Appendix: On Galois representations with values in G_2

Michael Dettweiler and Nicholas M. Katz

Let ℓ be a prime and let $\mathcal{H}(\mathbf{1},\mathbf{1})$ be the cohomologically rigid $\bar{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1_{\bar{\mathbb{Q}}}$ of rank 7 which is given in Thm. 1.3.1 of the article (whose notation we adopt). By loc. cit., the restriction of $\mathcal{H}(\mathbf{1},\mathbf{1})$ to $\mathbb{A}^1_{\bar{\mathbb{Q}}} \setminus \{0,1\}$ is a lisse $\bar{\mathbb{Q}}_\ell$-sheaf whose monodromy is dense in the exceptional algebraic group $G_2(\bar{\mathbb{Q}}_\ell)$ and whose local monodromy at $0,1,\infty$ is of the following type

$$\mathbf{1}^3 \oplus (-\mathbf{1})^4, \quad \mathbf{U}(2)^2 \oplus \mathbf{U}(3), \quad \mathbf{U}(7), \quad \text{respectively.} \tag{1}$$

By [5], Thm. 5.5.4, there exists a lisse $\mathbb{Q}_\ell$-sheaf $\mathcal{G}_\ell$ on $S_\ell := \mathbb{A}^1_{R_\ell} \setminus \{0,1\}$ ($R_\ell = \mathbb{Z}[\frac{1}{2\ell}]$) which, after the base change $R_\ell \to \bar{\mathbb{Q}}$ and the extension of scalars $\mathbb{Q}_\ell \to \bar{\mathbb{Q}}_\ell$ on the coefficients becomes the restriction of $\mathcal{H}(\mathbf{1},\mathbf{1})$ to $\mathbb{A}^1_{\bar{\mathbb{Q}}} \setminus \{0,1\}$. (The construction of $\mathcal{G}_\ell$ is given below.) The monodromy representation of the Tate twisted sheaf $\mathcal{G}_\ell(3)$ is denoted by

$$\rho_\ell : \pi_1(S_\ell) \longrightarrow \mathrm{GL}_7(\mathbb{Q}_\ell).$$

Let $s_0 \in S_\ell(\mathbb{Q})$. The morphism $s_0 \to S_\ell$ induces a homomorphism $\alpha : \pi_1(s_0, \bar{s}_0) \to \pi_1(S_\ell, \bar{s}_0)$. Since $\pi_1(s_0, \bar{s}_0)$ is isomorphic to $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ we can view α as a homomorphism $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \pi_1(S_\ell, \bar{s}_0)$. The *specialization of ρ_ℓ to s_0* is then defined as the composition

$$\rho_\ell^{s_0} := \rho_\ell \circ \alpha \; : \; \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_7(\mathbb{Q}_\ell).$$

Indeed, we may view s_0 as a point of S_ℓ with values in the ring $\mathbb{Z}[\frac{1}{2\ell}][s_0, \frac{1}{s_0}, \frac{1}{s_0-1}]$, so that $\rho_\ell^{s_0}$ is in fact unramified except possibly at $2, \ell$, and at those primes p such that either s_0 or $s_0 - 1$ fails to be a p-adic unit. Our main result is the following:

Theorem 1 (i) *The representation ρ_ℓ has values in $G_2(\mathbb{Q}_\ell)$.*

(ii) *Let a, b be two coprime integers which each have an odd prime divisor which is different from ℓ and let $s_0 := 1 + \frac{a}{b}$. Then the image of $\rho_\ell^{s_0}$ is Zariski dense in $G_2(\mathbb{Q}_\ell)$.*

For $s_0 \in S_\ell(\mathbb{Q})$, let M_{s_0} be the motive for motivated cycles which appears in Thm. 3.3.1. By construction, the above Galois representation $\rho_\ell^{s_0}$ is the Galois representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the ℓ-adic realization of the motive M_{s_0} (see the proof of Cor. 2 for this). As a corollary of Thm. 1, we obtain an explicit way to obtain motives with motivic Galois group of type G_2 :

