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Witt Vectors and a Question of Rudnick and Waxman

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This is Part III of the paper "Witt vectors and a question of Keating and Rudnick" [12]. We prove equidistribution results for the L-functions attached to "super-even" characters of the group of truncated "big" Witt vectors, and for the L-functions attached to the twists of these characters by the quadratic character.

1 Introduction: The Basic Setting

We work over a finite field $k = \mathbb{F}_q$ of characteristic p inside a fixed algebraic closure $\overline{k}$, and fix an *odd* integer $n \geq 3$. We form the k-algebra

$$B := k[X]/(X^{n+1}).$$

Following Rudnick and Waxman, we say that a character

$$\Lambda : B^\times \to \mathbb{C}^\times$$

is "super-even" if it is trivial on the subgroup $B^\times_{\text{even}} := (k[X^2]/(X^{n+1}))^\times$ of $B^\times$.

If Λ is nontrivial and super-even, one defines its L-function $L(\mathbb{A}^1/k, \Lambda, T)$, a priori a formal power series, by

$$L(\mathbb{A}^1/k, \Lambda, T) := (1-T)^{-1} \prod_{\substack{P \text{ monic irreducible} \\ P(0) \neq 0}} (1 - \Lambda(P) T^{\deg P})^{-1},$$

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where the product is over all monic irreducible polynomials $P \in k[X]$ other than X. In fact it is a polynomial. For Λ primitive (see Section 2), it is a polynomial of degree $n-1$, and there is a unique conjugacy class $\theta_{k,\Lambda}$ in the compact symplectic group $\mathrm{USp}(n-1)$ such that

$$\det(1 - T\theta_{k,\Lambda}) = L(\mathbb{A}^1/k, \Lambda, T/\sqrt{q}).$$

The question of the distribution of the symplectic conjugacy classes $\theta_{k,\Lambda}$ attached to variable super-even characters arises in the work of Rudnick and Waxman on (the variance in) a function field analogue of Hecke's theorem that Gaussian primes are equidistributed in angular sectors.

We will show (Theorem 5.1) that for odd $n \geq 7$, in any sequence of finite fields k_i of cardinalities tending to ∞ , the collections of conjugacy classes

$$\{\theta_{k_i,\Lambda}\}_{\Lambda \text{ primitive super-even}}$$

become equidistributed in the space $\mathrm{USp}(n-1)^\#$ of conjugacy classes of $\mathrm{USp}(n-1)$ for its induced Haar measure. For $n = 3, 5$ we need to exclude certain small characteristics, see Section 5.

Our second set of results deals with equidistribution in orthogonal groups. When the field k has odd characteristic, there is a quadratic character χ_2 of $k^\times$, which induces a quadratic character χ_2 of $B^\times$ given by $f \mapsto \chi_2(f(0))$. Given a super-even primitive character Λ mod X^{n+1} as above, we form the L-function $L(\mathbb{G}_m/k, \chi_2\Lambda, T)$ and get an associated conjugacy class $\theta_{k,\chi_2\Lambda}$ in the compact orthogonal group $\mathrm{O}(n,\mathbb{R})$. A natural question, although one which does not (yet) have applications to function field analogues of classical number-theoretic results, is whether these orthogonal conjugacy classes are suitably equidistributed in the compact orthogonal group.

We show (Theorem 7.1) that for a fixed odd integer $n \geq 5$, in any sequence k_i of finite fields of odd cardinalities tending to infinity, the conjugacy classes

$$\{\theta_{k_i,\chi_2\Lambda}\}_{\Lambda \text{ primitive super-even}}$$

become equidistributed in the space $\mathrm{O}(n,\mathbb{R})^\#$ of conjugacy classes of $\mathrm{O}(n,\mathbb{R})$. The same result holds for $n = 3$ if we restrict the characteristics of the finite fields to be different from 5.

With these two results about symplectic and orthogonal equidistribution established, a natural question is what one can say about the joint distribution.

We also show (Theorem 8.1) that the classes $\theta_{k,\Lambda}$ and $\theta_{k,\chi_2\Lambda}$ are independent, in the following sense. Fix an odd integer $n \geq 5$. In any sequence k_i of finite fields of odd cardinalities tending to infinity, the collections of pairs of conjugacy classes

$$\{(\theta_{k_i,\Lambda}, \theta_{k_i,\chi_2\Lambda})\}_{\Lambda \text{ primitive super-even}}$$

become equidistributed in the space $\mathrm{USp}(n-1)^{\#} \times \mathrm{O}(n,\mathbb{R})^{\#}$ of conjugacy classes of the product $\mathrm{USp}(n-1) \times \mathrm{O}(n,\mathbb{R})$. The same result holds for $n = 3$ if we restrict the characteristics of the finite fields to be different from 5.

This last result does not yet have applications to function field analogues of classical number-theoretic results, but is an instance of a natural question having a nice answer.

2 The Situation in Odd Characteristic

Throughout this section, we suppose that k has odd characteristic p. Then $B^{\times}_{\text{even}}$ is the subgroup of $B^{\times}$ consisting of those elements which are invariant under $X \mapsto -X$.

Let us denote by $B^{\times}_{\text{odd}} \subset B^{\times}$ the subgroup of elements $f(X) \in B^{\times}$ with constant term 1 which satisfy $f(-X) = 1/f(X)$ in $B^{\times}$.

Lemma 2.1. (p odd) The product $B^{\times}_{\text{even}} \times B^{\times}_{\text{odd}}$ maps isomorphically to $B^{\times}$ by the map $(f, g) \mapsto fg$. □

Proof. We first note that this map is injective. For if $g = 1/f$, then g is both even and odd and hence $g(-X)$ is both $g(X)$ and $1/g(X)$. Thus $g^2 = 1$ in $B^{\times}$. But the subgroup of elements of $B^{\times}$ with constant term 1 is a p-group. By assumption p is odd, hence $g = 1$. To see that the map is surjective, note first that $B^{\times}_{\text{even}}$ contains the constants $k^{\times}$. So it suffices to show that the image contains every element of $B^{\times}$ with constant term 1. This last group being a p-group, it suffices that the image contains the square of every such element. This results from writing

$$h(X)^2 = [h(X)h(-X)][h(X)/h(-X)].$$ ■

Recall from [12, & 2] that the quotient group $B^{\times}/k^{\times}$ is, via the Artin–Hasse exponential, isomorphic to the product

$$\prod_{m \geq 1 \text{ prime to } p,\ m \leq n} W_{\ell(m,n)}(A),$$

with $\ell(m,n)$ the integer defined by

$$\ell(m,n) = 1 + \text{the largest integer } k \text{ such that } mp^k \le n.$$

Via this isomorphism, the quotient $B^\times/B^\times_{\text{even}} \cong B^\times_{\text{odd}}$ becomes the sub-product

$$\prod_{m\ge1 \text{ prime to } p,\ m\le n,\ m \text{ odd}} W_{\ell(m,n)}(A).$$

Under these isomorphisms, the map from $\mathbb{A}^1(k)$ to $B^\times/k^\times$, $t \mapsto 1 - tX$, becomes the map

$$1 - tX \mapsto \prod_{m\ge1 \text{ prime to } p,\ m\le n} (t^m, 0, \ldots, 0) \in W_{\ell(m,n)}(A),$$

and its projection to $B^\times_{\text{odd}}$ becomes the map

$$1 - tX \mapsto \prod_{m\ge1 \text{ prime to } p,\ m\le n,\ m \text{ odd}} (t^m, 0, \ldots, 0) \in W_{\ell(m,n)}(A).$$

