L−functions and monodromy: four lectures on Weil II

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Lecture I

Introduction

It is now nearly 27 years since Deligne, in the summer of 1973, proved the last of the Weil Conjectures, the Riemann Hypothesis for projective smooth varieties over finite fields. That proof was the subject of his article [De−Weil I], often referred to as Weil I. In the fall of that same year, Deligne formulated and proved a vast generalization of his Weil I result. This generalization, which is the subject of Deligne's article [De−Weil II], is referred to as Weil II. As marvelous an achievement as Weil I was, it is Weil II which has turned out to be the fundamental tool. For example, both the "hard Lefschetz theorem" for projective smooth varieties over finite fields, and Deligne's general equidistribution theorem ("generalized Sato−Tate conjecture in the function field case") require Weil II.

Deligne's proof of Weil I was based upon combining Grothendieck's ℓ−adic cohomological theory of L−functions, the monodromy theory of Lefschetz pencils, and Deligne's own stunning transposition to the function field case of Rankin's method [Ran] of "squaring", developed by Rankin in 1939 to give the then best known estimates for the size of Ramanujan's τ(n). Deligne's proof of Weil II is generally regarded as being much deeper and more difficult that his proof of Weil I. In the spring of 1984, Laumon found a signifigant simplification [Lau−TF, 4, pp. 203−208] of Deligne's proof of Weil II, based upon Fourier Transform ideas.

In these lectures, we will present a further simplification of Laumon's simplification of Deligne's proof of Weil II. In order to explain the idea behind this simplification, suppose we want to prove the Riemann Hypothesis for a projective smooth hypersurface of some odd dimension 2n+1 and some degree d ≥ 2 (the case d=1 being trivial!) over a finite field of some characteristic p. Deligne in Weil I proves the Riemann Hypothesis **simultaneously** for **all** projective smooth hypersurfaces of dimension 2n+1 and degree d over all finite fields of characteristic p, cf. [Ka−ODP, pp. 288−297] The key point is that the family of all such hypersurfaces has big monodromy. In other words, he proves the Riemann Hypothesis for a particular hypersurface by putting it in a family with big monodromy.

Our underlying idea is to put the particular L−function for which we are trying to prove the Weil II estimate into a family of L−functions which has big monodromy, then to apply the Rankin squaring method to this family. So we end up with a proof of Weil II in the style of Weil I. We will not assume any familiarity with either Weil I or Weil II, but we will, after briefly recalling them, make use of the standard facts about ℓ−adic sheaves, their cohomology, and their L−functions. We will also make use of an elementary instance of the involutivity [Lau−TF, 1.2.2.1] of the Fourier Transform (in Step 1 of Lecture 4). Caveat emptor.

This paper is a fairly faithful written version of four lectures I gave in March, 2000 at the Arizona Winter School 2000. It is a pleasure to thank both the organizers and the students of the School.

Review of ℓ−adic sheaves and ℓ−adic cohomology

Recall that for any connected scheme X, and for any geometric point (:= point of X with values in an algebraically closed field) x of X, we have the profinite fundamental group $\pi_1(X, x)$, which classifies finite etale coverings of X. Just as in usual topology, if we change the base point x in X, say to another geometric point x' in X, then the choice of a "path" from x to x' gives a well−defined isomorphism from $\pi_1(X, x)$ to $\pi_1(X, x')$. If we vary the path, then this isomorphism will vary by an inner automorphism (of either source or target). If we have a second connected scheme Y and a morphism $f : X \to Y$, we get an induced homomorphism

$$f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$$

We will often suppress the base point, and write simply $\pi_1(X)$. Then "the" induced homomorphism

$$f_* : \pi_1(X) \to \pi_1(Y)$$

is only well−defined up to an inner automorphism of the target.

In what follows, we will only need to speak of $\pi_1(X)$ for X a normal connected scheme. Such a scheme X has a function field, say K. If we fix an algebraic closure $\overline{K}$ of K, and denote by K^{sep}/K the separable closure of K inside $\overline{K}$, then $\pi_1(X)$ [with base point given by the chosen $\overline{K}$] has an apparently simple description in terms of Galois theory. Namely, $\pi_1(X)$ is that quotient of $\mathrm{Gal}(K^{sep}/K)$ which classifies those finite separable extensions L/K inside K^{sep} with the property that the normalization of X in L is finite etale over X. [In fact in the pre−Grothendieck days, this is how people defined the fundamental group.] In particular, when X is the spectrum of a field k, then $\pi_1(\mathrm{Spec}(k))$ is just $\mathrm{Gal}(k^{sep}/k)$. When k is a finite field, $k^{sep} = \overline{k}$, and $\mathrm{Gal}(\overline{k}/k)$ is canonically the abelian group $\hat{\mathbb{Z}}$, with canonical generator "geometric Frobenius" F_k, defined as the **inverse**. of the "arithmetic Frobenius" automorphism $\alpha \mapsto \alpha^{\#k}$ of $\overline{k}$. Given two finite fields k and E, for any morphism from Spec(E) to Spec(k) (i.e., for any field inclusion $k \subset E$), the induced map from $\pi_1(\mathrm{Spec}(E))$ to $\pi_1(\mathrm{Spec}(k))$ sends F_E to $(F_k)^{\deg(E/k)}$.

Given a connected scheme X, a field E, and an E−valued point x in X(E), we can view x as a morphism from Spec(E) to X. So we get a group homomorphism

$$x_* : \pi_1(\mathrm{Spec}(E)) = \mathrm{Gal}(\overline{E}/E) \to \pi_1(X),$$

well defined up to inner automorphism. When E is a finite field, the image in $\pi_1(X)$ of the canonical generator F_E of $\pi_1(\mathrm{Spec}(E))$ is denoted $\mathrm{Frob}_{E,x}$: it is well−defined as a conjugacy class in $\pi_1(X)$. For any field automorphism σ of E, and any x in X(E), we may form the point $\sigma(x)$ in X(E); the Frobenius conjugacy classes of x and of $\sigma(x)$ are equal: $\mathrm{Frob}_{E,\sigma(x)} = \mathrm{Frob}_{E,x}$

If we start with a field k, and a k−scheme X/k, then for variable k−schemes S, we are generally interested not in the set $X(S) := \mathrm{Hom}_{\mathrm{Schemes}}(S, X)$ of all S−valued points of X as an abstract scheme, but rather in the subset $(X/k)(S) := \mathrm{Hom}_{\mathrm{Schemes}/k}(S, X)$ consisting of the k−scheme morphisms from S to X.

Now fix a prime number ℓ. Denote by $\mathbb{Q}_\ell$ the completion of $\mathbb{Q}$ for the ℓ−adic valuation, and by $\overline{\mathbb{Q}}_\ell$ an algebraic closure of $\mathbb{Q}_\ell$. Given a finite extension E_λ of $\mathbb{Q}_\ell$ inside $\overline{\mathbb{Q}}_\ell$, we will denote by $\mathcal{O}_\lambda$ its ring of integers, and by $\mathbb{F}_\lambda$ its residue field. There is the general notion of a constructible $\overline{\mathbb{Q}}_\ell$−sheaf on our connected normal X. In most of what follows, we will be concerned with the more restricted class of lisse $\overline{\mathbb{Q}}_\ell$−sheaves on X. For our purposes, a lisse $\overline{\mathbb{Q}}_\ell$−sheaf $\mathcal{F}$ on X "is" a finite−dimensional continuous $\overline{\mathbb{Q}}_\ell$−representation

$$\Lambda := \Lambda_{\mathcal{F}} : \pi_1(X) \to GL(r, \overline{\mathbb{Q}}_\ell).$$

For any given $\mathcal{F}$, there is a finite extension E_λ of $\mathbb{Q}_\ell$ such that $\Lambda_{\mathcal{F}}$ takes values in $GL(r, E_\lambda)$, cf. [Ka−Sar, 9.0.7]. In a suitable basis of the representation space, $\Lambda_{\mathcal{F}}$ takes values in $GL(r, \mathcal{O}_\lambda)$. The degree r of the representation $\Lambda_{\mathcal{F}}$ is called the rank of the lisse sheaf $\mathcal{F}$.

Suppose that k is a finite field, and that X is a (connected, normal) separated k−scheme of finite type. Inside a chosen $\overline{k}$, k has a unique extension k_n/k of each degree $n \geq 1$, and the sets $(X/k)(k_n)$ are all finite. Given a lisse $\overline{\mathbb{Q}}_\ell$−sheaf $\mathcal{F}$ on X, corresponding to a continuous $\overline{\mathbb{Q}}_\ell$−representation

$$\Lambda := \Lambda_{\mathcal{F}} : \pi_1(X) \to GL(n, \overline{\mathbb{Q}}_\ell),$$

the L−function attached to $\mathcal{F}$ on X/k is defined as a formal power series $L(X/k, \mathcal{F})(T)$ in $1 + T\overline{\mathbb{Q}}_\ell[[T]]$ by the following recipe. For each integer $n \geq 1$, one forms the sum

$$S_n(X/k, \mathcal{F}) := \Sigma_{x \text{ in } (X/k)(k_n)} \mathrm{Trace}(\Lambda_{\mathcal{F}}(\mathrm{Frob}_{k_n,x})).$$

[One sometimes writes $\mathrm{Trace}(\Lambda_{\mathcal{F}}(\mathrm{Frob}_{k_n,x}))$ as $\mathrm{Trace}(\mathrm{Frob}_{k_n,x} \mid \mathcal{F})$:

$$\mathrm{Trace}(\mathrm{Frob}_{k_n,x} \mid \mathcal{F}) := \mathrm{Trace}(\Lambda_{\mathcal{F}}(\mathrm{Frob}_{k_n,x})).$$

Then one defines

$$L(X/k, \mathcal{F})(T) = \exp(\Sigma_{n\geq 1} S_n(X/k, \mathcal{F})T^n/n).$$

There is also an Euler product for the L−function, over the closed points (:= orbits of $\mathrm{Gal}(k^{sep}/k)$ in $(X/k)(\overline{k})$) of X/k. The degree $\deg(\mathcal{P})$ of a closed point $\mathcal{P}$ is the cardinality of the corresponding orbit. For a closed point $\mathcal{P}$ of degree n, i.e., an orbit of $\mathrm{Gal}(k_n/k)$ in $(X/k)(k_n)$ of cardinality n, every point x in that orbit gives rise to the same Frobenius conjugacy class $\mathrm{Frob}_{k_n,x}$: this class is called $\mathrm{Frob}_{\mathcal{P}}$. One has the identity

$$L(X/k, \mathcal{F})(T) = \prod_{\mathcal{P}} 1/\det(1 - T^{\deg(\mathcal{P})}\Lambda_{\mathcal{F}}(\mathrm{Frob}_{\mathcal{P}})).$$

Suppose now and henceforth that X/k is separated of finite type, smooth and geometrically connected of dimension n over the finite field k. We have a short exact sequence

$$0 \to \pi_1(X\otimes_k\overline{k}) \to \pi_1(X) \xrightarrow{\text{degree}} \pi_1(\mathrm{Spec}(k)) = \mathrm{Gal}(\overline{k}/k) = \hat{\mathbb{Z}} \to 0,$$

in which the degree map has the following effect on Frobenii:

$$\mathrm{Frob}_{k_n,x} \mapsto (F_k)^n.$$

The first group $\pi_1(X\otimes_k\bar{k})$ is called $\pi_1{}^{geom}(X/k)$, the geometric fundamental group of X/k, and the middle group $\pi_1(X)$ is called $\pi_1{}^{arith}(X/k)$, the arithmetic fundamental group of X/k:

$$0 \to \pi_1{}^{geom}(X/k) \to \pi_1{}^{arith}(X/k) \to \hat{\mathbb{Z}} \to 0.$$

The inverse image in $\pi_1{}^{arith}(X/k)$ of the subgroup $\mathbb{Z}$ of $\hat{\mathbb{Z}}$ consisting of integer powers of F_k is called W(X/k), the Weil group of X/k:

$$0 \to \pi_1{}^{geom}(X/k) \to W(X/k) \to \mathbb{Z} \to 0.$$

We next define generalized Tate twists. Given an ℓ-adic unit α in $\bar{\mathbb{Q}}_\ell{}^\times$ (i.e., α lies in some $\mathcal{O}_\lambda{}^\times$), the group homomorphism

$$\mathbb{Z} \to \bar{\mathbb{Q}}_\ell{}^\times = GL(1, \bar{\mathbb{Q}}_\ell)$$
$$d \mapsto \alpha^d$$

extends uniquely to a continuous homomorphism

$$\hat{\mathbb{Z}} \to \bar{\mathbb{Q}}_\ell{}^\times.$$

We then view $\hat{\mathbb{Z}}$ as a quotient of $\pi_1{}^{arith}(X/k)$. The composite character

$$\pi_1{}^{arith}(X/k) \to \bar{\mathbb{Q}}_\ell{}^\times$$
$$\gamma \mapsto \alpha^{degree(\gamma)}$$

is called α^{deg}. The characters of $\pi_1{}^{arith}(X/k)$ which are trivial on $\pi_1{}^{geom}(X/k)$ are precisely these α^{deg} characters. Given a lisse $\bar{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on X, corresponding to a finite-dimensional representation $\Lambda_\mathcal{F}$, we denote by $\mathcal{F}\otimes\alpha^{deg}$ the lisse sheaf of the same rank defined to $\Lambda_\mathcal{F}\otimes$"$\alpha^{deg}$". For an integer r, and for $\alpha := (\#k)^{-r}$, we write $\mathcal{F}(r)$ for $\mathcal{F}\otimes\alpha^{deg}$

Suppose further that the prime number ℓ is invertible in k. For each integer $i \geq 0$, and for any constructible $\bar{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on X, we have both ordinary and compact cohomology groups

$$H^i(X\otimes_k\bar{k}, \mathcal{F}) \text{ and } H^i{}_c(X\otimes_k\bar{k}, \mathcal{F})$$

of $X\otimes_k\bar{k}$ with coefficients in $\mathcal{F}$. These are finite-dimensional $\bar{\mathbb{Q}}_\ell$-vector spaces on which $Gal(\bar{k}/k)$ acts continuously, and which vanish for $i > 2n$ (recall $n := \dim(X)$). If X/k is proper, then $H^i{}_c \cong H^i$. In general, there is a natural "forget supports" map

$$H^i{}_c(X\otimes_k\bar{k}, \mathcal{F}) \to H^i(X\otimes_k\bar{k}, \mathcal{F}),$$

which need not be an isomorphism. The group $H^{2n}{}_c(X\otimes_k\bar{k}, \mathcal{F})$ is a birational invariant, in the sense that for any dense open set U in X,

$$H^{2n}{}_c(U\otimes_k\bar{k}, \mathcal{F}) \cong H^{2n}{}_c(X\otimes_k\bar{k}, \mathcal{F}).$$

For $\mathcal{F}$ a lisse $\bar{\mathbb{Q}}_\ell$-sheaf on X/k (which we have assumed separated of finite type, smooth and geometrically connected of dimension n) its ordinary and compact cohomology are related by

Poincare duality. The group $H^{2n}_c(X\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell)$ is $\bar{\mathbb{Q}}_\ell(-n)$, the one–dimensional $\bar{\mathbb{Q}}_\ell$–vector space on which F_k acts as $(\#k)^n$. For $\mathcal{F}^\vee$ the linear dual (contragredient representation), the cup–product pairing

$$H^i_c(X\otimes_k\bar{k}, \mathcal{F})\times H^{2n-i}(X\otimes_k\bar{k}, \mathcal{F}^\vee) \to H^{2n}_c(X\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell) \cong \bar{\mathbb{Q}}_\ell(-n)$$

is a $\mathrm{Gal}(\bar{k}/k)$–equivariant pairing, which identifies each pairee with the $\bar{\mathbb{Q}}_\ell(-n)$–dual of the other.

For $\mathcal{F}$ lisse, the group $H^0(X\otimes_k\bar{k}, \mathcal{F})$ has a simple description: it is the space of $\pi_1{}^{geom}(X/k)$–invariants in the corresponding representation:

$$H^0(X\otimes_k\bar{k}, \mathcal{F}) = (\mathcal{F})^{\pi_1{}^{geom}(X/k)},$$

with the induced action of $\pi^{arith}/\pi_1{}^{geom} = \mathrm{Gal}(\bar{k}/k)$.

By Poincare duality, the group $H^{2n}_c(X\otimes_k\bar{k}, \mathcal{F})$ is the space of Tate–twisted $\pi_1{}^{geom}(X/k)$–coinvariants (largest quotient on which $\pi_1{}^{geom}(X/k)$ acts trivially) of the corresponding representation:

$$H^{2n}_c(X\otimes_k\bar{k}, \mathcal{F}) = (\mathcal{F})_{\pi_1{}^{geom}(X/k)}(-n).$$

If X is affine, then for any constructible $\mathcal{F}$ we have

$$H^i(X\otimes_k\bar{k}, \mathcal{F}) = 0 \text{ for } i > n.$$

Thus for $\mathcal{F}$ lisse we have the dual vanishing,

$$H^i_c(X\otimes_k\bar{k}, \mathcal{F}) = 0 \text{ for } i < n.$$

Thus for X/k an affine curve, and $\mathcal{F}$ lisse, the only possibly nonvanishing H^i_c are H^1_c and H^2_c, and of these two groups we know H^2_c: it is the Tate–twisted coinvariants.

For any constructible $\mathcal{F}$, its L–function is a rational function, given by its H^i_c

$$L(X/k, \mathcal{F})(T) = \prod_{i=0 \text{ to } 2n} \det(1 - TF_k|H^i_c(X\otimes_k\bar{k}, \mathcal{F}))^{(-1)^{i+1}}.$$

[This formula is equivalent (take logarithmic derivatives of both sides) to the Lefschetz Trace Formula [Gro–FL]: for every finite extension E/k,

$$\sum_{x \text{ in } (X/k)(E)}\mathrm{Trace}(\mathrm{Frob}_{E,x}|\mathcal{F}) = \sum_i (-1)^i\mathrm{Trace}(F_E \mid H^i_c(X\otimes_k\bar{k}, \mathcal{F})).]$$

In the fundamental case when X is an affine curve and $\mathcal{F}$ is lisse, the cohomological formula for L is simply

$$L(X/k, \mathcal{F})(T) = \det(1 - TF_k|H^1_c(X\otimes_k\bar{k}, \mathcal{F}))/\det(1 - TF_k|H^2_c(X\otimes_k\bar{k}, \mathcal{F})).$$

For X an affine curve, and $\mathcal{F}$ a constructible $\bar{\mathbb{Q}}_\ell$–sheaf on X, there is always a dense open set $j : U \to X$ on which $\mathcal{F}$ is lisse. We have a natural adjunction map $\mathcal{F} \to j_*j^*\mathcal{F}$, which is injective if and only if $H^0_c(X\otimes_k\bar{k}, \mathcal{F}) = 0$. [The point is that the kernel of the adjunction map, call it $\mathrm{Pct}_{\mathcal{F}}$, is punctual, supported at finitely many closed points. So $\mathrm{Pct}_{\mathcal{F}} = 0$ if and only if $H^0_c(X\otimes_k\bar{k}, \mathrm{Pct}_{\mathcal{F}}) = 0$. Because X is affine, $H^0_c(X\otimes_k\bar{k}, j_*j^*\mathcal{F}) = 0$, so the inclusion of $\mathrm{Pct}_{\mathcal{F}}$ into $\mathcal{F}$ induces an isomorphism on H^0_c, cf. [Ka−SE, 4.5.2].] If $H^0_c(X\otimes_k\bar{k}, \mathcal{F}) = 0$, then the largest open set on which $\mathcal{F}$ is lisse is precisely the set of points where its stalk has maximum rank.

