}/\mathcal{H}^{I(x)}).$$

For any of the P_i occurring in $D = \sum a_i P_i$, and any f with divisor of poles D, the $I(P_i)$–representation $(\mathcal{L}_{\chi(f)})(P_i)$ depends only on χ^{a_i}, as follows. Choose a uniformizing parameter at P_i, and use it to identify the complete local ring of C at P_i with the complete local ring k[[1/X]] (sic) of $\mathbb{P}^1$ at ∞, and to identify the inertia group $I(P_i)$ with $I(\infty)$. Consider the lisse sheaf $\mathcal{L}_{\chi^{a_i}} := \mathcal{L}_{\chi^{a_i}(X)}$ on $\mathbb{G}_m$. Then $(\mathcal{L}_{\chi(f)})(P_i)$ as $I(P_i)$–representation is just $(\mathcal{L}_{\chi^{a_i}})(\infty)$ as $I(\infty)$–representation. When we want to indicate unambiguously that we are thinking of $(\mathcal{L}_{\chi^{a_i}})(\infty)$ as an $I(P_i)$–representation by some choice of uniformizer as above, we will denote it $(\mathcal{L}_{\chi^{a_i}})(\infty, P_i)$.

Lemma 5.1.3 Hypotheses and notations as in 5.1.1 above, for any f in Fct(C, d, D, $Sing(\mathcal{F})_{finite}$), we have the dimension formula

(5.1.3.1) $$h^1(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})) = (2g-2 + deg(D))rank(\mathcal{F}) + \Sigma_{P_i \text{ in } D^{red}} Swan_{P_i}(\mathcal{F}) + \Sigma_{s \text{ in } Sing(\mathcal{F})_{finite}} Swan_s(\mathcal{F}). + \Sigma_{s \text{ in } Sing(\mathcal{F})_{finite}} drop_s(\mathcal{F}) + \Sigma_{P_i \text{ in } D^{red}} drop_{P_i}(\mathcal{F}(P_i)\otimes(\mathcal{L}_\chi a_i)(\infty, P_i)),$$

and the inequality

(5.1.3.2) $$h^1(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})) \geq (2g-2 + deg(D))rank(\mathcal{F}) + \#Sing(\mathcal{F})_{finite}.$$

proof The inequality 5.1.3.2 is an immediate consequence of the asserted dimension formula 5.1.3.1 and the observation that $drop_s(\mathcal{F}) \geq 1$ at each point in $Sing(\mathcal{F})_{finite}$. By Lemma 5.1.1, we have

$$h^1(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})) = -\chi(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})).$$

At each of the deg(D) distinct zeroes of f, $\mathcal{F}$ is lisse and $\mathcal{L}_{\chi(f)}$ is ramified, so $-\chi(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}))$ is equal to

$$= -\chi_c(C - f^{-1}(0) - D - Sing(\mathcal{F})_{finite}, \mathcal{F}\otimes\mathcal{L}_{\chi(f)}) - \Sigma_{s \text{ in } Sing(\mathcal{F})_{finite}} \dim(\mathcal{F}(s)^{I(s)}) - \Sigma_{P_i \text{ in } D^{red}} \dim((\mathcal{F}(P_i)\otimes(\mathcal{L}_\chi a_i)(\infty, P_i))^{I(P_i)}).$$

Now use the Euler–Poincaré formula to write this as

$$= (2g-2 + deg(D) + \#D^{red} + \#Sing(\mathcal{F})_{finite})rank(\mathcal{F}) + \Sigma_{P_i \text{ in } D^{red}} Swan_{P_i}(\mathcal{F}) + \Sigma_{s \text{ in } Sing(\mathcal{F})_{finite}} Swan_s(\mathcal{F}). - \Sigma_{s \text{ in } Sing(\mathcal{F})_{finite}} \dim(\mathcal{F}(s)^{I(s)}) - \Sigma_{P_i \text{ in } D^{red}} \dim((\mathcal{F}(P_i)\otimes(\mathcal{L}_\chi a_i)(\infty, P_i))^{I(P_i)})$$

$$= (2g-2 + deg(D))rank(\mathcal{F}) + \Sigma_{P_i \text{ in } D^{red}} Swan_{P_i}(\mathcal{F}) + \Sigma_{s \text{ in } Sing(\mathcal{F})_{finite}} Swan_s(\mathcal{F}). + \Sigma_{s \text{ in } Sing(\mathcal{F})_{finite}} drop_s(\mathcal{F}) + \Sigma_{P_i \text{ in } D^{red}} drop_{P_i}(\mathcal{F}(P_i)\otimes(\mathcal{L}_\chi a_i)(\infty, P_i)).$$ QED

5.2 Putting together the groups $H^1(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}))$

Construction–Proposition 5.2.1 (compare [Ka–RLS, 2.7.2]) Hypotheses and notations as in 5.0.1, 5.0.4, 5.0.8, and 5.0.10 above, There is a natural lisse $\bar{\mathbb{Q}}_\ell$–sheaf $\mathcal{G}$ on the space

$$Fct(C, d, D, Sing(\mathcal{F})_{finite})$$

whose stalk at f is the cohomology group $H^1(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}))$. More precisely, over the parameter space

$$X := Fct(C, d, D, Sing(\mathcal{F})_{finite}),$$

consider the proper smooth curve $\mathcal{C} := C\times X$, and in it the relative divisor $\mathcal{D}$ defined at "time f" by $D^{red} + Sing(\mathcal{F})finite + f^{-1}(0)$. Then $\mathcal{D}$ is finite etale over the base of constant degree

$$\#(D^{red}) + \#(Sing(\mathcal{F})finite) + d.$$

On $\mathcal{C} - \mathcal{D}$, we have the lisse sheaf $\mathcal{F}\otimes\mathcal{L}_{\chi(f)}$. Denote the projection

$$\pi : \mathcal{C} - \mathcal{D} \to Fct(C, d, D, Sing(\mathcal{F})_{finite}).$$

We have the following results.

1) The sheaves $R^i\pi_!(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})$ on $Fct(C, d, D, Sing(\mathcal{F})_{finite})$ vanish for $i\neq 1$, and $R^1\pi_!(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})$ is lisse [and, by proper base change, of formation compatible with arbitrary change of base].

2) The sheaves $R^i\pi_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})$ on $Fct(C, d, D, Sing(\mathcal{F})_{finite})$ vanish for $i\neq 1$, and $R^1\pi_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})$ is lisse, and of formation compatible with arbitrary change of base.

3) The image $\mathcal{G}$ of the natural "forget supports" map

$$R^1\pi_!(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}) \to R^1\pi_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})$$

is lisse, of formation compatible with arbitrary change of base. The stalk of $\mathcal{G}$ at the k–valued point "f" of $Fct(C, d, D, Sing(\mathcal{F})_{finite})$ is the cohomology group $H^1(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}))$.

4) If the irreducible middle extension $\mathcal{F}$ on C is orthogonally (respectively symplectically) self–dual, and χ has order two, then the lisse sheaf $\mathcal{G}$ on X is symplectically (respectively orthogonally) self–dual.

5) The rank of $\mathcal{G}$ is equal to

$$\begin{aligned}\mathrm{rank}(\mathcal{G}) = {} & (2g-2 + \deg(D))\mathrm{rank}(\mathcal{F}) \\ & + \textstyle\sum_{P_i \text{ in } D^{red}} \mathrm{Swan}_{P_i}(\mathcal{F}) + \sum_{s \text{ in } Sing(\mathcal{F})_{finite}} \mathrm{Swan}_s(\mathcal{F}) \\ & + \textstyle\sum_{s \text{ in } Sing(\mathcal{F})_{finite}} \mathrm{drop}_s(\mathcal{F}) \\ & + \textstyle\sum_{P_i \text{ in } D^{red}} \mathrm{drop}_{P_i}(\mathcal{F}(P_i)\otimes(\mathcal{L}_\chi a_i)(\infty, P_i)).\end{aligned}$$

6) We have the inequality

$$\mathrm{rank}(\mathcal{G}) \geq (2g-2 + \deg(D))\mathrm{rank}(\mathcal{F}) + \#\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}.$$

proof 1) By proper base change and the previous lemma, we have the vanishing of the $R^i\pi_!(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})$ for $i\neq1$. To show that $R^1\pi_!(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})$ is lisse, we apply Deligne's semicontinuity theorem [Lau–SC, 2.1.2], according to which it suffices to show the $\mathbb{Z}$–valued function which attaches to each k–valued point "f" of the base the sum of the Swan conductors of $\mathcal{F}\otimes\mathcal{L}_{\chi(f)}$ at all the points at infinity,

$$\begin{aligned} f \mapsto & \textstyle\sum_{P_i \text{ in } D^{\mathrm{red}}} \mathrm{Swan}_{P_i}(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}) \\ & + \textstyle\sum_{s \text{ in } \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}} \mathrm{Swan}_s(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}) \\ & + \textstyle\sum_{x \text{ in } f^{-1}(0)} \mathrm{Swan}_x(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}), \end{aligned}$$

is constant. As $\mathcal{L}_{\chi(f)}$ is rank one and everywhere tame, and $\mathcal{F}$ is lisse at every point of $f^{-1}(0)$, the terms at points of $f^{-1}(0)$ all vanish, and those at other points don't see the $\mathcal{L}_{\chi(f)}$. Thus the function is equal to the constant

$$\textstyle\sum_{P_i \text{ in } D^{\mathrm{red}}} \mathrm{Swan}_{P_i}(\mathcal{F}) + \sum_{s \text{ in } \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}} \mathrm{Swan}_s(\mathcal{F}).$$

Assertion 2) results by Poincaré duality from 1) for the dual sheaf $\mathcal{F}^\vee\otimes\mathcal{L}_{\overline{\chi}(f)}$. Once we have 1) and 2), $\mathcal{G}$ is lisse and of formation compatible with arbitrary change of base, being the image of a map of such sheaves on a smooth base X. That $\mathcal{G}$ has the asserted stalk at "f" amounts, by base change, to the fact that $H^1(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}))$ is the image of the "forget supports" map

$$\begin{aligned} & H^1_c(C - D - f^{-1}(0) - \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}, \mathcal{F}\otimes\mathcal{L}_{\chi(f)}) \\ & \quad \to H^1(C - D - f^{-1}(0) - \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}, \mathcal{F}\otimes\mathcal{L}_{\chi(f)}). \end{aligned}$$

Assertion 4) results from 1), 2), and 3), by Poincaré duality and standard properties of cup product. Because $\mathcal{G}$ is lisse, assertions 5) and 6) result from Lemma 5.1.3, applied to any single f in the parameter space $\mathit{Fct}(C, d, D, \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}})$. QED

Notation 5.2.2 When we want to keep in mind the twist genesis of the lisse sheaf $\mathcal{G}$ on $\mathit{Fct}(C, d, D, \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}})$ constructed in 5.2.1 above, we will denote it $\mathrm{Twist}_{\chi,C,D}(\mathcal{F})$:

(5.2.2.1) $$\mathcal{G} := \mathrm{Twist}_{\chi,C,D}(\mathcal{F}).$$

Remark 5.2.3 It will also be convenient to have the following variant on the above description of the sheaf $\mathcal{G} := \mathrm{Twist}_{\chi,C,D}(\mathcal{F})$ on the space

$$X := \mathit{Fct}(C, d, D, \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}).$$

Start as before with the lisse irreducible sheaf $\mathcal{F}\otimes\mathcal{L}_{\chi(f)}$ on $\mathcal{C} - \mathcal{D}$. The base X is itself lisse, of dimension $d + 1 - g$, so $\mathcal{F}\otimes\mathcal{L}_{\chi(f)}[d+2-g]$ is perverse irreducible on $\mathcal{C}-\mathcal{D}$. Denote by

$$j : \mathcal{C} - \mathcal{D} \to \mathcal{C}$$

the inclusion, and form the middle extension $j_{!*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}[d+2-g])$. Then according to [Ka–RLS, 2.7.2], if we denote by $\bar{\pi} : \mathcal{C} \to X$ the projection, we have

$$\mathcal{G}[d+1-g] = R\bar{\pi}_* j_{!*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}[d+2-g])$$
$$= \mathrm{image}(R\pi_!(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}[d+2-g]) \to R\pi_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}[d+2-g])),$$

where the image is taken in the category of perverse sheaves on X.

Lemma 5.2.4 With the notations of 5.2.1, denote by

$$j_1 : \mathcal{C} - \mathcal{D} \to \mathcal{C} - D^{red}\times X = (C - D)\times X$$

the inclusion. Then the middle extension of $\mathcal{F}\otimes\mathcal{L}_{\chi(f)}[d+2-g]$ by j_1 is the [shifted] literal tensor product

$$(j_1)_{!*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}[d+2-g]) = \mathcal{F}\otimes\mathcal{L}_{\chi(f)}[d+2-g]$$

on $(C-D)\times X$. Its formation commutes with arbitrary change of base on X.

proof We are forming the middle extension across two disjoint smooth divisors in $(C - D)\times X$, namely f=0 and $\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}\times X$. Consider the inclusions

$$j_2 : \mathcal{C} - \mathcal{D} \to \mathcal{C} - D^{red}\times X - \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}\times X,$$
$$j_3 : \mathcal{C} - D^{red}\times X - \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}\times X \to \mathcal{C} - D^{red}\times X.$$

Under j_2, we are extending across the divisor f=0. The sheaf $\mathcal{F}$ is lisse on the target $(\mathcal{C} - D^{red}\times X) - (\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}\times X)$, so we have

$$(j_2)_{!*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}[d+2-g]) \cong \mathcal{F}\otimes(j_2)_{!*}(\mathcal{L}_{\chi(f)}[d+2-g]).$$

To see that $(j_2)_{!*}(\mathcal{L}_{\chi(f)}[d+2-g]) = (j_{2!}\mathcal{L}_{\chi(f)})[d+2-g]$ amounts to showing that $j_{2*}\mathcal{L}_{\chi(f)}$ vanishes on f=0 (for then $j_{2*}\mathcal{L}_{\chi(f)}$ is lisse on f=0, and hence $(j_2)_{!*}(\mathcal{L}_{\chi(f)}[d+2-g]) = (j_{2*}\mathcal{L}_{\chi(f)})[d+2-g]$, but this latter is $(j_{2!}\mathcal{L}_{\chi(f)})[d+2-g]$). But near any point of f=0, f is part of a system of coordinates (f, coordinates for X), so by the Kunneth formula we are reduced to the fact that for $j : \mathbb{G}_m \to \mathbb{A}^1$ the inclusion, we have $j_!\mathcal{L}_\chi \cong j_*\mathcal{L}_\chi$.