Any super-even character takes values in the subfield $\mathbb{Q}(\mu_{p^\infty}) \subset \mathbb{C}$. We choose a prime number $\ell \ne p$, and an embedding of $\mathbb{Q}(\mu_{p^\infty}) \subset \overline{\mathbb{Q}_\ell}$. This allows us to view Λ as taking values in $\overline{\mathbb{Q}_\ell}^\times$, and will allow us to invoke ℓ-adic cohomology.

Corollary 2.2. (p odd) For Λ a super-even character of $B^\times$, and $\mathcal{L}_{\Lambda(1-tX)}$ the associated lisse rank one $\overline{\mathbb{Q}_\ell}$- sheaf on $\mathbb{A}^1/k$, we have

$$\mathcal{L}_{\Lambda^2(1-tX)} \cong \mathcal{L}_{\Lambda((1-tX)/(1+tX))}.$$ □

Proof. Indeed, we have

$$\begin{aligned}\Lambda^2(1-tX) &= \Lambda((1-tX)^2)\\ &= \Lambda([(1-tX)(1+tX)][(1-tX)/(1+tX)])\\ &= \Lambda((1-tX)/(1+tX)),\end{aligned}$$

the last equality because Λ is super-even. ■

Recall that a character Λ of $B^\times$ is called primitive if it is nontrivial on the subgroup $1 + kX^n$. The Swan conductor $Swan(\Lambda)$ of Λ is the largest integer $d \le n$ such that Λ is nontrivial on the subgroup $1 + kX^d$. One knows [12, Lemma 1.1] that the Swan conductor of Λ is equal to the Swan conductor at ∞ of the lisse, rank one sheaf $\mathcal{L}_{\Lambda(1-tX)}$ on the affine t-line.

When Λ is a nontrivial super-even character, its Swan conductor is an *odd* integer $1 \leq d \leq n$. Its L-function on $\mathbb{A}^1/k$ is given by

$$\det(1 - TFrob_k | H^1_c(\mathbb{A}^1 \otimes_k \overline{k}, \mathcal{L}_{\Lambda(1-tX)})),$$

a polynomial of degree $d-1$, which is "pure of weight one cf. [16]." In other words, it is of the form $\prod_{i=1}^{Swan(\Lambda)-1}(1-\beta_i T)$ with each β_i an algebraic integer all of whose complex absolute values are $\sqrt{q}$.

Lemma 2.3. (p odd) Suppose Λ is a nontrivial super-even character.

(1) The lisse sheaf $\mathcal{L}_{\Lambda(1-tX)}$ is isomorphic to its dual sheaf $\mathcal{L}_{\overline{\Lambda}(1-tX)}$; indeed it is the pullback $[t \mapsto -t]^{\star}(\mathcal{L}_{\overline{\Lambda}(1-tX)})$ of its dual.

(2) The resulting cup product pairing

$$H^1_c(\mathbb{A}^1 \otimes_k \overline{k}, \mathcal{L}_{\Lambda(1-tX)}) \times H^1_c(\mathbb{A}^1 \otimes_k \overline{k}, \mathcal{L}_{\Lambda(1-tX)})$$
$$\to H^2_c(\mathbb{A}^1 \otimes_k \overline{k}, \overline{\mathbb{Q}_\ell}) \cong \overline{\mathbb{Q}_\ell}(-1)$$

given by

$$(\alpha, \beta) \mapsto \alpha \cup [t \mapsto -t]^{\star}(\beta)$$

is a symplectic autoduality. □

Proof. As the group $B^{\times}_{\mathrm{odd}}$ is a p-group, its character group is a p-group, so every super-even character has a unique square root. So for [1] it suffices to treat the case of Λ^2, in which case the assertion is obvious from Corollary 2.2 above. For [2], we note first that both our $\mathcal{L}$'s are totally wildly ramified at ∞, so for each the forget supports map $H^1_c \to H^1$ is an isomorphism. Thus, the cup product pairing is an autoduality. Viewed inside the H^1 of $\mathcal{C}$, the cohomology group in question is the Λ-isotypical component of the H^1 of $\mathcal{C}$. The fact that the pairing is symplectic then results from the fact that cup-product is alternating on H^1 of $\mathcal{C}$; cf. [10, 3.10.1–2] for an argument of this type. ■

For Λ primitive and super-even, we define a conjugacy class $\theta_{k,\Lambda}$ in the compact symplectic group $\mathrm{USp}(n-1)$ in terms of its reversed characteristic polynomial by the formula

$$\det(1 - T\theta_{k,\Lambda}) = L(\mathbb{A}^1 \otimes_k \overline{k}, \mathcal{L}_{\Lambda(1-tX)})(T/\sqrt{q}).$$

We next recall how to realize these conjugacy classes in an algebro-geometric way. For each integer $r \geq 1$, choose a faithful character $\psi_r : W_r(\mathbb{F}_p) \cong \mathbb{Z}/p^r\mathbb{Z} \to \mu_{p^r}$. For convenience, choose these characters so that under the maps $x \mapsto px$ of $Z/p^r\mathbb{Z}$ to $Z/p^{r+1}\mathbb{Z}$, we have

$$\psi_r(x) = \psi_{r+1}(px).$$

[For example, take $\psi_r(x) := \exp(2\pi i x/p^r)$.]

Every character of $W_r(k)$ is of the form

$$w \mapsto \psi_r(\mathrm{Trace}_{W_r(k)/W_r(\mathbb{F}_p)}(aw))$$

for a unique $a \in W_r(k)$. We denote this character $\psi_{r,a}$.