For U/k an affine curve, $\mathcal{F}$ a lisse $\bar{\mathbb{Q}}_\ell$−sheaf on U, C/k the complete nonsingular model of U, and $j : U \to C$ the inclusion, we may form the constructible $\bar{\mathbb{Q}}_\ell$−sheaf $j_*\mathcal{F}$ on C. Its cohomology groups are related to those of $\mathcal{F}$ on U as follows:

$$H^0(C\otimes_k\bar{k}, j_*\mathcal{F}) = H^0(U\otimes_k\bar{k},\mathcal{F}), \text{ and } H^0_c(U\otimes_k\bar{k},\mathcal{F}) = 0,$$
$$H^1(C\otimes_k\bar{k}, j_*\mathcal{F}) = \mathrm{Image}(H^1_c(U\otimes_k\bar{k},\mathcal{F}) \to H^1(U\otimes_k\bar{k},\mathcal{F})),$$
$$H^2(C\otimes_k\bar{k}, j_*\mathcal{F}) = H^2_c(U\otimes_k\bar{k},\mathcal{F}), \text{ and } H^2(U\otimes_k\bar{k},\mathcal{F}) = 0.$$

Weights, and formulation of the target theorem

We now introduce archimedean considerations into our discussion. Denote by | | the usual complex absolute value on $\mathbb{C}$:

$$|x+iy| := \mathrm{Sqrt}(x^2 + y^2).$$

Fix a field embedding ι of $\bar{\mathbb{Q}}_\ell$ into $\mathbb{C}$. By means of ι, we may speak of the complex absolute value of an element α of $\bar{\mathbb{Q}}_\ell$:

$$|\alpha|_\iota := |\iota(\alpha)|.$$

Given a lisse $\bar{\mathbb{Q}}_\ell$−sheaf $\mathcal{F}$ on X/k as above (i.e. smooth and geometrically connected, of dimension n), and a real number w, we say that $\mathcal{F}$ is ι−pure of weight w if the following condition holds. For every finite extension field E/k, for every point x in (X/k)(E), we have

$$|\text{every eigenvalue of } \Lambda_{\mathcal{F}}(\mathrm{Frob}_{E,x})|_\iota = \mathrm{Sqrt}(\#E)^w.$$

We say that a finite−dimensional $\bar{\mathbb{Q}}_\ell$−representation V of $\mathrm{Gal}(\bar{k}/k)$ is ι−pure of weight w if

$$|\text{every eigenvalue of } F_k \text{ on } V|_\iota = \mathrm{Sqrt}(\#k)^w.$$

We say that V is ι−mixed of weight $\le$ w (resp. $\ge$ w) if we have the inequality

$$|\text{every eigenvalue of } F_k \text{ on } V|_\iota \le \mathrm{Sqrt}(\#k)^w,$$

respectively

$$|\text{every eigenvalue of } F_k \text{ on } V|_\iota \ge \mathrm{Sqrt}(\#k)^w,$$

In questions concerning ι−weights, we may, whenever convenient, extend scalars from the ground field k over which we started to any finite extension field.

The interpretation of H^0 (resp. of H^{2n}_c) in terms of $\pi_1{}^{\mathrm{geom}}(X/k)$−invariants (resp. Tate−twisted coinvariants) shows that if a lisse $\mathcal{F}$ is ι−pure of weight w, then $H^0(X\otimes_k\bar{k}, \mathcal{F})$ (and so also its subspace $H^0_c(X\otimes_k\bar{k}, \mathcal{F})$) is ι−pure of weight w, and $H^{2n}_c(X\otimes_k\bar{k}, \mathcal{F})$ is ι−pure of weight w +

2n. [For example, if $(X/k)(k_d)$ is nonempty, say containing a point x, then the action of $(F_k)^d$ on $H^0_c(X\otimes_k\bar{k}, \mathcal{F}))$ is induced by the action of $\mathrm{Frob}_{k_d,x}$ on $\mathcal{F}$, and the action of $(F_k)^d$ on $H^{2n}_c(X\otimes_k\bar{k}, \mathcal{F}))$ is induced by the action of $\mathrm{Frob}_{k_d,x}$ on $\mathcal{F}(-n)$.] Thus we always have

$H^0_c(X\otimes_k\bar{k}, \mathcal{F}))$ is ι−pure of weight w,

$H^{2n}_c(X\otimes_k\bar{k}, \mathcal{F})$ is ι−pure of weight w + 2n.

Given a lisse $\mathcal{F}$ which is ι−pure of some weight w, we can always find an ℓ−adic unit α such that $\mathcal{F}\otimes\alpha^{\deg}$ is ι−pure of weight zero. Indeed, if $\mathcal{F}$ has rank $r \geq 1$, and if E/k is a finite extension, say of degree d, for which (X/k)(E) is nonempty, pick a point x in (X/k)(E), and take for α any rd'th root of $1/\det(\Lambda_{\mathcal{F}}(\mathrm{Frob}_{E,x}))$. We have the trivial compatibility

$$H^i_c((X\otimes_k\bar{k}, \mathcal{F}\otimes\alpha^{\deg}) = H^i_c((X\otimes_k\bar{k}, \mathcal{F})\otimes\alpha^{\deg}.$$

We can now state our target theorem.

Target Theorem (Deligne) Let U/k be a smooth, geometrically connected curve over a finite field, ℓ a prime invertible in k, and $\mathcal{F}$ a lisse $\bar{\mathbb{Q}}_\ell$−sheaf on U which is ι−pure of weight w. Then

$H^1_c(U\otimes_k\bar{k}, \mathcal{F})$ is ι−mixed of weight $\leq$ w + 1.

As noted above, we already know that

$H^0_c(U\otimes_k\bar{k}, \mathcal{F})$ is ι−pure of weight w, and vanishes if U is affine,

$H^2_c(U\otimes_k\bar{k}, \mathcal{F})$ is ι−pure of weight w + 2.

Corollary [De−Weil II, 3.2.3]

1) The ordinary cohomology group $H^1(U\otimes_k\bar{k}, \mathcal{F})$ is ι−mixed of weight $\geq$ w + 1.

2) If the "forget supports" map is an isomorphism

$$H^1_c(U\otimes_k\bar{k}, \mathcal{F}) \cong H^1(U\otimes_k\bar{k}, \mathcal{F}),$$

then $H^1_c(U\otimes_k\bar{k}, \mathcal{F})$ is ι−pure of weight w+1.

3) For C/k the complete nonsingular model of U/k, for

$$j : U \to C$$

the inclusion, and for every integer $0 \leq i \leq 2$,

$H^i(C\otimes_k\bar{k}, j_*\mathcal{F})$ is ι−pure of weight w + i.

Proof of the corollary To prove 1), we argue as follows. For $\mathcal{F}$ lisse and ι−pure of weight w, the contragredient $\mathcal{F}^\vee$ is lisse and ι−pure of weight −w. By the target theorem applied to $\mathcal{F}^\vee$, $H^1_c(U\otimes_k\bar{k}, \mathcal{F}^\vee)$ is ι−mixed of weight $\leq$ −w + 1. But $H^1_c(U\otimes_k\bar{k}, \mathcal{F}^\vee)$ and $H^1(U\otimes_k\bar{k}, \mathcal{F})$ are Poincare−dually paired to $H^2_c(U\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell) \cong \bar{\mathbb{Q}}_\ell(-1)$, which is ι−pure of weight 2. Therefore $H^1(U\otimes_k\bar{k}, \mathcal{F})$ is ι−mixed of weight $\geq$ w+1.

To prove 2), notice that the target theorem and 1), both applied to $\mathcal{F}$, show that $H^1_c(U\otimes_k\bar{k}, \mathcal{F})$ is simultaneously ι−mixed of weight ≤ w+1 and ι−mixed of weight ≥ w+1.

We now prove 3). For i=0,

$$H^0(C\otimes_k\bar{k}, j_*\mathcal{F}) = H^0(U\otimes_k\bar{k}, \mathcal{F}),$$

and this last group is ι−pure of weight w. For i=2, we have

$$H^2(C\otimes_k\bar{k}, j_*\mathcal{F}) = H^2_c(C\otimes_k\bar{k}, j_*\mathcal{F}) \cong H^2_c(U\otimes_k\bar{k}, \mathcal{F}),$$

and this last group is ι−pure of weight w+2.

For i=1, $H^1(C\otimes_k\bar{k}, j_*\mathcal{F})$ is the image of $H^1_c(U\otimes_k\bar{k}, \mathcal{F})$ in $H^1(U\otimes_k\bar{k}, \mathcal{F})$. Thus $H^1(C\otimes_k\bar{k}, j_*\mathcal{F})$ is simultaneously a quotient of $H^1_c(U\otimes_k\bar{k}, \mathcal{F})$, so ι−mixed of weight ≤ w+1, and a subobject of $H^1(U\otimes_k\bar{k}, \mathcal{F})$, and so (by part 1)) ι−mixed of weight ≥ w+1. QED

First reductions in the proof of the target theorem

By an $\alpha^{\deg}$ twist, it suffices to prove the target theorem in the special case when $\mathcal{F}$ is ι−pure of weight zero. We henceforth assume that $\mathcal{F}$ is ι−pure of weight zero.

We next reduce to the case when the curve U/k is the affine line $\mathbb{A}^1$/k. We do this in several steps. To prove the theorem for the data (U/k, $\mathcal{F}$), we may always shrink U to a dense open set $U_1 \subset U$. Indeed, if we denote by $j_1: U_1 \to U$ the inclusion, we have a short exact sequence on U

$$0 \to (j_1)_!(j_1)^*\mathcal{F} \to \mathcal{F} \to \text{(a punctual sheaf)} \to 0.$$

The long exact cohomology sequence then exhibits $H^1_c(U\otimes_k\bar{k}, \mathcal{F})$ as a quotient of $H^1_c(U_1\otimes_k\bar{k}, \mathcal{F}|U_1)$. We may also extend scalars from k to any finite extension of k.

In this way, we may reduce to the case when U is affine, and at least one point at ∞, say P, is k−rational. We next reduce to the case when U is a dense open set in $\mathbb{A}^1$. For all large integers m, the Riemann−Roch space L(mP) contains a function f which has a pole of order precisely m at P, and no other poles. If we take m to be invertible in k, the differential df of any such f is nonzero. So f, viewed as a finite flat map of degree m from C to $\mathbb{P}^1$, is finite etale of degree m over some dense open set V of $\mathbb{P}^1$. Shrinking V, we may further assume that $V \subset \mathbb{A}^1$ and that $f^{-1}(V) \subset U$. Shrinking U, we may assume that U itself is finite etale over V, of degree m. Then $f_*\mathcal{F}$ is lisse on V, and ι−pure of weight zero, and we have

$$H^i_c(V\otimes_k\bar{k}, f_*\mathcal{F}) = H^i_c(U\otimes_k\bar{k}, \mathcal{F}),$$

by the (trivial) Leray spectral sequence for the map f: U → V.

Once we are reduced to the case of a dense open set U in $\mathbb{A}^1$/k, we may, by a further extension of scalars, reduce to the case when all the points in S := $\mathbb{A}^1$ − U are k−rational. Denote by Γ the additive subgroup of $\mathbb{A}^1(k)$ = (k, +) generated by the points in S. Thus Γ is an $\mathbb{F}_p$−vector space of dimension at most #S, so in particular it is a finite subgroup of $\mathbb{A}^1(k)$. Shrinking $U = \mathbb{A}^1 - S$, we reduce to the case when U is of the form $\mathbb{A}^1 - \Gamma$, for Γ a finite subgroup of $\mathbb{A}^1(k)$. The quotient $\mathbb{A}^1/\Gamma$ is itself an $\mathbb{A}^1/k$. The quotient map, say π,

$$\pi : \mathbb{A}^1 \to \mathbb{A}^1/\Gamma \cong \mathbb{A}^1,$$

is finite etale, and makes $\mathbb{A}^1 - \Gamma$ finite etale over $\mathbb{A}^1 - \{0\} := \mathbb{G}_m/k$. So it suffices to treat $\pi_*\mathcal{F}$ on $\mathbb{G}_m/k$.

To complete the reduction, we note that in positive characteristic p, the Abhyankar map [Ab, Thm. 1, page 830]

$$\text{Abhy}: \mathbb{G}_m \to \mathbb{A}^1,$$
$$x \mapsto x^p + 1/x,$$

makes $\mathbb{G}_m$ a finite etale covering of $\mathbb{A}^1$ of degree p+1. So it suffices to treat $\text{Abhy}_*\pi_*\mathcal{F}$ on $\mathbb{A}^1/k$.

So now we are reduced to proving the target theorem for a lisse $\mathcal{F}$ on $\mathbb{A}^1$ which is ι−pure of weight zero. We next reduce to the case when $\mathcal{F}$ is irreducible as a representation of $\pi_1^{\text{arith}}(\mathbb{A}^1/k)$. We proceed by induction on the length of a Jordan−Holder series. If $\mathcal{F}_1 \subset \mathcal{F}$ is an irreducible lisse subsheaf of $\mathcal{F}$, both $\mathcal{F}_1$ and $\mathcal{F}/\mathcal{F}_1$ are lisse and ι−pure of weight zero, and by induction both $H^1_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}_1)$ and $H^1_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}/\mathcal{F}_1)$ are ι−mixed of weight ≤ 1. The cohomology sequence for the short exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}/\mathcal{F}_1 \to 0$$

gives an exact sequence

$$H^1_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}_1) \to H^1_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}) \to H^1_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}/\mathcal{F}_1),$$

whose middle term is thus ι−mixed of weight ≤ 1 as well.

We now show that it suffices to prove the target theorem in the case when the lisse irreducible $\mathcal{F}$ on $\mathbb{A}^1$ is geometrically irreducible, i.e., irreducible as a representation of $\pi_1^{\text{geom}}(\mathbb{A}^1/k)$. We proceed by induction on the rank of $\mathcal{F}$. Consider the restriction of $\mathcal{F}$ to $\pi_1^{\text{geom}}(\mathbb{A}^1/k)$. Because $\mathcal{F}$ is irreducible for $\pi_1^{\text{arith}}(\mathbb{A}^1/k)$, and $\pi_1^{\text{geom}}(\mathbb{A}^1/k)$ is normal in $\pi_1^{\text{arith}}(\mathbb{A}^1/k)$, the restriction of $\mathcal{F}$ to $\pi_1^{\text{geom}}(\mathbb{A}^1/k)$ is completely reducible. So we may write it as a sum of irreducibles $\mathcal{F}_i$ with multiplicities n_i, say

$$\mathcal{F} \mid \pi_1^{\text{geom}}(\mathbb{A}^1/k) \cong \oplus_i n_i\mathcal{F}_i.$$

Consider first the case when $\mathcal{F} \mid \pi_1^{\text{geom}}(\mathbb{A}^1/k)$ is not isotypical, i.e., the case in which there are at least two distinct $\mathcal{F}_i$'s in the decomposition. In that case, the individual isotypical components $n_i\mathcal{F}_i$ are permuted among themselves by the action of the quotient group $\pi_1^{\text{arith}}(\mathbb{A}^1/k)/\pi_1^{\text{geom}}(\mathbb{A}^1/k) \cong \text{Gal}(\bar{k}/k)$. As there are only finitely many such isotypical components, an open subgroup of finite index in $\text{Gal}(\bar{k}/k)$ stabilizes each isotypical component separately. So after extending scalars from k to some finite extension field, $\mathcal{F}$ becomes reducible as a representation of $\pi_1{}^{\mathrm{arith}}$. Each irreducible summand has lower rank, so the target theorem holds for it, by induction.

It remains to treat the case in which $\mathcal{F}|\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k)$ is isotypical, say

$$\mathcal{F}|\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k) \cong n_1\mathcal{F}_1,$$

with $\mathcal{F}_1$ corresponding to some irreducible representation ρ of $\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k)$. We will show that in this case $\mathcal{F}$ is already geometrically irreducible, i.e., that n=1. Because $n\rho$ is a representation of $\pi_1{}^{\mathrm{arith}}(\mathbb{A}^1/k)$, for each fixed element γ in $\pi_1{}^{\mathrm{arith}}(\mathbb{A}^1/k)$, the representation ρ of $\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k)$ has the same trace function as the representation $\rho^{(\gamma)}$: $g \mapsto \rho(\gamma g\gamma^{-1})$ of $\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k)$. So for fixed γ, we can choose A in GL(degree(ρ), $\overline{\mathbb{Q}}_\ell$) such that for every g in $\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k)$, we have

$$\rho(\gamma g\gamma^{-1}) = A\rho(g)A^{-1}.$$

Now take for γ an element of degree one. Then the Weil group $W(\mathbb{A}^1/k)$ (:= the subgroup of $\pi_1{}^{\mathrm{arith}}(\mathbb{A}^1/k)$ consisting of elements of integer degree) is the semidirect product

$$W(\mathbb{A}^1/k) = \pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k) \ltimes (\text{the } \mathbb{Z} \text{ generated by } \gamma),$$

with γ acting on $\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k)$ by conjugation. Then ρ extends to a representation $\tilde{\rho}$ of the Weil group $W(\mathbb{A}^1/k)$, defined by

$$\tilde{\rho}(g\gamma^n) := \rho(g)A^n.$$

Now consider $\mathcal{F}|W(\mathbb{A}^1/k)$. What is its relation to $\tilde{\rho}$? Well, on the normal subgroup $\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k)$ of $W(\mathbb{A}^1/k)$, $\mathcal{F}$ agrees with $n\tilde{\rho}$. So the natural map

$$\tilde{\rho}\otimes\mathrm{Hom}_{\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k)}(\tilde{\rho}, \mathcal{F}) \to \mathcal{F}$$

is an isomorphism of $W(\mathbb{A}^1/k)$−representations (because it is $W(\mathbb{A}^1/k)$−equivariant, and it is an isomorphism of $\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k)$−representations). Now $\mathcal{F}$, being irreducible on $\pi_1{}^{\mathrm{arith}}(\mathbb{A}^1/k)$, remains irreducible on the dense subgroup $W(\mathbb{A}^1/k)$. Therefore the n−dimensional representation $\mathrm{Hom}_{\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k)}(\tilde{\rho}, \mathcal{F})$ must be an irreducible representation of $W(\mathbb{A}^1/k)$. But $\mathrm{Hom}_{\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k)}(\tilde{\rho}, \mathcal{F})$ is in fact a representation of the quotient group

$$W(\mathbb{A}^1/k)/\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k) = \mathbb{Z}.$$

Being irreducible, its dimension must be 1. Thus n=1, which means precisely that $\mathcal{F}|\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k) \cong \mathcal{F}_1$ is irreducible.