When we extend by j_3, across $\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}\times X$, $\mathcal{L}_{\chi(f)}$ is lisse in a neighborhood of this divisor, we may pull it out, and then we are reduced, by Kunneth, to the fact that $\mathcal{F}$ on C–D is its own middle extension across $\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}$. QED

Variant Construction of $\mathcal{G}$:= Twist$_{\chi,C,D}(\mathcal{F})$ 5.2.5 (compare [Ka–RLS, 2.7.2]) Notations as in 5.2.1 above, over the parameter space

$$X := \mathit{Fct}(C, d, D, \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}),$$

consider the proper smooth curve $\mathcal{C} := C\times X$ over X and in it the product divisor $D^{red}\times X$. On the open curve $\mathcal{C} - D^{red}\times X = (C - D)\times X$, form the literal tensor product sheaf $\mathcal{F}\otimes\mathcal{L}_{\chi(f)}$. Denote by

$$j_\infty : (C - D)\times X \to C\times X$$

the inclusion.

Denote by

$$pr_2 : (C - D)\times X \to X = \mathit{Fct}(C, d, D, \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}})$$

and

$$\bar{\pi} : C\times X \to X$$

the projections. Then

1) The sheaves $R^i pr_{2!}(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})$ on $\mathit{Fct}(C, d, D, \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}})$ vanish for $i\neq 1$, and $R^1 pr_{2!}(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})$ is lisse.

2) The sheaves $R^i pr_{2*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})$ on $\mathit{Fct}(C, d, D, \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}})$ vanish for $i\neq 1$, and $R^1 pr_{2*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})$ is lisse, and of formation compatible with arbitrary change of base.

3) The perverse object $\mathcal{G}[d+1-g]$ on X is given by

$$\mathcal{G}[d+1-g] = R\bar{\pi}_* j_{\infty!*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}[d+2-g])$$
$$= \mathrm{image}(Rpr_{2!}(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})[d+2-g] \to Rpr_{2*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})[d+2-g]).$$

proof For 1), we see the vanishing fibre by fibre. The lisseness results from part 1) of the 5.2.1 via the long cohomology sequence for $Rpr_{2!}$ attached to the short exact sequence of sheaves

$$0 \to j_{1!}j_1{}^*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}) \to \mathcal{F}\otimes\mathcal{L}_{\chi(f)} \to \mathcal{F}\otimes\mathcal{L}_{\chi(f)} \mid (\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}\times X \to 0.$$

For 2), denote by $\mathcal{F}^\vee$ the middle extension sheaf dual to $\mathcal{F}$. By Lemma 5.2.4 above, applied to $\mathcal{F}^\vee$ and $\bar{\chi}$, $\mathcal{F}^\vee\otimes\mathcal{L}_{\bar{\chi}(f)}[d+2-g]$ is its own middle extension from $\mathcal{C}-\mathcal{D}$, so it is the Verdier dual of $\mathcal{F}\otimes\mathcal{L}_{\chi(f)}[d+2-g]$. So 2) for $\mathcal{F}\otimes\mathcal{L}_{\chi(f)}$ results from 1) for $\mathcal{F}^\vee\otimes\mathcal{L}_{\bar{\chi}(f)}$ by Poincaré duality. For 3), we already know (5.2.3) that

$$\mathcal{G}[d+1-g] = R\bar{\pi}_* j_{!*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}[d+2-g])$$

for j the inclusion of $\mathcal{C} - \mathcal{D}$ into $\mathcal{C}$. So by the transitivity of middle extension ($j_{!*} = j_{\infty!*}\circ j_{1!*}$) and Lemma 5.2.4, we get

$$\mathcal{G}[d+1-g] = R\bar{\pi}_* j_{\infty!*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}[d+2-g]).$$

That $R\bar{\pi}_* j_{\infty!*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}[d+2-g])$ is the image of the canonical map

$$Rpr_{2!}((\mathcal{F}\otimes\mathcal{L}_{\chi(f)})[d+2-g]) \to Rpr_{2*}((\mathcal{F}\otimes\mathcal{L}_{\chi(f)})[d+2-g])$$

is [Ka–RLS, 2.7.2]. QED

5.3 First properties of twist families: relation to middle additive convolution on $\mathbb{A}^1$

(5.3.1) We begin with a direct image formula, which, although elementary, is a fundamental reduction tool in what is to follow.

(5.3.2) Fix f in Fct(C, d, D, $\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}$). Thus f is a finite flat map from C−D to $\mathbb{A}^1$ = Spec(k[X]) of degree d, whose fibre over 0 consists of d distinct points, none of which lies in $\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}$. Denote by CritPt(f) ⊂ C−D the finite set of points in C−D at which df vanishes. Define

(5.3.2.1) $$\mathrm{CritVal}(f, \mathcal{F}) := f(\mathrm{CritPt}(f)) \cup f(\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}),$$

a finite subset of $\mathbb{A}^1$. Then for t in $\mathbb{A}^1$ − CritVal(f, $\mathcal{F}$), the function t−f lies in Fct(C, d, D, $\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}$), and so we have a morphism

(5.3.2.2) $$\mathbb{A}^1 - \mathrm{CritVal}(f, \mathcal{F}) \to \mathit{Fct}(C, d, D, \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}})$$

given by t ↦ t−f.

(5.3.3) What is the relation to convolution? We first explain the idea. For a good value t_0 of t, the stalk of $\mathcal{G}$ at t_0−f is the cohomology group

$$H^1(C, j_{\infty*}(\mathcal{F}\otimes\mathcal{L}_{\chi(t_0 - f)})) = \text{image of the "forget supports" map}$$
$$H_c^1(C-D, \mathcal{F}\otimes\mathcal{L}_{\chi(t_0 - f)}) \to H^1(C-D, \mathcal{F}\otimes\mathcal{L}_{\chi(t_0 - f)}).$$

Compute these cohomology groups on C−D by first mapping C−D to $\mathbb{A}^1$ by f. Since $\mathcal{L}_{\chi(t_0 - f)}$ is $f^*\mathcal{L}_{\chi(t_0-X)}$, the projection formula gives

$$H_c^1(C-D, \mathcal{F}\otimes\mathcal{L}_{\chi(t_0 - f)}) = H_c^1(\mathbb{A}^1, (f_*\mathcal{F})\otimes\mathcal{L}_{\chi(t_0 -X)}),$$
$$H^1(C-D, \mathcal{F}\otimes\mathcal{L}_{\chi(t_0 - f)}) = H^1(\mathbb{A}^1, (f_*\mathcal{F})\otimes\mathcal{L}_{\chi(t_0 -X)}).$$

So we get

$$H^1(C, j_{\infty*}(\mathcal{F}\otimes\mathcal{L}_{\chi(t_0 - f)}) = \text{image of the "forget supports" map}$$
$$H_c^1(\mathbb{A}^1, (f_*\mathcal{F})\otimes\mathcal{L}_{\chi(t_0 -X)}) \to H^1(\mathbb{A}^1, (f_*\mathcal{F})\otimes\mathcal{L}_{\chi(t_0 -X)}).$$

If we denote by $j_\infty : \mathbb{A}^1 \to \mathbb{P}^1$ the inclusion, this image is just $H^1(\mathbb{P}^1, j_{\infty*}((f_*\mathcal{F})\otimes\mathcal{L}_{\chi(t_0 -X)}))$. According to [Ka−RLS, 2.8.5], there is an open dense set in $\mathbb{A}^1$ such that for t_0 in this open dense set, $H^1(\mathbb{P}^1, j_{\infty*}((f_*\mathcal{F})\otimes\mathcal{L}_{\chi(t_0 -X)}))$ is the stalk at t_0 of the [shifted] middle additive convolution of $f_*\mathcal{F}$ with $\mathcal{L}_\chi$.

(5.3.4) Here is the precise result.

Proposition 5.3.5 Hypotheses and notations as in 5.2.1, fix f in Fct(C, d, D, $\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}$), viewed as a map from C−D to $\mathbb{A}^1$. Form the direct image sheaf $f_*(\mathcal{F}|C-D)$ on $\mathbb{A}^1$. The object

$$f_*(\mathcal{F}|C-D)[1]$$

on $\mathbb{A}^1$ is perverse. For $j : \mathbb{G}_m \to \mathbb{A}^1$ the inclusion, form the sheaf $j_*\mathcal{L}_\chi = j_!\mathcal{L}_\chi$ on $\mathbb{A}^1$, and the perverse object $j_*\mathcal{L}_\chi[1]$ on $\mathbb{A}^1$. Consider the middle additive convolution [Ka–RLS, 2.9]

$$f_*(\mathcal{F}|C-D)[1]*_{mid+}j_*\mathcal{L}_\chi[1]$$

on $\mathbb{A}^1$. On $\mathbb{A}^1 - \mathrm{CritVal}(f, \mathcal{F})$ we have a canonical isomorphism

$$([t \to t - f]^*\mathcal{G})[1] \cong (f_*(\mathcal{F}|C-D)[1])*_{mid+}j_*\mathcal{L}_\chi[1].$$

proof The sheaf $\mathcal{F}$ on C–D is a middle extension, so $\mathcal{F}[1]$ on C–D is perverse. Since f is a finite map, f_*(perverse) is perverse.

We use the description of $\mathcal{G}[d+1-g]$ as

$$\mathrm{image}(Rpr_{2!}((\mathcal{F}\otimes\mathcal{L}_{\chi(f)})[d+1-g]) \to Rpr_{2*}((\mathcal{F}\otimes\mathcal{L}_{\chi(f)})[d+1-g]))$$

on $Fct(C, d, D, \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}})$.

This description commutes with arbitrary change of base, so $([t \to t - f]^*\mathcal{G})[1]$ is

$$\mathrm{image}(Rpr_{2!}((\mathcal{F}\otimes\mathcal{L}_{\chi(t-f)})[1]) \to Rpr_{2*}((\mathcal{F}\otimes\mathcal{L}_{\chi(t-f)})[1])),$$

pr_2 the projection of $(C-D)\times(\mathbb{A}^1 - \mathrm{CritVal}(f, \mathcal{F}))$ to $\mathbb{A}^1 - \mathrm{CritVal}(f, \mathcal{F})$. Now factor this projection the composition of

$$f\times id: (C-D)\times(\mathbb{A}^1 - \mathrm{CritVal}(f, \mathcal{F})) \to \mathbb{A}^1\times(\mathbb{A}^1 - \mathrm{CritVal}(f, \mathcal{F}))$$

with the projection

$$pr_{2,\mathbb{A}} : \mathbb{A}^1\times(\mathbb{A}^1 - \mathrm{CritVal}(f, \mathcal{F}))\to (\mathbb{A}^1 - \mathrm{CritVal}(f, \mathcal{F})).$$

Since f is finite, we have $f_! = f_* = Rf_*$. The key point is that

$$\mathcal{L}_{\chi(t-f)} = (f\times id)^*\mathcal{L}_{\chi(t-X)}$$

and hence by the projection formula we find

$$\begin{aligned} Rpr_{2!}(\mathcal{F}\otimes\mathcal{L}_{\chi(t-f)}) &= Rpr_{2!}(\mathcal{F}\otimes(f\times id)^*\mathcal{L}_{\chi(t-X)}) \\ &= Rpr_{2,\mathbb{A}!}((f\times id)_!(\mathcal{F}\otimes(f\times id)^*\mathcal{L}_{\chi(t-X)})) \\ &= Rpr_{2,\mathbb{A}!}((f_!\mathcal{F})\otimes\mathcal{L}_{\chi(t-X)}) \\ &= Rpr_{2,\mathbb{A}!}((f_*\mathcal{F})\otimes\mathcal{L}_{\chi(t-X)}). \end{aligned}$$

Similarly we find

$$Rpr_{2*}(\mathcal{F}\otimes\mathcal{L}_{\chi(t-f)}) = Rpr_{2,\mathbb{A}*}((f_*\mathcal{F})\otimes\mathcal{L}_{\chi(t-X)}).$$

Thus we get that $([t \to t - f]^*\mathcal{G})[1]$ is

$$\begin{aligned} &\mathrm{image}(Rpr_{2!}((\mathcal{F}\otimes\mathcal{L}_{\chi(t-f)})[1]) \to Rpr_{2*}((\mathcal{F}\otimes\mathcal{L}_{\chi(t-f)})[1])) \\ &= \mathrm{image}(Rpr_{2,\mathbb{A}!}((f_*\mathcal{F})\otimes\mathcal{L}_{\chi(t-X)})[1]) \to Rpr_{2,\mathbb{A}*}((f_*\mathcal{F})\otimes\mathcal{L}_{\chi(t-X)})[1])). \end{aligned}$$

This last image is the restriction to $\mathbb{A}^1$ – CritVal(f, $\mathcal{F}$) of the middle additive convolution of $f_*\mathcal{F}$ and $\mathcal{L}_\chi$, thanks to [Ka–RLS, 2.7.2 and 2.8.4]. QED

Proposition 5.3.6 Hypotheses and notations as in 5.2.1, suppose we are in one of the following situations:
1a) Sing$(\mathcal{F})_{\text{finite}}$ is nonempty, deg(D) ≥ 2g+1, and char(k) ≠ 2.
1b) Sing$(\mathcal{F})_{\text{finite}}$ is nonempty, deg(D) ≥ 2g+3, and char(k) = 2.
2a) deg(D) ≥ 4g+2, and char(k) ≠ 2.
2b) deg(D) ≥ 4g+6, and char(k) = 2.
Then the lisse $\overline{\mathbb{Q}}_\ell$–sheaf $\mathcal{G}$ on *Fct*(C, d, D, Sing$(\mathcal{F})_{\text{finite}}$) is irreducible (or zero).

proof Suppose first that Sing$(\mathcal{F})_{\text{finite}}$ is nonempty. If char(k) ≠ 2 [resp. if char(k) = 2] pick a function f in Fct(C, d, D, Sing$(\mathcal{F})_{\text{finite}}$) which also lies in the dense open set U of Theorem 2.2.6 [resp. Theorem 2.4.2], applied with S taken to be Sing$(\mathcal{F})_{\text{finite}}$. Thus f as map from C–D to $\mathbb{A}^1$ is of Lefschetz type, and for each s in Sing$(\mathcal{F})_{\text{finite}}$, the fibre $f^{-1}(s)$ consists of d distinct points, only one of which lies in Sing$(\mathcal{F})_{\text{finite}}$. By the Irreducible Induction Criterion 3.3.1, $f_*(\mathcal{F}|C–D)$ is an irreducible middle extension on $\mathbb{A}^1$. By [Ka–RLS, 2.9.7], the middle additive convolution $(f_*(\mathcal{F}|C–D)[1])*_{\text{mid+}}j_*\mathcal{L}_\chi[1]$ on $\mathbb{A}^1$ is perverse irreducible. Hence its restriction to any dense open set of $\mathbb{A}^1$ is perverse irreducible (or zero).