A super-even character Λ of $B^\times$, under the isomorphism

$$B^\times_{\mathrm{odd}} \cong \prod_{m\geq 1 \text{ prime to } p,\ m\leq n,\ m \text{ odd}} W_{\ell(m,n)}(k),$$

becomes a character of $\prod_{m\geq 1 \text{ prime to } p,\ m\leq n,\ m \text{ odd}} W_{\ell(m,n)}(k)$, where it is of the form

$$(w(m))_m \mapsto \prod_m \psi_{\ell(m,n),a(m)}(w(m))$$

for uniquely defined elements $a(m) \in W_{\ell(m,n)}(k)$.

The lisse sheaf $\mathcal{L}_{\Lambda(1-tX)}$ on $\mathbb{A}^1/k$ thus becomes the tensor product

$$\mathcal{L}_{\Lambda(1-tX)} \cong \otimes_m \mathcal{L}_{\psi_{\ell(m,n)}(a(m)(t^m,0'\mathrm{s}))},$$

over the $m \geq 1$ prime to p, $m \leq n$, m odd.

Recall from [12, Lemma 3.2] the following lemma, which will be applied here to super-even characters Λ.

Lemma 2.4. (p odd) Write the odd integer $n = n_0 p^{r-1}$ with n_0 prime to p and $r \geq 1$. Then, we have the following results about a super-even character Λ of $B^\times$.

(1) We have $Swan_\infty(\otimes_m \mathcal{L}_{\psi_{\ell(m,n)}(a(m)(t^m,0'\mathrm{s}))}) = n$ if and only if the Witt vector $a(n_0) \in W_{\ell(n_0,n)}(k) = W_r(k)$ has its initial component $a(n_0)_0 \in k^\times$.

(2) We have $Swan_\infty(\mathcal{L}_{\Lambda(1-tX)}) = n$ if and only if Λ is a primitive super-even character of $B^\times$ □

We continue with our odd $n \geq 3$ written as $n = n_0 p^{r-1}$ with n_0 prime to p and $r \geq 1$. As explained above, the sheaves $\mathcal{L}_{\Lambda(1-tX)}$ with Λ primitive are exactly the sheaves

$$\otimes_m \mathcal{L}_{\psi_{\ell(m,n)}(a(m)(t^m, 0's))}$$

for which the Witt vector $a(n_0) \in W_{\ell(n_0,n)}(k) = W_r(k)$ has its initial component $a(n_0)_0 \in k^\times$. Let us denote by

$$W_r^\times \subset W_r$$

the open subscheme of W_r defined by the condition that the initial component a_0 be invertible.

Let us denote by $\mathbb{W}$ the product space $\prod_{m\geq 1 \text{ prime to } p,\ m\leq n,\ m \text{ odd}} W_{\ell(m,n)}$. Thus $\mathbb{W}$ is a unipotent group over $\mathbb{F}_p$, with $\mathbb{W}(k) = B^\times_{\text{odd}}$, whose k-valued points are the super-even characters of $B^\times$.

On the space $\mathbb{A}^1 \times_k \mathbb{W}$, with coordinates $(t, (a(m)_m)$, we have the lisse rank one $\overline{\mathbb{Q}_\ell}$-sheaf

$$\mathcal{L}_{\text{univ, odd}} := \otimes_m \mathcal{L}_{\psi_{\ell(m,n)}(a(m)(t^m, 0's))}.$$

Denoting by

$$pr_2 : \mathbb{A}^1 \times_k \mathbb{W} \to \mathbb{W}$$

the projection on the second factor, we form the sheaf

$$\mathcal{F}_{\text{univ, odd}} := R^1(pr_2)_!(\mathcal{L}_{\text{univ, odd}})$$

on $\mathbb{W}$. This is a sheaf of perverse origin in the sense of [8].

For E/k a finite extension, and $\Lambda_{((a(m))_m}$ a super-even nontrivial character of $(E[X]/(X^{n+1}))^\times$ given by a non-zero point $a = (a(m))_m \in \mathbb{W}(E)$, we have

$$\begin{aligned}
&\det(1 - TFrob_{E,((a(m))_m)} | \mathcal{F}_{\text{univ, odd}}) \\
&\quad = \det(1 - TFrob_E, H^1_c(\mathbb{A}^1 \otimes_k \overline{k}, \mathcal{L}_{\Lambda_{(a(m))_m}(1-tX)})) \\
&\quad = L(\mathbb{A}^1/E, \Lambda_{(a(m))_m})(T).
\end{aligned}$$

Let us denote by

$$\mathrm{Prim}_{n,\mathrm{odd}} \subset \prod_{m\geq 1 \text{ prime to } p,\ m\leq n,\ m \text{ odd}} W_{\ell(m,n)}$$

the open set defined by the condition that the n_0 component lie in $W_r^\times$. Exactly as in [12], we see that the restriction of $\mathcal{F}_{\text{univ, odd}}$ to $\mathrm{Prim}_{\mathrm{n,odd}}$ is lisse of rank $n-1$, pure of weight one. By Lemma 2.3 above, it is symplectically self-dual toward $\overline{\mathbb{Q}_\ell}(-1)$. Moreover, the Tate-twisted sheaf $\mathcal{F}_{\text{univ, odd}}(1/2)$, restricted to $\mathrm{Prim}_{\mathrm{n,odd}}$, is pure of weight zero and symplectically self-dual.

We now state an equicharacteristic version of our equidistribution theorem in odd characteristic.

Theorem 2.5. Suppose either

(1) $n \geq 3$ and $p \geq 7$
or
(2) $n \geq 7$ and $p \geq 3$
or
(3) $n = 3$ and $p = 3$
or
(4) $n = 5$ and $p = 3$ or $p = 5$.

The geometric and arithmetic monodromy groups of the lisse sheaf $\mathcal{F}_{\text{univ, odd}}(1/2)|\mathrm{Prim}_{n,\mathrm{odd}}$ are given by $G_{\mathrm{geom}} = G_{\mathrm{arith}} = \mathrm{Sp}(n-1)$. □

3 Analysis of the Situation in Characteristic 2 and a Variant Situation in Arbitrary Characteristic p

We work over a finite field $k = \mathbb{F}_q$ of arbitrary characteristic p inside a fixed algebraic closure $\overline{k}$, and fix an integer $n \geq 3$ which is prime to p. We choose a prime number $\ell \neq p$, and an embedding of $\mathbb{Q}(\mu_{p^n}) \subset \overline{\mathbb{Q}_\ell}$. We form the k-algebra

$$B := k[X]/(X^{n+1}).$$

Inside $B^\times$, we have the subgroup $B^\times_{p'\text{th powers}}$ consisting of p'th powers of elements of $B^\times$. Concretely, $B^\times_{p'\text{th powers}}$ is the image of $k[[X^p]]^\times$ in $B^\times$. When $p=2$, $B^\times_{p'\text{th powers}}$ is the subgroup $B^\times_{\text{even}}$.