Corollary 2 (i) *Let $s_0 = 1+\frac{a}{b}$ be as in Thm. 1. Then the motive for motivated cycles M_{s_0} has a motivic Galois group of type G_2.*

(ii) *Let (a,b) and (a',b') be pairs of squarefree odd coprime integers, each ≥ 3, such that $(a,b) \neq (a',b')$. Let $s_0 = 1+\frac{a}{b}$ and $s_0' = 1+\frac{a'}{b'}$. For any prime ℓ not dividing the product $aba'b'$, the ℓ-adic representations $\rho_\ell^{s_0}$ and $\rho_\ell^{s_0'}$ are not isomorphic. In particular, the motives M_{s_0} and $M_{s_0'}$ are not isomorphic.*

(iii) *There exist infinitely many non-isomorphic motives M_{s_0} whose motivic Galois group is of type G_2.*

The proof of Thm. 1 and Cor. 2 is given below. Let us first recall the construction of $\mathcal{G}_\ell$: The group $\mu_2(R_\ell)$ of the second roots of unity of $R_\ell = \mathbb{Z}[\frac{1}{2\ell}]$ acts on the étale covers f_1, f_2 of $S_\ell = \mathbb{A}^1_{R_\ell} \setminus \{0,1\}$ which are defined by the equations $y^2 = x$, resp. $y^2 = x-1$. The covers f_1 and f_2 therefore define surjections

$$\eta_i : \pi_1(S_\ell) \longrightarrow \mu_2(R_\ell) \quad \text{for} \quad i = 1,2.$$

The composition of the embedding $\chi : \mu_2(R_\ell) \to \mathbb{Q}_\ell$ with η_i, $i = 1,2$, define lisse $\mathbb{Q}_\ell$-sheaves $\mathcal{L}_{\chi(x)}$, resp. $\mathcal{L}_{\chi(x-1)}$, on S_ℓ. Let $j : S_\ell \to \mathbb{A}^1_{R_\ell}$ denote the tautological inclusion and let

$$\mathcal{F}_3 = \mathcal{F}_5 = \mathcal{F}_7 := j_*(\mathcal{L}_{\chi(x)}) \quad \text{and} \quad \mathcal{F}_2 = \mathcal{F}_4 = \mathcal{F}_6 := j_*(\mathcal{L}_{\chi(x-1)}).$$

Let $\mathcal{H}_0 := j_*\left(\mathcal{L}_{\chi(x)} \otimes \mathcal{L}_{\chi(x-1)}\right)$ and define inductively

$$\mathcal{H}_i := j_*\left(\mathcal{F}_{i+1} \otimes j^*(\mathrm{MC}_\chi(\mathcal{H}_{i-1}))\right), \quad \text{for} \quad i = 1, \ldots, 6, \tag{2}$$

where $\mathrm{MC}_\chi(\mathcal{H}_i)$ is as defined in [5], Section 4.3 (see also Rem. 5 below). We remark that on each geometric fibre $\bar{S}_\ell$ of S_ℓ, one has

$$\mathrm{MC}_\chi(\mathcal{H}_{i-1})|_{\bar{S}_\ell} = \mathrm{MC}_\chi(\mathcal{H}_{i-1}|_{\bar{S}_\ell}), \tag{3}$$

where on the right hand side, MC_χ is the middle convolution functor defined in [5], Chap. 5 (or in Section 1.1 of the article). We then define $\mathcal{G}_\ell$ to be the lisse sheaf $\mathcal{H}_6|_{S_\ell}$. It follows from the construction of $\mathcal{H}(\mathbf{1},\mathbf{1})$ in the proof of Thm. 1.3.1 and from Formula (3) that

$$\left(\mathcal{G}_\ell \otimes \bar{\mathbb{Q}}_\ell\right)|_{\mathbb{A}^1_{\bar{\mathbb{Q}}}\setminus\{0,1\}} = \mathcal{H}(\mathbf{1},\mathbf{1})|_{\mathbb{A}^1_{\bar{\mathbb{Q}}}\setminus\{0,1\}}. \tag{4}$$

Remark 3 By [5], 5.5.4 part (3), the weight of $\mathcal{G}_\ell$ is equal to 6, which implies that the Tate twist $\mathcal{G}_\ell(3)$ has weight zero. By loc cit., 5.5.4 part (2), the restriction of $\mathcal{H}_6$ to any geometric fibre is irreducible and cohomologically rigid of the same type of local monodromy. Moreover, by the Specialization Theorem (cf. loc. cit. 4.2.4), the geometric monodromy group (of the restriction of $\mathcal{G}_\ell$) on any geometric fibre of S_ℓ is also Zariski dense in G_2.