So we are now reduced to proving the target theorem for a lisse, geometrically irreducible $\mathcal{F}$ on $\mathbb{A}^1/k$ which is ι−pure of weight zero. We now look to see if $\mathcal{F}$ happens to be geometrically self−dual, i.e, we look to see if $\mathcal{F}^\vee$ is isomorphic to $\mathcal{F}$ as a representation of $\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k)$. If $\mathcal{F}$ is geometrically self dual, we claim that some $\alpha^{\deg}$ twist of $\mathcal{F}$ is arithmetically self−dual, i.e., isomorphic to its contragredient as a representation of $\pi_1{}^{\mathrm{arith}}(\mathbb{A}^1/k)$, and still ι−pure of weight zero. To see this, we argue as follows. Pick a $\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k)$−invariant non−zero bilinear form < , > on the representation space, say V, of the representation $\rho := \Lambda_{\mathcal{F}}$:

$$<\rho(g)v, \rho(g)w> = <v, w>$$

for every g in $\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k)$. Because $\mathcal{F}$ is geometrically irreducible, the form < , > is unique up to a $\overline{\mathbb{Q}}_\ell{}^\times$ factor. We claim that for any fixed element γ in $\pi_1{}^{\mathrm{arith}}(\mathbb{A}^1/k)$, the nonzero bilinear form

$$T_\gamma[v, w] := <\rho(\gamma)v, \rho(\gamma)w>$$

is also $\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k)$−invariant. This holds simply because $\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k)$ is a normal subgroup of $\pi_1{}^{\mathrm{arith}}(\mathbb{A}^1/k)$: for g in $\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k)$, we readily calculate

$$\begin{aligned}
&T_\gamma[\rho(g)v, \rho(g)w] := <\rho(\gamma)\rho(g)v, \rho(\gamma)\rho(g)w> \\
&= <\rho(\gamma g\gamma^{-1})\rho(\gamma)v, \rho(\gamma g\gamma^{-1})\rho(\gamma)w> \\
&= <\rho(\gamma)v, \rho(\gamma)w> := T_\gamma[v, w].
\end{aligned}$$

Therefore the form T_γ is a $\overline{\mathbb{Q}}_\ell{}^\times$−multiple of < , >, say

$$T_\gamma = c_\gamma< , >.$$

The map $\gamma \mapsto c_\gamma$ is a continuous homomorphism from $\pi_1{}^{\mathrm{arith}}(\mathbb{A}^1/k)$ to $\overline{\mathbb{Q}}_\ell{}^\times$ which is trivial on $\pi_1{}^{\mathrm{geom}}(\mathbb{A}^1/k)$, so it is of the form $\beta^{\deg}$ for some ℓ−adic unit in $\overline{\mathbb{Q}}_\ell{}^\times$. If we take for α either square root of $1/\beta$, then $\mathcal{F}\otimes\alpha^{\deg}$ is arithmetically self dual, and ι−pure of some weight w (namely the real number w such that $|\iota(\alpha)| = (\#k)^{w/2}$). The contragredient of $\mathcal{F}\otimes\alpha^{\deg}$ is therefore ι−pure of weight −w. Thus w = −w, so w = 0, as required.

Lecture II

To summarize: we are reduced to proving the target theorem for a lisse, geometrically irreducible $\mathcal{F}$ on $\mathbb{A}^1/k$ which is ι−pure of weight zero, and which, if geometrically self−dual, is arithmetically self dual. We may further assume that $\mathcal{F}$ is geometrically nontrivial. For if $\mathcal{F}$ is geometrically trivial, then $H^1{}_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}) = 0$ (simply because $H^1{}_c(\mathbb{A}^1\otimes_k\bar{k}, \overline{\mathbb{Q}}_\ell) = 0$), so in this case there is nothing to prove.

Review of the Artin−Schreier sheaf $\mathcal{L}_\psi$: wild twisting and the auxiliary sheaf $\mathcal{G}$ on $\mathbb{A}^2$: the purity theorem

Let us fix a nontrivial additive $\overline{\mathbb{Q}}_\ell{}^\times$−valued character ψ of our ground field k,

$$\psi : (k, +) \to \overline{\mathbb{Q}}_\ell{}^\times.$$

Then $\mathbb{A}^1/k$ becomes a finite etale galois covering of itself with galois group (k, +), by the (sign−changed) Artin−Schreier map $x \mapsto x - x^{\#k}$:

$$\begin{array}{l} \mathbb{A}^1/k \\ \downarrow x \mapsto x - x^{\#k} \\ \mathbb{A}^1/k \end{array}$$

This covering exhibits (k, +) as a finite quotient group of $\pi_1{}^{\mathrm{arith}}(\mathbb{A}^1/k)$:

$$\pi_1{}^{\mathrm{arith}}(\mathbb{A}^1/k) \to (k, +) \to 0.$$

If we compose this surjection with ψ, we get a continuous homomorphism

$$\pi_1{}^{\mathrm{arith}}(\mathbb{A}^1/k) \to (k, +) \to \overline{\mathbb{Q}}_\ell{}^\times,$$

i.e., we get a lisse $\overline{\mathbb{Q}}_\ell$−sheaf of rank one on $\mathbb{A}^1/k$. This by definition is $\mathcal{L}_\psi$. Its trace function is given as follows. For any finite extension E/k, and for any point x in $(\mathbb{A}^1/k)(E) = E$, $\mathrm{Frob}_{E,x}$ acts on $\mathcal{L}_\psi$ as the scalar $\psi(\mathrm{Trace}_{E/k}(x))$. [It is to get this trace to come out correctly that we use the sign−changed Artin−Schreier map.] Since the trace is always a p'th root of unity for p:= char(k), $\mathcal{L}_\psi$ is ι−pure of weight zero.

Given any k−scheme X/k, and any function f on X, we can view f as a k−morphism from X to $\mathbb{A}^1/k$, and form the pullback sheaf $f^*\mathcal{L}_\psi$ on X, which we denote $\mathcal{L}_{\psi(f)}$. Thus $\mathcal{L}_{\psi(f)}$ is a lisse rank one $\overline{\mathbb{Q}}_\ell$−sheaf on X, ι−pure of weight zero, and its trace function is given as follows. For any finite extension E/k, and any point x in (X/k)(E), $\mathrm{Frob}_{E,x}$ acts on $\mathcal{L}_{\psi(f)}$ as the scalar $\psi(\mathrm{Trace}_{E/k}(f(x)))$. For f the zero function, the sheaf $\mathcal{L}_{\psi(0)}$ is canonically the constant sheaf $\overline{\mathbb{Q}}_{\ell,X}$ on X.

Suppose now that X is $\mathbb{A}^1/k$ itself, and f is a polynomial function. To fix ideas, think of $\mathbb{A}^1/k$ as Spec(k[T]). For f in k[T] whose degree is invertible in k, $\mathcal{L}_{\psi(f)}$ on $\mathbb{A}^1/k$ has Swan conductor at ∞ equal to the degree of f:

$$\mathrm{Swan}_\infty(\mathcal{L}_{\psi(f)}) = \deg(f).$$

For $\mathcal{F}$ a lisse $\overline{\mathbb{Q}}_\ell$−sheaf on $\mathbb{A}^1/k$, the Euler−Poincare formula [Ray] of Grothendieck−Neron−Ogg−Shafarevic gives

$$\begin{aligned} \chi_c(\mathbb{A}^1\otimes_k\overline{k}, \mathcal{F}) &:= h^2{}_c(\mathbb{A}^1\otimes_k\overline{k}, \mathcal{F}) - h^1{}_c(\mathbb{A}^1\otimes_k\overline{k}, \mathcal{F}) \\ &= \mathrm{rank}(\mathcal{F}) - \mathrm{Swan}_\infty(\mathcal{F}). \end{aligned}$$

Given a lisse $\mathcal{F}$, take f in k[T] with deg(f) invertible in k and $\deg(f) > \mathrm{Swan}_\infty(\mathcal{F})$. Then $\mathcal{F}\otimes\mathcal{L}_{\psi(f)}$ is totally wild at ∞, all its "upper numbering breaks" are equal to deg(f), and

$$\mathrm{Swan}_\infty(\mathcal{F}\otimes\mathcal{L}_{\psi(f)}) = \deg(f)\times\mathrm{rank}(\mathcal{F}).$$

The total wildness forces

$$H^2{}_c(\mathbb{A}^1\otimes_k\overline{k}, \mathcal{F}\otimes\mathcal{L}_{\psi(f)}) = 0.$$

Thus for such an f we have

$$h^1{}_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}\otimes\mathcal{L}_{\psi(f)})) = (\deg(f) - 1)\times\mathrm{rank}(\mathcal{F}),$$
$$H^i{}_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}\otimes\mathcal{L}_{\psi(f)})) = 0 \text{ for } i \neq 1.$$

Moreover, the total wildness at ∞ implies that under the "forget supports map" is an isomorphism

$$H^1{}_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}\otimes\mathcal{L}_{\psi(f)})) \cong H^1(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}\otimes\mathcal{L}_{\psi(f)})).$$

We now specialize to the case where $\mathcal{F}$ is our lisse, geometrically irreducible and geometrically nontrivial sheaf on $\mathbb{A}^1/k$ which is ι−pure of weight zero, and which, if geometrically self−dual, is arithmetically self−dual. We will now describe an auxiliary lisse sheaf $\mathcal{G}$ on $\mathbb{A}^2/k$.

Fix an integer N which satisfies

N is invertible in k,
$N > \mathrm{Swan}_\infty(\mathcal{F})$,
$N \geq 3$ if $\mathrm{char}(k) \neq 2$,
$N \geq 5$ if $\mathrm{char}(k) = 2$.

Choose a polynomial f(T) in k[T] of degree N.

Suppose first that k has odd characteristic. Our idea is to consider the two parameter family of polynomials of degree N given by

$$(a, b) \mapsto f(T) + aT + bT^2,$$

and to form the sheaf on $\mathbb{A}^2/k$ which incarnates the assignment

$$(a, b) \mapsto H^1{}_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}\otimes\mathcal{L}_{\psi(f(T) + aT + bT^2)})).$$

To make the formal definition, consider the space $\mathbb{A}^3/k$, with coordinates T, a, b. On this space we have the polynomial function

$$(T, a, b) \to f(T) + aT + bT^2,$$

so we may speak of $\mathcal{L}_{\psi(f(T) + aT + bT^2)}$ on $\mathbb{A}^3/k$. Using the first projection

$$\mathrm{pr}_1 : \mathbb{A}^3/k \to \mathbb{A}^1/k,$$
$$(T, a, b) \mapsto T,$$

we pull back $\mathcal{F}$, and form $(\mathrm{pr}_1)^*\mathcal{F}$ on $\mathbb{A}^3/k$. We then tensor this with $\mathcal{L}_{\psi(f(T) + aT + bT^2)}$, obtaining

$$((\mathrm{pr}_1)^*\mathcal{F})\otimes\mathcal{L}_{\psi(f(T) + aT + bT^2)} \text{ on } \mathbb{A}^3/k.$$

Using the projection onto (a, b)−space

$$\mathrm{pr}_{2,3} : \mathbb{A}^3/k \to \mathbb{A}^2/k,$$
$$(T, a, b) \mapsto (a, b),$$

we define

$$\mathcal{G} = \mathcal{G}_{\mathcal{F},\psi,f} := R^1(\mathrm{pr}_{2,3})_!(((\mathrm{pr}_1)^*\mathcal{F})\otimes\mathcal{L}_{\psi(f(T) + aT + bT^2)}).$$

If char(k) is two, we use the two parameter family

$$(a, b) \mapsto f(T) + aT + bT^3,$$

and define

$$\mathcal{G} = \mathcal{G}_{\mathcal{F},\psi,\mathrm{f}} := \mathrm{R}^1(\mathrm{pr}_{2,3})_!(((\mathrm{pr}_1)^*\mathcal{F})\otimes\mathcal{L}_{\psi(\mathrm{f(T)} + \mathrm{aT} + \mathrm{bT}^3)}).$$

We will use the Weil I idea (which in terms of Weil II amounts to a "baby case" of the purity criterion 1.5.1 of Weil II) to prove

Purity Theorem Notations as in the three preceding paragraphs, the auxiliary sheaf $\mathcal{G} = \mathcal{G}_{\mathcal{F},\psi,\mathrm{f}}$ on $\mathbb{A}^2/\mathrm{k}$ is lisse of rank $(\mathrm{N}-1)\mathrm{rank}(\mathcal{F})$, and ι−pure of weight one.

We will prove this Purity Theorem in a later lecture. For now, we will admit its truth, and explain how it allows us to prove the target theorem. To begin, we extract an obvious corollary.

Purity Corollary $\mathrm{H}^1{}_c(\mathbb{A}^1\otimes_{\mathrm{k}}\bar{\mathrm{k}}, \mathcal{F}\otimes\mathcal{L}_{\psi(\mathrm{f})})$ is ι−pure of weight one.

proof By proper base change, the stalk of $\mathcal{G}_{\mathcal{F},\psi,\mathrm{f}}$ at the origin is the cohomology group in question. QED

Reduction of the target theorem to the purity theorem

This purity corollary holds for every polynomial f of the fixed degree N. In particular, for every finite extension field E/k, and for every s in $\mathrm{E}^\times$, $\mathrm{H}^1{}_c(\mathbb{A}^1\otimes_{\mathrm{E}}\bar{\mathrm{k}}, \mathcal{F}\otimes\mathcal{L}_{\psi(\mathrm{sT}^\mathrm{N})})$ is ι−pure of weight one. If we put s = 0, the sheaf $\mathcal{F}\otimes\mathcal{L}_{\psi(\mathrm{sT}^\mathrm{N})}$ becomes the sheaf $\mathcal{F}$. The rough idea now is to "take the limit as $\mathrm{s} \to 0$" and show that weights can only decrease. Then we will get the target theorem, namely that $\mathrm{H}^1{}_c(\mathbb{A}^1\otimes_{\mathrm{E}}\bar{\mathrm{k}}, \mathcal{F}\otimes\mathcal{L}_{\psi(\mathrm{sT}^\mathrm{N})})$ is ι−mixed of weight ≤ 1 for s = 0.

To make sense of this rough idea, we work on the $\mathbb{A}^2/\mathrm{k}$ with coordinates (T, s), on which we have the lisse sheaves $(\mathrm{pr}_1)^*\mathcal{F}$, $\mathcal{L}_{\psi(\mathrm{sT}^\mathrm{N})}$, and $((\mathrm{pr}_1)^*\mathcal{F})\otimes\mathcal{L}_{\psi(\mathrm{sT}^\mathrm{N})}$. We form the sheaf $\mathcal{H}$ on the s−line $\mathbb{A}^1/\mathrm{k}$ defined as

$$\mathcal{H} := \mathrm{R}^1(\mathrm{pr}_2)_!(((\mathrm{pr}_1)^*\mathcal{F})\otimes\mathcal{L}_{\psi(\mathrm{sT}^\mathrm{N})}).$$

By proper base change, for every finite extension field E/k, and every s in E, the stalk $\mathcal{H}_\mathrm{s}$ of $\mathcal{H}$ at (a geometric point over) s is the cohomology group $\mathrm{H}^1{}_c(\mathbb{A}^1\otimes_{\mathrm{E}}\bar{\mathrm{k}}, \mathcal{F}\otimes\mathcal{L}_{\psi(\mathrm{sT}^\mathrm{N})})$.

Degeneration Lemma The sheaf $\mathcal{H}$ on $\mathbb{A}^1/\mathrm{k}$ has

$$\mathrm{H}^0{}_c(\mathbb{A}^1\otimes_{\mathrm{k}}\bar{\mathrm{k}}, \mathcal{H}) = 0.$$

Restricted to $\mathbb{G}_m/\mathrm{k}$, $\mathcal{H}$ is lisse of rank $(\mathrm{N}-1)\mathrm{rank}(\mathcal{F})$, and ι−pure of weight one. Denoting by $\mathrm{j} : \mathbb{G}_m \to \mathbb{A}^1$ the inclusion, the adjunction map

$$\mathcal{H} \to \mathrm{j}_*\mathrm{j}^*\mathcal{H}$$

is injective.

proof Consider the Leray spectral sequence for the map

$$\mathrm{pr}_2 : \mathbb{A}^2/k \to \mathbb{A}^1/k$$

and the lisse sheaf $\mathcal{K} := ((\mathrm{pr}_1)^*\mathcal{F})\otimes\mathcal{L}_{\psi(sT^N)}$ on the source:

$$E_2^{a,b} = H^a_c(\mathbb{A}^1\otimes_k\bar{k}, R^b(\mathrm{pr}_2)_!\mathcal{K}) \Rightarrow H^{a+b}_c(\mathbb{A}^2\otimes_k\bar{k}, \mathcal{K}).$$

By proper base change, and looking fibre by fibre, we see that $R^i(\mathrm{pr}_2)_!\mathcal{K}$ vanishes for $i\neq 1$ [Remember $\mathcal{F}$ is geometrically irreducible and nontrivial, so $H^2_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F})$ vanishes.] So the spectral sequence degenerates, and gives, for every a,

$$H^a_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{H}) = H^{a+1}_c(\mathbb{A}^2\otimes_k\bar{k}, \mathcal{K}).$$

Taking a=0, we find

$$H^0_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{H}) = H^1_c(\mathbb{A}^2\otimes_k\bar{k}, \mathcal{K}).$$

But as $\mathcal{K}$ is lisse, we have $H^1_c(\mathbb{A}^2\otimes_k\bar{k}, \mathcal{K}) = 0$, so we get the vanishing of $H^0_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{H})$. Therefore $\mathcal{H}$ is lisse wherever its stalks have maximal rank. By proper base change, these stalks have rank $(N-1)\mathrm{rank}(\mathcal{F})$ at every point of $\mathbb{G}_m/k$. At 0, $\mathcal{H}_0 = H^1_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F})$ has strictly lower rank, equal to

$$-\chi_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}) = \mathrm{Swan}_\infty(\mathcal{F}) - \mathrm{rank}(\mathcal{F}).$$

Once $\mathcal{H}|\mathbb{G}_m$ is lisse, it is ι−pure of weight one by proper base change and the purity corollary. As already noted, the vanishing of $H^0_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{H})$ gives the injectivity of $\mathcal{H} \to j_*j^*\mathcal{H}$. QED

Let us spell out more explicitly what the injectivity of

$$\mathcal{H} \to j_*j^*\mathcal{H}$$

gives us. The sheaf $j^*\mathcal{H}$ is lisse on $\mathbb{G}_m/k = \mathrm{Spec}(k[T, 1/T])$, so "is" a representation of $\pi_1^{\mathrm{arith}}(\mathbb{G}_m/k)$. The inclusion of rings

$$k[T, 1/T] \subset k[[T]][1/T]$$

gives a map $\mathrm{Spec}(k[[T]][1/T]) \to \mathbb{G}_m/k$, and a map of π_1's

$$\pi_1(\mathrm{Spec}(k[[T]][1/T])) \to \pi_1^{\mathrm{arith}}(\mathbb{G}_m/k).$$

The group $\pi_1(\mathrm{Spec}(k[[T]][1/T]))$ is D(0), the "decomposition group" at 0 (:= the absolute galois group of the fraction field $k[[T]][1/T]$ of the complete local ring at the origin in $\mathbb{A}^1$). Its subgroup

$$I(0) := \pi_1(\mathrm{Spec}(\bar{k}[[T]][1/T]))$$

is the inertia group at 0; it sits in a short exact sequence

$$0 \to I(0) \to D(0) \to \mathrm{Gal}(\bar{k}/k) \to 0.$$

Thus $\mathcal{H}|\mathbb{G}_m$ gives a representation of D(0). The stalk of $j_*j^*\mathcal{H}$ at 0 is the space of I(0)−invariants $(\mathcal{H}|\mathbb{G}_m)^{I(0)}$, with its induced action of $D(0)/I(0) = \mathrm{Gal}(\bar{k}/k)$. So the injectivity of

$$\mathcal{H} \to j_*j^*\mathcal{H},$$

read on the stalks at 0, gives us nf F_k−equivariant inclusion

$$\mathcal{H}_0 = H^1{}_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}) \subset (j_*j^*\mathcal{H})_0 = (\mathcal{H}|\mathbb{G}_m)^{I(0)}.$$

How do we exploit this inclusion? To prove the target theorem, namely that $H^1{}_c(\mathbb{A}^1\otimes_E\bar{k}, \mathcal{F})$ is ι−mixed of weight ≤ 1, it suffices now to show that $(\mathcal{H}|\mathbb{G}_m)^{I(0)}$ is ι−mixed of weight ≤ 1. We will deduce this from the fact that $\mathcal{H}|\mathbb{G}_m$ is lisse and ι−pure of weight one.