A character

$$\Lambda : B^\times \to \mathbb{C}^\times$$

is trivial on the subgroup $B^\times_{p'\text{th powers}}$ of $B^\times$ if and only if $\Lambda^p = \mathbb{1}$.

Lemma 3.1. Via the Artin–Hasse exponential, the quotient group $B^\times / B^\times_{p'\text{th powers}}$ is isomorphic to the additive group consisting of all polynomials $f(X) = \sum_i a_m X^m$ in $k[X]$ such that

$$\text{degree}(f) \le n,\ a_0 = 0, a_m = 0 \text{ if } p|m. \qquad \square$$

Proof. The Artin–Hasse is the formal series, a priori in $1 + X\mathbb{Q}[[X]]$, defined by

$$AH(X) := \exp(-\sum_{n\ge 0} X^{p^n}/p^n) = 1 - X + \cdots .$$

The "miracle" is that in fact $AH(X)$ has p-integral coefficients, that is, it lies in $1 + X\mathbb{Z}_{(p)}[[X]]$.

For R any $\mathbb{Z}_{(p)}$ algebra, that is, any ring in which every prime number other than p is invertible, in particular for k, one knows that every element of the multiplicative group $1 + XR[[X]]$ has a unique representation as an infinite product

$$\prod_{m\ge 1 \text{ prime to } p,\ a\ge 0} AH(a_{mp^a} X^{mp^a})^{1/m}$$

with coefficients $a_{mp^a} \in R$.

In the quotient group $(1 + XR[[X]])/(1 + X^p R[[X^p]])$, the factors with $a \ge 1$ die, so every element in this quotient group is of the form

$$\prod_{m\ge 1 \text{ prime to } p} AH(a_m X^m)^{1/m}$$

for some choice of coefficients $a_m \in R$. The key observation is that for any two elements $a, b \in R$, we have

$$AH(aX)AH(bX)/AH((a+b)X) \in 1 + X^p R[[X^p]]).$$

To see this, we argue as follows. The quotient lies in $1 + XR[[X]]$. By reduction to the universal case (when R is the polynomial ring $\mathbb{Z}_{(p)}[a, b]$ in two variables a, b), it suffices to treat the case when R lies in a $\mathbb{Q}$-algebra, where we must show that only powers of X^p occur. It suffices to check this after extension of scalars from R to the $\mathbb{Q}$-algebra $R \otimes_{\mathbb{Z}} \mathbb{Q}$. So we reduce to the case when R is a $\mathbb{Q}$-algebra, in which case the assertion is obvious, as

$$AH(aX)AH(bX)/AH((a+b)X) = \exp(-\sum_{n\geq 1}(a^{p^n} + b^{p^n} - (a+b)^{p^n})X^{p^n}/p^n)$$

is visibly a series in X^p.

Thus, the map

$$\prod_{m\geq 1 \text{ prime to } p} R \to (1 + XR[[X]])/(1 + X^p R[[X^p]])$$

given by

$$(a_m)_m \mapsto \prod_{m\geq 1 \text{ prime to } p} AH(a_m X^m)^{1/m} \bmod 1 + X^p R[[X^p]]$$

is a surjective group homomorphism with source the additive group $\prod_{m\geq 1 \text{ prime to } p} R$. Truncating mod X^{n+1}, and taking $R = k$, we get a surjective homomorphism from the additive group consisting of all polynomials $f(X) = \sum_i a_m X^m$ in $k[X]$ such that

$$\text{degree}(f) \leq n,\ a_0 = 0, a_m = 0 \text{ if } p|m,$$

to $B^\times/B^\times_{p'\text{th powers}}$. This map is an isomorphism, because source and target have the same cardinality. ■

Let us denote by $\mathbb{W}[p]$ the additive groupscheme over $\mathbb{F}_p$ whose R-valued points are the Artin–Schreier reduced polynomials of degree $\leq n$ over R which are strongly odd [10, 3.10.4], that is, those polynomials $f(X) = \sum_i a_m X^m$ in $R[X]$ such that

$$\text{degree}(f) \leq n,\ a_0 = 0, a_m = 0 \text{ if either } p|m \text{ or } 2|m.$$

Let us denote by $B^\times_{\text{even},p'\text{th powers}}$ the subgroup of $B^\times$ generated by both $B^\times_{\text{even}}$ and $B^\times_{p'\text{th powers}}$.

Corollary 3.2. The quotient $B^\times/B^\times_{\text{even},p'\text{th powers}}$ is isomorphic to the additive group $\mathbb{W}[p](k)$. □

The group $\mathbb{W}[p](k)$ is its own Pontrayagin dual, by the pairing

$$(f,g) \mapsto \psi_1(\text{constant term of } f(X)g(1/X)).$$

For Λ a character of $B^\times/B^\times_{\text{even },p'\text{th powers}}$, the corresponding lisse, rank one sheaf $\mathcal{L}_\Lambda(1-tX)$ on $\mathbb{A}^1$ is of the form $\mathcal{L}_{\psi_1(f(t))}$ for a unique $f(t) \in k[t]$ which is strongly odd and Artin–Schreier reduced of degree $\leq n$. This Λ is primitive if and only if f has degree n. For such Λ, we define a conjugacy class $\theta_{k,\Lambda}$ in the compact symplectic group $\text{USp}(n-1)$ in terms of its reversed characteristic polynomial by the formula

$$\det(1 - T\theta_{k,\Lambda}) = L(\mathbb{A}^1 \otimes_k \overline{k}, \mathcal{L}_{\Lambda(1-tX)})(T/\sqrt{q}).$$

When $p = 2$, these are precisely the conjugacy classes attached to the super-even characters which are primitive.

On the product $\mathbb{A}^1 \times \mathbb{W}[p]$, with coordinates (t,f), we have the lisse, rank one Artin–Schreier sheaf

$$\mathcal{L}_{\text{univ,odd,AS}} := \mathcal{L}_{\psi_1(f(t))},$$

and the projection

$$pr_2 : \mathbb{A}^1 \times \mathbb{W}[p] \to \mathbb{W}[p].$$

We then define the sheaf $\mathcal{F}_{\text{univ,AS}}$ by

$$\mathcal{F}_{\text{univ,odd,AS}} := R^1(pr_2)_!(\mathcal{L}_{\text{univ,AS}}).$$

This is a sheaf of perverse origin on $\mathbb{W}[p]$.