Proposition 4 *Let s_0 be a rational number, not 0 or 1, and let p be an odd prime number different from ℓ. Then the following holds:*

(i) *If $\mathrm{ord}_p(s_0) < 0$, then the restriction of $\rho_\ell^{s_0}$ to the inertia subgroup $I_p \leq \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ at p factors through the tame inertia group at p, $I_p^{\mathrm{tame}} \cong \hat{\mathbb{Z}}^{(not\ p)}(1)$, and is an indecomposable unipotent block of length 7.*

(ii) *If $\mathrm{ord}_p(s_0 - 1) > 0$, then the restriction of $\rho_\ell^{s_0}$ to I_p factors through the tame inertia group $I_p^{\mathrm{tame}} \cong \hat{\mathbb{Z}}^{(\mathrm{not\ p})}(1)$ and is the direct sum of an indecomposable unipotent block of length 3 and of two indecomposable unipotent blocks of length 2.*

(iii) *If $\mathrm{ord}_p(s_0) > 0$, then I_p acts tamely, by automorphisms of order at most 2.*

(iv) *If both s_0 and $s_0 - 1$ are p-adic units, then I_p acts trivially.*

Proof of Proposition 4: We first prove (i). Let W_p denote the ring of Witt vectors of an algebraic closure of $\mathbb{F}_p$. Let t be the standard parameter on $\mathbb{A}^1_{R_\ell}$, let $z := \frac{1}{t}$ denote the parameter at infinity, and consider the formal punctured disc $\Delta_p := \mathrm{Spec}\,(W_p[[z]][\frac{1}{z}])$. Since $\mathrm{Spec}\,(W_p)$ is simply connected, one knows that $\pi_1(\Delta_p)$ is the group $\hat{\mathbb{Z}}^{(\mathrm{not\ p})}(1)$ (this follows from Abhyankar's Lemma, cf. [4], Ex. on Page 120). In more concrete terms: all finite connected étale covers of Δ_p are obtained by taking the N-th root of z for some N prime to p. We can read the effect of a topological generator of this group in our representation after extension of scalars from W_p to the complex numbers, so we know a topological generator gives a single unipotent block of size 7 (since this is the local monodromy of $\mathcal{G}_\ell$ around ∞ on every geometric fibre of S_ℓ over R_ℓ). If we specialize z to a nonzero point z_0 (here $\frac{1}{s_0}$) in the maximal ideal pW_p of W_p, the resulting ring homomorphism $W_p[[z]][\frac{1}{z}] \to K_p := \mathrm{Frac}(W_p)$ induces a homomorphism of fundamental groups $I_p \to \pi_1(\Delta_p)$, which, in view of Abhyankar's Lemma, factors through $I_p^{\mathrm{tame}} \cong \hat{\mathbb{Z}}^{(\mathrm{not\ p})}(1)$. Identifying both source and target of this map $I_p^{\mathrm{tame}} \to \pi_1(\Delta_p)$ with the group $\hat{\mathbb{Z}}^{(\mathrm{not\ p})}(1)$, we see that this map is nonzero (simply because in W_p, z_0 does not have an N-th root for any N not dividing $\mathrm{ord}_p(z_0)$). So after pullback to such a point, the specialized representation of I_p^{tame} remains unipotent and indecomposable (simply because in characteristic zero, if A is a unipotent automorphism of a finite-dimensional vector space, then A and any nonzero power of A have the same Jordan decomposition). To prove (ii) and (iii), we repeat these arguments, but now with z taken to be $t-1$ (resp., z taken to be t) and using the fact that for our sheaf $\mathcal{G}_\ell$, local monodromy around 1 (resp., around 0) is unipotent of the asserted shape (resp., involutory). Claim (iv) was already noted at the beginning of the Appendix. $\square$