We now turn to the case in which either char(k) ≠ 2 and deg(D) ≥ 4g+2, or char(k) = 2 and deg(D) ≥ 4g+6. Write D as the sum of two effective divisors $D = D_1 + D_2$, with both D_i having degree ≥ 2g+1 (resp. ≥ 2g+3 if char(k) = 2).

Since $\deg(D_1)$ ≥ 2g+1 (resp. ≥ 2g+3 if char(k) = 2), we may choose a function f_1 in Fct(C, $\deg(D_1)$, D_1, Sing$(\mathcal{F})\cup D^{\text{red}}$). Thus f_1 lies in $L(D_1)$, its divisor of poles is D_1, and it has $\deg(D_1)$ distinct zeroes, none of which lies in either Sing$(\mathcal{F})$ or in D. Fix one such f_1.

As $\deg(D_2)$ ≥ 2g+1 if char(k) ≠ 2 [resp. ≥ 2g+3 if char(k) = 2], we may pick a function f_2 in Fct(C, $\deg(D_2)$, D_2, Sing$(\mathcal{F})\cup D^{\text{red}}\cup f_1^{-1}(0)$) which lies in the open set U of Theorem 2.2.6 if char(k) ≠ 2 [resp. in the open set U of Theorem 2.4.2 if char(k) = 2] with respect to S the set $(\text{Sing}(\mathcal{F})\cup D^{\text{red}}\cup f_1^{-1}(0))\cap(C - D_2)$.

Thus f_2 has divisor of poles D_2, it has $\deg(D_2)$ distinct zeroes, none of which lies in Sing$(\mathcal{F})\cup D^{\text{red}}\cup f_1^{-1}(0)$, and for each zero α of f_1, the f_2–fibre containing it, $f_2^{-1}(f_2(\alpha))$, consists of $\deg(D_2)$ distinct points, of which only α is a zero of f_1, and none of which lies in D or in Sing$(\mathcal{F})$. In particular, the middle extension of $\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)}$ is lisse at every point of this fibre $f_2^{-1}(f_2(\alpha))$ other than at α itself. For any such f_2, the product f_1f_2 lies in the space $Fct(C, d, D, Sing(\mathcal{F})_{finite})$. Moreover, for most scalars t, the product $f_1(t - f_2)$ lies in the space $Fct(C, d, D, Sing(\mathcal{F})_{finite})$. Indeed, we need that $f_2 - t$ have d_2 distinct zeroes, all of which are disjoint from $div(f_1)$ and from $Sing(\mathcal{F})$. Let us denote

$$CritPt(f_2, f_1, \mathcal{F}) := (div(f_1) \cup Sing(\mathcal{F}) \cup (\text{zeroes of } df_2)) \cap (C - D_2),$$
$$CritVal(f_2, f_1, \mathcal{F}) := f_2(CritPt(f_2, f_1, \mathcal{F})).$$

Then for fixed f_1 and f_2 as above, we have a map

$$\mathbb{A}^1 - CritVal(f_2, f_1, \mathcal{F}) \to Fct(C, d, D, Sing(\mathcal{F})_{finite}),$$
$$t \mapsto f_1(t - f_2).$$

Proposition 5.3.7 Given an effective D of degree $d \geq 4g+2$ (resp. $d \geq 4g+6$ if char(k) = 2), write it as D_1+D_2 with both D_i effective of $deg(D_i) \geq 2g+1$ (resp. $\geq 2g+3$ if char(k) = 2). Fix

$$f_1 \text{ in } Fct(C, deg(D_1), D_1, Sing(\mathcal{F}) \cup D^{red}).$$

Fix a function f_2 in $Fct(C, deg(D_2), D_2, Sing(\mathcal{F}) \cup D^{red} \cup f_1^{-1}(0))$ which also lies in the open set U of Theorem 2.2.6 if char(k) $\neq$ 2 [resp. in the open set U of Theorem 2.4.2 if char(k) = 2] with respect to the set $S := (Sing(\mathcal{F}) \cup D^{red} \cup f_1^{-1}(0)) \cap (C - D_2)$. View f_2 as a finite flat map from $C - D_2$ to $\mathbb{A}^1$. For i=1,2, denote by

$$j_i : C - D \to C - D_i$$

the inclusion. Start with the sheaf $\mathcal{F} \otimes \mathcal{L}_{\chi(f_1)}$ on C– D, form its direct image $j_{2*}(\mathcal{F} \otimes \mathcal{L}_{\chi(f_1)})$ on $C-D_2$, and take its direct image $f_{2*}j_{2*}(\mathcal{F} \otimes \mathcal{L}_{\chi(f_1)})$ on $\mathbb{A}^1$. The object $f_{2*}j_{2*}(\mathcal{F} \otimes \mathcal{L}_{\chi(f_1)})[1]$ on $\mathbb{A}^1$ is perverse. For $j : \mathbb{G}_m \to \mathbb{A}^1$ the inclusion, form the sheaf $j_*\mathcal{L}_\chi = j_!\mathcal{L}_\chi$ on $\mathbb{A}^1$, and the perverse object $j_*\mathcal{L}_\chi[1]$ on $\mathbb{A}^1$. Consider the middle additive convolution [Ka–RLS, 2.9]

$$f_{2*}j_{2*}(\mathcal{F} \otimes \mathcal{L}_{\chi(f_1)})[1] *_{mid+} j_*\mathcal{L}_\chi[1]$$

on $\mathbb{A}^1$. On $\mathbb{A}^1 - CritVal(f_2, f_1, \mathcal{F})$, we have a canonical isomorphism

$$([t \mapsto f_1(t - f_2)]^*\mathcal{G})[1] \cong (f_{2*}j_{2*}j_1{}^*(\mathcal{F} \otimes \mathcal{L}_{\chi(f_1)})[1]) *_{mid+} j_*\mathcal{L}_\chi[1].$$

proof of 5.3.7 We work over the space

$$T := \mathbb{A}^1 - CritVal(f_2, f_1, \mathcal{F}).$$

For i=1, 2, denote by $j_{i,\infty}$ the inclusion

$$j_{i,\infty} : C - D_i \to C.$$

We know that $([t \mapsto f_1(t - f_2)]^*\mathcal{G})[1]$ on T is given in terms of the projections

$$pr_{2,D} : (C-D)\times T \to T$$

and

$$pr_2 : C\times T \to T$$

as

$$\mathrm{image}(Rpr_{2,D!}((\mathcal{F}\otimes\mathcal{L}_{\chi(f_1(t-f_2))})[2]) \to Rpr_{2,D*}((\mathcal{F}\otimes\mathcal{L}_{\chi(f_1(t-f_2))})[2]))$$
$$= Rpr_{2*}((j_\infty\times id)_{!*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1(t-f_2))}[2]))$$
$$= Rpr_{2*}((j_{2,\infty}\times id)_{!*}(j_2\times id)_{!*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1(t-f_2))}[2])).$$

Now $(j_2\times id)_{!*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1(t-f_2))}[1])$ means extending across points which are in D_1 but not in D_2, and $\mathcal{L}_{\chi(t-f_2)}$ is lisse near such points [simply because t does not lie in $f_2((\mathrm{div}(f_1)\cap(C - D_2))]$. So

$$(j_2\times id)_{!*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1(t-f_2))}[1]) = j_{2*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)})[1]\otimes\mathcal{L}_{\chi(t-f_2)}.$$

Thus $([t \mapsto f_1(t - f_2)]^*\mathcal{G})[1]$ on T is

$$Rpr_{2*}(j_{2,\infty}\times id)_{!*}((j_{2*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)})[1]\otimes\mathcal{L}_{\chi(t-f_2)}[1]).$$

Denote $\overline{f_2} := f_2$ viewed as a map of C to $\mathbb{P}^1$. Compute Rpr_{2*} by factoring pr_2 as

$$\overline{f_2}\times id: C\times T \to \mathbb{P}^1\times T$$

followed by

$$pr_{2,\mathbb{P}}: \mathbb{P}^1\times T \to T.$$

Thus

$$Rpr_{2*}(j_{2,\infty}\times id)_{!*}(j_{2*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)})[1]\otimes\mathcal{L}_{\chi(t-f_2)}[1]).$$
$$= Rpr_{2,\mathbb{P}*}(f_2\times id)_*(j_{2,\infty}\times id)_{!*}(j_{2*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)})[1]\otimes\mathcal{L}_{\chi(t-f_2)}[1])$$

In terms of the inclusion

$$j_{\mathbb{A}} : \mathbb{A}^1 \to \mathbb{P}^1$$

and

$$f_2 : C-D_2 \to \mathbb{A}^1$$

we have a cartesian diagram

$$\begin{array}{ccc} C - D_2 & \xrightarrow{j_{2,\infty}} & C \\ f_2 \downarrow & & \downarrow \overline{f_2} \\ \mathbb{A}^1 & \xrightarrow[j_{\mathbb{A}}]{} & \mathbb{P}^1 \end{array}$$

in which the horizontal maps are affine open immersions, and the vertical maps are finite. So we have

$$(\overline{f}_2\times id)_*(j_{2,\infty}\times id)_{!*} = (j_{\mathbb{A}}\times id)_{!*}(f_2\times id)_{!*}.$$

So we get

$$Rpr_{2,\mathbb{P}*}(\overline{f}_2\times id)_*(j_{2,\infty}\times id)_{!*}(j_{2*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)})[1]\otimes\mathcal{L}_{\chi(t-f_2)}[1])$$
$$= Rpr_{2,\mathbb{P}*}(j_{\mathbb{A}}\times id)_{!*}(f_2\times id)_*(j_{2*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)})[1]\otimes\mathcal{L}_{\chi(t-f_2)}[1]).$$

Now $\mathcal{L}_{\chi(t-f_2)}$ is $f_2{}^*\mathcal{L}_{\chi(t-X)}$, so by the projection formula we may rewrite this last expression as

$$= Rpr_{2,\mathbb{P}*}(j_{\mathbb{A}}\times id)_{!*}(\mathcal{L}_{\chi(t-X)}[1]\otimes(f_{2*}j_{2*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)})[1])).$$

By [Ka–RLS, 2.9.2], this is (restriction to T of) the asserted middle convolution. QED for 5.3.7

Once we have Proposition 5.3.7, then to prove the irreducibility of $\mathcal{G}$ it suffices to show that $f_{2*}j_{2*}j_1{}^*(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)})$ is an irreducible middle extension. This is immediate from the Irreducible Induction Criterion 3.3.1, since the singularities of $j_{2*}j_1{}^*(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)})$ on $C - D_2$ include the $\deg(D_1)$ distinct zeroes of f_1, and the f_2–fibre containing each of these zeroes consists of $\deg(D_2)$ distinct points, precisely one of which, namely that zero, is a singularity of $j_{2*}j_1{}^*(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)})$. QED

5.4 Theorems of big monodromy in characteristic not 2

Theorem 5.4.1 Let k be an algebraically closed field of characteristic not 2, C/k a proper, smooth connected curve of genus g. Suppose that $D = \sum a_iP_i$ is an effective divisor of degree $d \geq 2g+1$, with all a_i invertible in k. Let $\mathcal{F}$ be an irreducible middle extension sheaf on C with $Sing(\mathcal{F})_{finite} := Sing(\mathcal{F})\cap(C-D)$ nonempty. Suppose that either $\mathcal{F}$ is everywhere tame, or that $\mathcal{F}$ is tame at all points of D and that the characteristic p is either zero or a prime $p \geq rank(\mathcal{F}) + 2$. Suppose that the following inequalities hold:

if $rank(\mathcal{F}) = 1$, $\quad 2g-2+d \geq Max(2\#Sing(\mathcal{F})_{finite}, 4rank(\mathcal{F}))$,

if $rank(\mathcal{F}) \geq 2$, $\quad 2g-2+d \geq Max(2\#Sing(\mathcal{F})_{finite}, 72rank(\mathcal{F}))$.

Fix a nontrivial character χ whose finite order $n \geq 2$ is invertible in k. Form the lisse sheaf

$$\mathcal{G} := Twist_{\chi,C,D}(\mathcal{F})$$

on the space *Fct*(C, d, D, $Sing(\mathcal{F})_{finite}$).

Pick a function f in Fct(C, d, D, $Sing(\mathcal{F})_{finite}$) which also lies in the dense open set U of Theorem 2.2.6 applied with S taken to be $Sing(\mathcal{F})_{finite}$. Thus f as map from C–D to $\mathbb{A}^1$ is of Lefschetz type, and for each s in $Sing(\mathcal{F})_{finite}$, the fibre $f^{-1}(s)$ consists of d distinct points, only one of which lies in $\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}$. Consider the lisse $\overline{\mathbb{Q}}_\ell$–sheaf $\mathcal{H}$ on $\mathbb{A}^1 - \mathrm{CritVal}(f, \mathcal{F})$ given by

$$\mathcal{H} := [t \mapsto t-f]^* \mathcal{G},$$

i.e., by

$$t \mapsto H^1(C, j_*(\mathcal{F} \otimes \mathcal{L}_{\chi(t-f)}).$$

Its geometric monodromy group G_{geom} is either Sp or SO or O, or G_{geom} contains SL. If $\mathcal{F}$ is orthogonally (respectively symplectically) self–dual, and χ has order 2, then G_{geom} is Sp (respectively SO or O). If χ has order ≥ 3, then G_{geom} contains SL.

proof Let us put $r := \mathrm{rank}(\mathcal{F})$, $m := \#\, \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}$. We have seen (5.3.5) that $\mathcal{H}$ is the restriction to $\mathbb{A}^1 - \mathrm{CritVal}(f, \mathcal{F})$ of the middle additive convolution of $f_*\mathcal{F}$ and $\mathcal{L}_\chi$.

Let us put

$$\mathcal{F}_1 := f_*\mathcal{F}.$$

We have seen above (in the proof of 5.3.6) that $\mathcal{F}_1$ is an irreducible middle extension on $\mathbb{A}^1$. Notice that $\mathcal{F}_1$ lies in the class $\mathcal{P}_{\mathrm{conv}}$, cf. 4.0.2, because its rank is ≥ 3. [Indeed, its rank is $d \times \mathrm{rank}(\mathcal{F}) \geq d$. If $g > 0$, then the hypothesis that $d \geq 2g+1$ gives $d \geq 3$. If $g = 0$, the hypothesis

$$2g-2+d \geq \mathrm{Max}(2\#\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}, 4\mathrm{rank}(\mathcal{F})).$$

gives $d \geq 6$.]