On the open set $\text{Prim}_{n,\text{odd}}[p] \subset \mathbb{W}[p]$ where the coefficient a_n of X^n is invertible, $\mathcal{F}_{\text{univ,odd,AS}}$ is lisse of rank $n-1$, pure of weight one, and symplectically self-dual.

The following theorem is essentially proven in [10, 3.10.7], cf. the remark below.

Theorem 3.3. Fix an odd integer $n \geq 3$ which is prime to p. If either $n \geq 7$ or $p \geq 7$, the geometric and arithmetic monodromy groups of $\mathcal{F}_{\text{univ,odd,AS}}(1/2)|\text{Prim}_{n,\text{odd}}[p]$ are given by $G_{\text{geom}} = G_{arith} = \text{Sp}(n-1)$. □

Remark 3.4. We say "essentially" because in [10, 3.10.7], the parameter space $\mathcal{D}(1,n,odd)$ consists of all strictly odd polynomials of degree n; the requirement of being Artin–Schreier reduced is not imposed. When $p=2$, the Artin–Schreier reducedness is automatic, implied by strict oddness. When p is odd, $\mathcal{D}(1,n,odd)$ contains the image of the space of strongly odd polynomials of degree $\leq n/p$ under the map $g \mapsto g-g^p$, and is the product of $\mathrm{Prim}_{n,\mathrm{odd}}[p]$ with this subspace. But one knows that $\mathcal{L}_{\psi_1(f(t)+g(t)-g(t)^p)}$ is isomorphic to $\mathcal{L}_{\psi_1(f(t))}$. Thus, the universal $\mathcal{F}$ on $\mathcal{D}(1,n,odd)$ is the pullback of $\mathcal{F}_{\mathrm{univ,odd,AS}}|\mathrm{Prim}_{n,\mathrm{odd}}[p]$ by the "Artin–Schrier reduction" map of $\mathcal{D}(1,n,odd)$ on to $\mathrm{Prim}_{n,\mathrm{odd}}[p]$. □

4 Proof of Theorem 2.5

We have a priori inclusions $G_{\mathrm{geom}} \subset G_{arith} \subset Sp(n-1)$, so it suffices to show that $G_{\mathrm{geom}} = \mathrm{Sp}(n-1)$.

We first treat the case (Cases (1) and (2)) when either $n \geq 7$ or $p \geq 7$. In this case, we exploit the fact that if n is prime to p, then $\mathrm{Prim}_{n,\mathrm{odd,AS}}$ lies in $\mathrm{Prim}_{n,\mathrm{odd}}$, and the restriction of $\mathcal{F}_{\mathrm{univ,\ odd}}|\mathrm{Prim}_{n,\mathrm{odd}}$ to $\mathrm{Prim}_{n,odd,AS}$ is the sheaf $\mathcal{F}_{\mathrm{univ,\ odd,AS}}|\mathrm{Prim}_{n,\mathrm{odd,AS}}$.

Thus if n is prime to p, already a pullback of $\mathcal{F}_{\mathrm{univ,\ odd}}|\mathrm{Prim}_{n,\mathrm{odd}}$ has $G_{\mathrm{geom}} = \mathrm{Sp}(n-1)$.

We must now treat the case when $p|n$. Because n is odd, $p \geq 3$. We first apply the "low p-adic ordinal" argument of [12, Lemma 7.2.], which, when n and p are both odd, conveniently produces a super-even primitive character Λ whose $\mathbb{F}_p$-character sum has low p-adic ordinal. This insures that the Fourier Transform $\mathrm{NFT}(\mathcal{L}_\Lambda)$, which is the restriction of $\mathcal{F}_{\mathrm{univ,\ odd}}|\mathrm{Prim}_{n,\mathrm{odd}}$ to a line in $\mathrm{Prim}_{n,\mathrm{odd}}$, has a G_{geom} which is not finite. This $\mathrm{NFT}(\mathcal{L}_\Lambda)$ is an irreducible Airy sheaf in the sense of [15, 11.1], according to which it either has finite G_{geom}, or is Artin–Schrier induced, or is Lie irreducible. As $\mathrm{NFT}(\mathcal{L}_\Lambda)$ has rank $n-1$ prime to p, it cannot be Artin–Schrier induced. Therefore $\mathrm{NFT}(\mathcal{L}_\Lambda)$ is Lie-irreducible. According to [15, 11.6], its G^0_{geom} is either $\mathrm{Sp}(n-1)$ or $SL(n-1)$. As we have an a priori inclusion of its G_{geom} in $\mathrm{Sp}(n-1)$, $\mathrm{NFT}(\mathcal{L}_\Lambda)$ has $G_{\mathrm{geom}} = \mathrm{Sp}(n-1)$. So also in this case, already a pullback of $\mathcal{F}_{\mathrm{univ,\ odd}}|\mathrm{Prim}_{n,\mathrm{odd}}$ has $G_{\mathrm{geom}} = \mathrm{Sp}(n-1)$.

Suppose now that (n,p) is either $(3,3)$ or $(5,3)$ or $(5,5)$. In these cases,$n \geq p \geq 3$ and $\ell(1,n) = 2$, so the "low p-adic ordinal" argument of [12, Lemma 7.2.] again produces a super-even primitive character Λ whose $\mathbb{F}_p$-character sum has low p-adic ordinal. Again here $n-1$ is prime to p, and we conclude as in the previous paragraph.

This concludes the proof of Theorem 2.5.

5 The Target Theorem

Our goal is to prove the following equidistribution theorem. Endow the space $\mathrm{USp}(n-1)^{\#}$ of conjugacy classes of $\mathrm{USp}(n-1)$ with the direct image of the total mass one Haar measure on $\mathrm{USp}(n-1)$. Equidistribution in the theorem below is with respect to this measure.

Theorem 5.1. We have the following results.

(1) Fix an odd integer $n \geq 7$. In any sequence of finite fields k_i of (possibly varying) characteristics p_i, whose cardinalities q_i are archimedeanly increasing to ∞, the collections of conjugacy classes

$$\{\theta_{k_i,\Lambda}\}_{\Lambda \text{ primitive super-even}}$$

become equidistributed in $\mathrm{USp}(n-1)^{\#}$.

(2) For $n = 3$, we have the same result if every k_i has characteristic $p_i = 3$ or $p_i \geq 7$.

(3) For $n = 5$, we have the same result if every k_i has characteristic $p_i \geq 3$. □

Proof. Whenever p is an allowed characteristic, then by Theorem 3.3 for $p = 2$ and by Theorem 2.5 for odd p, the relevant monodromy groups are $G_{\mathrm{geom}} = G_{arith} = \mathrm{Sp}(n-1)$.