Weight Drop Lemma [De−Weil II, 1.8.1] Let $\mathcal{J}$ be a lisse sheaf on $\mathbb{G}_m/k$ which is ι−pure of weight w. Then $(\mathcal{J})^{I(0)}$ is ι−mixed of weight $\le$ w.

proof Step 1. By an $\alpha^{\deg}$ twist, it suffices to treat the case when w = 0.

Step 2. We first establish the weak estimate, that $(\mathcal{J})^{I(0)}$ is ι−mixed of weight ≤ 2. For this, we consider the L−functions both of $\mathcal{J}$ on $\mathbb{G}_m/k$ and of $j_*\mathcal{J}$ on $\mathbb{A}^1/k$. For the first, we have

$$L(\mathbb{G}_m/k, \mathcal{J})(T)$$
$$= \det(1 - TF_k|H^1{}_c(\mathbb{G}_m\otimes_k\bar{k}, \mathcal{J}))/\det(1 - TF_k|H^2{}_c(\mathbb{G}_m\otimes_k\bar{k}, \mathcal{J})).$$

For the second, we have

$$L(\mathbb{A}^1/k, j_*\mathcal{J})(T)$$
$$= \det(1 - TF_k|H^1{}_c(\mathbb{A}^1\otimes_k\bar{k}, j_*\mathcal{J}))/\det(1 - TF_k|H^2{}_c(\mathbb{A}^1{}_k\bar{k}, j_*\mathcal{J})),$$

[remember $H^0{}_c(\mathbb{A}^1\otimes_k\bar{k}, j_*\mathcal{J}) = 0$, because $j_*\mathcal{J}$ has no nonzero punctual sections]. By the birational invariance of $H^2{}_c$, we have

$$H^2{}_c(\mathbb{G}_m\otimes_k\bar{k}, \mathcal{J}) = H^2{}_c(\mathbb{A}^1{}_k\bar{k}, j_*\mathcal{J}).$$

From the Euler products for the two L−functions, we see that

$$L(\mathbb{G}_m/k, \mathcal{J})(T) = \det(1 - TF_k| (\mathcal{J})^{I(0)})\times L(\mathbb{A}^1/k, j_*\mathcal{J})(T).$$

So comparing their cohomological expressions ,we find

$$\det(1 - TF_k|H^1{}_c(\mathbb{G}_m\otimes_k\bar{k}, \mathcal{J}))$$
$$= \det(1 - TF_k| (\mathcal{J})^{I(0)})\times\det(1 - TF_k|H^1{}_c(\mathbb{A}^1\otimes_k\bar{k}, j_*\mathcal{J}).$$

The key point here is that

$$\det(1 - TF_k| (\mathcal{J})^{I(0)}) \text{ divides } \det(1 - TF_k|H^1{}_c(\mathbb{G}_m\otimes_k\bar{k}, \mathcal{J})).$$

So it suffices to show that $H^1{}_c(\mathbb{G}_m\otimes_k\bar{k}, \mathcal{J})$ is ι−mixed of weight ≤ 2. Consider the exponential sum formula for $L(\mathbb{G}_m/k, \mathcal{J})(T)$:

$$L(\mathbb{G}_m/k, \mathcal{J})(T) = \exp(\Sigma_{n\ge 1} S_n(\mathbb{G}_m/k, \mathcal{J})T^n/n),$$

where

$$S_n(\mathbb{G}_m/k, \mathcal{J}) := \Sigma_{x \text{ in } (\mathbb{G}_m/k)(k_n)} \mathrm{Trace}(\Lambda_{\mathcal{J}}(\mathrm{Frob}_{k_n},x)).$$

Since $\mathcal{J}$ is ι−pure of weight zero, and $(\mathbb{G}_m/k)(k_n)$ has $(\#k)^n - 1$ points, we have the trivial estimate

$$|\iota(S_n(\mathbb{G}_m/k, \mathcal{J}))| \le (\#k)^n \mathrm{rank}(\mathcal{J}).$$

So the inner sum $\sum_{n\ge 1} \iota(S_n(\mathbb{G}_m/k, \mathcal{J}))T^n/n$ is dominated term by term in absolute value by the series

$$\mathrm{rank}(\mathcal{J})\times\sum_{n\ge 1} (\#k)^n|T|^n/n = \log((1/(1 - (\#k)|T|))^{\mathrm{rank}(\mathcal{J})}),$$

and hence converges absolutely in $|T| < 1/\#k$. Therefore its exponential $L(\mathbb{G}_m/k, \mathcal{J})(T)$ is an invertible function in $|T| < 1/\#k$. In other words, the ratio

$$\det(1 - TF_k|H^1{}_c(\mathbb{G}_m\otimes_k\bar{k}, \mathcal{J}))/\det(1 - TF_k|H^2{}_c(\mathbb{G}_m\otimes_k\bar{k}, \mathcal{J}))$$

is invertible in $|T| < 1/\#k$. But $H^2{}_c(\mathbb{G}_m\otimes_k\bar{k}, \mathcal{J})$ is ι−pure of weight two, so the denominator

$$\det(1 - TF_k|H^2{}_c(\mathbb{G}_m\otimes_k\bar{k}, \mathcal{J}))$$

is invertible in $|T| < 1/\#k$. Therefore

$$\det(1 - TF_k|H^1{}_c(\mathbb{G}_m\otimes_k\bar{k}, \mathcal{J}))$$

is itself invertible in $|T| < 1/\#k$, and this means precisely that $H^1{}_c(\mathbb{G}_m\otimes_k\bar{k}, \mathcal{J})$ is ι−mixed of weight ≤ 2.

Step 3. We apply the result of Step 2 to the tensor powers $\mathcal{J}^{\otimes n}$ of $\mathcal{J}$. These are all lisse on $\mathbb{G}_m/k$, and ι−pure of weight zero. If α is an eigenvalue of F_k on $(\mathcal{J})^{I(0)}$, then α^n is an eigenvalue of F_k on $(\mathcal{J}^{\otimes n})^{I(0)}$. So by step 2 we have the estimate

$$|\iota(\alpha^n)| \le \#k.$$

Letting $n \to \infty$, we find $|\iota(\alpha)| \le 1$, as required. QED

Lecture III

To summarize: in order to prove the target theorem for our $\mathcal{F}$ on $\mathbb{A}^1$, it suffices to prove the purity theorem for the auxiliary sheaf $\mathcal{G} = \mathcal{G}_{\mathcal{F},\psi,f}$ on $\mathbb{A}^2/k$.

Why the auxiliary sheaf $\mathcal{G}$ is lisse on $\mathbb{A}^2/k$

Recall that $\mathcal{G} = \mathcal{G}_{\mathcal{F},\psi,f}$ was defined as

$$\mathcal{G} := R^1(pr_{2,3})_!\mathcal{K}$$

for $\mathcal{K}$ the lisse sheaf on $\mathbb{A}^3/k$ defined as

$$\mathcal{K} := ((pr_1)^*\mathcal{F})\otimes\mathcal{L}_{\psi(f(T) + aT + bT^2)}, \text{ if char(k) is odd,}$$

$$\mathcal{K} := ((pr_1)^*\mathcal{F})\otimes\mathcal{L}_{\psi(f(T) + aT + bT^3)}, \text{ if char(k) =2.}$$

Looking fibre by fibre, we see that all the stalks of $\mathcal{G}$ have the same rank, namely $(N-1)\times\mathrm{rank}(\mathcal{F})$. Moreover, for $i \ne 1$, we have $R^i(pr_{2,3})_!\mathcal{K} = 0$, as we see using proper base change and looking fibre by fibre, on each of which we have a lisse sheaf on $\mathbb{A}^1$ which is totally wild at ∞.

To show that a constructible $\bar{\mathbb{Q}}_\ell$−sheaf $\mathcal{G}$ on $\mathbb{A}^2$ is lisse, it suffices to show that for every smooth connected affine curve $C/\bar{k}$ and for every $\bar{k}$−morphism $\varphi : C \to \mathbb{A}^2$, $\varphi^*\mathcal{G}$ is lisse on C. Since $\varphi^*\mathcal{G}$ has stalks of constant rank on C, it is lisse on C if and only if it has no nonzero punctual sections, i.e, if and only if

$$H^0_c(C, \varphi^*\mathcal{G}) = 0?$$

Form the cartesian diagram

$$\begin{array}{ccc} S & \xrightarrow{\tilde{\varphi}} & \mathbb{A}^3\otimes_k\bar{k} \\ \pi\downarrow & & \downarrow \mathrm{pr}_{2,3} \\ C & \xrightarrow{\varphi} & \mathbb{A}^2\otimes_k\bar{k}. \end{array}$$

Because $\mathrm{pr}_{2,3}$ is a smooth affine map of relative dimension one with geometrically connected fibres, so is the map π: $S \to C$. As C is a smooth affine connected curve over $\bar{k}$, S is a smooth affine connected surface over $\bar{k}$. On S we have the lisse sheaf $\tilde{\varphi}^*\mathcal{K}$. By proper base change we know that $\varphi^*\mathcal{G}$ is just $R^1\pi_!(\tilde{\varphi}^*\mathcal{K})$, and that for $i \neq 1$, we have $R^i\pi_!(\tilde{\varphi}^*\mathcal{K}) = 0$. So the Leray spectral sequence for the map $\pi : S \to C$ and the lisse sheaf $\tilde{\varphi}^*\mathcal{K}$ degenerates, and gives

$$H^a_c(C, R^1\pi_!(\tilde{\varphi}^*\mathcal{K})) = H^{a+1}_c(S, \tilde{\varphi}^*\mathcal{K}),$$

i.e.,

$$H^a_c(C, \tilde{\varphi}^*\mathcal{G}) = H^{a+1}_c(S, \tilde{\varphi}^*\mathcal{K}).$$

Taking a=0, we find

$$H^0_c(C, \tilde{\varphi}^*\mathcal{G}) = H^1_c(S, \tilde{\varphi}^*\mathcal{K}) = 0,$$

this last vanishing because $\tilde{\varphi}^*\mathcal{K}$ is lisse on the smooth connected affine surface $S/\bar{k}$.

Geometric monodromy: the monodromy theorem for $\mathcal{G}$

Let us return momentarily to a more general situation: k is our finite field, ℓ is invertible in k, X/k is smooth and geometrically connected, of dimension $n \geq 1$, and $\mathcal{G}$ is a lisse $\bar{\mathbb{Q}}_\ell$−sheaf on X, of rank $r \geq 1$, corresponding to a continuous r−dimensional $\bar{\mathbb{Q}}_\ell$−representation (ρ, V) of $\pi_1^{\mathrm{arith}}(X/k)$, say

$$\rho : \pi_1^{\mathrm{arith}}(X/k) \to GL(V) \cong GL(r, \bar{\mathbb{Q}}_\ell).$$

The Zariski closure in GL(V) of the image $\rho(\pi_1^{\mathrm{geom}}(X/k))$ of the **geometric** fundamental group of X/k is called the geometric monodromy group G_{geom} attached to $\mathcal{G}$. Thus G_{geom} is, by definition, the smallest Zariski closed subgroup of GL(V) whose $\bar{\mathbb{Q}}_\ell$−points contain $\rho(\pi_1^{\mathrm{geom}}(X/k))$.

The key property for us of this algebraic subgoup G_{geom} of GL(V) is this. Take any finite–dimensional $\overline{\mathbb{Q}}_\ell$–representation (Λ, W) of GL(V). We may view W as a representation of G_{geom}, then as a representation of its "abstract" subgroup $\rho(\pi_1{}^{geom}(X/k))$, and finally as a representation of $\pi_1{}^{geom}(X/k)$. Then we have an equality of spaces of coinvariants

$$W_{G_{geom}} = W_{\pi_1{}^{geom}(X/k)}.$$

We now return to our concrete situation. We have the lisse sheaf $\mathcal{G} = \mathcal{G}_{\mathcal{F},\psi,f}$ on $\mathbb{A}^2/k$ attached to a lisse, geometrically irreducible and geometrically nontrivial $\overline{\mathbb{Q}}_\ell$–sheaf $\mathcal{F}$ on $\mathbb{A}^1/k$ which is ι–pure of weight zero, and which, if geometrically self–dual, is arithmetically self dual.

Monodromy Theorem The lisse sheaf $\mathcal{G} = \mathcal{G}_{\mathcal{F},\psi,f}$ on $\mathbb{A}^2/k$, corresponding to the representation (ρ, V) of $\pi_1{}^{arith}(\mathbb{A}^2/k)$, is geometrically irreducible. Its G_{geom} is constrained as follows.

1) If char(k) is odd, then G_{geom} is either a finite irreducible subroup of GL(V), or G_{geom} contains SL(V) as a subgroup of finite index.

2) If char(k) = 2 and $\mathcal{F}$ is not geometrically self–dual, then just as in case 1) G_{geom} is either a finite irreducible subroup of GL(V), or G_{geom} contains SL(V) as a subgroup of finite index.

3) If char(k) = 2 and $\mathcal{F}$ is geometrically self–dual by an orthogonal autoduality, then $\mathcal{G}$ is geometrically self–dual by a symplectic autoduality, and G_{geom} is either a finite irreducible subgroup of Sp(V), or it is the full group Sp(V).

4) If char(k) = 2 and $\mathcal{F}$ is geometrically self–dual by a symplectic autoduality, then $\mathcal{G}$ is geometrically self–dual by a orthogonal autoduality, and G_{geom} is either a finite irreducible subgroup of O(V), or it is either SO(V) or O(V).

Reduction of the purity theorem to the monodromy theorem

We will prove the monodromy theorem in the next lecture. Let us explain how, using it, we can prove the Purity Theorem.

Because $\mathcal{F}$ is lisse and ι–pure of weight zero, the sheaves $\mathcal{F}$ and $\mathcal{F}^\vee$ on $\mathbb{A}^1/k$ have complex conjugate trace functions: for any finite extension E/k, and for any x in $(\mathbb{A}^1/k)(E)$,

$$\iota(\mathrm{Trace}(\mathrm{Frob}_{E,x} \mid \mathcal{F})) \text{ and } \iota(\mathrm{Trace}(\mathrm{Frob}_{E,x} \mid \mathcal{F}^\vee))$$

are complex conjugates. For this reason, we will allow ourselves to denote

$$\overline{\mathcal{F}} := \mathcal{F}^\vee.$$

Denote by $\overline{\psi}$ the character $x \mapsto \psi(-x)$. Thus after our complex embedding ι, ψ and $\overline{\psi}$ are complex conjugates of each other. What is the relation of the lisse sheaf $\mathcal{G}_{\overline{\mathcal{F}},\overline{\psi},f}$ to the lisse sheaf $\mathcal{G}_{\mathcal{F},\psi,f}$?

Duality/Conjugacy Lemma

1) The dual sheaf $(\mathcal{G}_{\mathcal{F},\psi,f})^\vee$ on $\mathbb{A}^2/k$ is canonically the sheaf $\mathcal{G}_{\overline{\mathcal{F}},\overline{\psi},f}(1)$.

2) The sheaves $\mathcal{G}_{\mathcal{F},\psi,f}$ and $\mathcal{G}_{\overline{\mathcal{F}},\overline{\psi},f}$ have complex conjugate trace functions.

proof 1) The definition of $\mathcal{G}_{\mathcal{F},\psi,f}$ was of the form

$$\mathcal{G}_{\mathcal{F},\psi,f} := R^1(pr_{2,3})_!(\text{a certain lisse sheaf,say } \mathcal{K}, \text{ on } \mathbb{A}^3/k),$$

where $\mathcal{K}$ was

$$((pr_1)^*\mathcal{F})\otimes\mathcal{L}_{\psi(f(T) + aT + bT^2)}, \text{ in characteristic not 2},$$

$$((pr_1)^*\mathcal{F})\otimes\mathcal{L}_{\psi(f(T) + aT + bT^3)}, \text{ in characteristic 2}.$$

The definition of $\mathcal{G}_{\overline{\mathcal{F}},\overline{\psi},f}$ is then

$$\mathcal{G}_{\overline{\mathcal{F}},\overline{\psi},f} := R^1(pr_{2,3})_!(\text{the contragredient } \mathcal{K}^\vee \text{ of this same } \mathcal{K}).$$

So have a cup−product pairing of lisse sheaves on $\mathbb{A}^2/k$

$$\mathcal{G}_{\mathcal{F},\psi,f} \times \mathcal{G}_{\overline{\mathcal{F}},\overline{\psi},f} \to R^2(pr_{2,3})_!(\overline{\mathbb{Q}}_\ell) \cong \overline{\mathbb{Q}}_\ell(-1).$$

To see that this pairing identifies each pairee with the $\overline{\mathbb{Q}}_\ell(-1)$−dual of the other, it suffices to check on fibres At the point (a, b), put

$$f_{a,b}(T) := f(T) + aT + bT^2, \text{ in characteristic not 2},$$

$$:= f(T) + aT + bT^3, \text{ in characteristic 2}.$$

We need the cup product pairing

$$H^1{}_c(\mathbb{A}^1\otimes_k\overline{k}, \mathcal{F}\otimes\mathcal{L}_{\psi(f_{a,b})}) \times H^1{}_c(\mathbb{A}^1\otimes_k\overline{k}, \mathcal{F}^\vee\otimes\mathcal{L}_{\psi(-f_{a,b})}) \to \overline{\mathbb{Q}}_\ell(-1)$$

to be a duality. But we have already noted that

$$H^1{}_c(\mathbb{A}^1\otimes_k\overline{k}, \mathcal{F}\otimes\mathcal{L}_{\psi(f_{a,b})}) \cong H^1(\mathbb{A}^1\otimes_k\overline{k}, \mathcal{F}\otimes\mathcal{L}_{\psi(f_{a,b})}),$$

so this is just usual Poincare duality.