The sheaf $\mathcal{F}_1$ is tame at ∞, because $\mathcal{F}$ is tame at all the poles of f, and the poles of f all have order prime to p. Moreover, the $I(\infty)$–invariants are given by

$$\mathcal{F}_1(\infty)^{I(\infty)} \cong \oplus_{\text{points } P_i \text{ in } D}\, \mathcal{F}(P_i)^{I(P_i)}.$$

Over each critical value α of f, $\mathcal{F}$ is lisse, and $f-\alpha$ has one and only one double zero, so the local monodromy of $\mathcal{F}_1$ at α is quadratic of drop r, with scale the unique character of order 2:

$$\mathcal{F}_1(\alpha)/\mathcal{F}_1(\alpha)^{I(\alpha)} \cong \text{r copies of} \mathcal{L}_{\chi_2(x-\alpha)}.$$

Over the m images $\delta = f(\beta)$ of points β in $\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}$, f is finite etale, and β is the unique point of $\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}$ in the fibre, so the local monodromy of $\mathcal{F}_1$ at δ has drop $\leq r$. More precisely, we have

$$\mathcal{F}_1(\delta)/\mathcal{F}_1(\delta)^{I(\delta)} \cong \mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)},$$

where we use f to identify $I(\delta)$ with $I(\beta)$.

At all other points of $\mathbb{A}^1$, i.e., on $\mathbb{A}^1 - \mathrm{CritVal}(f, \mathcal{F})$, $\mathcal{F}_1$ is lisse. Moreover, if $\mathcal{F}$ is everywhere tame on C, then $\mathcal{F}_1$ is everywhere tame. Now form $\mathcal{H}$, the middle additive convolution of $\mathcal{F}_1$ with $\mathcal{L}_\chi$. Thus $\mathcal{H}$ is tame at ∞ (by 4.1.10, part 2d)), and it is everywhere tame if $\mathcal{F}$ is everywhere tame (by 4.1.10, parts 1b) and 2d)). By 5.2.1, part 6), we have the inequality

$$\mathrm{rank}(\mathcal{H}) \geq (2g-2+d)r + \#\mathrm{Sing}_{\mathrm{finite}}(\mathcal{F}) > (2g-2+d)r.$$

The local monodromy of $\mathcal{H}$ at each of the m images $\delta = f(\beta)$ of points β in $\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}$ has drop $\leq$ r, by 4.1.10, part 1c), and is given by

$$\mathcal{H}(\delta)/\mathcal{H}(\delta)^{I(\delta)} \cong MC_{\chi}\mathrm{loc}(\delta)(\mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)} \text{ as } I(\delta)\text{–rep'n.}$$

The local monodromy of $\mathcal{H}$ at each critical value α of f is quadratic of drop r, with scale the character $\chi\chi_2$:

$$\mathcal{H}(\alpha)/\mathcal{H}(\alpha)^{I(\alpha)} \cong \mathcal{L}_{\chi(x-\alpha)}\otimes(\text{r copies of } \mathcal{L}_{\chi_2(x-\alpha)})$$
$$\cong \text{r copies of } \mathcal{L}_{\chi\chi_2(x-\alpha)}.$$

The key observation here is that $\chi\chi_2$ is **not** of order two, and that f **has** critical points (because their number, the number of zeroes of df, is

$$2g-2+ \Sigma_i(1+a_i) > 2g-2+d > 2\#\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}} > 2 > 0).$$

The conclusion follows from Theorem 1.5.1 (and Theorem 1.7.1, if r=1), applied to the data (r, m, $\mathcal{H}$). QED

Proposition 5.4.2 Hypotheses and notations as in Theorem 5.4.1 above, suppose that χ has order 2, but $\mathcal{F}$ is not self–dual. Then G_{geom} contains SL.

proof If not, then by the paucity of choice, G_{geom} is contained in either Sp or O, and hence $\mathcal{H}$ is self–dual. But $\mathcal{H}$ is the middle convolution of $f_*\mathcal{F}$ and $\mathcal{L}_\chi$. As χ has order 2, we recover $f_*\mathcal{F}$ as the middle convolution of $\mathcal{H}$ and $\mathcal{L}_\chi$. As χ has order 2, $\mathcal{L}_\chi$ is self–dual. As both $\mathcal{H}$ and $\mathcal{L}_\chi$ are self–dual, so is their middle convolution, $f_*\mathcal{F}$. By Proposition 3.4.1, the autoduality of $f_*\mathcal{F}$ implies that of $\mathcal{F}$, contradiction. QED

Proposition 5.4.3 Hypotheses and notations as in Theorem 5.4.1 above, suppose that χ has order 2, and that $\mathcal{F}$ is symplectically self–dual.

1) Suppose there exists a finite singularity β of $\mathcal{F}$, i.e., a point β in $\mathrm{Sing}(\mathcal{F})\cap(C-D)$, such that the following two conditions hold:

1a) $\mathcal{F}$ is tame at β,

1b) $\mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)}$ has odd dimension.

Then the group G_{geom} for the sheaf $\mathcal{H}$ is the full orthogonal group O.

2) Suppose that $\mathcal{F}$ is everywhere tame. Then G_{geom} for $\mathcal{H}$ is the special orthogonal group SO if and only if $\mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)}$ has even dimension for every finite singularity β of $\mathcal{F}$.

proof In terms of $\mathcal{F}_1 := f_*\mathcal{F}$, $\mathcal{H}$ on $\mathbb{A}^1 - \mathrm{CritVal}(f, \mathcal{F})$ is (the restriction from $\mathbb{A}^1$ of) the middle convolution $\mathcal{F}_1 *_{\mathrm{mid}+} \mathcal{L}_\chi$. We already know that G_{geom} for $\mathcal{H}$ is either SO or O, so we have only to see whether $\det(\mathcal{H})$ is trivial or not. Since $\det(\mathcal{H})$ is either trivial or of order 2, it is **tame** on $\mathbb{A}^1$ – CritVal(f, $\mathcal{F}$). Hence $\det(\mathcal{H})$ is trivial if and only if it is trivial on every **finite** inertia group $I(\gamma)$, γ in CritVal(f, $\mathcal{F}$).

At γ which is a critical value α of f, we have seen that the local monodromy of $\mathcal{F}_1$ at α is quadratic of drop r := rank($\mathcal{F}$), with scale the unique character of order 2:

$$\mathcal{F}_1(\alpha)/\mathcal{F}_1(\alpha)^{I(\alpha)} \cong \text{r copies of} \mathcal{L}_{\chi_2(x-\alpha)}.$$

The local monodromy of $\mathcal{H} = \mathcal{F}_1 *_{\text{mid}+} \mathcal{L}_\chi$ at α is given by

$$\mathcal{H}(\alpha)/\mathcal{H}(\alpha)^{I(\alpha)} \cong (\mathcal{F}_1(\alpha)/\mathcal{F}_1(\alpha)^{I(\alpha)}) \otimes \mathcal{L}_{\chi(x-\alpha)}$$
$$\cong \text{r copies of } \mathbb{1},$$

this last equality because χ is the quadratic character χ_2. From this we calculate

$$\det(\mathcal{H}(\alpha)) = \det(\mathcal{H}(\alpha)/\mathcal{H}(\alpha)^{I(\alpha)}) = \mathbb{1}.$$

Thus the local monodromy of $\det(\mathcal{H})$ is trivial at all the critical values of f.

At γ which is the image $\delta = f(\beta)$ of a point β in $\mathrm{Sing}(\mathcal{F})_{\text{finite}}$, we have seen that

$$\mathcal{F}_1(\delta)/\mathcal{F}_1(\delta)^{I(\delta)} \cong \mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)}$$

where we use f to identify $I(\delta)$ with $I(\beta)$. Using this identification, the local monodromy of $\mathcal{H} = \mathcal{F}_1 *_{\text{mid}+} \mathcal{L}_\chi$ at δ is

$$\mathcal{H}(\delta)/\mathcal{H}(\delta)^{I(\delta)} \cong \mathrm{MC}_\chi\mathrm{loc}(\delta)(\mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)}) \text{ as } I(\delta)\text{–rep'n}.$$

If $\mathcal{F}$ is tame at β, we have

$$\mathcal{H}(\delta)/\mathcal{H}(\delta)^{I(\delta)} \cong (\mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)}) \otimes \mathcal{L}_{\chi(x-\delta)}.$$

We then readily compute

$$\det(\mathcal{H}(\delta)) = \det(\mathcal{H}(\delta)/\mathcal{H}(\delta)^{I(\delta)})$$
$$= \det((\mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)}) \otimes \mathcal{L}_{\chi(x-\delta)})$$
$$= \det((\mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)})) \otimes (\mathcal{L}_{\chi(x-\delta)})^{\dim(\mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)})}.$$

But we have

$$\det((\mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)})) = \det(\mathcal{F}(\beta)) = \mathbb{1},$$

this last equality because $\mathcal{F}$ is symplectic, and $\mathrm{Sp} \subset \mathrm{SL}$. Thus we find

$$\det(\mathcal{H}(\delta)) = (\mathcal{L}_{\chi(x-\delta)})^{\dim(\mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)})}.$$

Thus $\det(\mathcal{H})$ is nontrivial at the image $\delta = f(\beta)$ of a point β in $\mathrm{Sing}(\mathcal{F})_{\text{finite}}$ at which $\mathcal{F}$ is tame, if and only if $\mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)}$ has odd dimension. This proves 1).

Suppose now that $\mathcal{F}$ is everywhere tame. We already know that $\det(\mathcal{H})$ is trivial at all the critical values of f, so $\det(\mathcal{H})$ is trivial if and only if it is trivial at every $\delta = f(\beta)$, β in $\mathrm{Sing}(\mathcal{F})_{\text{finite}}$. For $\mathcal{F}$ everywhere tame, this triviality at every $\delta = f(\beta)$ means precisely that $\mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)}$ has even dimension for all finite singularities β of $\mathcal{F}$. QED

Remark 5.4.4 Here is an example to show that part 2) of the above proposition can fail if we drop the hypothesis that $\mathcal{F}$ be everywhere tame. We fix an even integer $2n \geq 2$, and work over $\overline{\mathbb{F}}_p$ for any prime $p \geq 2n+2$. Fix a nontrivial $\overline{\mathbb{Q}}_\ell$–valued additive character ψ of $\mathbb{F}_p$. Denote by Kl_{2n} the standard Kloosterman sheaf in 2n variables: thus Kl_{2n} is the lisse sheaf of rank 2n on $\mathbb{G}_m/\mathbb{F}_p$ whose trace function at a point α in $E^\times$, E a finite extension E of $\mathbb{F}_p$, is

$$\mathrm{Trace}(\mathrm{Frob}_{\alpha,E} \mid Kl_{2n}) = -\Sigma_{x_1x_2...x_{2n}=\alpha \text{ in } E}\, \psi(\Sigma x_i).$$

One knows that Kl_{2n} is symplectically self–dual.

Take $\mathcal{F}$ the middle extension of the lisse sheaf $[x \mapsto 1/x]^*Kl_{2n}$ on $\mathbb{G}_m$. One knows that $Kl_{2n}(\infty)$ is a totally wild irreducible representation of $I(\infty)$, all of whose slopes are 1/2n. Thus $\mathcal{F}$ is totally wild at zero, and hence $\mathcal{F}(0)/\mathcal{F}(0)^{I(0)}$ has even dimension 2n.

We take C to be $\mathbb{P}^1/\overline{\mathbb{F}}_p$, D to be $d\infty$ for a sufficiently large integer d prime to p, χ to be the quadratic character χ_2, and $\mathcal{F}$ as above. Then $\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}$ is {0}, and, as noted above, $\mathcal{F}(0)/\mathcal{F}(0)^{I(0)}$ has even dimension 2n. Nonetheless, we will see that G_{geom} for $\mathcal{H}$ is the full orthogonal group O. More precisely, with $\delta := f(0)$, we will show that $\det(\mathcal{H})$ is nontrivial at δ. To simplify the notations, let us replace f by $f - \delta$, so that $f(0) = 0$. Then we have

$$\mathcal{H}(0)/\mathcal{H}(0)^{I(0)} \cong MC_\chi loc(0)(\mathcal{F}(0)/\mathcal{F}(0)^{I(0)}).$$

We will show that $\det(\mathcal{H}(0))$ is $\mathcal{L}_\chi$. We have

$$\det(\mathcal{H}(0)) = \det(\mathcal{H}(0)/\mathcal{H}(0)^{I(0)}) = \det(MC_\chi loc(0)(\mathcal{F}(0)/\mathcal{F}(0)^{I(0)})).$$

We will calculate $MC_\chi loc(0)(\mathcal{F}(0)/\mathcal{F}(0)^{I(0)})$ by a global argument. The sheaf $\mathcal{F}$ on $\mathbb{A}^1$ lies in $\mathcal{P}_{\mathrm{conv}}$ of 4.0.2. We define

$$\mathcal{G} := \mathcal{F} *_{\mathrm{mid}+} \mathcal{L}_\chi \text{ in } \mathcal{P}_{\mathrm{conv}}.$$

Then by Theorem 4.1.10, part 1) we have

$$\mathcal{G}(0)/\mathcal{G}(0)^{I(0)} \cong MC_\chi loc(0)(\mathcal{F}(0)/\mathcal{F}(0)^{I(0)}).$$

Thus

$$\begin{aligned}\det(\mathcal{H}(0)) &= \det(MC_\chi loc(0)(\mathcal{F}(0)/\mathcal{F}(0)^{I(0)}))\\ &= \det(\mathcal{G}(0)/\mathcal{G}(0)^{I(0)})\\ &= \det(\mathcal{G}(0)).\end{aligned}$$

Hence we are reduced to showing that $\det(\mathcal{G}(0))$ is $\mathcal{L}_\chi$.