Fix the odd integer $n \geq 3$. By the Weyl criterion, it suffices show that for each irreducible nontrivial representation Ξ of $\mathrm{USp}(n-1)$, there exists a constant $C(\Xi)$ such that for any allowed characteristic p and any finite field k of characteristic p, we have the estimate

$$|\sum_{\Lambda \text{ super-even and primitive}} \mathrm{Trace}(\Xi(\theta_{k,\Lambda}))| \leq \#\mathrm{Prim}_{n,\mathrm{odd}}(k)C(\Xi)/\sqrt{\#k}.$$

For a given allowed characteristic p, Deligne's equidistribution theorem [1, 3.5.3], as explicated in [13, 9.2.6, part (2)], we can take

$$C(\Xi,p) := \sum_i h^i_c(\mathrm{Prim}_{n,\mathrm{odd}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}, \Xi(\mathcal{F}_{\mathrm{univ,odd}})).$$

This sum of Betti numbers is uniformly bounded as p varies. In fact, we have the following estimate.

Lemma 5.2. Fix an irreducible nontrivial representation Ξ of $\mathrm{USp}(n-1)$. Let $M \geq 1$ be an integer such that Ξ occurs in $std_{n-1}^{\otimes M}$. [For example, if the highest weight of Ξ is

$\sum_i r_i\omega_i$ in Bourbaki numbering, then ω_i occurs in $\Lambda^i(std_{n-1}) \subset std_{n-1}^{\otimes i}$, and so we may take $M := \sum_i ir_i$.] In characteristic $p > n$, we have the estimate

$$\sum_i h_c^i(\mathrm{Prim}_{n,\mathrm{odd}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}, \Xi(\mathcal{F}_{\mathrm{univ,odd}}))$$
$$\leq \sum_i h_c^i(\mathrm{Prim}_{n,\mathrm{odd}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}, \mathcal{F}_{\mathrm{univ,odd}}^{\otimes M})$$
$$\leq 3(n+2)^{M+1+(n+3)/2} \leq 3(n+2)^{M+n+1}. \qquad \square$$

Proof. The first asserted inequality is obvious, since $\Xi(\mathcal{F}_{\mathrm{univ,odd}})$ is a direct summand of $(\mathcal{F}_{\mathrm{univ,odd}}|\mathrm{Prim}_{n,\mathrm{odd}})^{\otimes M}$.

When $p > n$, the space $\mathbb{W}$ is the space of odd polynomials f of degree $\leq n$, the sheaf $\mathcal{L}_{\mathrm{univ,odd}}$ on $\mathbb{A}^1 \times \mathbb{W}$ with coordinates (t, f) is $\mathcal{L}_{\psi_1(f(t))}$, and $\mathcal{F}_{\mathrm{univ,odd}}$ on $\mathbb{W}$ is $R^1(pr_2)_!(\mathcal{L}_{\psi_1(f(t))})$. The space $\mathrm{Prim}_{n,\mathrm{odd}} \subset \mathbb{W}$ is the space of odd polynomials of degree n, that is, the open set of $\mathbb{W}$ where the coefficient a_n of $f = \sum_{i\ odd, i\leq n} a_i t^i$ is invertible. The key point is that over $\mathrm{Prim}_{n,\mathrm{odd}}$, the $R^i(pr_2)_!(\mathcal{L}_{\psi_1(f(t))})$ vanish for $i \neq 1$ (as one sees looking fiber by fiber). By the Kunneth formula [14, Exp. XVII, 5.4.3], the Mth tensor power of $\mathcal{F}_{\mathrm{univ,odd}}|\mathrm{Prim}_{n,\mathrm{odd}}$ is $R^M(pr_2)_!(\mathcal{L}_{\psi_1(f(t_1)+f(t_2)+\cdots+f(t_M))})$ for the projection of $\mathbb{A}^M \times \mathrm{Prim}_{n,\mathrm{odd}}$ on to $\mathrm{Prim}_{n,\mathrm{odd}}$, and the $R^i(pr_2)_!(\mathcal{L}_{\psi_1(f(t_1)+f(t^2)+\cdots+f(t^M))})$ vanish for $i \neq M$. [One might note that $f(t_1) + f(t_2) + \cdots + f(t_M)$ is, for each f, a Deligne polynomial [10, 3.5.8] of degree n in M variables.] So the cohomology groups which concern us are

$$H_c^i(\mathrm{Prim}_{n,\mathrm{odd}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}, \mathcal{F}_{\mathrm{univ,odd}}^{\otimes M})$$
$$= H_c^{i+M}(\mathbb{A}^M \times \mathrm{Prim}_{n,\mathrm{odd}}, \mathcal{L}_{\psi_1(f(t_1)+f(t_2)+\cdots+f(t_M))}).$$

Here the space $A^M \times \mathrm{Prim}_{n,\mathrm{odd}}$ is the open set in $\mathbb{A}^{M+(n+1)/2}$, coordinates $(t_1, \ldots, t_M, a_1, a_3, \ldots, a_n)$ where a_n is invertible, so defined in $\mathbb{A}^{M+1+(n+1)/2}$, with one new coordinate z, by one equation $za_n = 1$. The function $f(t_1)+f(t_2)+\cdots+f(t_M)$ is a polynomial in the $M+(n+1)/2$ variables the t_i and the a_j of degree $n+1$. The asserted estimate is then a special case of [7, Theorem 12]. ■

Here is another method, which avoids the problem of finding good bounds for the sum of the Betti numbers in large characteristic, but which itself only applies when $p > 2(n-1)+1$. As above, the primitive super-even Λ's give precisely the Artin–Schreier sheaves $\mathcal{L}_{\psi_1(f(t))}$ for f running over the strictly odd polynomials of degree n. Each of these sheaves has its Fourier Transform, call it

$$\mathcal{G}_f := \mathrm{NFT}(\mathcal{L}_{\psi_1(f(t))}),$$

lisse of rank $n-1$ on $\mathbb{A}^1$, with all ∞-slopes equal to $n/(n-1)$, and one knows [3, Theorem 19] that its G_{geom} is $\mathrm{Sp}(n-1)$. [In the reference [3, Theorem 19], the hypothesis is stated as $p > 2n+1$, but what is used is that $p > 2\mathrm{rank}(\mathcal{G}_f)+1$.] This $\mathcal{G}_f$ is just the restriction of $\mathcal{F}_{\mathrm{univ,odd}}$ to the line $a \mapsto f(t)+at$, and the restriction of $\Xi(\mathcal{F}_{\mathrm{univ,odd}})$ to this line is $\Xi(\mathcal{G}_f)$. Because $\mathcal{G}_f$ has $G_{\mathrm{geom}} = \mathrm{Sp}(n-1)$, and has all ∞-slopes $\le n/(n-1)$, we have the estimate

$$h^1_c(\mathbb{A}^1 \otimes \overline{\mathbb{F}_p}, \Xi(\mathcal{G}_f)) \le \dim(\Xi)/(n-1), \text{ other } h^i_c = 0,$$

cf. the proof of [12, 8.2]. Thus, we get

$$|\sum_{a\in k, \Lambda \cong f(t)+at} \mathrm{Trace}(\Xi(\theta_{k,\Lambda}))| \le (\dim(\Xi)/(n-1))\#k/\sqrt{\#k}.$$