2) Let E/k be a finite extension, (a, b) in $(\mathbb{A}^2/k)(E)$. Then as already noted we have

$$H^2{}_c(\mathbb{A}^1\otimes_k\overline{k}, \mathcal{F}\otimes\mathcal{L}_{\psi(f_{a,b})}) = 0,$$

$$H^2{}_c(\mathbb{A}^1\otimes_k\overline{k}, \mathcal{F}^\vee\otimes\mathcal{L}_{\psi(-f_{a,b})}) = 0.$$

So by proper base change and the Lefschetz Trace Formula we have

$$\begin{aligned}&\mathrm{Trace}(\mathrm{Frob}_{E,(a,b)} \mid \mathcal{G}_{\mathcal{F},\psi,f})\\ &= \mathrm{Trace}(F_E \mid H^1{}_c(\mathbb{A}^1\otimes_k\overline{k}, \mathcal{F}\otimes\mathcal{L}_{\psi(f_{a,b})}))\\ &= -\textstyle\sum_{t \text{ in } E} \mathrm{Trace}(\mathrm{Frob}_{E,t} \mid \mathcal{F})\psi(\mathrm{Trace}_{E/k}(f_{a,b}(t))).\end{aligned}$$

Similarly, we find

$$\begin{aligned}&\mathrm{Trace}(\mathrm{Frob}_{E,(a,b)} \mid \mathcal{G}_{\overline{\mathcal{F}},\overline{\psi},f})\\ &= \mathrm{Trace}(F_E \mid H^1{}_c(\mathbb{A}^1\otimes_k\overline{k}, \mathcal{F}^\vee\otimes\mathcal{L}_{\psi(-f_{a,b})}))\\ &= -\textstyle\sum_{t \text{ in } E} \mathrm{Trace}(\mathrm{Frob}_{E,t} \mid \mathcal{F}^\vee)\psi(-\mathrm{Trace}_{E/k}(f_{a,b}(t))),\end{aligned}$$

and this last sum is, term by term, the complex conjugate of the sum which gives the trace of $\mathrm{Frob}_{E,(a,b)}$ on $\mathcal{G}_{\mathcal{F},\psi,f}$ at the same point (a, b). QED

Positive Trace Corollary The trace function of $\mathcal{G}\otimes\mathcal{G}^\vee$ takes values, after ι, in $\mathbb{R}_{\geq 0}$.

proof Let us write

$$\mathcal{G} := \mathcal{G}_{\mathcal{F},\psi,\mathrm{f}},$$
$$\overline{\mathcal{G}} := \mathcal{G}_{\overline{\mathcal{F}},\overline{\psi},\mathrm{f}}.$$

Then $\mathcal{G}\otimes\mathcal{G}^\vee$ is $\mathcal{G}\otimes\overline{\mathcal{G}}(1)$. Its trace at (a, b) in $(\mathbb{A}^2/k)(E)$ is thus

$$(1/\#E)|\iota(\mathrm{Trace}(\mathrm{Frob}_{E,(a,b)} \mid \mathcal{G}_{\mathcal{F},\psi,\mathrm{f}}))|^2. \text{ QED}$$

How do we bring to bear the monodromy theorem? Recall that (ρ, V) is the representation of $\pi_1^{\mathrm{arith}}(\mathbb{A}^2/k)$ corresponding to $\mathcal{G}$. Because $\pi_1^{\mathrm{geom}}(\mathbb{A}^2/k)$ is a normal subgroup of $\pi_1^{\mathrm{arith}}(\mathbb{A}^2/k)$, for any element γ in $\pi_1^{\mathrm{arith}}(\mathbb{A}^2/k)$, $\rho(\gamma)$ normalizes G_{geom} (inside GL(V)).

Consider first the case in which G_{geom} for $\mathcal{G}$ is a finite irreducible subgroup of GL(V). Then $\mathrm{Aut}(G_{\mathrm{geom}})$ is finite, say of order M. So given γ in $\pi_1^{\mathrm{arith}}(\mathbb{A}^2/k)$, $\rho(\gamma)^M$, acting on G_{geom} by conjugation in the ambient GL(V), acts trivially. This means that $\rho(\gamma)^M$ commutes with G_{geom}, and as G_{geom} is an irreducible subgroup of GL(V), this in turn implies that $\rho(\gamma)^M$ is a scalar.

Now consider the case when G_{geom} contains SL(V). Then trivially we have the inclusion

$$\mathrm{GL}(V) \subset \mathbb{G}_m G_{\mathrm{geom}}.$$

So in this case, given γ in $\pi_1^{\mathrm{arith}}(\mathbb{A}^2/k)$, there exists a scalar α in $\overline{\mathbb{Q}}_\ell^\times$ and an element g in G_{geom} such that

$$\rho(\gamma) = \alpha g.$$

If G_{geom} is Sp(V), then $\rho(\gamma)$ is an element of GL(V) which normalizes Sp(V), and is therefore a symplectic similitude [because Sp(V) acts irreducibly on V, cf. the twisting autoduality discussion]. So in this case also there exists a scalar α in $\overline{\mathbb{Q}}_\ell^\times$ and an element g in Sp(V) = G_{geom} such that

$$\rho(\gamma) = \alpha g.$$

If G_{geom} is SO(V) or O(V), then char(k) = 2 and $\mathcal{F}$ is symplectically self−dual, So $N \geq 5$, and $\mathcal{F}$ has even rank. Thus

$$\dim(V) := \mathrm{rank}(\mathcal{G}) = (N-1)\times\mathrm{rank}(\mathcal{F}) \geq 4\times 2 = 8 > 2.$$

Therefore SO(V) acts irreducibly on V. Just as above, $\rho(\gamma)$ is an element of GL(V) which normalizes G_{geom} and hence normaizes $(G_{\mathrm{geom}})^0$ = SO(V), and is therefore an orthogonal similitude. Thus there exists a scalar α_0 in $\overline{\mathbb{Q}}_\ell^\times$ and an element g_0 in O(V) such that $\rho(\gamma) = \alpha_0 g$. Squaring, we find that there exists a scalar α (:= $(\alpha_0)^2$) and an element g (:= $(g_0)^2$) in SO(V) $\subset$ G_{geom} such that

$$\rho(\gamma)^2 = \alpha g.$$

So in all cases, there exists an integer M such that given any element γ in $\pi_1^{\mathrm{arith}}(\mathbb{A}^2/k)$,

$\rho(\gamma)^M$ is of the form αg, with α a scalar, and g in G_{geom}. Therefore on $V\otimes V^\vee$ and on all its tensor powers $(V\otimes V^\vee)^{\otimes a}$, γ^M acts as an element of G_{geom}. Therefore for every integer $a \geq 1$, γ^M acts trivially on the space of G_{geom}−coinvariants

$$((V\otimes V^\vee)^{\otimes a})_{G_{geom}}.$$

In particular, for ever integer $a \geq 1$, every eigenvalue of γ acting on

$$((V\otimes V^\vee)^{\otimes a})_{G_{geom}}.$$

is a root of unity (of order dividing M, but we don't care about this).

In order to exploit this result, we need to restrict $\mathcal{G}$ to cleverly chosen curves in $\mathbb{A}^2/k$, cf. [Ka−Spacefill]. Let us write

$$q := \#k.$$

For each integer $n \geq 1$, consider the curve C_n/k in $\mathbb{A}^2/k$ defined by the equation

$$C_n : y^{q^n} - y = x(x^{q^n} - x).$$

Then C_n/k is a smooth, geometrically connected curve, and it visibly goes through all the k_n−rational points of $\mathbb{A}^2/k$. As explained in [Ka−Spacefill, Cor. 7], if we restrict $\mathcal{G}$ to C_n, then for n sufficiently large and sufficiently divisible, $\mathcal{G}|C_n$ on C_n/k has the same G_{geom} as $\mathcal{G}$ did on $\mathbb{A}^2/k$. Let us say that such a C_n is good for $\mathcal{G}$.

Application of Rankin's method

Let us now prove that $\mathcal{G}$ is ι−pure of weight one. To say that $\mathcal{G}$ is ι−pure of weight one is to say that for every finite extension E/k and every point (A, B) in $(\mathbb{A}^2/k)(E)$, $\rho(\mathrm{Frob}_{E, (A,B)})$ has, via ι, all its eigenvalues of complex absolute value equal to $(\#E)^{1/2}$. Take a C_d which is good for $\mathcal{G}$ and which contains all the E−valued points of $\mathbb{A}^2/k$ (i.e., take d divisible by deg(E/k)). Pick an integer $a \geq 1$, and consider the L−function of $(\mathcal{G}\otimes\mathcal{G}^\vee)^{\otimes a}$ on C_d. The denominator of its cohomological expression is

$$\det(1-TF_k|H^2{}_c(C_d\otimes_k\bar{k}, (\mathcal{G}\otimes\mathcal{G}^\vee)^{\otimes a})).$$

In terms of the representation (ρ, V) corresponding to $\mathcal{G}$, we have

$$H^2{}_c(C_d\otimes_k\bar{k}, (\mathcal{G}\otimes\mathcal{G}^\vee)^{\otimes a}) = ((V\otimes V^\vee)^{\otimes a})_{G_{geom}}(-1),$$

and the action of F_k on this space is induced by the action of an Frobenius element $\mathrm{Frob}_{k, x}$ at any rational point in C_d/k. [It is here that we use the fact that $\mathcal{G}|C_d$ has the same G_{geom} as $\mathcal{G}$ did.] Therefore every eigenvalue of F_k on $H^2{}_c(C_d\otimes_k\bar{k}, (\mathcal{G}\otimes\mathcal{G}^\vee)^{\otimes a})$ is of the form

$$(\#k)\times(\text{a root of unity}).$$

Of this, we retain only that

$$H^2_c(C_d \otimes_k \bar{k}, (\mathcal{G} \otimes \mathcal{G}^\vee)^{\otimes a}) \text{ is } \iota\text{–pure of weight 2.}$$

Then from the cohomological formula for the L–function, we infer that $L(C_d/k, (\mathcal{G} \otimes \mathcal{G}^\vee)^{\otimes a})(T)$ is holomorphic in $|T| < 1/\#k$.

Now think of this same L–function as a power series in T, given by

$L(C_d/k, (\mathcal{G} \otimes \mathcal{G}^\vee)^{\otimes a})(T) = \exp(\sum_{n\geq 1} S_n(C_d/k, (\mathcal{G} \otimes \mathcal{G}^\vee)^{\otimes a})T^n/n)$,

where

$$\begin{aligned}
&\iota(S_n(C_d/k, (\mathcal{G} \otimes \mathcal{G}^\vee)^{\otimes a}))) \\
&:= \sum_{x \text{ in } (C_d/k)(k_n)} \iota(\mathrm{Trace}(\mathrm{Frob}_{k_n,x} \mid (\mathcal{G} \otimes \mathcal{G}^\vee)^{\otimes a})) \\
&= \sum_{x \text{ in } (C_d/k)(k_n)} \iota(\mathrm{Trace}(\mathrm{Frob}_{k_n,x} \mid (\mathcal{G} \otimes \bar{\mathcal{G}}(1))^{\otimes a})) \\
&= (1/\#k)^{na} \sum_{x \text{ in } (C_d/k)(k_n)} \iota(\mathrm{Trace}(\mathrm{Frob}_{k_n,x} \mid (\mathcal{G} \otimes \bar{\mathcal{G}})^{\otimes a})) \\
&= (1/\#k)^{na} \sum_{x \text{ in } (C_d/k)(k_n)} |\iota(\mathrm{Trace}(\mathrm{Frob}_{k_n,x} \mid \mathcal{G})|^{2a}.
\end{aligned}$$

Thus the coefficients $S_n(C_d/k, (\mathcal{G} \otimes \mathcal{G}^\vee)^{\otimes a}))$ lie, via ι, in $\mathbb{R}_{\geq 0}$.

Positive Coefficient Lemma Let F(T) and G(T) be elements of $T\mathbb{R}[[T]]$ whose coefficients all lie in $\mathbb{R}_{\geq 0}$. Suppose that F(T) – G(T) has all its coefficients in $\mathbb{R}_{\geq 0}$. Then

1) exp(F(T)) and exp(G(T)) as series in $1 + T\mathbb{R}[[T]]$ both have coefficients in $\mathbb{R}_{\geq 0}$.

2) exp(F(T)) – exp(G(T)) has all its coefficients in $\mathbb{R}_{\geq 0}$.

proof Assertion 1) holds because $\exp(T) = \sum_n T^n/n!$ has all its coefficients in $\mathbb{R}_{\geq 0}$. For 2), put H(T) := F(T) – G(T). By 1) applied to H(T), we have

$$\exp(H(T)) \in 1 + T\mathbb{R}_{\geq 0}[[T]],$$

say

$$\exp(H(T)) = 1 + K(T),$$

with K in $T\mathbb{R}_{\geq 0}[[T]]$.

Then

$\exp(F(T)) = \exp(G(T)) \times \exp(H(T)) = \exp(G(T)) \times (1 + K(T))$

$= \exp(G(T)) + \exp(G(T)) \times K(T)$.

The final product $\exp(G(T)) \times K(T)$ lies in $\mathbb{R}_{\geq 0}[[T]]$, being the product of two elements of $\mathbb{R}_{\geq 0}[[T]]$. QED

Now let us return to the point (A, B) in $(C_d/k)(E)$ at which we are trying to establish the ι–purity of $\mathcal{G}$. For each strictly positive multiple

$$m = n \times \deg(E/k)$$

of deg(E/k), one of the non–negative summands of

$$\iota(S_m(C_d/k, (\mathcal{G}\otimes\mathcal{G}^\vee)^{\otimes a})))$$
$$= (1/\#k)^{ma} \sum_{x \text{ in } (C_d/k)(k_m)} |\iota(\mathrm{Trace}(\mathrm{Frob}_{k_m,x} \mid \mathcal{G})|^{2a}.$$

is the single term

$$(1/\#k)^{ma} |\iota(\mathrm{Trace}(\mathrm{Frob}_{k_m,(A,B)} \mid \mathcal{G})|^{2a}.$$

We now apply the positive coefficient lemma to

$$F(T) := \sum_{n\geq 1} \iota(S_n(C_n/k, (\mathcal{G}\otimes\mathcal{G}^\vee)^{\otimes a}))T^n/n$$

and to

$$G(T)$$
$$:= \sum_{m\geq 1,\ \deg(E/k) \mid m}(1/\#k)^{ma} |\iota(\mathrm{Trace}(\mathrm{Frob}_{k_m,(A,B)} \mid \mathcal{G})|^{2a}T^m/m.$$

We have

$$\exp(F(T)) = \iota L(C_d/k, (\mathcal{G}\otimes\mathcal{G}^\vee)^{\otimes a})(T),$$
$$\exp(G(T)) = 1/\iota\det(1 - T^{\deg(E/k)}\mathrm{Frob}_{E,(A,B)} \mid (\mathcal{G}\otimes\mathcal{G}^\vee)^{\otimes a}).$$

By part 1) of the positive coefficient lemma, the L−function is a series with positive coefficients. We have shown that the L−function is holomorphic in $|T| < 1/\#k$. Therefore its power series around the origin, $\exp(F(T))$, is convergent in $|T| < 1/\#k$. By part 2) of the positive coefficient lemma, $\exp(G(T))$ has positive coefficients, and it is dominated by $\exp(F(T))$, coefficient by coefficient. Therefore the series $\exp(G(T))$ must also converge in $|T| < 1/\#k$. But

$$\exp(G(T)) = 1/\iota\det(1 - T^{\deg(E/k)}\mathrm{Frob}_{E,(A,B)} \mid (\mathcal{G}\otimes\mathcal{G}^\vee)^{\otimes a}),$$

so the polynomial

$$\iota\det(1 - T^{\deg(E/k)}\mathrm{Frob}_{E,(A,B)} \mid (\mathcal{G}\otimes\mathcal{G}^\vee)^{\otimes a})$$

has no zeroes in $|T| < 1/\#k$. In other words, the polynomial

$$\iota\det(1 - T\mathrm{Frob}_{E,(A,B)} \mid (\mathcal{G}\otimes\mathcal{G}^\vee)^{\otimes a})$$

has no zeroes in $|T| < 1/\#E$, i.e., every eigenvalue of $\mathrm{Frob}_{E,(A,B)}$ on $(\mathcal{G}\otimes\mathcal{G}^\vee)^{\otimes a} = (\mathcal{G}\otimes\overline{\mathcal{G}}(1))^{\otimes a}$ has, via ι, complex absolute value $\leq \#E$.

If α is an eigenvalue of $\mathrm{Frob}_{E,(A,B)}$ on $\mathcal{G}\otimes\mathcal{G}^\vee$, then α^a is an eigenvalue of of $\mathrm{Frob}_{E,(A,B)}$ on $(\mathcal{G}\otimes\mathcal{G}^\vee)^{\otimes a}$. Therefore

$$|\iota(\alpha)|^a \leq \#E.$$

Letting $a \to \infty$, we infer that

$$|\iota(\text{any eigenvalue of } \mathrm{Frob}_{E,(A,B)} \text{ on } \mathcal{G}\otimes\mathcal{G}^\vee)| \leq 1.$$

Since $\mathrm{Frob}_{E,(A,B)}$ on $\mathcal{G}\otimes\mathcal{G}^\vee$ has trivial determinant, these inequalities are necessarily equalities:

$$|\iota(\text{any eigenvalue of } \mathrm{Frob}_{E,(A,B)} \text{ on } \mathcal{G}\otimes\mathcal{G}^\vee)| = 1.$$

If δ is (the image under ι of) an eigenvalue of $\mathrm{Frob}_{E,(A,B)}$ on $\mathcal{G}$, then $\delta\overline{\delta}/\#E$ is (the image under ι of) an eigenvalue of $\mathrm{Frob}_{E,(A,B)}$ on $\mathcal{G}\otimes\mathcal{G}^\vee \cong \mathcal{G}\otimes\overline{\mathcal{G}}(1)$. Thus we get

$$|\delta\overline{\delta}/\#E| = 1,$$

which is to say,

$$|\iota(\text{any eigenvalue of } \mathrm{Frob}_{E,(A,B)} \text{ on } \mathcal{G})| = (\#E)^{1/2}.$$

This concludes the proof of the purity theorem, modulo the monodromy theorem.

Lecture IV

In this lecture, we will prove the monodromy theorem. Let us recall its statement.

Monodromy Theorem The lisse sheaf $\mathcal{G} = \mathcal{G}_{\mathcal{F},\psi,f}$ on $\mathbb{A}^2/k$, corresponding to the representation (ρ, V) of $\pi_1^{\mathrm{arith}}(\mathbb{A}^2/k)$, is geometrically irreducible. Its G_{geom} is constrained as follows.

1) If char(k) is odd, then G_{geom} is either a finite irreducible subroup of GL(V), or G_{geom} contains SL(V) as a subgroup of finite index.

2) If char(k) = 2 and $\mathcal{F}$ is not geometrically self–dual, then just as in case 1) G_{geom} is either a finite irreducible subroup of GL(V), or G_{geom} contains SL(V) as a subgroup of finite index.

3) If char(k) = 2 and $\mathcal{F}$ is geometrically self–dual by an orthogonal autoduality, then $\mathcal{G}$ is geometrically self–dual by a symplectic autoduality, and G_{geom} is either a finite irreducible subgroup of Sp(V), or it is the full group Sp(V).

4) If char(k) = 2 and $\mathcal{F}$ is geometrically self–dual by a symplectic autoduality, then $\mathcal{G}$ is geometrically self–dual by a orthogonal autoduality, and G_{geom} is either a finite irreducible subgroup of O(V), or it is either SO(V) or O(V).

Proof of the monodromy theorem

Step 1: Geometric irreducibility of $\mathcal{G}$

In order to show that $\mathcal{G} = \mathcal{G}_{\mathcal{F},\psi,f}$ on $\mathbb{A}^2/k$ is geometrically irreducible, it suffices to exhibit some smooth, geometrically connected curve C/k and a k–map $\varphi : C \to \mathbb{A}^2$ such that the pullback $\varphi^*\mathcal{G}$ on C is geometrically irreducible. We will take for C the straight line "b = 0" in (a, b) space. The pullback of $\mathcal{G}$ to this line is, by proper base change, the higher direct image

$$\begin{aligned}\mathcal{G}|C &= R^1(pr_2)_!((pr_1{}^*\mathcal{F})\otimes\mathcal{L}_{\psi(f(T) + aT)})\\ &=R^1(pr_2)_!((pr_1{}^*(\mathcal{F}\otimes\mathcal{L}_{\psi(f(T))})\otimes\mathcal{L}_{\psi(aT)}).\end{aligned}$$

Here pr_1 and pr_2 are the two projections of the $\mathbb{A}^2/k$ with coordinates (T, a) onto the two factors.