Proof of Theorem 1: For fields of cohomological dimension ≤ 2 (and hence for ℓ-adic fields) it is known that there exists only one form of the algebraic group G_2: the split form (see [8] and [6]). It follows thus from Thm. 1.3.1 of the article and from Formula (4) that the geometric monodromy of $\mathcal{G}_\ell(3)$ is Zariski dense in the group $G_2(\mathbb{Q}_\ell)$. Poincaré duality, applied in each step of the convolution construction of $\mathcal{G}_\ell$ given in (2) implies that the sheaf $\mathcal{G}_\ell(3)$ is orthogonally self dual, and hence ρ_ℓ respects a nondegenerate orthogonal form. The normalizer of $G_2(\mathbb{Q}_\ell)$ in the orthogonal group $\mathrm{O}_7(\mathbb{Q}_\ell)$ consists of the scalars $\langle \pm 1\rangle$ only. Since the representation ρ_ℓ has degree 7, it follows therefore that there exists a character $\epsilon_\ell : \pi_1(S_\ell) \to \langle \pm 1\rangle$ such that $\rho_\ell \otimes \epsilon_\ell$ has values in $G_2(\mathbb{Q}_\ell)$.

We have to show that ϵ_ℓ is trivial. To see this we argue as follows: Because on each fiber the geometric monodromy group is G_2 by Rem. 3, the character ϵ_ℓ is actually a character of $\pi_1(\mathbb{Z}[\frac{1}{2\ell}])$. As ℓ varies, the characters ϵ_ℓ form a compatible system (this follows from the compatibility of ρ_ℓ which follows from the compatibility of MC_χ which is proved in [5], 5.5.4 (4)). So taking ℓ to be 2, one sees that ϵ_ℓ is a quadratic character whose conductor is a power of 2. Given the structure of 2-adic units as the product of $\langle \pm 1\rangle$ with the pro-cyclic group $1+4\mathbb{Z}_2$, one sees that any homomorphism from this group to $\langle \pm 1\rangle$ actually factors through the units modulo 8. Therefore it suffices to show that for p in a set of primes whose reduction modulo 8 meets each nontrivial class of units mod 8 and for one $t \in \mathbb{F}_p \setminus \{0,1\}$), the Frobenius element $\rho_\ell(\mathrm{Frob}_{p,t})$ is contained in $G_2(\mathbb{Q}_\ell)$.

Since the weight of ρ_ℓ is 0 by Rem. 3, the eigenvalues of $\rho_\ell(\mathrm{Frob}_{p,t})\,(p \neq 2,\ell)$ are Weil numbers of complex absolute value equal to 1. Moreover, any Frobenius element is contained either in $G_2(\mathbb{Q}_2)$ or in the coset $-G_2(\mathbb{Q}_\ell)$. Since any semisimple element in $G_2(\bar{\mathbb{Q}}_\ell) \leq \mathrm{GL}_7(\bar{\mathbb{Q}}_\ell)$ is conjugate to a diagonal matrix of the form $\mathrm{diag}(x,y,xy,1,(xy)^{-1},y^{-1},x^{-1})$, it follows (from elementary arguments on trigonometric functions) that the trace of $\rho_\ell(\mathrm{Frob}_{p,t})$ lies in the interval $[-2,7]$ if $\rho_\ell(\mathrm{Frob}_{p,t})$ is contained in $G_2(\mathbb{Q}_\ell)$, or it lies in the interval $[-7,2]$ if $\rho_\ell(\mathrm{Frob}_{p,t})$ is contained in $-G_2(\mathbb{Q}_\ell)$. By compatibility and the discussion above, it therefore suffices to show that for p in a set of primes whose reduction modulo 8 meets each nontrivial class of units mod 8 and for some $t \in \mathbb{F}_p \setminus \{0,1\}$, the trace of $\rho_\ell(\mathrm{Frob}_{p,t})$ lies in the left open interval $]2,7]$ if $p \neq \ell$. Using the computer system Mathematica, the authors have checked that in fact, for the primes $p = 137, 139, 149, 151$, the trace of some $\rho_\ell(\mathrm{Frob}_{p,t})$ is contained in $]2,7]$ if $\ell \neq p$. (Details of the actual computation are discussed in Rem. 5 below.) This implies that ϵ_ℓ is trivial for all primes ℓ, proving the first claim.