Summing this estimate over equivalence classes of strictly odd f's of degree n (for the equivalence relation $f \cong g$ if $\deg(f-g) \le 1$), we get, in characteristic $p > 2(n-1)+1$, the estimate

$$|\sum_{\Lambda \text{ super-even and primitive}} \mathrm{Trace}(\Xi(\theta_{k,\Lambda}))|$$
$$\le \#\mathrm{Prim}_{n,\mathrm{odd}}(k)(\dim(\Xi)/(n-1))/\sqrt{\#k}.$$

Thus, we may take

$$C(\Xi) := \mathrm{Max}\,(\dim(\Xi)/(n-1), \mathrm{Max}_{\,p\le 2n-1,\text{ allowed}} C(\Xi,p)). \qquad \blacksquare$$

6 Twisting by the Quadratic Character

In this section, $k = \mathbb{F}_q$ is a finite field of odd characteristic, and $\chi_2 : k^\times \to \pm 1$ denotes the quadratic character, extended to k by $\chi_2(0) := 0$. We can view χ_2 as the character of $B^\times$ given by $f(X) \mapsto \chi_2(f(0))$.

For Λ any nontrivial super-even character of $B^\times$, the L-function

$$\det(1 - T\mathit{Frob}_k | H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\Lambda(1-tX)}))$$

is a polynomial of degree $\mathit{Swan}(\Lambda)$, which is pure of weight one. For any nontrivial additive character ψ of k, with Gauss sum

$$G(\psi,\chi_2) := \sum_{t\in k^\times} \psi(t)\chi_2(t),$$

the product

$$(-1/G(\psi, \chi_2))(-\sum_{t \in k^\times} \chi_2(t)\Lambda(1 - tX))$$

is easily checked to be real.

On the space $\mathbb{G}_m \times_k \mathbb{W}$, with coordinates $(t, (a(m)_m))$, we have the lisse rank one $\overline{\mathbb{Q}_\ell}$-sheaf

$$\mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\text{univ, odd}} := \mathcal{L}_{\chi_2(t)} \otimes \otimes_m \mathcal{L}_{\psi_{\ell(m,n)}(a(m)(t^m, 0's))}.$$

Denoting by

$$pr_2 : \mathbb{G}_m \times_k \mathbb{W} \to \mathbb{W}$$

the projection on the second factor, we form the sheaf

$$\mathcal{F}_{\text{univ,odd},\chi_2} := R^1(pr_2)_!(\mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\text{univ, odd}})$$

on $\mathbb{W}$. This is a sheaf of perverse origin in the sense of [8].

For E/k a finite extension, and $\Lambda_{(a(m))_m}$ a super-even nontrivial character of $(E[X]/(X^{n+1}))^\times$ given by a non-zero point $a = (a(m))_m \in \mathbb{W}(E)$, we have

$$\begin{aligned} &\det(1 - TFrob_{E,((a(m))_m)}|\mathcal{F}_{\text{univ,odd}\chi_2}) \\ &\quad = \det(1 - TFrob_E, H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\Lambda_{((a(m))_m)}(1-tX)})). \end{aligned}$$

The restriction of $\mathcal{F}_{\text{univ,odd},\chi_2}$ to $\text{Prim}_{n,\text{odd}}$ is lisse of rank n, pure of weight one. It is geometrically irreducible, because for any super-even primitive Λ, its restriction to a suitable line is $\text{NFT}(\mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\Lambda(1-tX)})$. The sheaf

$$\mathcal{F}_{\text{univ,odd},\chi_2}(-G(\psi, \chi_2))^{-\text{degree}}|\text{Prim}_{n,\text{odd}}$$

is thus geometrically irreducible, and pure of weight zero. Its trace function is $\mathbb{R}$-valued, so this sheaf is isomorphic to its dual. Since its rank is the odd integer n, the resulting autoduality must be orthogonal. Thus, the G_{geom} and G_{arith} of $\mathcal{F}_{\text{univ,odd},\chi_2}(-G(\psi, \chi_2))^{-\text{degree}}|\text{Prim}_{n,\text{odd}}$ have

$$G_{\text{geom}} \subset G_{\text{arith}} \subset O(n).$$

Lemma 6.1. $G_{\text{geom}} \not\subset SO(n)$. □

Proof. If G_{geom} were contained in $\mathrm{SO}(n)$, then $\det(\mathcal{F}_{\text{univ,odd},\chi_2}|\text{Prim}_{n,\text{odd}})$ would be geometrically constant. In particular, for any two primitive super-even characters Λ_0 and Λ_1 of $B^\times$, we would have

$$\det(Frob_k|H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\Lambda_0(1-tX)})) \\ = \det(Frob_k|H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\Lambda_1(1-tX)})).$$

Fix a primitive super-even Λ_0. Choose a nonsquare $a \in k^\times$, and take

$$\Lambda_1(1-tX) = \Lambda_0(1-atX).$$

[Concretely, if Λ_0 has "coordinates" $a(m)$, with

$$\mathcal{L}_{\Lambda(_0 1-tX)} \cong \otimes_m \mathcal{L}_{\psi_{\ell(m,n)}(a(m)(t^m, 0's))},$$

then Λ_1 has coordinates $\text{Teich}(a^m)a(m)$.]

We will show that the two determinants have opposite signs. The sums

$$-\sum_{t \in k^\times} \chi_2(t)\Lambda_0(1-atX)$$

and

$$-\sum_{t \in k^\times} \chi_2(t)\Lambda_0(1-tX)$$

have opposite signs; make the change of variable $t \mapsto t/a$ in the first sum, and remember that $\chi_2(a) = -1$. These sums over odd degree extensions of k continue to have opposite signs, while these sums over even degree extensions coincide. In terms of the eigenvalues $\alpha_i, i = 1, \ldots, n$ and $\beta, i = 1, \ldots, n$ of $Frob_k$ on the cohomology groups in question, this means precisely that for the Newton symmetric functions, we have

$$N_i(\alpha's) = (-1)^i N_i(\beta's)$$

for all $i \geq 1$. But

$$(-1)^i N_i(\beta's) = N_i(-\beta's).$$

Thus, the α's and the $-\beta$'s have the same Newton symmetric functions. As we are in $\overline{\mathbb{Q}_\ell}$, a field of characteristic zero, the α's and the $-\beta$'s have the same elementary symmetric functions, hence agree as sets with multiplicity. Since n is odd,

$$\prod_{j=1}^{n} \alpha_j = \prod_{j=1}^{n}(-\beta_j) = -\prod_{j=1}^{n} \beta_j.$$

Thus, the two determinants in question have opposite signs. ■

Theorem 6.2. Suppose either

(1) $n \geq 5$ and $p \geq 5$, or
(2) $n \geq 3$ and $p \geq 7$, or
(3) $n = 3$ and $p = 3$, or
(4) $n \geq 5$ and $p \geq 3$.