Applying Fourier transform FT ($:= FT_\psi$) to the defining equation

$$\mathcal{G} := \mathcal{F} *_{\mathrm{mid}+} \mathcal{L}_\chi,$$

we obtain

$$FT(\mathcal{G}) = j_*(FT(\mathcal{F}) \otimes \mathcal{L}_\chi \mid \mathbb{G}_m).$$

The key observation is that, because $\mathcal{F}$ is $[x \mapsto 1/x]^*\mathrm{Kl}_{2n}$, we have

$$\mathrm{FT}(\mathcal{F}) \cong \mathrm{Kl}_{2n+1},$$

a remark due to Deligne [De–AFT, 7.1.4] and developed in [Ka–ESDE, 8.1.12 and 8.4.3]. Thus we find

$$\mathrm{FT}(\mathcal{G}) = j_*(\mathrm{FT}(\mathcal{F})\otimes\mathcal{L}_\chi \mid \mathbb{G}_m) = j_!\mathrm{Kl}_{2n+1}(\chi, \chi, ..., \chi).$$

We can calculate $\mathrm{FT}(j_!\mathrm{Kl}_{2n+1}(\chi, \chi, ..., \chi))$ as a hypergeometric sheaf of type (1, 2n+1), cf. [Ka–ESDE, 9.3.2 with d=1]. The result is

$$\mathrm{FT}(j_!\mathrm{Kl}_{2n+1}(\chi, \chi, ..., \chi)) \cong j_*\mathcal{H}\mathrm{yp}(\mathbb{1}; \chi, ..., \chi).$$

Since FT is involutive, we find a geometric isomorphism

$$[x \mapsto -x]^*\mathcal{G} \cong j_*\mathcal{H}\mathrm{yp}(\mathbb{1}; \chi, ..., \chi).$$

So to show that $\det(\mathcal{G}(0))$ is $\mathcal{L}_\chi$, it is equivalent to show that $\det(\mathcal{H}\mathrm{yp}(\mathbb{1}; \chi, ..., \chi))(0)$ is $\mathcal{L}_\chi$.

The sheaf $\mathcal{H}\mathrm{yp}(\mathbb{1}; \chi, ..., \chi)$ is lisse on $\mathbb{G}_m$. Its local monodromy at ∞ is $\mathcal{L}_\chi\otimes\mathrm{Unip}(2n+1)$, whose determinant is $\mathcal{L}_\chi$ (remember χ is χ_2). Its local monodromy at 0 is $\mathbb{1}\oplus W$, where W has rank 2n and all slopes 1/2n. Since all slopes at 0 are < 1, $\det(\mathcal{H}\mathrm{yp}(\mathbb{1}; \chi, ..., \chi))$ is tame at 0. Thus $\det(\mathcal{H}\mathrm{yp}(\mathbb{1}; \chi, ..., \chi))$ is lisse on $\mathbb{G}_m$, tame at both 0 and ∞, and agrees with $\mathcal{L}_\chi$ at ∞. Therefore we have a global isomorphism

$$\det(\mathcal{H}\mathrm{yp}(\mathbb{1}; \chi, ..., \chi)) \cong \mathcal{L}_\chi \text{ on } \mathbb{G}_m/\overline{\mathbb{F}}_p.$$

In particular, $\det(\mathcal{H}\mathrm{yp}(\mathbb{1}; \chi, ..., \chi))(0)$ is $\mathcal{L}_\chi$.

Here is a further elaboration on this sort of counterexample. With 2n, p, and d fixed as above, choose further an **odd** integer $k \geq 1$ which is prime to p. Now define $\mathcal{F}$ to be the middle extension of the lisse sheaf $[x \mapsto 1/x^k]^*\mathrm{Kl}_{2n}$ on $\mathbb{G}_m$. Then $\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}$ is $\{0\}$, $\mathcal{F}$ is totally wild at 0, and $\mathcal{F}(0)/\mathcal{F}(0)^{I(0)}$ has even dimension 2n. Using [Ka–ESDE, 9.3.2 with d=k], a similar argument now shows that $\mathcal{H}$ has G_{geom} the full orthogonal group, and that $\det(\mathcal{H})$ is nontrivial at 0.

(5.4.5) We will now give another one–parameter family of twists with big monodromy. Before stating the result, we need an elementary lemma.

Lemma 5.4.6 Let k be an algebraically closed field of any characteristic, C/k a proper, smooth connected curve of genus g. Suppose that $D = \sum a_i P_i$ is an effective divisor of degree d. Suppose d_1 and d_2 are positive integers with $d_1 + d_2 = d$. If k has characteristic $p > 0$, suppose further that $d_2/d \leq (p-1)/p$. Then we can write D as a sum of effective divisors $D_1 + D_2$ with D_2 of degree either d_2 or $d_2 + 1$, such that $D_2 = \sum c_i P_i$, has all its nonzero c_i invertible in k. Moreover, if $d_2/d > 1/3$, then D_2 can be chosen so that, in addition, for any point P_i which occurs in D but not in D_2, we have $a_i \leq 2$.

proof If k has characteristic zero, any writing of D as a sum of effective divisors $D_1 + D_2$ with D_2 of degree d_2 does the job.

If k has characteristic $p > 0$, put $\lambda := d_2/d$. For real $x \geq 0$, we denote its "floor" and "ceiling"

$$[x]_{fl} := \text{the greatest integer} \leq x,$$
$$[x]_{ce} := \text{the least integer} \geq x.$$

Since $\lambda \leq 1$, we have, for each i,

$$a_i \geq [\lambda a_i]_{ce} \geq \lambda a_i \geq [\lambda a_i]_{fl}.$$

We define effective divisors D_{fl} and D_{ce} by

$$D_{fl} := \Sigma_i [\lambda a_i]_{fl} P_i, \; D_{ce} := \Sigma_i [\lambda a_i]_{ce} P_i.$$

Thus $D \geq D_{ce} \geq D_{fl}$, and $\deg(D_{ce}) \geq d_2 \geq \deg(D_{fl})$. For each i, the coefficients $[\lambda a_i]_{ce}$ and $[\lambda a_i]_{fl}$ are either equal or differ by 1. So we can choose, for each i, either $[\lambda a_i]_{ce}$ and $[\lambda a_i]_{fl}$, call it b_i, so that the "intermediate" divisor $D_{int} := \Sigma_i b_i P_i$ has degree d_2. Clearly

$$D_{ce} \geq D_{int} \geq D_{fl}.$$

If D_{int} has all its nonzero b_i invertible in k, we take D_2 to be D_{int}. Then D_2 will have degree d_2.

If some of the nonzero b_i are divisible by p, we modify D_{int} as follows. First of all, if p divides a nonzero b_i, then $b_i \geq p$, so $b_i - 1$ is positive and prime to p. What about $b_i + 1$? It is prime to p, but is $b_i + 1 \leq a_i$? In other words, is $b_i < a_i$? The answer is yes, because if not, then $b_i = a_i$. But $a_i \geq [\lambda a_i]_{ce} \geq b_i$, so we would have $a_i = [\lambda a_i]_{ce}$. This means in turn that $\lambda a_i > a_i - 1$, i.e., $1 > a_i(1-\lambda)$. But p divides b_i, so $a_i \geq p$, and so $1 > p(1-\lambda)$, which contradicts the hypothesis $\lambda \leq (p-1)/p$.

So each nonzero b_i that is divisible by p can be either increased by 1 or decreased by 1 and continue to lie in the range $[0, a_i]$. If there are evenly many indices i whose b_i is divisible by p, increase half of them by 1 and decrease the other half by 1, to get the desired D_2: it has degree d_2. If there are oddly many b_i divisible by p, group all but one in pairs, and in each pair increase one member by 1 and decrease the other by 1. Increase the leftover by 1. This gives a D_2 of degree $1+d_2$.

It remains to prove the "moreover". Suppose now that $\lambda := d_2/d > 1/3$. Let P_i be a point which occurs in D but which has $c_i = 0$. Recall that

1) b_i is either $[\lambda a_i]_{ce}$ or $[\lambda a_i]_{ce} - 1$,

2) c_i is b_i unless we are in positive characteristic p and b_i is a nonzero multiple of p, in which case c_i is $b_i \pm 1$.
If we are in positive characteristic p and b_i is a nonzero multiple of p, then $c_i \geq p-1$, so c_i is nonzero. Thus $c_i = 0$ implies $b_i = 0$, which in turn implies that either $[\lambda a_i]_{ce} = 0$ or $[\lambda a_i]_{ce} = 1$. So if $c_i = 0$, we have $\lambda a_i \leq 1$, and hence $a_i \leq 1/\lambda$. Since $\lambda > 1/3$, we have $1/\lambda < 3$, and so $a_i < 3$. As a_i is an integer, we have $a_i \leq 2$, as asserted. QED

Remark 5.4.7 The example of a divisor D of the form $\Sigma_i pP_i$, which has all its $a_i = p$, shows that the hypothesis $d_2/d \leq (p-1)/p$ cannot be relaxed. The example of a divisor D of the form dP, and the choice $d_2 = p$, shows that we cannot insist that D_2 have degree d_2.

Corollary 5.4.8 Let k be an algebraically closed field, C/k a proper, smooth connected curve of genus g.
1) Suppose that $D = \Sigma a_iP_i$ is an effective divisor of degree $d \geq 4g+5$. Then we can write D as a sum of effective divisors $D_1 + D_2$ with degrees $d_1 \geq 2g+2$ and $d_2 \geq 2g+2$, such that $D_2 = \Sigma c_iP_i$ has all its nonzero c_i invertible in k and such that for any point P_i which occurs in D but not in D_2, we have $a_i \leq 2$.
2) Suppose that $D = \Sigma a_iP_i$ is an effective divisor of degree $d \geq 4g+4$. Then we can write D as a sum of effective divisors $D_1 + D_2$ with degrees $d_1 \geq 2g+2$ and $d_2 \geq 2g+1$, such that $D_2 = \Sigma c_iP_i$ has all its nonzero c_i invertible in k and such that for any point P_i which occurs in D but not in D_2, we have $a_i \leq 2$.
3) Fix an integer $A \geq 0$. Suppose that $D = \Sigma a_iP_i$ is an effective divisor of degree

$$d \geq \mathrm{Max}(6g+9, 6A + 11),$$

and that the characteristic is not two. Then we can write D as a sum of effective divisors $D_1 + D_2$ both of whose degrees d_1 and d_2 are at least 2g+2, such that $D_2 = \Sigma c_iP_i$ has all its nonzero c_i invertible in k, such that for any point P_i which occurs in D but not in D_2, we have $a_i \leq 2$, and such that $2g - 2 + d > 2(A+d_1)$.

proof 1) Take $d_2 = [d/2]_{fl}$. Because $d \geq 4g + 5$, $d - d_2 \geq 2g+3$, and $d_2/d > 1/3$. So we simply apply 5.4.6 to this situation.
2) If $d \geq 4g+5$, apply part 1). If $d = 4g+4$, we argue as follows. If $g \geq 1$, then take $d_2 = 2g+1$. We have $d_2/d > 1/3$, and again we simply apply 5.4.6. If $g = 0$, then $d = 4$, and we resort to an ad hoc argument,

depending on how many distinct P_i there are in D. Here is a table of the various types of effective D of degree 4, and appropriate choices of D_1 and D_2 for each.

D	D_1	D_2
4P	3P	P
3P + Q	2P	P + Q
2P + 2P	P+Q	P+Q
2P + Q + R	P+R	P+Q
P+Q+R+S	P+Q	R+S.

For 3), we apply the lemma with the initial choice $d_2 := [2d/3]$, allowed because the characteristic is not two. We end up with D_2 of degree d_2 either $[2d/3]$ or $[2d/3]+1$, both of which are $\geq (2d-2)/3$ and both of which are $\leq (2d+3)/3$. Then D_1 has degree d_1 either $d - [2d/3]$ or $d - 1 - [2d/3]$, both of which are $\geq (d-3)/3$, and both of which are $\leq (d+2)/3$. So both D_1 and D_2 have degree at least $(d-3)/3 \geq 2g+2$. We also have

$$\begin{aligned} 2g - 2 + d - 2(A+d_1) &= 2g - 2 + d_1 + d_2 - 2(A+d_1) \\ &= d_2 - d_1 + 2g - 2 - 2A \\ &\geq d_2 - d_1 - 2A - 2 \\ &\geq (2d-2)/3 - (d+2)/3 - 2A - 2 \\ &= (d-4)/3 - 2A - 2 \\ &\geq (6A+7)/3 - 2A - 2 > 0, \end{aligned}$$

as required. It remains to check that $d_2/d > 1/3$. This holds simply because $d_2 :=[2d/3] \geq (2d-2)/3$, and hence $d_2/d > (2d-2)/3d > 1/3$, the last inequality because $d > 2$. QED

Theorem 5.4.9 Let k be an algebraically closed field of characteristic not 2, C/k a proper, smooth connected curve of genus g. Suppose that $D = \sum a_i P_i$ is an effective divisor of degree $d \geq 4g+4$. Write D as a sum of effective divisors $D_1 + D_2$ of degrees $d_1 \geq 2g+2$ and $d_2 \geq 2g+1$, such that $D_2 = \sum c_i P_i$ has all its nonzero c_i invertible in k and such that for any point P_i which occurs in D but not in D_2, we have $a_i \leq 2$.

Let $\mathcal{F}$ be an irreducible middle extension sheaf on C. Suppose that either $\mathcal{F}$ is everywhere tame, or that $\mathcal{F}$ is tame at all points of D and that the characteristic p is either zero or a prime $p \geq \mathrm{rank}(\mathcal{F}) + 2$. Suppose that the following inequalities hold:

$$\text{if } \mathrm{rank}(\mathcal{F}) = 1,\ 2g - 2 + d > \mathrm{Max}(2\#(\mathrm{Sing}(\mathcal{F})\cap(C-D_2)), 4\mathrm{rank}(\mathcal{F})),$$

if $\mathrm{rank}(\mathcal{F}) \geq 2$, $2g - 2 + d > \mathrm{Max}(2\#(\mathrm{Sing}(\mathcal{F})\cap(C-D_2)), 72\mathrm{rank}(\mathcal{F}))$.

Fix a nontrivial character χ of finite order $n \geq 2$. If $n=4$, suppose also that

$$2g - 2 + d > 2(\#(\mathrm{Sing}(\mathcal{F})\cap(C-D_2)) + d_1).$$

If $n=4$ and the curve C has genus $g=0$, suppose in addition that D_1 and D_2 are chosen so that $d_2 \geq 2$. (Such a choice is always possible if $g=0$ by Corollary 5.4.8, part 1), because $d-2 = 2g-2+d > 72\mathrm{rank}(\mathcal{F}) \geq 72$, hence $d \geq 75 > 4g+5$.)