By Scholl [7], Prop. 3, a pure ℓ-adic Galois representation ρ_ℓ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is irreducible if the following conditions are satisfied: there exists a prime $p \neq \ell$ and an open subgroup $I \leq I_p$ such that the restriction of ρ_ℓ to I is unipotent and indecomposable, and the restriction of ρ to $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ is Hodge-Tate. By Prop. 4 (i) and the assumption on $s_0 = 1 + \frac{a}{b}$, the restriction of $\rho_\ell^{s_0} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_7(\mathbb{Q}_\ell)$ to I_p is unipotent and indecomposable, where p is any odd prime divisor of b which is different from ℓ. It follows from the motivic interpretation of $\mathcal{G}_\ell$ given in Cor. 2.4.2 of the article that $\rho_\ell^{s_0}$ is a Galois submodule of the 6-th étale cohomology group of a smooth projective variety over $\mathbb{Q}$. Since the étale cohomology groups of a smooth projective variety over $\mathbb{Q}_\ell$ are Hodge-Tate by Faltings [3], the restriction of $\rho_\ell^{s_0}$ to $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ has the Hodge-Tate property. Since $\rho_\ell^{s_0}$ is pure of weight 0, Scholl's result implies that the representation $\rho_\ell^{s_0}$ is absolutely irreducible. Let q be an odd prime divisor of a which is different from ℓ and let J_q be the image of a topological generator of I_q^{tame} under $\rho_\ell^{s_0}$. By Prop. 4, the Jordan canonical form of J_q has two Jordan blocks of length 2 and one of length 3. By [2], Cor. 12, a Zariski closed proper maximal subgroup of $G_2(\bar{\mathbb{Q}}_\ell)$ is either reducible or G is isomorphic to the group $\mathrm{PSL}_2(\bar{\mathbb{Q}}_\ell)$. In the latter case, the non-trivial unipotent elements of the image of $\mathrm{PSL}_2(\bar{\mathbb{Q}}_\ell)$ are conjugate in $\mathrm{GL}_7(\bar{\mathbb{Q}}_\ell)$ to a Jordan block of length 7. Thus the existence of J_q implies that the Zariski closure of $\mathrm{Im}(\rho_\ell^{s_0})$ in $G_2(\bar{\mathbb{Q}}_\ell)$ is equal to $G_2(\bar{\mathbb{Q}}_\ell)$. It follows that $\mathrm{Im}(\rho_\ell^{s_0})$ is Zariski dense in $G_2(\mathbb{Q}_\ell)$, finishing the proof of the second claim of Thm. 1. $\square$

Proof of Corollary 2: By construction, the Galois representation $\rho_\ell^{s_0} : G_\mathbb{Q} \to \mathrm{GL}_7(\mathbb{Q}_\ell)$ is isomorphic to the Galois representation on the stalk $(\mathcal{G}_\ell(3))_{\bar{s}_0}$. Moreover, the stalk $(\mathcal{G}_\ell(3))_{\bar{s}_0}$ is the ℓ-adic realization of the motive M_{s_0} which appears in Section 3.3 of the article. The motivic Galois group $G_{M_{s_0}}$ of M_{s_0} can be characterized as the stabilizer of the spaces of motivated cycles in the realizations of every subobject of the Tannakian category $\langle M_{s_0} \rangle$ generated by M_{s_0} (this can be seen using the arguments in [1], Chap. 6.3). By Chevalley's theorem, there exists one object $M \in \langle M_{s_0} \rangle$ such that the motivic Galois group $G_{M_{s_0}}$ is characterized as the stabilizer of a line in the realization of M which is spanned by a motivated cycle. This line is fixed by an open subgroup of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Therefore, the group $G_{M_{s_0}}(\mathbb{Q}_\ell)$ contains the image of an open subgroup of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ under $\rho_\ell^{s_0}$. Since the group $G_2(\mathbb{Q}_\ell)$ is connected and since the Zariski closure of $\rho_\ell^{s_0}(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}))$ is dense in $G_2(\mathbb{Q}_\ell)$ by Thm. 1 (ii), the group $G_2(\mathbb{Q}_\ell)$ is contained in $G_{M_{s_0}}(\mathbb{Q}_\ell)$. By construction, the motivic Galois group $G_{M_{s_0}}$ of M_{s_0} is contained in the group G_2 (see the proof of Thm. 3.3.1 of the article). With what was said before, one concludes that $G_{M_{s_0}}(\mathbb{Q}_\ell) = G_2(\mathbb{Q}_\ell)$, so $G_{M_{s_0}}(\mathbb{Q}_\ell)$ is of type G_2, proving the first claim.