In more down to earth terms, $\mathcal{G}|C$ is simply the "naive Fourier transform" $\mathrm{NFT}_\psi((\mathcal{F}\otimes\mathcal{L}_{\psi(f(T))})$ in the notation of [Ka–GKM, 8.2]. The input sheaf $\mathcal{F}\otimes\mathcal{L}_{\psi(f(T))}$ has all its ∞–breaks equal to N > 1. As noted in [Ka–ESDE, 7.8, "Class(1)"] as a consequence of Laumon's general theory of Fourier Transform, NFT_ψ induces an autoequivalence of the abelian category of those lisse sheaves on $\mathbb{A}^1/k$ all of whose ∞–breaks are > 1. Its quasi–inverse is $\mathrm{NFT}_{\overline{\psi}}(1)$. In particular, NFT_ψ preserves geometrically irreducibility of objects in this category. Thus $\mathcal{G}|C$ is geometrically irreducible, and hence $\mathcal{G}$ itself is geometrically irreducible.

Step 2: det($\mathcal{G}$) is geometrically of finite order

Indeed, any $\overline{\mathbb{Q}}_\ell^\times$valued character χ of $\pi_1(\mathbb{A}^2\otimes_k\bar{k})$ has finite order. To see this, note first that χ takes values in some $\mathcal{O}_\lambda^\times$, for $\mathcal{O}_\lambda$ the ring of integers in some finite extension of $\mathbb{Q}_\ell$. The quotient group $\mathcal{O}_\lambda^\times/(1 + 2\ell\mathcal{O}_\lambda)$ is finite, say of order M. Then χ^M takes values in $1 + 2\ell\mathcal{O}_\lambda$, a multiplicative group that is isomorphic, by the log, to the additive group $2\ell\mathcal{O}_\lambda$. Thus $(1/2\ell)\log(\chi^M)$ lies in

$$\mathrm{Hom}(\pi_1(\mathbb{A}^2\otimes_k\bar{k}), \mathcal{O}_\lambda) = H^1(\mathbb{A}^2\otimes_k\bar{k}, \mathcal{O}_\lambda) = 0.$$

Therefore $\log(\chi^M) = 0$, and hence χ^M is trivial.

Step 3: the moment criterion ("Larsen's Alternative")

We have seen so far that the representation (ρ, V) of $\pi_1^{\mathrm{arith}}(\mathbb{A}^2/k)$, is geometrically irreducible, and that $\det(\rho)$ is geometrically of finite order. Therefore the group G_{geom} is an irreducible, and hence reductive, Zariski closed subgroup of GL(V), whose determinant is of finite order.

Suppose we are in characteristic 2, and $\mathcal{F}$ is geometrically self−dual. Because $\mathcal{F}$ is geometrically irreducible, the autoduality is either orthogonal or symplectic. In characteristic 2, the sheaf $\mathcal{L}_{\psi(\text{anything})}$ is itself orthogonally self−dual. So by standard properties of cup−product, $\mathcal{G}$ will be geometrically self−dual (toward $\overline{\mathbb{Q}}_\ell(-1)$, but geometrically this is isomorphic to $\overline{\mathbb{Q}}_\ell$), with an autoduality of the opposite sign, i.e., if $\mathcal{F}$ is orthogonal, $\mathcal{G}$ will be symplectic, and if $\mathcal{F}$ is symplectic, then $\mathcal{G}$ will be orthogonal.

We now apply the following moment criterion, originally due to Michael Larsen, cf. [Lar−Normal] and [Lar−Char]. See [Ka−MCG] for some "G finite" cases of the theorem.

Moment Criterion ("Larsen's Alternative") Let V be a finite−dimensional $\overline{\mathbb{Q}}_\ell$−vector space with $\dim(V) \geq 2$, and $G \subset GL(V)$ a Zariski closed irreducible subgroup of GL(V) with det(G) finite. Define the "fourth absolute moment" $M_4(G, V)$ to be the dimension of

$$(V\otimes V\otimes V^\vee\otimes V^\vee)^G.$$

1) If $M_4(G, V) = 2$, then either $G \supset SL(V)$ with finite index, or G is finite.

2) Suppose <,> is a nondegenerate symmetric bilinear form on V, and suppose G lies in the orthogonal group O(V) := Aut(V, <,>). If $M_4(G, V) = 3$, then either G = O(V), or G = SO(V), or G is finite.

3) Suppose <,> is a nondegenerate alternating bilinear form on V, and suppose G lies in the symplectic group Sp(V) := Aut(V, <,>). Suppose also dim(V) > 2. If $M_4(G, V) = 3$, then either G = Sp(V), or G is finite.

proof We may view $V\otimes V\otimes V^\vee\otimes V^\vee$ as being either End(End(V)) or as being $\mathrm{End}(V^{\otimes 2})$. We can decompose End(V) as a sum of irreducible G−modules with multiplicities (because G, having a faithful irreducible representation, is reductive). Say

$$\mathrm{End}(V) = \sum n_i \mathrm{Irred}_i.$$

Then

$$M_4(G, V) = \sum (n_i)^2.$$

Or if we decompose

$$V\otimes V = \sum m_i \mathrm{Irred}_i$$

as a sum of irreducible G−modules with multiplicities, then

$$M_4(G, V) = \sum (m_i)^2.$$

Suppose first that $M_4(G, V) = 2$. Then looking at End(V), we see that End(V) must be the sum of two distinct irreducible G−modules. But End(V) has the a priori GL(V)−decomposition

$$\mathrm{End}(V) = \mathbb{1} \oplus \mathrm{End}^0(V) = \mathbb{1} \oplus \mathrm{Lie}(SL(V)).$$

Therefore both of these pieces must already be G−irreducible. Thus Lie(SL(V)) is G−irreducible. But Lie($G\cap SL(V)$) is a G−stable submodule of Lie(SLV)). So either Lie($G\cap SL(V)$) = 0, or Lie($G\cap SL(V)$) = Lie(SL(V)). In the first case, $G\cap SL(V)$ is finite. In the second case, $G\cap SL(V) = SL(V)$, so $SL(V) \subset G$. Since det(G) is finite, $G/G\cap SL(V)$ is finite. So in the first case, G is finite, and in the second case G/SL(V) is finite.

Suppose now that $G \subset O(V)$ and that $M_4(G, V) = 3$. In this case, we decompose $V\otimes V$ into G−irreducibles. Since $M_4(G, V) = 3$, $V\otimes V$ must be the sum of three distinct G−irreducibles. We have the GL(V)−decomposition

$$V\otimes V = \Lambda^2(V) \oplus \mathrm{Sym}^2(V).$$

We have the further O(V)−decomposition

$$\mathrm{Sym}^2(V) = \mathbb{1} \oplus \mathrm{SphHarm}^2(V).$$

And we have

$$\Lambda^2(V) = \mathrm{Lie}(SO(V)).$$

So all in all we have an a priori O(V)−decomposition

$$V\otimes V = \mathrm{Lie}(SO(V)) \oplus \mathbb{1} \oplus \mathrm{SphHarm}^2(V).$$

Therefore all three of these pieces must be G−irreducible. In particular, Lie(SO(V)) is G−irreducible. But Lie($G\cap SO(V)$) is a G−submodule, so Lie($G\cap SO(V)$) is either 0 or Lie(SO(V)), and we conclude as before.

If $G \subset Sp(V)$ and $M_4(G, V) = 3$, we decompose $V\otimes V$ into G−irreducibles. Since $M_4(G, V) = 3$, $V\otimes V$ must be the sum of three distinct G−irreducibles. This time we have

$$\Lambda^2(V) = \mathbb{1} \oplus (\Lambda^2(V)/\mathbb{1}),$$

and

$$\mathrm{Sym}^2(V) = \mathrm{Lie}(Sp(V)).$$

So now Lie(Sp(V)) must be G−irreducible, and we conclude as in the previous case. QED

Step 4 Cohomological version of the moment criterion

For G a reductive subgroup of GL(V), and W any representation of GL(V), the natural map

$$W^G \to W_G$$

from G−invariants to G−coinvariants is an isomorphism (just write W as a sum of G−irreducibles). So we can think of $M_4(G, V)$ as being the dimension of the space of G−coinvariants:

$$M_4(G, V) = \text{dimension of } (V\otimes V\otimes V^\vee\otimes V^\vee)_G.$$

Thus for (ρ, V) corresponding to $\mathcal{G}$, $M_4(G, V)$ is the dimension of

$$H^4_c(\mathbb{A}^2\otimes_k\bar{k}, \mathcal{G}\otimes\mathcal{G}\otimes\mathcal{G}^\vee\otimes\mathcal{G}^\vee)(2)$$
$$= H^4_c(\mathbb{A}^2\otimes_k\bar{k}, \mathcal{G}\otimes\mathcal{G}\otimes\mathcal{G}^\vee(-1)\otimes\mathcal{G}^\vee(-1))(4)$$
$$= H^4_c(\mathbb{A}^2\otimes_k\bar{k}, \mathcal{G}\otimes\mathcal{G}\otimes\bar{\mathcal{G}}\otimes\bar{\mathcal{G}})(4).$$

So using the moment criterion, we see that the monodromy theorem results from the following

Cohomological Moment Calculation

1) Suppose that char(k) is odd. Then

$$\dim H^4_c(\mathbb{A}^2\otimes_k\bar{k}, \mathcal{G}\otimes\mathcal{G}\otimes\bar{\mathcal{G}}\otimes\bar{\mathcal{G}}) = 2.$$

2) Suppose that char(k) = 2, and that $\mathcal{F}$ is not geometrically self−dual. Then

$$\dim H^4_c(\mathbb{A}^2\otimes_k\bar{k}, \mathcal{G}\otimes\mathcal{G}\otimes\bar{\mathcal{G}}\otimes\bar{\mathcal{G}}) = 2.$$

3) Suppose that char(k) = 2, and that $\mathcal{F}$ is geometrically self−dual. Then

$$\dim H^4_c(\mathbb{A}^2\otimes_k\bar{k}, \mathcal{G}\otimes\mathcal{G}\otimes\bar{\mathcal{G}}\otimes\bar{\mathcal{G}}) = 3.$$

Interlude: The idea behind the calculation

Suppose we already knew all the results of Weil II. Then $\mathcal{G}$ is ι−pure of weight one, $\mathcal{G}\otimes\mathcal{G}\otimes\bar{\mathcal{G}}\otimes\bar{\mathcal{G}}$ is ι−pure of weight 4,

$$H^4_c(\mathbb{A}^2\otimes_k\bar{k}, \mathcal{G}\otimes\mathcal{G}\otimes\bar{\mathcal{G}}\otimes\bar{\mathcal{G}}) \text{ is } \iota\text{−pure of weight 8,}$$

and for $i \le 3$

$$H^i_c(\mathbb{A}^2\otimes_k\bar{k}, \mathcal{G}\otimes\mathcal{G}\otimes\bar{\mathcal{G}}\otimes\bar{\mathcal{G}}) \text{ is } \iota\text{−mixed of weight} \le 4 + i.$$

By the Lefschetz Trace Formula, we have, for every finite extension E/k,

$$\Sigma_{(a,b) \text{ in } E^2} |\iota(\mathrm{Trace}(\mathrm{Frob}_{E,(a,b)}|\mathcal{G}))|^4 =$$
$$\Sigma_i (-1)^i\iota(\mathrm{Trace}(F_E \mid H^i_c(\mathbb{A}^2\otimes_k\bar{k}, \mathcal{G}\otimes\mathcal{G}\otimes\bar{\mathcal{G}}\otimes\bar{\mathcal{G}}))).$$

Using Weil II, the second sum is

$$= \iota(\mathrm{Trace}(F_E \mid H^4_c(\mathbb{A}^2\otimes_k\bar{k}, \mathcal{G}\otimes\mathcal{G}\otimes\bar{\mathcal{G}}\otimes\bar{\mathcal{G}}))) + O((\#E)^{7/2}).$$

Since the H^4_c is ι−pure of weight 8, we could recover the dimension of $H^4_c(\mathbb{A}^2\otimes_k\bar{k}, \mathcal{G}\otimes\mathcal{G}\otimes\bar{\mathcal{G}}\otimes\bar{\mathcal{G}})$ as the limsup over larger and larger finite extensions E/k of the sums

$(1/\#E)^4\Sigma_{(a,b) \text{ in } E^2} |\iota(\mathrm{Trace}(\mathrm{Frob}_{E,(a,b)}|\mathcal{G}))|^4$.

We now try to evaluate these sums. To fix ideas, suppose first that the characteristic is odd. For each (a, b) in E^2, we have

$$\mathrm{Trace}(\mathrm{Frob}_{E,(a,b)}|\mathcal{G}) = -\Sigma_{x \text{ in } E} \mathrm{Trace}(F_{E,x}|\mathcal{F})\psi_E(f(x)+ax+bx^2),$$

where we have written ψ_E for the additive character of E given by

$$\psi_E := \psi\circ\mathrm{Trace}_{E/k}.$$

Then

$$\Sigma_{(a,b) \text{ in } E^2} |\iota(\mathrm{Trace}(\mathrm{Frob}_{E,(a,b)}|\mathcal{G}))|^4$$

is the image under ι of

$$\Sigma_{(a,b) \text{ in } E^2} \Sigma_{x,y,z,w \text{ in } E} \text{ of}$$
$$\mathrm{Trace}(F_{E,x}|\mathcal{F})\mathrm{Trace}(F_{E,y}|\mathcal{F})\mathrm{Trace}(F_{E,z}|\overline{\mathcal{F}})\mathrm{Trace}(F_{E,w}|\overline{\mathcal{F}})\times$$
$$\psi_E(f(x)+ax+bx^2 + f(y)+ay+by^2 - f(z)-az-bz^2 - f(w)-aw-bw^2).$$

If we exchange the order of summation, this is

$$\Sigma_{x,y,z,w \text{ in } E} \mathrm{Trace}(F_{E,x}|\mathcal{F})\mathrm{Trace}(F_{E,y}|\mathcal{F})\mathrm{Trace}(F_{E,z}|\overline{\mathcal{F}})\mathrm{Trace}(F_{E,w}|\overline{\mathcal{F}})$$
$$\times\psi_E(f(x) + f(y) - f(z) - f(w))$$
$$\times\Sigma_{(a,b) \text{ in } E^2} \psi_E(a(x + y - z - w))\psi_E(b(x^2 + y^2 - z^2 - w^2)).$$

Now the innermost sum vanishes unless the point (x, y, z, w) satisfies the two equations

$$x + y = z + w,$$
$$x^2 + y^2 = z^2 + w^2.$$

[If both equations hold, the inner sum is $(\#E)^2$.] These equations say the first two Newton symmetric functions of (x, y) and of (z, w) coincide. Since we are over a field E in which 2 is invertible, the agreement of the first two Newton symmetric functions implies the agreement of the first two elementary symmetric functions. Concretely, $xy = (1/2)[(x + y)^2 - (x^2 + y^2)]$.
So the inner sum vanishes unless either

$$x = z \text{ and } y=w, \text{ or}$$
$$x = w \text{ and } y = z.$$

The only points satisfying both these conditions are the points x=y=z=w. At any point satisfying either condition,

$$\psi_E(f(x) + f(y) - f(z) - f(w)) = 1.$$

So our sum is equal to $(\#E)^2$ times

$$\Sigma_{x,y \text{ in } E} \mathrm{Trace}(F_{E,x}|\mathcal{F})\mathrm{Trace}(F_{E,y}|\mathcal{F})\mathrm{Trace}(F_{E,x}|\overline{\mathcal{F}})\mathrm{Trace}(F_{E,y}|\overline{\mathcal{F}})$$
$$+\Sigma_{x,y \text{ in } E} \mathrm{Trace}(F_{E,x}|\mathcal{F})\mathrm{Trace}(F_{E,y}|\mathcal{F})\mathrm{Trace}(F_{E,y}|\overline{\mathcal{F}})\mathrm{Trace}(F_{E,x}|\overline{\mathcal{F}})$$
$$-\Sigma_{x \text{ in } E} \mathrm{Trace}(F_{E,x}|\mathcal{F})\mathrm{Trace}(F_{E,x}|\mathcal{F})\mathrm{Trace}(F_{E,x}|\overline{\mathcal{F}})\mathrm{Trace}(F_{E,x}|\overline{\mathcal{F}}).$$

Each of the first two sums is equal to

$(\Sigma_{x \text{ in } E} \mathrm{Trace}(F_{E,x}|\mathcal{F})\mathrm{Trace}(F_{E,x}|\overline{\mathcal{F}}))^2$
$= (\Sigma_{x \text{ in } E} |\mathrm{Trace}(F_{E,x}|\mathcal{F})|^2)^2$

and the third sum is equal to

$-\Sigma_{x \text{ in } E} \mathrm{Trace}(F_{E,x}|\mathcal{F}\otimes\overline{\mathcal{F}}\otimes\mathcal{F}\otimes\overline{\mathcal{F}})$.

Recall that $\mathcal{F}$ is ι-pure of weight zero and geometrically irreducible. The third sum is trivially bounded by $(\#E)\times(\mathrm{rank}(\mathcal{F}))^4$. If we knew Weil II, the quantity being squared in the first two sums would be

$$\Sigma_{x \text{ in } E} |\mathrm{Trace}(F_{E,x}|\mathcal{F})|^2 = (\#E) + O((\#E)^{1/2}).$$

So our overall sum would be

$$2(\#E)^4 + O((\#E)^{7/2}),$$

and we would conclude that $M_4(G, V) = 2$.

What happens if we are in characteristic two? The only difference is that, in the above calculation of the sum, the innermost sum would be

$$\Sigma_{(a,b) \text{ in } E^2} \psi_E(a(x + y - z - w))\psi_E(b(x^3 + y^3 - z^3 - w^3)),$$

which vanishes unless

$$x + y = z + w,$$
$$x^3 + y^3 = z^3 + w^3.$$

Because we are in characteristic two, there is a "new" way these equations can be satisfied, namely we might have

$$x + y = z + w = 0.$$

If x+y and z+w are invertible, then these equations give

$$(x^3 + y^3)/(x + y) = (z^3 + w^3)/(z + w),$$

i.e.,

$$x^2 - xy + y^2 = z^2 - zw + w^2,$$

i.e.,

$$(x + y)^2 - xy = (z + w)^2 - zw.$$

Since we already know that $x + y = z + w$, we find that

$$xy = zw.$$

So once again we have the two previous families of solutions

$$x = z \text{ and } y=w, \text{ or}$$
$$x = w \text{ and } y = z.$$

The calculation now proceeds as before, except that now we have an additional term coming from the "new" family of solutions

$$x + y = z + w = 0.$$

As before, we can ignore cases where any two of the families intersect. The additional main term we get is $(\#E)^2$ times

$\Sigma_{x,z \text{ in } E}$ Trace$(F_{E,x}|\mathcal{F})$Trace$(F_{E,x}|\mathcal{F})$Trace$(F_{E,z}|\overline{\mathcal{F}})$Trace$(F_{E,z}|\overline{\mathcal{F}})$
$= (\Sigma_{x \text{ in } E}$ Trace$(F_{E,x}|\mathcal{F})$Trace$(F_{E,x}|\mathcal{F}))^2$
$= (\Sigma_{x \text{ in } E}$ Trace$(F_{E,x}|\mathcal{F}\otimes\mathcal{F}))^2$.

If $\mathcal{F}$ is geometrically self dual, (and if we do a preliminary $\alpha^{\deg}$ twist so that $\mathcal{F} \cong \overline{\mathcal{F}}$ arithmetically, possible by our previous discussion), then this term is once again equal to

$$(\Sigma_{x \text{ in } E} |\mathrm{Trace}(F_{E,x}|\mathcal{F}|^2)^2,$$

the overall sum is

$$3(\#E)^4 + O((\#E)^{7/2}),$$

and we conclude that $M_4(G, V) = 3$.