Fix a function

$$f_1 \text{ in } \mathrm{Fct}(C, \deg(D_1), D_1, \mathrm{Sing}(\mathcal{F})\cup D^{\mathrm{red}}).$$

Fix a function f_2 in $\mathrm{Fct}(C, \deg(D_2), D_2, \mathrm{Sing}(\mathcal{F})\cup D^{\mathrm{red}}\cup f_1^{-1}(0))$ which also lies in the open set U of Theorem 2.2.6 with respect to the set $S := (\mathrm{Sing}(\mathcal{F})\cup D^{\mathrm{red}}\cup f_1^{-1}(0))\cap(C - D_2)$. Consider the lisse $\overline{\mathbb{Q}}_\ell$–sheaf $\mathcal{H}$ on $\mathbb{A}^1 - \mathrm{CritVal}(f_2, f_1, \mathcal{F})$ given by $[t \mapsto f_1(t-f_2)]^*\mathcal{G}$, i.e., by

$$t \mapsto H^1(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1(t-f_2))})).$$

Its geometric monodromy group G_{geom} is either Sp or SO or O or a group between SL and GL. If $\mathcal{F}$ is orthogonally (respectively symplectically) self–dual, and χ has order 2, then G_{geom} is Sp (respectively SO or O). If χ has order ≥ 3, then G_{geom} contains SL.

proof Suppose first $n \neq 4$. Put $r := \mathrm{rank}(\mathcal{F})$, $m := \#(\mathrm{Sing}(\mathcal{F})\cap(C-D_2))$. We have seen in Proposition 5.3.7 that $\mathcal{H}$ is the restriction to $\mathbb{A}^1 - \mathrm{CritVal}(f_2, f_1, \mathcal{F})$ of the middle additive convolution of $f_{2*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)})$ and $\mathcal{L}_\chi$.

Let us put

$$\mathcal{F}_1 := f_{2*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)}).$$

As already noted at the end of the proof of 5.3.6, the Irreducible Induction Criterion 3.3.1 shows that $\mathcal{F}_1$ is an irreducible middle extension sheaf. The sheaf $\mathcal{F}_1$ lies in the class $\mathcal{P}_{\mathrm{conv}}$, because it has at least $d_1 \geq 2g+2 \geq 2$ finite singularities, namely the d_1 distinct images by f_2 of the d_1 distinct zeroes of f_1. It is tame at ∞, because $\mathcal{F}$ is tame at all the poles of f_2, and the poles of f_2 all have order prime to p.

Over each critical value α of f_2, $\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)}$ is lisse, and $f_2-\alpha$ has one and only one double zero, so the local monodromy of $\mathcal{F}_1$ at α is quadratic of drop r, with scale the unique character of order 2:

$$\mathcal{F}_1(\alpha)/\mathcal{F}_1(\alpha)^{I(\alpha)} \cong \text{r copies of} \mathcal{L}_{\chi_2(x-\alpha)}.$$

Over the m images $\delta = f_2(\beta)$ of points β in $C-D_2$ which are singularities of $\mathcal{F}$, f_2 is finite etale, and β is the only point in its f_2-fibre at which $\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)}$ can possibly fail to be lisse. So $\mathcal{F}_1$ has drop $\le r$ at each of these m points. More precisely, if we define

$$\mathcal{F}_0 := \mathcal{F}\otimes\mathcal{L}_{\chi(f_1)},$$

we have

$$\mathcal{F}_1(\delta)/\mathcal{F}_1(\delta)^{I(\delta)} \cong \mathcal{F}_0/\mathcal{F}_0^{I(\beta)},$$

where we use f_2 to identify $I(\delta)$ with $I(\beta)$.

Where **else** on $C - D_2$ can $\mathcal{F}_0 := \mathcal{F}\otimes\mathcal{L}_{\chi(f_1)}$ fail to be lisse? Only at a pole of zero of f_1 which is not in $\mathrm{Sing}(\mathcal{F})$ or in D_2. Let us first consider the case of a pole, say ρ, of f_1 which lies in $C - D_2$. Thus ρ is a point of D which is not in D_2. There may of course be no such points. But if ρ is such a point, then f_1 has a pole of order either 1 or 2 at ρ. By the choice of f_2, f_2 is finite etale over $\sigma = f_2(\rho)$, and ρ is the only point in its f_2-fibre at which $\mathcal{F}_0 := \mathcal{F}\otimes\mathcal{L}_{\chi(f_1)}$ can fail to be lisse. Let us define

$$a(\rho) := \mathrm{ord}_\rho(f_1).$$

Thus $a(\rho)$ is either -1 or -2. Since $\mathcal{F}$ itself is lisse at ρ, we see that the local monodromy of $\mathcal{F}_1$ at $\sigma = f_2(\rho)$ is quadratic of drop r, with scale the character $\mathcal{L}_{\chi^{a(\rho)}(x-\sigma)}$ of $I(\sigma)$. Notice that if $a(\rho) = -1$, this scale is just χ^{-1}, while if $a(\rho) = -2$, this scale is χ^{-2}. Notice that in the single exceptional case when both $a(\rho) = -2$ and when χ has order 2, $\mathcal{F}_0 := \mathcal{F}\otimes\mathcal{L}_{\chi(f_1)}$ is lisse at ρ.

We now turn to the zeroes of f_1. Over each of the d_1 images $\gamma = f_2(\zeta)$ of the zeroes of f_1, f_2 is finite etale, ζ is the only zero of f_1 in its f_2–fibre, and $\mathcal{F}$ is lisse. Thus $\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)}$ is lisse at all but the point ζ in the fibre $f_2^{-1}(\gamma)$. At ζ the local monodomy of $\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)}$ is quadratic of drop r, with scale the character $\mathcal{L}_{\chi(\text{uniformizer at }\zeta)}$ of $I(\zeta)$. Thus the local monodromy of $\mathcal{F}_1$ at γ is quadratic of drop r, with scale the character $\mathcal{L}_{\chi(x-\gamma)}$ of $I(\gamma)$.

At all other points of $\mathbb{A}^1$, i.e., on $\mathbb{A}^1 - \mathrm{CritVal}(f_2, f_1, \mathcal{F})$, $\mathcal{F}_1$ is lisse. Moreover, if $\mathcal{F}$ is everywhere tame on C, then $\mathcal{F}_1$ is everywhere tame. Now form $\mathcal{H}$, the middle additive convolution of $\mathcal{F}_1$ with $\mathcal{L}_\chi$:

$$\mathcal{H} := \mathcal{F}_1 *_{\mathrm{mid}+} \mathcal{L}_\chi.$$

Thus (by 4.1.10, 2d) and 1b)) $\mathcal{H}$ is tame at ∞, and it is everywhere tame if $\mathcal{F}$ is everywhere tame. Its rank is given by (5.2.1, part 5))

$$\begin{aligned}\mathrm{rank}(\mathcal{H}) &= (2g-2+d)r \\ &+ \Sigma_{P_i \text{ in } D^{red}} \mathrm{Swan}_{P_i}(\mathcal{F}) + \Sigma_{s \text{ in } \mathrm{Sing}(\mathcal{F})_{finite}} \mathrm{Swan}_s(\mathcal{F}) \\ &+ \Sigma_{s \text{ in } \mathrm{Sing}(\mathcal{F})_{finite}} \mathrm{drop}_s(\mathcal{F}) \\ &+ \Sigma_{P_i \text{ in } D^{red}} \mathrm{drop}_{P_i}(\mathcal{F}(P_i)\otimes(\mathcal{L}_{\chi}a_i)(\infty, P_i)),\end{aligned}$$

where we have written $\mathrm{Sing}(\mathcal{F})_{finite}$ for $\mathrm{Sing}(\mathcal{F})\cap(C-D)$.

In particular, we have the inequality (5.2.1, part 6))

$$\mathrm{rank}(\mathcal{H}) \geq (2g-2+d)r.$$

The local monodromy of $\mathcal{H}$ at the m images $\delta = f_2(\beta)$ of points β in $\mathrm{Sing}(\mathcal{F})\cap(C-D_2)$ has drop $\leq$ r, by (4.1.10, part 1c), applied to $\mathcal{F}_1$).

What about the local monodromy of $\mathcal{H}$ at the image $\sigma = f_2(\rho)$ of any pole ρ of f_1 which lies neither in D_2 nor in $\mathrm{Sing}(\mathcal{F})$? In the single exceptional case when $a(\rho) = -2$ and when χ has order 2, $\mathcal{H}$ is lisse at σ. Otherwise, the local monodromy of $\mathcal{H}$ at σ is quadratic of drop r, with scale the character $\chi^{1+a(\rho)}$.

$$\begin{aligned}\mathcal{H}(\sigma)/\mathcal{H}(\sigma)^{I(\sigma)} &\cong \mathcal{L}_{\chi(x-\sigma)}\otimes(\text{r copies of } \mathcal{L}_{\chi^{a(\rho)}(x-\sigma)}) \\ &\cong \text{r copies of } \mathcal{L}_{\chi^{1+a(\rho)}(x-\sigma)}.\end{aligned}$$

Since $a(\rho)$ is either -1 or -2, the local monodromy of $\mathcal{H}$ at σ satisfies

$$\mathcal{H}(\sigma)/\mathcal{H}(\sigma)^{I(\sigma)} \cong \text{r copies of } \mathbb{1}, \text{ if } a(\rho) = -1,$$
$$\mathcal{H}(\sigma)/\mathcal{H}(\sigma)^{I(\sigma)} \cong \text{r copies of } \chi^{-1}, \text{ if } a(\rho) = -2 \text{ and } \chi^2 \neq \mathbb{1},$$
$$\mathcal{H}(\sigma)/\mathcal{H}(\sigma)^{I(\sigma)} = 0, \text{ if } a(\rho) = -2, \text{ and } \chi^2 = \mathbb{1}.$$

In particular, the local monodromy of $\mathcal{H}$ at σ is either trivial, or it is quadratic of drop r, with scale a character which is not of order 2.

The local monodromy of $\mathcal{H}$ at each critical value α of f_2 is quadratic of drop r, with scale the character $\chi\chi_2$:

$$\begin{aligned}\mathcal{H}(\alpha)/\mathcal{H}(\alpha)^{I(\alpha)} &\cong \mathcal{L}_{\chi(x-\alpha)}\otimes(\text{r copies of } \mathcal{L}_{\chi_2(x-\alpha)}) \\ &\cong \text{r copies of } \mathcal{L}_{\chi\chi_2(x-\alpha)}.\end{aligned}$$

Over each of the d_1 images $\gamma = f_2(\zeta)$ of the zeroes of f_1, the local monodromy of $\mathcal{H}$ at γ is quadratic of drop r, with scale the character $\mathcal{L}_{\chi^2(x-\gamma)}$ of $I(\gamma)$.

With the exception of at most m points of $\mathbb{A}^1$, namely the images by f_2 of points in $\mathrm{Sing}(\mathcal{F})\cap(C-D_2)$, the local monodromy of $\mathcal{H}$ is quadratic of drop r, with scale a character not of order 2. Indeed, at the critical values of f_2, $\chi\chi_2$ is not of order 2 (χ being nontrivial), and at the d_1 images of the zeroes of f_1, χ^2 is not of order 2 (because the order n of χ is assumed to be not 4). At the images $\sigma = f_2(\rho)$ of the poles, if any, of f_1 which lie neither in Sing($\mathcal{F}$) not in D_2, we have already noted that the the local monodromy of $\mathcal{H}$ at σ is either trivial, or it is quadratic of drop r, with scale a character which is not of order 2.

The conclusion now follows from Theorem 1.5.1 (and Theorem 1.7.1, if r=1), applied to the data (r, m, $\mathcal{H}$).

Suppose now that n is 4. Our $\mathcal{F}_1$ is still perverse irreducible, and in the class $\mathcal{P}_{\mathrm{conv}}$. The difficulty with the case n=4 is this: at the d_1 images $\gamma = f_2(\zeta)$ of the zeroes of f_1, the local monodromy of $\mathcal{H}$ at γ is quadratic of drop r, with scale the character $\mathcal{L}_{\chi^2(x-\gamma)}$ of $I(\gamma)$. But for χ of order 4, χ^2 is the quadratic character, and so these d_1 points will be part of the excluded "at all but at most m points" in hypothesis 4) of Theorem 1.5.1. To overcome this difficulty, we assume that

$$2g - 2 + d > 2(\#(\mathrm{Sing}(\mathcal{F})\cap(C-D_2)) + d_1).$$

We put r := rank($\mathcal{F}$), m := $\#(\mathrm{Sing}(\mathcal{F})\cap(C-D_2)) + d_1$. We have noted above that

$$\mathrm{rank}(\mathcal{H}) \geq (2g-2 + d)r,$$

so we have

$$\mathrm{rank}(\mathcal{H}) > \mathrm{Max}(2mr, 72r^2).$$

With the exception of at most m points of $\mathbb{A}^1$, namely the images by f_2 of points in $\mathrm{Sing}(\mathcal{F})\cap(C-D_2)$ and the d_1 images by f_2 of the zeroes of f_1, the local monodromy of $\mathcal{H}$ is quadratic of drop r, with scale a character not of order 2 (in fact, of order 1 or 4). The key point is that the remaining finite singularities of $\mathcal{H}$ are of two types, the critical values of f_2 and the image $\sigma = f_2(\rho)$ of the poles, if any, of f_1 which lie neither in Sing($\mathcal{F}$) not in D_2. At the image $\sigma = f_2(\rho)$ of the poles, if any, of f_1 which lie neither in Sing($\mathcal{F}$) not in D_2, we have seen above that the local monodromy of $\mathcal{H}$ is quadratic of drop r, with scale either $\mathbb{1}$ (the case of a simple pole) or χ^{-1} (the case of a double pole). [The possibility that $\mathcal{H}$ be lisse at σ, which arose above in the $n \neq 4$ case, does not arise here, because χ has order n = 4.] At the critical values of f_2, where the local monodromy is quadratic of drop r, with scale $\chi\chi_2$, which has order 4. [The number of critical values is

$$2g-2 + \Sigma_i (1+c_i).$$

This number is strictly positive unless g=0 and $d_2 = 1$. This exceptional case (g=0, d_2=1) is not allowed if n is 4.]