To prove the second claim, we argue as follows: Fix a prime ℓ which does not divide the product $aba'b'$. By Prop. 4, we recover the odd primes p which divide a, resp. b, as those odd primes p where $I_p^{\rm tame}$ acts unipotently with a block of length 3 and two blocks of length 2, resp. where $I_p^{\rm tame}$ acts unipotently with a single block of length 7. Since $(a,b) \neq (a',b')$, the Galois representations $\rho_\ell^{s_0}$ and $\rho_\ell^{s_0'}$ have a different ramification behaviour at at least one prime divisor p of $a \cdot b$ or of $a' \cdot b'$. Thus the Galois representations $\rho_\ell^{s_0}$ and $\rho_\ell^{s_0'}$ are not isomorphic, so long as ℓ does not divide the product $aba'b'$. For any such ℓ, the ℓ-adic realizations of M_{s_0} and $M_{s_0'}$ are not isomorphic as Galois representations, which implies that the motives M_{s_0} and $M_{s_0'}$ are not isomorphic. This concludes the proof of (ii). Assertion (iii) is an immediate consequence of (ii). □

Remark 5 Let $\pi : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ denote the addition map. For $\mathcal{H}_i$, $i = 0, \ldots, 6$, and χ as above, the sheaf $\mathrm{MC}_\chi(\mathcal{H}_i)$ is the image of the !-*convolution*

$$\mathcal{H}_i *_! j_*(\mathcal{L}_\chi) = R\pi_!(\mathcal{H}_i \boxtimes j_*(\mathcal{L}_\chi))$$

in the $*$-convolution $R\pi_*(\mathcal{H}_i \boxtimes j_*(\mathcal{L}_\chi))$ under the forget supports map (cf. [5], Section 4.3). In the case at hand, it happens that each of the sheaves $\mathcal{H}_i$ has unipotent local monodromy at ∞ (in fact a single Jordan block of length $i+1$). It then results from [5], 2.9.4 part 3), that the canonical map

$$\mathcal{H}_i *_! j_*(\mathcal{L}_\chi) \longrightarrow \mathcal{H}_i *_{\rm mid} j_*(\mathcal{L}_\chi) = \mathrm{MC}_\chi(\mathcal{H}_i)$$

is an isomorphism. At each $\mathbb{F}_p$-rational point $t \in S_\ell(\mathbb{F}_p)$, one may then use the Grothendieck-Lefschetz Trace Formula to see that the trace of the Frobenius $\mathrm{Frob}_{p,t}$ at t on the stalk $(\mathcal{H}_i *_! j_*(\mathcal{L}_\chi))_{\bar t}$ is given by the convolution

$$f_i * f_2(t) := -\sum_{x \in \mathbb{F}_p} f_i(x) f_2(t-x), \quad i = 1, \ldots, 6, \tag{5}$$

where $f_i(x)$ gives the trace of $\mathrm{Frob}_{p,x}$ on $\mathcal{H}_i$ and $f_2(x)$ gives the trace of $\mathrm{Frob}_{p,x}$ on $\mathcal{L}_\chi$. Using standard computer algebra systems, like Mathematica, it is easy to derive from Formula (5) the trace of $\mathrm{Frob}_{p,t}$ (for small primes p) for the sequence $\tilde{\mathcal{H}}_0 = \mathcal{H}_0, \tilde{\mathcal{H}}_1, \ldots, \tilde{\mathcal{H}}_6$ of constructible sheaves which is defined as follows: the "middle tensor" operation