If $\mathcal{F}$ is not geometrically self dual, then Weil II gives

$$\Sigma_{x \text{ in } E} \mathrm{Trace}(F_{E,x}|\mathcal{F}\otimes\mathcal{F}) = O((\#E)^{1/2}),$$

the overall sum is

$$2(\#E)^4 + O((\#E)^{7/2}),$$

and we conclude that $M_4(G, V) = 2$.

Step 5 The cohomological moment calculation

Having explained how we knew what the fourth moment should be, we now give the cohomological translation of our calculation. In our heuristic calculation above, we used Weil II to be able to "detect" the group

$$H^4_c(\mathbb{A}^2\otimes_k\overline{k}, \mathcal{G}\otimes\mathcal{G}\otimes\overline{\mathcal{G}}\otimes\overline{\mathcal{G}})$$

through the character sum which which is the alternating sum of traces of Frobenius on all the $H^i_c(\mathbb{A}^2\otimes_k\overline{k}, \mathcal{G}\otimes\mathcal{G}\otimes\overline{\mathcal{G}}\otimes\overline{\mathcal{G}})$. In the calculation we are about to perform, we will "find" the H^4_c sitting alone in various spectral sequences.

The other main tool used in the heuristic calculation was the orthogonality relation for characters

$$\Sigma_{a,b \text{ in } E}\, \psi_E(a\lambda + b\mu) = (\#E)^2 \text{ if } (\lambda, \mu) = 0,$$
$$= 0 \text{ if not.}$$

The cohomological translation of this orthogonality relation is that on the $\mathbb{A}^2/k$ with coordinates a, b, we have

$$H^i_c(\mathbb{A}^2\otimes_k\overline{k}, \mathcal{L}_{\psi(\lambda a + \mu b)}) = 0 \text{ for all i, if } (\lambda, \mu) \neq (0, 0),$$

while on any X/k, $\mathcal{L}_{\psi(f)}$ is trivial is f is identically zero.

We wish to compute $H^4_c(\mathbb{A}^2\otimes_k\overline{k}, \mathcal{G}\otimes\mathcal{G}\otimes\overline{\mathcal{G}}\otimes\overline{\mathcal{G}})$. We first treat the case of odd characteristic. The sheaves $\mathcal{G}$ and $\overline{\mathcal{G}}$ on $\mathbb{A}^2/k$ are

$$\mathcal{G} := R^1(pr_{2,3})_!((pr_1{}^*\mathcal{F})\otimes\mathcal{L}_{\psi(f(T)+ aT + bT^2)}),$$

$$\overline{\mathcal{G}} := R^1(pr_{2,3})_!((pr_1{}^*\overline{\mathcal{F}})\otimes\mathcal{L}_{\overline{\psi}(f(T)+ aT + bT^2)}),$$

and, in both cases, all the other $R^i(pr_{2,3})_!$ vanish. Let us denote by

$$\mathcal{K} := ((pr_1{}^*\mathcal{F})\otimes\mathcal{L}_{\psi(f(T)+ aT + bT^2)}) \text{ on } \mathbb{A}^3,$$
$$\overline{\mathcal{K}} := ((pr_1{}^*\overline{\mathcal{F}})\otimes\mathcal{L}_{\overline{\psi}(f(T)+ aT + bT^2)}) \text{ on } \mathbb{A}^3.$$

Consider the four−fold fibre product of

$$pr_{2,3} : \mathbb{A}^3 \to \mathbb{A}^2$$

with itself over $\mathbb{A}^2$, with the four sources endowed respectively with $\mathcal{K}$, $\mathcal{K}$, $\overline{\mathcal{K}}$, and $\overline{\mathcal{K}}$. The total space is $\mathbb{A}^6$, with coordinates x, y, z, w, a, b, the projection to $\mathbb{A}^2$ is $pr_{5,6}$. On this total space, denote by $\mathcal{J}$ the relative tensor product of the sheaves $\mathcal{K}$, $\mathcal{K}$, $\overline{\mathcal{K}}$, and $\overline{\mathcal{K}}$ on the four sources. By the relative Kunneth formula with compact supports, the sheaf $\mathcal{G}\otimes\mathcal{G}\otimes\overline{\mathcal{G}}\otimes\overline{\mathcal{G}}$ on $\mathbb{A}^2/k$ is given by

$$\mathcal{G}\otimes\mathcal{G}\otimes\overline{\mathcal{G}}\otimes\overline{\mathcal{G}} = R^4(pr_{5,6})_!\mathcal{J},$$

and all other $R^i(pr_{5,6})_!\mathcal{J}$ vanish. So the Leray spectral sequence for $\mathcal{J}$ and the map

$$pr_{5,6} : \mathbb{A}^6/k \to \mathbb{A}^2/k$$

degenerates, and gives

$$H^i{}_c(\mathbb{A}^2\otimes_k\overline{k}, \mathcal{G}\otimes\mathcal{G}\otimes\overline{\mathcal{G}}\otimes\overline{\mathcal{G}}) = H^{i+4}{}_c(\mathbb{A}^6\otimes_k\overline{k}, \mathcal{J}).$$

In particular, we have

$$H^4{}_c(\mathbb{A}^2\otimes_k\overline{k}, \mathcal{G}\otimes\mathcal{G}\otimes\overline{\mathcal{G}}\otimes\overline{\mathcal{G}}) = H^8{}_c(\mathbb{A}^6\otimes_k\overline{k}, \mathcal{J}).$$

This first step corresponds to the "opening" of the sum at the beginning of the character sum calculation.

The step which corresponds to interchanging the order of summation is to compute $H^8{}_c(\mathbb{A}^2\otimes_k\overline{k}, \mathcal{J})$ by using the Leray spectral sequence for $\mathcal{J}$ and the map

$$pr_{1,2,3,4} : \mathbb{A}^6/k \to \mathbb{A}^4/k.$$

At this point, we write out what $\mathcal{J}$ is on the total space $\mathbb{A}^6/k$. It is of the form $\mathcal{A}\otimes\mathcal{B}$, with

$$\mathcal{A}:=pr_1{}^*(\mathcal{F}\otimes\mathcal{L}_{\psi(f)})\otimes pr_2{}^*(\mathcal{F}\otimes\mathcal{L}_{\psi(f)})\otimes pr_3{}^*(\overline{\mathcal{F}}\otimes\mathcal{L}_{\overline{\psi}(f)})\otimes pr_4{}^*(\overline{\mathcal{F}}\otimes\mathcal{L}_{\overline{\psi}(f)})$$

and

$$\mathcal{B}:= \mathcal{L}_{\psi(a(x + y - z - w)}\otimes\mathcal{L}_{\psi(b(x^2 + y^2 - z^2 - w^2)}.$$

The sheaf $\mathcal{A}$ is a pullback from the base space $\mathbb{A}^4/k$. So we have

$$R^i(pr_{1,2,3,4})_!(\mathcal{A}\otimes\mathcal{B}) = \mathcal{A}\otimes R^i(pr_{1,2,3,4})_!\mathcal{B}.$$

Let us denote by

$$Z \subset \mathbb{A}^4/k$$

the closed subscheme defined by the two equations

$$x + y - z - w = 0,$$
$$x^2 + y^2 - z^2 - w^2 = 0.$$

Over the open set $\mathbb{A}^4 - Z$, all the sheaves $R^i(pr_{1,2,3,4})_!\mathcal{B}$ vanish, as we see looking fibre by fibre and using the cohomological version of the orthogonality relation for characters. Over the closed set Z, the sheaf $\mathcal{B}$ is $\mathcal{L}_{\psi(0)} \cong \overline{\mathbb{Q}}_\ell$, and the space over $\mathbb{A}^2$ is $Z\times\mathbb{A}^2$. So over Z we have

$$R^4(pr_{1,2,3,4})_!\mathcal{B} \mid Z = \overline{\mathbb{Q}}_\ell(-2),$$

all other $R^i(pr_{1,2,3,4})_!\mathcal{B} \mid Z$ vanish.

So the sheaves $R^i(pr_{1,2,3,4})_!\mathcal{B}$ on $\mathbb{A}^4/k$ are given by

$$R^4(pr_{1,2,3,4})_!\mathcal{B} = \overline{\mathbb{Q}}_\ell(-2)_Z \text{ } (:= \overline{\mathbb{Q}}_\ell(-2) \text{ on } Z, \text{ extended by } 0),$$

all other $R^i(pr_{1,2,3,4})_!\mathcal{B}$ vanish.

Thus for the sheaf $\mathcal{J} = \mathcal{A}\otimes\mathcal{B}$ we have

$$R^4(pr_{1,2,3,4})_!\mathcal{J} = \mathcal{A}(-2)_Z, \text{ extended by zero,}$$

$$R^i(pr_{1,2,3,4})_!\mathcal{J} = 0 \text{ for } i \neq 4.$$

So the Leray spectral sequence for $\mathcal{J}$ and the map

$$pr_{1,2,3,4} : \mathbb{A}^6/k \to \mathbb{A}^4/k$$

degenerates, and gives, for every i,

$$H^i_c(Z\otimes_k\overline{k}, \mathcal{A})(-2) = H^{4+i}_c(\mathbb{A}^6\otimes_k\overline{k}, \mathcal{J}).$$

In particular,

$$H^8_c(\mathbb{A}^6\otimes_k\overline{k}, \mathcal{J}) = H^4_c(Z\otimes_k\overline{k}, \mathcal{A})(-2).$$

The scheme Z is the union of two irreducible components

$$Z_1 : x = w \text{ and } y = z,$$

$$Z_2: x = z \text{ and } y = w,$$

each of which is an $\mathbb{A}^2/k$, and whose intersection $Z_1\cap Z_2$ is of dimension < 2. So by excision we have

$$H^4_c(Z\otimes_k\overline{k}, \mathcal{A}) = H^4_c(Z_1\otimes_k\overline{k}, \mathcal{A}) \oplus H^4_c(Z_2\otimes_k\overline{k}, \mathcal{A}).$$

We can use the coordinates x, y to identify each Z_i with $\mathbb{A}^2/k$. Let us write $\mathcal{F}_x$ for $pr_1{}^*\mathcal{F}$, $\mathcal{F}_y$ for $pr_2{}^*\mathcal{F}$.

On Z_1, the sheaf $\mathcal{A}$ is

$$\mathcal{F}_x\otimes\mathcal{F}_y\otimes\overline{\mathcal{F}}_x\otimes\overline{\mathcal{F}}_y\otimes\mathcal{L}_{\psi}(f(x) + f(y) - f(x) - f(y))$$
$$\cong \mathcal{F}_x\otimes\mathcal{F}_y\otimes\overline{\mathcal{F}}_x\otimes\overline{\mathcal{F}}_y.$$

Thus

$$H^4_c(Z_1\otimes_k\overline{k}, \mathcal{A}) = H^4_c(\mathbb{A}^2\otimes_k\overline{k}, \mathcal{F}_x\otimes\mathcal{F}_y\otimes\overline{\mathcal{F}}_x\otimes\overline{\mathcal{F}}_y).$$

By the Kunneth formula, we have

$$H^4_c(\mathbb{A}^2\otimes_k\bar{k}, \mathcal{F}_x\otimes\mathcal{F}_y\otimes\bar{\mathcal{F}}_x\otimes\bar{\mathcal{F}}_y)$$
$$\cong H^2_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}\otimes\bar{\mathcal{F}})\otimes H^2_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}\otimes\bar{\mathcal{F}}).$$

Each tensoree
$H^2_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}\otimes\bar{\mathcal{F}}) = H^2_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}\otimes\mathcal{F}^\vee) = H^2_c(\mathbb{A}^1\otimes_k\bar{k}, \underline{\mathrm{End}}(\mathcal{F}))$
is one−dimensional, precisely because $\mathcal{F}$ on $\mathbb{A}^1/k$ is geometrically irreducible (Schur's Lemma!). Similarly, $H^4_c(Z_2\otimes_k\bar{k}, \mathcal{A})$ is one−dimensional. So all in all $H^4_c(Z\otimes_k\bar{k}, \mathcal{A})$ has dimension 2, and tracing back we find that

$$H^4_c(\mathbb{A}^2\otimes_k\bar{k}, \mathcal{G}\otimes\mathcal{G}\otimes\bar{\mathcal{G}}\otimes\bar{\mathcal{G}}) \text{ has dimension 2.}$$

If we are in characteristic 2, the calculation is very much the same. We end up with the same $\mathcal{A}$, but now $\mathcal{B}$ is

$$\mathcal{B} := \mathcal{L}_{\psi(a(x + y - z - w)}\otimes\mathcal{L}_{\psi(b(x^3 + y^3 - z^3 - w^3)},$$

and Z is defined by the equations

$$x + y = z + w,$$
$$x^3 + y^3 = z^3 + w^3.$$

Now Z is the union of three irreducible components, the Z_1 and Z_2 we had in the previous case, and one "new" component

$$Z_0 : x=y,\ z=w.$$

Just as above, excision gives
$H^4_c(Z\otimes_k\bar{k}, \mathcal{A}) = H^4_c(Z_0\otimes_k\bar{k}, \mathcal{A}) \oplus H^4_c(Z_1\otimes_k\bar{k}, \mathcal{A}) \oplus H^4_c(Z_2\otimes_k\bar{k}, \mathcal{A}).$
And exactly as above, both the terms $H^4_c(Z_i\otimes_k\bar{k}, \mathcal{A})$ for i = 1, 2 are one−dimensional. On Z_0, which is $\mathbb{A}^2/k$ with coordinates x, z, the sheaf $\mathcal{A}$ is

$$\mathcal{F}_x\otimes\mathcal{F}_x\otimes\bar{\mathcal{F}}_z\otimes\bar{\mathcal{F}}_z.$$

By Kunneth we have

$$H^4_c(\mathbb{A}^2\otimes_k\bar{k}, \mathcal{F}_x\otimes\mathcal{F}_x\otimes\bar{\mathcal{F}}_z\otimes\bar{\mathcal{F}}_z)$$
$$\cong H^2_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}\otimes\mathcal{F})\otimes H^2_c(\mathbb{A}^1\otimes_k\bar{k}, \bar{\mathcal{F}}\otimes\bar{\mathcal{F}})$$
$$= H^2_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}\otimes\mathcal{F})\otimes H^2_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}^\vee\otimes\mathcal{F}^\vee)$$

Because $\mathcal{F}$ is geometrically irreducible, $\mathcal{F}\otimes\mathcal{F}$ has nonzero π_1^{geom}−invariants (or equivalently coinvariants) if and only if $\mathcal{F}$ is geometrically self−dual, in which case both $H^2_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}\otimes\mathcal{F})$ and $H^2_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}^\vee\otimes\mathcal{F}^\vee)$ are one−dimensional. If $\mathcal{F}$ is not geometrically self dual, then $H^2_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{F}\otimes\mathcal{F})$ vanishes. So in characteristic 2 we find the asserted dimension for

$$H^4_c(\mathbb{A}^2\otimes_k\bar{k}, \mathcal{G}\otimes\mathcal{G}\otimes\bar{\mathcal{G}}\otimes\bar{\mathcal{G}}),$$

namely 2 if $\mathcal{F}$ is not geometrically self−dual, and 3 if $\mathcal{F}$ is geometrically self dual. QED for the cohomological moment calculation.

Applications of the target theorem

To summarize so far: we have completed the proof of the target theorem and its corollary. Let us recall their statements.

Target Theorem (Deligne) Let U/k be a smooth, geometrically connected curve over a finite field, ℓ a prime invertible in k, and $\mathcal{F}$ a lisse $\overline{\mathbb{Q}}_\ell$−sheaf on U which is ι−pure of weight w. Then

$H^1_c(U\otimes_k\overline{k}, \mathcal{F})$ is ι−mixed of weight $\le w + 1$.

As noted above, we already know that

$H^0_c(U\otimes_k\overline{k}, \mathcal{F})$ is ι−pure of weight w, and vanishes if U is affine,

$H^2_c(U\otimes_k\overline{k}, \mathcal{F})$ is ι−pure of weight w + 2.

Corollary [De−Weil II, 3.2.3]

1) The ordinary cohomology group $H^1(U\otimes_k\overline{k}, \mathcal{F})$ is ι−mixed of weight $\ge w + 1$.

2) If the "forget supports" map is an isomorphism

$$H^1_c(U\otimes_k\overline{k}, \mathcal{F}) \cong H^1(U\otimes_k\overline{k}, \mathcal{F}),$$

then $H^1_c(U\otimes_k\overline{k}, \mathcal{F})$ is ι−pure of weight w+1.

3) For C/k the complete nonsingular model of U/k, for

$$j : U \to C$$

the inclusion, and for every integer $0 \le i \le 2$,

$$H^i(C\otimes_k\overline{k}, j_*\mathcal{F}) \text{ is } \iota\text{−pure of weight } w + i.$$

Although the target theorem does not by itself imply the main theorem 3.3.1 of Weil II, the target theorem alone has many striking applications.

Proof of Weil I (using a bit of Lefschetz pencil theory)

Theorem [De−Weil I, 1.7] Let X/k be projective smooth and geometrically connected over a finite field k, ℓ a prime number invertible in k. For any ι, and any integer i, $H^i(X\otimes_k\overline{k}, \overline{\mathbb{Q}}_\ell)$ is ι−pure of weight i.

proof We proceed by induction on n := dim(X). The case n = 0 is trivial, and the case n = 1 is a special case of part 3) of the above Corollary (take $\mathcal{F}$ the constant sheaf).

Fix an auxiliary integer $d \ge 2$. After a finite extension of the ground field, we can find a Lefschetz pencil $\lambda F + \mu G$ on X of hypersurface sections of the chosen degree d. The axis Δ of the pencil is the smooth codimension two subvariety of X where both F and G vanish.

After we pass to the blowup X' of X along Δ, we get a projective morphism

$$\begin{array}{c} X' \\ \pi\downarrow \\ \mathbb{P}^1 \end{array}$$

of relative dimension n−1, whose fibre over (λ, μ) is the intersection of X with the hypersurface of equation $\lambda F + \mu G = 0$. The cohomology of X injects into that of X', cf. [SGA7, XVII, 4.2]. So it suffices to prove the purity of the groups $H^i(X'\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell)$.

There is a maximal dense open set $\mathbb{P}^1 - S$ of $\mathbb{P}^1$ over which the morphism π is (projective and) smooth. Let us denote by

$$j: \mathbb{P}^1 - S \to \mathbb{P}^1$$

the inclusion. Extending scalars, we may assume that S consists of k−rational points. By induction, the theorem already holds for the fibres of π over the open set $\mathbb{P}^1 - S$. For each integer i, form the sheaf

$$\mathcal{F}_i := R^i\pi_*\bar{\mathbb{Q}}_\ell$$

on $\mathbb{P}^1$. Then $j^*\mathcal{F}_i$ is lisse on $\mathbb{P}^1 - S$, and by induction (and proper base change), we know that

$$j^*\mathcal{F}_i \text{ is } \iota\text{−pure of weight i on } \mathbb{P}^1 - S.$$

Suppose first that n := dim(X) is odd. Then [SGA7, XVIII, 6.3.1] for every i, the adjunction map is an isomorphism

$$\mathcal{F}_i \cong j_*j^*\mathcal{F}_i.$$

If n := dim(X) is even, then **either** we have

$$\mathcal{F}_i \cong j_*j^*\mathcal{F}_i \text{ for every i,}$$

cf. [SGA7, XVIII, 6.3.6.3.2], **or** we have

$$\mathcal{F}_i \cong j_*j^*\mathcal{F}_i \text{ for every } i \neq n,$$

and a short exact sequence

$$0 \to \oplus_{s \text{ in } S}\ \bar{\mathbb{Q}}_\ell(-n/2)_s \to \mathcal{F}_n \to j_*j^*\mathcal{F}_n \to 0,$$

cf [SGA7, XVIII, 6.3.2 and 5.3.4] and [SGA7, XV, 3.4].