The result now follows from Theorem 1.5.1 (and Theorem 1.7.1, if r=1), applied to the data (r, m, $\mathcal{H}$). QED

Exactly as in Proposition 5.4.2 above, we have

Proposition 5.4.10 Hypotheses and notations as in Theorem 5.4.9 above, suppose that χ has order 2, but $\mathcal{F}$ is not self–dual. Then G_{geom} contains SL.

proof If not, then exactly as in the proof of Proposition 5.4.2, we infer that $f_{2*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)})$ is self–dual, and then that $\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)}$, and hence $\mathcal{F}$, are self–dual. QED

Proposition 5.4.11 Hypotheses and notations as in Theorem 5.4.9 above, suppose that χ has order 2, and that $\mathcal{F}$ is symplectically self–dual.
1) Suppose there exists a D_2–finite singularity β of $\mathcal{F}$, i.e., a point β in $\mathrm{Sing}(\mathcal{F})\cap(C-D_2)$, such that the following three conditions hold:
  1a) $\mathcal{F}$ is tame at β,
  1b) $\mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)}$ has odd dimension.
  1c) β occurs in D with even (possibly zero) multiplicity.
Then the group G_{geom} for the sheaf $\mathcal{H}$ is the full orthogonal group O.
2) Suppose that $\mathcal{F}$ is everywhere tame. Suppose further that every D_2-finite singularity β of $\mathcal{F}$ occurs in D with even (possibly zero) multiplicity. Then G_{geom} for $\mathcal{H}$ is the special orthogonal group SO if and only if $\mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)}$ has even dimension for every D_2–finite singularity β of $\mathcal{F}$.

proof This is proven by essentially recopying the proof of 5.4.3, applied to the sheaf $\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)}$ and the function f_2 . To prove 1), remember that f_1 is invertible at β, if β lies in $\mathrm{Sing}(\mathcal{F})\cap(C-D)$, and f_1 has a double pole at β, if β lies in $\mathrm{Sing}(\mathcal{F})\cap D\cap(C - D_2)$, so in both cases $\mathcal{L}_{\chi(f_1)}$ is lisse at β). To prove 2), we must examine what happens at all the points of $\mathrm{CritVal}(f_2, f_1, \mathcal{F})$. At points $\gamma = f_2(\beta)$, β in $f_2(\mathrm{Sing}(\mathcal{F})\cap(C-D_2))$, we have

$$\mathcal{H}(\gamma)/\mathcal{H}(\gamma)^{I(\gamma)} \cong (\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)})(\beta)/(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)})(\beta)^{I(\beta)} \cong \mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)},$$

the last equality because, as noted just above, $\mathcal{L}_{\chi(f_1)}$ is lisse at β.
To complete the proof, along the model of 5.4.3, it suffices to show

that at all other points of CritVal(f_2, f_1, $\mathcal{F}$), det($\mathcal{H}$) is lisse. For this, it suffices to observe that $\mathcal{H}$ has unipotent local monodromy at those points.

At a point lying under a (necessarily simple) zero of f_1, $\mathcal{H}$ has unipotent local monodromy, cf. the proof of 5.4.9. At a point lying under a pole of f_1 in C - D_2 at which $\mathcal{F}$ is lisse, either $\mathcal{H}$ is unipotent (the case of a simple pole) or $\mathcal{H}$ is lisse (the case of a double pole: remember that χ has order 2), cf. the proof of 5.4.9. At a critical value of f_2, $\mathcal{H}$ is unipotent, cf. the proof of 5.4.3. QED

5.5 Theorems of big monodromy for $\mathcal{G}$:= $\mathrm{Twist}_{\chi,C,D}(\mathcal{F})$ on *Fct*(C, d, D, $\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}$) in characteristic not 2

Theorem 5.5.1 Let k be an algebraically closed field in which 2 is invertible. Fix a prime number ℓ which is invertible in k. Fix a character χ of finite order $n \geq 2$ of the tame fundamental group of $\mathbb{G}_m$/k. Let C/k be a proper smooth connected curve of genus g. Fix an irreducible middle extension $\overline{\mathbb{Q}}_\ell$–sheaf $\mathcal{F}$ on C. Let $D = \sum a_i P_i$ be an effective divisor of degree d on C. Suppose that either

1a) $d \geq 2g+1$, all a_i are invertible in k, $\mathrm{Sing}(\mathcal{F}) \cap (C-D)$ is nonempty, and the following inequalities hold:

if $\mathrm{rank}(\mathcal{F}) = 1$, $2g - 2 + d \geq \mathrm{Max}(2\#(\mathrm{Sing}(\mathcal{F})\cap(C-D)), 4\mathrm{rank}(\mathcal{F}))$,
if $\mathrm{rank}(\mathcal{F}) \geq 2$, $2g - 2 + d \geq \mathrm{Max}(2\#(\mathrm{Sing}(\mathcal{F})\cap(C-D)), 72\mathrm{rank}(\mathcal{F}))$,

or

1b) $d \geq 4g+4$, the following inequalities hold:

if $\mathrm{rank}(\mathcal{F}) = 1$, $2g - 2 + d > \mathrm{Max}(2\#\mathrm{Sing}(\mathcal{F}), 4\mathrm{rank}(\mathcal{F}))$,
if $\mathrm{rank}(\mathcal{F}) \geq 2$, $2g - 2 + d > \mathrm{Max}(2\#\mathrm{Sing}(\mathcal{F}), 72\mathrm{rank}(\mathcal{F}))$,

and, if n=4,

$d \geq \mathrm{Max}(6g+9, 6\#\mathrm{Sing}(\mathcal{F}) + 11)$.

Suppose further that

2) either $\mathcal{F}$ is everywhere tame, or $\mathcal{F}$ is tame at all points of D and the characteristic p is either zero or $p \geq \mathrm{rank}(\mathcal{F}) + 2$.

Then the lisse sheaf $\mathcal{G}$ on *Fct*(C, d, D, $\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}$) given by

$$f \mapsto H^1(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})$$

has G_{geom} given as follows:

a) If $\mathcal{F}$ is orthogonally self–dual, and χ has order 2, then G_{geom} is Sp.

b) If $\mathcal{F}$ is symplectically self–dual, and χ has order 2, then G_{geom} is either SO or O.

c) If either $\mathcal{F}$ is not self–dual or if χ has order > 2, then G_{geom} contains SL.

proof If χ has order two and $\mathcal{F}$ is orthogonally (respectively symplectically) self–dual, then $\mathcal{G}$ is symplectically (resp. orthogonally) self–dual, and we have a priori inclusions

$$G_{geom} \subset Sp \text{ (resp. } G_{geom} \subset O).$$

In general, we have an a priori inclusion

$$G_{geom} \subset GL.$$

Given a smooth connected curve U/k and a map

$$\pi : U \to \mathit{Fct}(C, d, D, Sing(\mathcal{F})_{finite}),$$

we have an a priori inclusion

$$G_{geom}(\pi^*\mathcal{G} \text{ on } U) \subset G_{geom}(\mathcal{G} \text{ on } \mathit{Fct}(C, d, D, Sing(\mathcal{F})_{finite})).$$

So it suffices to produce a π such that $G_{geom}(\pi^*\mathcal{G}$ on U) contains, in the three cases, the groups Sp, SO, and SL respectively. This is precisely what we have done in Theorem 5.4.1 (under hypotheses 1a) and 2)) and in Theorem 5.4.9 (under hypotheses 1b) and 2)). QED

Proposition 5.5.2 Hypotheses and notations as in Theorem 5.5.1 above, suppose that χ has order 2, and that $\mathcal{F}$ is symplectically self–dual.

1) Suppose that there exists a finite singularity β of $\mathcal{F}$, i.e., a point β in $Sing(\mathcal{F})\cap(C-D)$, such that the following two conditions hold:

1a) $\mathcal{F}$ is tame at β,

1b) $\mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)}$ has odd dimension.

Then the group G_{geom} for the sheaf $\mathcal{G}$ is the full orthogonal group O.

2) Suppose we are in case 1b) of Theorem 5.5.1, and that there exists a singularity β of $\mathcal{F}$ (but here we do **not** assume that β lies in C–D) such that the following two conditions hold:

2a) $\mathcal{F}$ is tame at β,

2b) $\mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)}$ has odd dimension.

Suppose further that we can write D as the sum of two effective divisors $D_1 + D_2$ of degrees $d_1 \geq 2g+2$ and $d_2 \geq 2g+1$, such that $D_2 = \Sigma c_i P_i$ has all its nonzero c_i invertible in k, and such that every P_i in D which is absent from D_2 occurs with multiplicity at most 2 in D. Suppose further that $\beta \in C - D_2$, and that β occurs in D with even (possibly zero) multiplicity. Then the group G_{geom} for the sheaf $\mathcal{G}$ is the full orthogonal group O.

3) Suppose that the sheaf $\mathcal{G}$ has odd rank. Then the group G_{geom} for the sheaf $\mathcal{G}$ is the full orthogonal group O.

proof If we are in case 1a) of Theorem 5.5.1, then Assertion 1) results from Propostion 5.4.3. If we are in case 1b) of Theorem 5.5.1, then Assertion 1) is a special case of Assertion 2), thanks to Corollary 5.4.8, part 2). Assertion 2) results from Proposition 5.4.11. For assertion 3), we argue as follows. We know that G_{geom} for $\mathcal{G}$ contains SO and is contained in O. To show that G_{geom} is O, it suffices to find a one−parameter family

$$\pi : \mathbb{G}_m \to Fct(C, d, D, Sing(\mathcal{F})_{finite})$$

such that $\det(\pi^*\mathcal{G})$ is nontrivial on $\mathbb{G}_m$.

Fix **any** f in Fct(C, d, D, $Sing(\mathcal{F})_{finite}$), and consider the map

$$\pi : \mathbb{G}_m \to Fct(C, d, D, Sing(\mathcal{F})_{finite})$$

defined by

$$t \mapsto tf.$$

Thus $\pi^*\mathcal{G}$ is the lisse sheaf on $\mathbb{G}_m$ given by

$$t \mapsto H^1(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(tf)}) = \mathcal{L}_{\chi(t)}\otimes H^1(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)})).$$

If $\mathcal{G}$ has odd rank, then $\pi^*\mathcal{G}$ is the direct sum of an odd number of copies of $\mathcal{L}_{\chi(t)}$, and hence, χ being χ_2, $\det(\pi^*\mathcal{G}) \cong \mathcal{L}_{\chi(t)}$. QED

Question 5.5.3 Outside the cases covered by Proposition 5.5.2, we do not know a general, a priori way to distinguish the SO and O cases. The sheaf $\det(\mathcal{G})$ on Fct(C, d, D, $Sing(\mathcal{F})_{finite}$) is a character of order dividing 2 of $\pi_1(Fct(C, d, D, Sing(\mathcal{F})_{finite}))$, or, if we like, an element in

$$H^1(Fct(C, d, D, Sing(\mathcal{F})_{finite}), \boldsymbol{\mu}_2).$$

What is it?

5.6 Theorems of big monodromy in characteristic 2

Theorem 5.6.1 Let k be an algebraically closed field of characteristic 2, C/k a proper, smooth connected curve of genus g. Suppose that $D = \sum a_i P_i$ is an effective divisor of degree $d \geq 6g+3$, with all a_i odd. Let $\mathcal{F}$ be an irreducible middle extension sheaf on C with $Sing(\mathcal{F})_{finite} :=$ $Sing(\mathcal{F})\cap(C-D)$ nonempty. Suppose that $\mathcal{F}$ is everywhere tame. Suppose that the degree d is so large that the following inequalities hold:

if $rank(\mathcal{F}) = 1$, $2g - 2 + d \geq Max(2\#(Sing(\mathcal{F})\cap(C-D)), 4rank(\mathcal{F}))$,

if $rank(\mathcal{F}) \geq 2$, $2g - 2 + d \geq Max(2\#(Sing(\mathcal{F})\cap(C-D)), 72rank(\mathcal{F}))$,

Fix a nontrivial character χ of odd finite order $n \geq 3$. Pick an f in Fct(C, d, D, $Sing(\mathcal{F})_{finite}$) which also lies in the dense open set U of Theorem 2.4.4 applied with S taken to be $Sing(\mathcal{F})_{finite}$. Thus f as map from C−D to $\mathbb{A}^1$ is of Lefschetz type, each finite monodromy of $f_*\overline{\mathbb{Q}}_\ell$ is a reflection of Swan conductor 1 (by 2.7.1), and for each s in $Sing(\mathcal{F})_{finite}$, the fibre $f^{-1}(s)$ consists of d distinct points, only one of which lies in $Sing(\mathcal{F})_{finite}$. Consider the lisse $\overline{\mathbb{Q}}_\ell$−sheaf $\mathcal{H}$ on $\mathbb{A}^1$ − CritVal(f, $\mathcal{F}$) given by

$$\mathcal{H} := [t \mapsto t-f]^*\mathcal{G},$$

i.e., by

$$t \mapsto H^1(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(t-f)})).$$

Its geometric monodromy group G_{geom} contains SL.

proof The argument is quite similar to the one given for Theorem 5.4.1. Thus r := rank($\mathcal{F}$), m := # $\mathrm{Sing}(\mathcal{F})_{finite}$, $\mathcal{F}_1 := f_*\mathcal{F}$, and $\mathcal{H}$ is the restriction to $\mathbb{A}^1$ – CritVal(f, $\mathcal{F}$) of the middle additive convolution of $\mathcal{F}_1$ and $\mathcal{L}_\chi$. We know that the function f has

$$g-1 + \Sigma(1+a_i)/2 \geq (d+1)/2 - 1 \geq (6g+4)/2 - 1 \geq 1$$

critical points, and as many critical values. Over each critical value α of f, $\mathcal{F}$ is lisse, so the local monodromy of $\mathcal{F}_1$ at α is quadratic of drop r, with scale a character ρ_α of I(α) of order 2 and Swan conductor 1:

$$\mathcal{F}_1(\alpha)/\mathcal{F}_1(\alpha)^{I(\alpha)} \cong \text{r copies of } \rho_\alpha.$$

Over the m images $\delta = f(\beta)$ of points β in $\mathrm{Sing}(\mathcal{F})_{finite}$, f is finite etale, and β is the unique point of $\mathrm{Sing}(\mathcal{F})_{finite}$ in the fibre, so the local monodromy of $\mathcal{F}_1$ at δ has drop ≤ r. More precisely, we have

$$\mathcal{F}_1(\delta)/\mathcal{F}_1(\delta)^{I(\delta)} \cong \mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)},$$

where we use f to identify I(δ) with I(β).