$$j_*(\mathcal{F}_{i+1} \otimes j^*(\mathrm{MC}_\chi(\mathcal{H}_{i-1}))), \quad i = 1, \ldots, 6, \tag{6}$$

on the right hand side of Formula (2) is replaced by literal tensor product

$$\tilde{\mathcal{H}}_i = \mathcal{F}_{i+1} \otimes (\tilde{\mathcal{H}}_{i-1} *_! j_*(\mathcal{L}_\chi)), \quad i = 1, \ldots, 6, \tag{7}$$

of $\mathcal{F}_{i+1}$ with the !-convolution $\tilde{\mathcal{H}}_{i-1} *_! j_*(\mathcal{L}_\chi)$. We derive the traces of the following Frobenius elements on $\tilde{\mathcal{H}}_6(3)$:

$\mathrm{Trace}(\mathrm{Frob}_{137,85})$	$\mathrm{Trace}(\mathrm{Frob}_{139,18})$	$\mathrm{Trace}(\mathrm{Frob}_{149,59})$	$\mathrm{Trace}(\mathrm{Frob}_{151,73})$
$2.88\ldots$	$3.59\ldots$	$3.51\ldots$	$3.03\ldots$

How well do these traces of Frobenii on $\tilde{\mathcal{H}}_6(3)$ approximate the traces of the same Frobenii on $\mathcal{H}_6(3)$? Although the canonical map $\mathcal{H}_i *_! j_*(\mathcal{L}_\chi) \to \mathcal{H}_i *_{\mathrm{mid}} j_*(\mathcal{L}_\chi) = \mathrm{MC}_\chi(\mathcal{H}_i)$ is an isomorphism at each stage, the middle tensor product in (6) may differ, by a δ-function at either 0 or 1, from the literal tensor product used in (7). Keeping careful track of these δ-functions and their progeny under later stages of the algorithmic construction of $\mathcal{H}_6(3)$ and $\tilde{\mathcal{H}}_6(3)$ leads to the conclusion that the largest error in computing traces at $\mathbb{F}_p$-points when working with !-convolution and literal tensoring instead of middle convolution and middle tensoring is bounded in absolute value by $\frac{8}{\sqrt{p}} + \frac{4}{p}$. Thus for $p > 100$, the largest error in trace at an $\mathbb{F}_p$-rational point of $\mathbb{A}^1 \setminus \{0,1\}$ is 0.84. So from the table above, we see that for each p listed, the trace of Frobenius on $\mathcal{H}_6(3)$ at the indicated $\mathbb{F}_p$-rational point does indeed lie in $]2,7]$.

References

[1] Y. André. *Une introduction aux motifs*. Panoramas et Synthèses 17. Soc. Math. de France, 2004.

[2] M. Aschbacher. Chevalley groups of type G_2 as the group of a trilinear form. *J. Algebra*, 109:193–259, 1987.

[3] G. Faltings. p-adic Hodge theory. *J. Am. Math. Soc.*, 1(1): 255–299, 1988.

[4] N.M. Katz. *Sommes Exponentielles*. Astérisque 79. Soc. Math. de France, 1980.

[5] N.M. Katz. *Rigid Local Systems*. Annals of Mathematics Studies 139. Princeton University Press, 1996.

[6] M. Kneser. Galois-Kohomologie halbeinfacher algebraischer Gruppen über p-adischen Körpern I, II. *Math. Zeitschrift 88 and 89*, pages 259–276 and 250–272, 1965.

[7] A. Scholl. On some ℓ-adic representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ attached to noncongruence subgroups. *Bull. London Math. Soc.*, 38:561–567, 2006.

[8] J.-P. Serre. Cohomologie galoisienne: progrès et problèmes. In *Séminaire Bourbaki 1993/94*, pages 229–257. Astérisque 227, 1995.