Now consider the Leray spectral sequence for π:

$$E_2^{a,b} = H^a(\mathbb{P}^1\otimes_k\bar{k}, \mathcal{F}_b) \Rightarrow H^{a+b}(X'\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell).$$

It suffices to show that $E_2^{a,b}$ is ι−pure of weight a+b.

For $b \neq n$,

$$E_2^{a,b} = H^a(\mathbb{P}^1\otimes_k\bar{k}, \mathcal{F}_b) = H^a(\mathbb{P}^1\otimes_k\bar{k}, j_*j^*\mathcal{F}_b)$$

is ι−pure of weight a+b, because $j^*\mathcal{F}_b$ is ι−pure of weight b. For b = n, either $\mathcal{F}_n \cong j_*j^*\mathcal{F}_n$ and this same argument applies, or we have a short exact sequence

$$0 \to \oplus_{s \text{ in } S}\ \bar{\mathbb{Q}}_\ell(-n/2)_s \to \mathcal{F}_n \to j_*j^*\mathcal{F}_n \to 0.$$

Taking the long exact cohomology sequence, we get

$$0 \to \oplus_{s \text{ in } S}\ \bar{\mathbb{Q}}_\ell(-n/2) \to H^0(\mathbb{P}^1\otimes_k\bar{k}, \mathcal{F}_n) \to H^0(\mathbb{P}^1\otimes_k\bar{k}, j_*j^*\mathcal{F}_n) \to 0,$$

and isomorphisms

$$H^a(\mathbb{P}^1\otimes_k\bar{k}, \mathcal{F}_b) \cong H^a(\mathbb{P}^1\otimes_k\bar{k}, j_*j^*\mathcal{F}_b) \text{ for } a \geq 1.$$

So in all cases, $E_2^{a,b}$ is ι−pure of weight a+b, as required. QED

Geometric semisimplicity of lisse pure sheaves

Theorem [De−Weil II, 3.4.1 (iii)] Let X/k be smooth and geometrically connected over a finite field k, ℓ a prime number invertible in k, and $\mathcal{F}$ a lisse, ι−pure $\bar{\mathbb{Q}}_\ell$−sheaf on X. Then $\mathcal{F}$ as a representation of $\pi_1^{geom}(X/k)$ is semisimple (:=completely reducible), i.e., the algebraic group G_{geom} is reductive.

proof By pulling back $\mathcal{F}$ to a suitable spacefilling curve, we do not change its G_{geom}, cf.[Ka−Spacefill, Corollary 7]. So it suffices to treat the case when X/k is a curve C/k. We proceed by induction on the length of a Jordan Holder series for $\mathcal{F}$ as a representation of $\pi_1^{arith}(C/k)$. If $\mathcal{F}$ is $\pi_1^{arith}(C/k)$−irreducible, then it is $\pi_1^{geom}(C/k)$−semisimple, just because $\pi_1^{geom}(C/k)$ is a normal subgroup of $\pi_1^{arith}(C/k)$. If $\mathcal{F}$ is not $\pi_1^{arith}(C/k)$−irreducible, let $\mathcal{A}$ be a nonzero $\pi_1^{arith}(C/k)$−irreducible subsheaf, and $\mathcal{B}$ the quotient $\mathcal{F}/\mathcal{A}$. By induction, both $\mathcal{A}$ and $\mathcal{B}$ are $\pi_1^{geom}(C/k)$−semisimple, so it suffices to show that as $\pi_1^{geom}(C/k)$−modules we have $\mathcal{F} \cong \mathcal{A} \oplus \mathcal{B}$. So we want to show the existence of a $\pi_1^{geom}(C/k)$−splitting of the short exact sequence

$$0 \to \mathcal{A} \to \mathcal{F} \to \mathcal{B} \to 0$$

Tensor this sequence with $\mathcal{B}^\vee$:

$$0 \to \mathcal{A}\otimes\mathcal{B}^\vee \to \mathcal{F}\otimes\mathcal{B}^\vee \to \mathcal{B}\otimes\mathcal{B}^\vee \to 0.$$

The identity endomorphism of $\mathcal{B}$ is an element, say ξ, in $H^0(C\otimes_k\bar{k}, \mathcal{B}\otimes\mathcal{B}^\vee)$ which is fixed by F_k. Finding a $\pi_1^{geom}(C/k)$−splitting means finding an element in $H^0(C\otimes_k\bar{k}, \mathcal{F}\otimes\mathcal{B}^\vee)$ which maps to ξ. Such an element exists if and only if ξ dies under the coboundary map

$$H^0(C\otimes_k\bar{k}, \mathcal{B}\otimes\mathcal{B}^\vee) \to H^1(C\otimes_k\bar{k}, \mathcal{A}\otimes\mathcal{B}^\vee).$$

Because this map is $\mathrm{Gal}(\bar{k}/k)$−equivariant, the image of ξ is an element of $H^1(C\otimes_k\bar{k}, \mathcal{A}\otimes\mathcal{B}^\vee)$ which is fixed by F_k. But $\mathcal{A}\otimes\mathcal{B}^\vee$ is ι−pure of weight zero, so by part 1) of the Corollary to the target theorem, $H^1(C\otimes_k\bar{k}, \mathcal{A}\otimes\mathcal{B}^\vee)$ is ι−mixed of weight ≥ 1. In particular, 1 is not an eigenvalue of F_k on $H^1(C\otimes_k\bar{k}, \mathcal{A}\otimes\mathcal{B}^\vee)$. Therefore ξ dies in $H^1(C\otimes_k\bar{k}, \mathcal{A}\otimes\mathcal{B}^\vee)$, and hence ξ is the image of some element $H^0(C\otimes_k\bar{k}, \mathcal{F}\otimes\mathcal{B}^\vee)$. QED

The Hard Lefschetz Theorem [De−Weil II, 4.1.1] Let X/k be projective, smooth, and connected over an algebraically closed field K, of dimension n. Fix a prime number ℓ invertible in k. Fix a projective embedding $X \subset \mathbb{P}$, and denote by L in $H^2(X, \bar{\mathbb{Q}}_\ell)(-1)$ the cohomology class of a hyperplane. Then for every integer i with $1 \le i \le n$, the iterated cup product maps

$$L^i : H^{n-i}(X, \overline{\mathbb{Q}}_\ell) \to H^{n+i}(X, \overline{\mathbb{Q}}_\ell)(-i)$$

are isomorphisms.

sketch of proof By standard spreading out techniques, one reduces to the case when K is the algebraic closure of a finite field k, and X begins life over k. Take a Lefshetz pencil on X. By the previous result, the lisse sheaf $j^*\mathcal{F}_{n-1}$ on $\mathbb{P}^1 - S$ is $\pi_1{}^{geom}((\mathbb{P}^1 - S)/k)$−semisimple. If we already know Hard Lefschetz for all smooth fibres of the pencil, then Hard Lefschetz for X itself results from the $\pi_1{}^{geom}((\mathbb{P}^1 - S)/k)$−semisimplicity of $j^*\mathcal{F}_{n-1}$, c.f. [De−Weil II, 4.1.1−4]. QED

Semisimplicity of G_{geom} for a lisse ι−pure sheaf

Theorem (Deligne, Weil II, 1.3.9) Let X/k be smooth and geometrically connected over a finite field k, ℓ a prime number invertible in k, and $\mathcal{F}$ a lisse, ι−pure $\overline{\mathbb{Q}}_\ell$−sheaf on X. Then the algebraic group G_{geom} is semisimple. More generally, for $\mathcal{F}$ any lisse $\overline{\mathbb{Q}}_\ell$−sheaf on X/k, if G_{geom} is reductive, then it is semisimple.

proof By pulling back to a suitable spacefilling curve, it suffices to treat the case when X/k is a curve C/k. Extending scalars and pulling back to a suitable finite etale covering of C, we replace G_{geom} by its identity component. So we may further assume that G_{geom} is connected and reductive. Then G_{geom} is the "almost product" (the two factors have finite intersection) of its derived group and its connected center:

$$G_{geom} = (G_{geom})^{der} \cdot Z(G_{geom})^0.$$

The derived group is semisimple, and the connected center is a torus. Because their intersection is finite, say of order N, the map

$$g = g_{der} \cdot z \mapsto z^N$$

exhibits $Z(G_{geom})^0$ as a quotient of G. So to show that G_{geom} is semisimple, it suffices to show that any homomorphism from G_{geom} to $\mathbb{G}_m$, say $\chi : G_{geom} \to \mathbb{G}_m$, is of finite order. Think of χ as a one−dimensional representation of G_{geom} (or equivalently, of $\pi_1{}^{geom}(C/k)$). Because G_{geom} is reductive, and the representation (ρ, V) of it corresponding to $\mathcal{F}$ is faithful, every irreducible representation of G_{geom} occurs in some tensor space $V^{\otimes a} \otimes (V^\vee)^{\otimes b}$. Then χ corresponds to a rank one lisse subsheaf $\mathcal{L}$ of some $\mathcal{F}^{\otimes a} \otimes (\mathcal{F}^\vee)^{\otimes b}$ on $C \otimes_k \overline{k}$. Since $\mathcal{F}^{\otimes a} \otimes (\mathcal{F}^\vee)^{\otimes b}$ on $C \otimes_k \overline{k}$ contains only finitely isomorphism classes of irreducibles, after extending scalars from k to some finite extension E/k, we reduce to the case where each isotypical component of $\mathcal{F}^{\otimes a} \otimes (\mathcal{F}^\vee)^{\otimes b}$ as $\pi_1{}^{geom}(C/k)$−representation is a $\pi_1{}^{arith}(C/k)$−subrepresentation of $\mathcal{F}^{\otimes a} \otimes (\mathcal{F}^\vee)^{\otimes b}$. In particular, the isotypical component $n\mathcal{L}$ of $\mathcal{L}$ is a $\pi_1{}^{arith}(C/k)$−subrepresentation. Therefore $\mathcal{L}^{\otimes n} = \Lambda^n(n\mathcal{L})$

extends to a character of $\pi_1^{\mathrm{arith}}(C/k)$. So replacing χ by χ^n, we may assume that χ as character of $\pi_1^{\mathrm{geom}}(C/k)$ extends to a continuous character

$$\chi : \pi_1^{\mathrm{arith}}(C/k) \to \overline{\mathbb{Q}}_\ell^\times.$$

Such a character lands in some $\mathcal{O}_\lambda^\times$. Replacing χ by a further power of itself, we may further assume that χ takes values in the multiplicative subgroup $1+2\ell\mathcal{O}_\ell$, which maps injectively to the additive group of $\overline{\mathbb{Q}}_\ell$ by the logarithm. Then $\log(\chi)$ restricted to $\pi_1^{\mathrm{geom}}(C/k)$ is an element of

$$\mathrm{Hom}(\pi_1^{\mathrm{geom}}(C/k), \overline{\mathbb{Q}}_\ell) := H^1(C\otimes_k\overline{k}, \overline{\mathbb{Q}}_\ell)$$

which is invariant by $\pi_1^{\mathrm{arith}}(C/k)$−conjugation, i.e, it is a fixed point of the action of F_k on $H^1(C\otimes_k\overline{k}, \overline{\mathbb{Q}}_\ell)$. But $H^1(C\otimes_k\overline{k}, \overline{\mathbb{Q}}_\ell)$ has weight ≥ 1. So 1 is not an eigenvalue of F_k on $H^1(C\otimes_k\overline{k}, \overline{\mathbb{Q}}_\ell)$. Thus $\log(\chi)$ and with it χ are trivial on $\pi_1^{\mathrm{geom}}(C/k)$. Thus χ is trivial on G_{geom}. So G_{geom} has no nontrivial quotient torus, hence is semisimple. QED

Equidistribution: Sato−Tate [De−Weil II, 3.5.3]

Once we have the target theorem, and the semisimplicity of G_{geom}, we get Deligne's equidistribution theorem ("Sato−Tate conjecture in the function field case") over a curve, cf. [Ka−GKM, Chapter 3].

Ramanujan Conjecture [De−FMRℓ] and [De−Weil II, 3.7.1]

Take an integer $N \geq 4$. Denote by $\mathcal{M} := \mathcal{M}_{\Gamma_1(N)}$ the modular curve over $\mathbb{Z}[1/N]$ which represents the functor "elliptic curves plus a point of exact order N". In Deligne's working out [De−FMRℓ] of Sato's idea that "Weil implies Ramanujan", the key technical assertion is this. Consider the univeral family of elliptic curves carried by $\mathcal{M}$:

$$\begin{array}{c} \mathcal{E} \\ f\downarrow \\ \mathcal{M}. \end{array}$$

Take ℓ any prime dividing N. Form the sheaf

$$\mathcal{F} := R^1f_*\overline{\mathbb{Q}}_\ell$$

on $\mathcal{M}$. It is a lisse sheaf on $\mathcal{M}$ of rank 2, which is ι−pure of weight one for any ι [Hasse's theorem [H], fibre by fibre]. For every integer $k \geq 1$, $\mathrm{Sym}^k(\mathcal{F})$ is then ι−pure of weight k. Fix a prime p which does not divide N. Denote by $\overline{\mathcal{M}}\otimes\mathbb{F}_p$ the complete nonsingular model of $\mathcal{M}\otimes\mathbb{F}_p$. Denote by

$$j : \mathcal{M}\otimes\mathbb{F}_p \to \overline{\mathcal{M}}\otimes\mathbb{F}_p$$

the inclusion. The action of Frobenius on the cohomology group

$$H^1(\overline{\mathcal{M}}\otimes\mathbb{F}_p, j_*\mathrm{Sym}^k(\mathcal{F}))$$

is related to the action of T_p on cusp forms of weight k+2 on $\Gamma_1(N)$ by an equality of characteristic polyomials

$$\det(1 - XT_p + X^2p^{k+1}\langle p\rangle \mid S_{k+2}(\Gamma_1(N)))$$
$$= \det(1 - XF_{\mathbb{F}_p} \mid H^1(\bar{\mathcal{M}}\otimes\mathbb{F}_p, j_*\mathrm{Sym}^k(\mathcal{F})),$$

cf. [De, FMRℓ, Theorem 4]. By Part 3) of the corollary to the target theorem,

$$H^1(\bar{\mathcal{M}}\otimes\mathbb{F}_p, j_*\mathrm{Sym}^k(\mathcal{F}))$$

is ι−pure of weight k+1, for every ι. This ι−purity implies that the eigenvalues of T_p on $S_{k+2}(\Gamma_1(N))$ have their absolute values bounded by $2\mathrm{Sqrt}(p^{k+1})$, which is the Ramanujan conjecture.

What we don't get from the target theorem alone

In our target theorem, we say nothing about the weights which actually occur in $H^1{}_c(U\otimes_k\bar{k}, \mathcal{F})$, only that they are $\leq w + 1$. In fact, the weights that occur all differ by integers from w + 1, and the weight drops reflect the structure of the local monodromies at the missing points. Moreover, if $\mathcal{F}$ is ι−pure of the same weight w for **all** ι's, then the ι−weight of any eigenvalue α of F_k on $H^1{}_c(U\otimes_k\bar{k}, \mathcal{F})$ is independent of ι.

To prove these finer results requires Deligne's detailed analysis [De−Weil II, 1.8.4] of local monodromy on curves, and of its interplay with weights. His further analysis of the variation of local monodromy in **families** of curves [De−Weil II, 1.6 through 1.8, especially 1.8.6−8] is the essential ingredient in his beautiful deduction of the main theorem 3.3.1 of Weil II from the target theorem.

Another topic we have not discussed here is Deligne's theorem [De−Weil II, 1.5.1] that for any lisse sheaf $\mathcal{F}$ on C/k whose trace function is ι−real (i.e., all traces of all Frobenii land, via ι, in $\mathbb{R}$), each of its Jordan−Holder constituents is ι−pure of some weight. As Kiehl and Weissauer point out in [KW, 9.2 and 9.3], one can use this result, together with Deligne's monodromy analysis for individual curves over finite fields (i.e."just" [De−Weil II, 1.8.4]), to give an alternate derivation of the main theorem 3.3.1 of Weil II from the target theorem.

Appedix : statement of the main theorem 3.3.1 of Weil II

Fix a prime number ℓ, a real number w, and a collection $\mathcal{I}$ of embeddings ι of $\bar{\mathbb{Q}}_\ell$ into $\mathbb{C}$. Let Z be a separated scheme of finite type over $\mathbb{Z}[1/\ell]$ which is normal and connected. A lisse $\bar{\mathbb{Q}}_\ell$−sheaf $\mathcal{F}$ on Z, corresponding to a representation (ρ, V) of $\pi_1(Z)$, is said to be $\mathcal{I}$−pure of weight w if, for every finite field k and every point z in Z(k), every eigenvalue of $\rho(\mathrm{Frob}_{k,z})$ has, via every ι in $\mathcal{I}$, complex absolute value $(\#k)^{w/2}$. [In the case when k is a finite field, $\mathcal{I}$ is a single ι, and Z/k is smooth and geometrically connected, this is equivalent to the earlier definition, in terms of points in (Z/k)(E) for all finite extension fields E/k. The point is that given any finite extension field E/k, and any point z in Z(E), there is an automorphism σ of E such that σ(z) lies in (Z/k)(E), (because Aut(E) acts transitively on the set of inclusions of k into E).]

Let X be a separated scheme of finite type over $\mathbb{Z}[1/\ell]$, and $\mathcal{F}$ a constructible $\bar{\mathbb{Q}}_\ell$−sheaf $\mathcal{F}$

on X. Then there exists a partition of X^{red} into a finite disjoint union of locally closed subschemes Z_i, each of which is normal and connected, such that $\mathcal{F} | Z_i$ is a lisse $\overline{\mathbb{Q}}_\ell$−sheaf on Z_i for each i. The sheaf $\mathcal{F}$ on X is said to be punctually $\mathcal{I}$−pure of weight w on X if, for some partition as above, the restriction of $\mathcal{F}$ to each Z_i is ι−pure of weight w in the sense of the preceding paragraph, for every ι in $\mathcal{I}$.

A constructible $\overline{\mathbb{Q}}_\ell$−sheaf $\mathcal{F}$ on X is said to be $\mathcal{I}$−mixed of weight ≤ w on X if it is a successive extension of finitely many constructible $\overline{\mathbb{Q}}_\ell$−sheaves $\mathcal{F}_i$ on X, with each $\mathcal{F}_i$ punctually $\mathcal{I}$−pure of some weight $w_i \le w$.

The main theorem of Weil II is the following.

Theorem [De−Weil II, 3.3.1] Let X and Y be separated $\mathbb{Z}[1/\ell]$−schemes of finite type, $f : X \to Y$ a morphism. Let $\mathcal{F}$ be a constructible $\overline{\mathbb{Q}}_\ell$−sheaf $\mathcal{F}$ on X which is $\mathcal{I}$−mixed of weight ≤ w. Then for every i, the constructible $\overline{\mathbb{Q}}_\ell$−sheaf $R^i f_! \mathcal{F}$ on Y is $\mathcal{I}$−mixed of weight ≤ w + i.

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