At all other points of $\mathbb{A}^1$, i.e., on $\mathbb{A}^1$ – CritVal(f, $\mathcal{F}$), $\mathcal{F}_1$ is lisse. As $\mathcal{F}$ is everywhere tame on C, $\mathcal{F}_1$ is tame except at the critical values of $\mathcal{F}$. Now form $\mathcal{H}$, the middle additive convolution of $\mathcal{F}_1$ with $\mathcal{L}_\chi$. Thus by 4.1.10, 2d), 1b), and 1c), $\mathcal{H}$ is tame at ∞, it is tame outside the critical values of f, and it is lisse outside ∞, the critical values of f, and the m images $\delta = f(\beta)$ of points β in $\mathrm{Sing}(\mathcal{F})_{finite}$. Its rank is given by (5.2.1 part 5))

$$\begin{aligned}\mathrm{rank}(\mathcal{H}) = {} & (2g-2 + d)r \\ & + \Sigma_{P_i \text{ in } D^{red}} \mathrm{Swan}_{P_i}(\mathcal{F}) + \Sigma_{s \text{ in } \mathrm{Sing}(\mathcal{F})_{finite}} \mathrm{Swan}_s(\mathcal{F}) \\ & + \Sigma_{s \text{ in } \mathrm{Sing}(\mathcal{F})_{finite}} \mathrm{drop}_s(\mathcal{F}) \\ & + \Sigma_{P_i \text{ in } D^{red}} \mathrm{drop}_{P_i}(\mathcal{F}(P_i)\otimes(\mathcal{L}_{\chi}a_i)(\infty, P_i)).\end{aligned}$$

In particular, we have the inequality (5.2.1, part 6))

$$\mathrm{rank}(\mathcal{H}) \geq (2g-2 + d)r + \#\mathrm{Sing}_{finite}(\mathcal{F}) > (2g-2 + d)r.$$

The local monodromy of $\mathcal{H}$ at the m images $\delta = f(\beta)$ of points β in $\mathrm{Sing}(\mathcal{F})_{finite}$ is tame and has drop ≤ r, by 4.1.10, part 1c). It is given by

$$\mathcal{H}(\delta)/\mathcal{H}(\delta)^{I(\delta)} \cong MC_\chi\mathrm{loc}(\delta)(\mathcal{F}(\beta)/\mathcal{F}(\beta)^{I(\beta)} \text{ as } I(\delta)\text{–rep'n}).$$

The local monodromy of $\mathcal{H}$ at each critical value α of f is quadratic of drop r, with scale a character $MC_{\chi}loc(\alpha)(\rho_{\alpha})$ whose order, twice the order of χ by 4.2.2, is ≥ 6. Thus

$$\mathcal{H}(\alpha)/\mathcal{H}(\alpha)^{I(\alpha)} \cong \text{r copies of a character of order} \geq 6.$$

The conclusion follows from Theorem 1.5.1 (and Theorem 1.7.1 if r=1), applied to (r, m, $\mathcal{H}$), with $S - S_0$ the critical values of f, and S_0 the m images $\delta = f(\beta)$ of points β in $\mathrm{Sing}(\mathcal{F})_{\mathrm{finite}}$.QED

Theorem 5.6.2 Let k be an algebraically closed field of characteristic 2, C/k a proper, smooth connected curve of genus g. Suppose that $D = \sum a_i P_i$ is an effective divisor of degree d ≥ 12g+7. Write D as a sum of effective divisors $D_1 + D_2$ both of whose degrees d_1 and d_2 are at least 6g+3, such that $D_2 = \sum c_i P_i$ has all its nonzero c_i odd and such that for any point P_i which occurs in D but not in D_2, we have $a_i \leq 2$.. Let $\mathcal{F}$ be an irreducible middle extension sheaf on C. Suppose that $\mathcal{F}$ is everywhere tame. Suppose that the following inequalities hold:

if $\mathrm{rank}(\mathcal{F}) = 1$, $2g - 2 + d \geq \mathrm{Max}(2\#(\mathrm{Sing}(\mathcal{F})\cap(C-D_2)), 4\mathrm{rank}(\mathcal{F}))$,

if $\mathrm{rank}(\mathcal{F}) \geq 2$, $2g - 2 + d > \mathrm{Max}(2\#(\mathrm{Sing}(\mathcal{F})\cap(C-D_2)), 72\mathrm{rank}(\mathcal{F}))$.

Fix a nontrivial character χ of odd finite order n ≥ 3.

Fix a function

$$f_1 \text{ in } \mathrm{Fct}(C, \deg(D_1), D_1, \mathrm{Sing}(\mathcal{F})\cup D^{\mathrm{red}}).$$

Fix a function f_2 in $\mathrm{Fct}(C, \deg(D_2), D_2, \mathrm{Sing}(\mathcal{F})\cup D^{\mathrm{red}}\cup f_1^{-1}(0))$ which also lies in the open set U of Theorem 2.4.4 with respect to the set $S := (\mathrm{Sing}(\mathcal{F})\cup D^{\mathrm{red}}\cup f_1^{-1}(0))\cap(C - D_2)$.

Consider the lisse $\overline{\mathbb{Q}}_{\ell}$–sheaf $\mathcal{H}$ on $\mathbb{A}^1 - \mathrm{CritVal}(f_2, f_1, \mathcal{F})$ given by $[t \mapsto f_1(t-f_2)]^*\mathcal{G}$, i.e., by

$$t \mapsto H^1(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1(t-f_2))})).$$

Its geometric monodromy group G_{geom} contains SL.

proof The argument is quite similar to the one given for Theorem 5.4.9.We will indicate the modifications which must be made.

Put $r := \mathrm{rank}(\mathcal{F})$, $m := \#(\mathrm{Sing}(\mathcal{F})\cap(C-D_2))$, $\mathcal{F}_1 := f_{2*}(\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)})$. We have seen in Proposition 5.3.7 that $\mathcal{H}$ is the restriction to

$$\mathbb{A}^1 - \mathrm{CritVal}(f_2, f_1, \mathcal{F})$$

of the middle additive convolution of $\mathcal{F}_1$ and $\mathcal{L}_{\chi}$.

We have seen above (end of the proof of 5.3.6) that by the Irreducible Induction Criterion 3.3.1, $\mathcal{F}_1$ is an irreducible middle extension sheaf. It is tame at ∞, because $\mathcal{F}$ is tame at all the poles of f_2, and the poles of f_2 all have odd order.

We know that the function f_2 has

$$g-1 + \Sigma(1+c_i)/2 \geq (d_2+1)/2 - 1 \geq (6g+4)/2 - 1 \geq 1$$

critical points, and as many critical values. Over each critical value α of f_2, $\mathcal{F}$ and $\mathcal{L}_{\chi(f_1)}$ are, by the choice of f_2 in the open set U of Theorem 2.4.4, both lisse, so the local monodromy of $\mathcal{F}_1$ at α is quadratic of drop r, with scale a character ρ_α of $I(\alpha)$ of order 2 and Swan conductor 1:

$$\mathcal{F}_1(\alpha)/\mathcal{F}_1(\alpha)^{I(\alpha)} \cong \text{r copies of } \rho_\alpha.$$

Over the m images $\delta = f_2(\beta)$ of points β in $\mathrm{Sing}(\mathcal{F}) \cap (C-D_2)$, f_2 is finite etale, and β is, by the choice of f_2 in the open set U of Theorem 2.4.4, the unique point of $\mathrm{Sing}(\mathcal{F}) \cap (C-D_2)$ in the fibre, so the local monodromy of $\mathcal{F}_1$ at δ has drop $\leq$ r. More precisely, if we put

$$\mathcal{F}_0 := \mathcal{F} \otimes \mathcal{L}_{\chi(f_1)},$$

we have

$$\mathcal{F}_1(\delta)/\mathcal{F}_1(\delta)^{I(\delta)} \cong \mathcal{F}_0(\beta)/\mathcal{F}_0(\beta)^{I(\beta)},$$

where we use f_2 to identify $I(\delta)$ with $I(\beta)$.

What happens over a pole, say ρ, of f_1 which lies in neither D_2 nor in $\mathrm{Sing}(\mathcal{F})$? Thus ρ is a point of D which is not in D_2. If ρ is such a point, then f_1 has a pole of order either 1 or 2 at ρ. By the choice of f_2 in the open set U of Theorem 2.4.4, f_2 is finite etale over $\sigma = f_2(\rho)$, and ρ is the only point in its f_2-fibre at which $\mathcal{F}_0 := \mathcal{F} \otimes \mathcal{L}_{\chi(f_1)}$ can fail to be lisse.The local monodromy of $\mathcal{F}_1$ at $\sigma = f_2(\rho)$ is quadratic of drop r, with scale the character χ^{-1} (case of a simple pole) or χ^{-2} (case of a double pole).

Over each of the d_1 images $\gamma = f_2(\zeta)$ of the zeroes of f_1, f_2 is finite etale, ζ is, by the choice of f_2 in the open set U of Theorem 2.4.4, the only zero of f_1 in its f_2–fibre, and $\mathcal{F}$ is lisse. Thus $\mathcal{F} \otimes \mathcal{L}_{\chi(f_1)}$ is lisse at all but the point ζ in the fibre $f_2^{-1}(\gamma)$. At ζ the local monodomy of $\mathcal{F} \otimes \mathcal{L}_{\chi(f_1)}$ is quadratic of drop r, with scale the character $\mathcal{L}_{\chi(\text{uniformizer at } \zeta)}$ of $I(\zeta)$. Thus the local monodromy of $\mathcal{F}_1$ at γ is quadratic of drop r, with scale the character $\mathcal{L}_{\chi(x-\gamma)}$ of $I(\gamma)$.

At all other points of $\mathbb{A}^1$, i.e., on $\mathbb{A}^1 - \mathrm{CritVal}(f_2, f_1, \mathcal{F})$, $\mathcal{F}_1$ is lisse. As $\mathcal{F}$ is everywhere tame on C, $\mathcal{F}_1$ is tame outside the critical values of f_2. Now form $\mathcal{H}$, the middle additive convolution of $\mathcal{F}_1$ with $\mathcal{L}_\chi$. Thus (by 4.1.10, 2d), 1b), and 1c)) $\mathcal{H}$ is tame at ∞, it is tame outside the critical values of f_2, and it is lisse on $\mathbb{A}^1$ – CritVal(f_2, $\mathcal{F}\otimes\mathcal{L}_{\chi(f_1)}$). Its rank is given by (5.2.1, part 5))

$$\begin{aligned}\mathrm{rank}(\mathcal{H}) = (2g-2 + d)r \\ + \textstyle\sum_{P_i \text{ in } D^{red}} \mathrm{Swan}_{P_i}(\mathcal{F}) + \sum_{s \text{ in } \mathrm{Sing}(\mathcal{F})_{finite}} \mathrm{Swan}_s(\mathcal{F}) \\ + \textstyle\sum_{s \text{ in } \mathrm{Sing}(\mathcal{F})_{finite}} \mathrm{drop}_s(\mathcal{F}) \\ + \textstyle\sum_{P_i \text{ in } D^{red}} \mathrm{drop}_{P_i}(\mathcal{F}(P_i)\otimes(\mathcal{L}_\chi a_i)(\infty, P_i)),\end{aligned}$$

where we have written $\mathrm{Sing}(\mathcal{F})_{finite}$ for $\mathrm{Sing}(\mathcal{F})\cap(C-D)$.
In particular, we have the inequality (5.2.1, part 6))

$$\mathrm{rank}(\mathcal{H}) \geq (2g-2 + d)r.$$

The local monodromy of $\mathcal{H}$ at the m images $\delta = f_2(\beta)$ of points β in $\mathrm{Sing}(\mathcal{F})\cap(C-D_2)$ is tame and has drop ≤ r, by 4.1.10 parts 1b) and 1c).

At the image $f_2(\rho)$ of a pole ρ of f_1 which lies in neither D_2 nor in Sing($\mathcal{F}$), the local monodromy of $\mathcal{H}$ is quadratic of drop r, with scale either $\mathbb{1}$ (the case of a simple pole) or χ^{-1} (the case of a double pole).

The local monodromy of $\mathcal{H}$ at each critical value α of f_2 is quadratic of drop r, with scale a character $MC_\chi loc(\alpha)(\rho_\alpha)$ whose order, twice the order of χ by 4.2.2, is ≥ 6. Thus

$$\mathcal{H}(\alpha)/\mathcal{H}(\alpha)^{I(\alpha)} \cong \text{r copies of a character of order} \geq 6.$$

Over each of the d_1 images $\gamma = f_2(\zeta)$ of the zeroes of f_1, the local monodromy of $\mathcal{H}$ at γ is quadratic of drop r, with scale the character $\mathcal{L}_{\chi^2(x-\gamma)}$ of $I(\gamma)$, whose order, that of χ, is ≥ 3.

With the exception of at most m points of $\mathbb{A}^1$, namely the images by f_2 of points in $\mathrm{Sing}(\mathcal{F})\cap(C-D_2)$, the local monodromy of $\mathcal{H}$ is quadratic of drop r, with scale a character not of order 2. The conclusion follows from Theorem 1.5.1 (and Theorem 1.7.1, if r=1), applied to (r, m, $\mathcal{H}$), with S taken to be CritVal(f_2, f_1, $\mathcal{F}$), and S_0 the m images $\delta = f(\beta)$ of points β in $\mathrm{Sing}(\mathcal{F})\cap(C-D_2)$. QED

5.7 Theorems of big monodromy for $\mathcal{G}$:= $\mathrm{Twist}_{\chi,C,D}(\mathcal{F})$ on *Fct*(C, d, D, $\mathrm{Sing}(\mathcal{F})_{finite}$) in characteristic 2

Theorem 5.7.1 Let k be an algebraically closed field of characteristic 2. Fix a prime number ℓ which is invertible in k. Fix a nontrivial character χ of finite odd order n ≥ 3. Let C/k be a proper smooth connected curve of genus g. Fix an irreducible middle extension $\overline{\mathbb{Q}}_\ell$–sheaf $\mathcal{F}$ on C. Let $D = \sum a_i P_i$

be an effective divisor of degree d on C. Suppose that either

1a) $d \geq 6g+3$, all a_i are odd, $\mathrm{Sing}(\mathcal{F})\cap(C-D)$ is nonempty, and the following inequalities hold:

$$\text{if } \mathrm{rank}(\mathcal{F}) = 1,\ 2g - 2 + d \geq \mathrm{Max}(2\#(\mathrm{Sing}(\mathcal{F})\cap(C-D)),\ 4\mathrm{rank}(\mathcal{F})),$$
$$\text{if } \mathrm{rank}(\mathcal{F}) \geq 2,\ 2g - 2 + d \geq \mathrm{Max}(2\#(\mathrm{Sing}(\mathcal{F})\cap(C-D)),\ 72\mathrm{rank}(\mathcal{F})),$$

or
1b) $d \geq 12g+7$, and the following inequalities hold:

$$\text{if } \mathrm{rank}(\mathcal{F}) = 1,\ 2g - 2 + d > \mathrm{Max}(2\#\mathrm{Sing}(\mathcal{F}),\ 4\mathrm{rank}(\mathcal{F})).$$
$$\text{if } \mathrm{rank}(\mathcal{F}) \geq 2,\ 2g - 2 + d > \mathrm{Max}(2\#\mathrm{Sing}(\mathcal{F}),\ 72\mathrm{rank}(\mathcal{F})).$$

Suppose further that
2) $\mathcal{F}$ is everywhere tame.

Then for the lisse sheaf $\mathcal{G}$ on $\mathit{Fct}(C, d, D, \mathrm{Sing}(\mathcal{F})_{\mathrm{finite}})$ given by

$$f \mapsto H^1(C, j_*(\mathcal{F}\otimes\mathcal{L}_{\chi(f)}),$$

G_{geom} contains SL.

proof This follows from Theorems 5.6.1 and 5.6.2 above in exactly the same way that Theorem 5.5.1 followed from Theorems 5.4.1 and 5.4.9. QED