In short, $n \geq 3$ and p are odd, and $(n, p) \neq (3, 5)$.

Then $\mathcal{F}_{\text{univ,odd},\chi_2}(-G(\psi, \chi_2))^{-degree}|\text{Prim}_{n,\text{odd}}$ has

$$G_{\text{geom}} = G_{\text{arith}} = \text{O}(n). \qquad \square$$

Proof. From the inclusions

$$G_{\text{geom}} \subset G_{\text{arith}} \subset O(n),$$

it suffices to prove that $G_{\text{geom}} = \text{O}(n)$.

Suppose first that $p \geq 5$ and $n \geq 5$. For any super-even primitive Λ, we consider the lisse sheaf $\mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\Lambda(1-tX)} \otimes \mathcal{L}_{\psi_1(at+bt^3)}$ on $\mathbb{G}_m \times \mathbb{A}^2$ (parameters (t, a, b)), and its cohomology along the fibers

$$\mathcal{G}_\Lambda := R^1(pr_2)_!(\mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\Lambda(1-tX)} \otimes \mathcal{L}_{\psi_1(at+bt^3)}).$$

This $\mathcal{G}_\Lambda$ is the restriction of $\mathcal{F}_{\text{univ,odd},\chi_2}$ to an $\mathbb{A}^2$ in $\text{Prim}_{n,\text{odd}}$. The moment calculation of [6, pp. 115–119] or [10, 3.11.4] shows that $\mathcal{G}_\Lambda$ has fourth moment 3. As we have the a priori inclusion $G_{geom,\mathcal{G}_\Lambda} \subset O(n)$, Larsen's Alternative [6, p. 113] shows that either $G_{geom,\mathcal{G}_\Lambda}$ is finite, or it is $\text{SO}(n)$ or $\text{O}(n)$.

The group $G_{geom,\mathcal{G}_\Lambda}$ is a subgroup of G_{geom}. Thus if $G_{geom,\mathcal{G}_\Lambda}$ is not finite, then $G_{geom,\mathcal{G}_\Lambda}$ contains $\text{SO}(n)$, and hence G_{geom} contains $\text{SO}(n)$. By the previous lemma, we must have $G_{\text{geom}} = \text{O}(n)$.

It remains to show that there exists at least one super-even primitive Λ for which $G_{geom,\mathcal{G}_\Lambda}$ is not finite. If $G_{geom,\mathcal{G}_\Lambda}$ is always finite, then by the diophantine criterion [4, 8.14.6] for finiteness, for every finite extension E/k and for every super-even primitive character Λ of $(B \otimes_k E)^\times$, the sum

$$-\sum_{t \in E^\times} \chi_2(t)\Lambda(1 - tX)$$

is divisible by $\sqrt{\#E}$ as an algebraic integer. If this holds for all Λ, then the diophantine criterion, applied to $\mathcal{F}_{\text{univ,odd},\chi_2}|\text{Prim}_{n,\text{odd}}$, shows that G_{geom} is finite. However, $\mathcal{F}_{\text{univ,odd},\chi_2}$ is a sheaf of perverse origin. Restricting it to the subspace of super-even characters of conductor 5, it would result from [8] that we have finite G_{geom} in the $n = 5$ case.

For $p \geq 7$, one knows [9, 3.12] that $G_{geom,n=5}$ is not finite, indeed it contains SO(5). For $p = 5 = n$, we show that $G_{geom,n=5}$ is not finite by the “low ordinal" method. Take the character of conductor 5 given by $t \mapsto \psi_2(t, 0)$ (concretely, the character $t \mapsto \exp(2\pi i t^p/p^2)$ of the Heilbronn sum in the case p = 5). Then, the sum

$$-\sum_{t \in \mathbb{F}_5^\times} \chi_2(t)\psi_2(t, 0)$$

has $\text{ord}_p = 1/10 < 1/2$. Indeed, the Teichmuller representatives of $1, 2, 3, 4$ mod 25 are $1, 7, -7, -1$. Denote by ζ_{25} the primitive 25th root of unity which is the value $\psi_2(1, 0)$. Then minus our sum is

$$\begin{aligned}\zeta_{25} - \zeta_{25}^{7} - \zeta_{25}^{-7} + \zeta_{25}^{-1} &= \zeta_{25}(1 - \zeta_{25}^{6}) - \zeta_{25}^{-7}(1 - \zeta_{25}^{6}) \\ &= (\zeta_{25} - \zeta_{25}^{-7})(1 - \zeta_{25}^{6}) = -\zeta_{25}^{-7}(1 - \zeta_{25}^{8})(1 - \zeta_{25}^{6})\end{aligned}$$

is the product of two uniformizing parameters in $\mathbb{Z}_p[\zeta]$, each with $\text{ord}_p = 1/20$.

Suppose now $n = 3$ and $p \geq 7$. In this case, it is shown in [9, 3.7] that G_{geom} contains SO(3). In view of Lemma 6.1, we have $G_{\text{geom}} = O(3)$.

Suppose that $n = 3 = p$. It suffices to show that G_{geom} is not finite. For then the identity component G^0_{geom} is a nontrivial semisimple (because $\mathcal{F}_{\text{univ,odd},\chi_2}|\text{Prim}_{3,odd}$ is pure) connected subgroup of SO(3). The only such subgroup is SO(3) itself. Indeed, such a subgroup is the image of $SL(2)$ in a three-dimensional orthogonal representation, and the only such representation is $\text{Sym}^2(std_2)$, whose image is SO(3). We show that G_{geom} is not finite by the “low ordinal" argument. For ζ_9 the primitive ninth root of unity $\zeta_9 := \psi_2(1, 0)$, the sum

$$-\sum_{t \in \mathbb{F}_3^\times} \chi_2(t)\psi_2(t, 0) = -(\zeta_9 - \zeta_9^{-1}) = \zeta_9^{-1}(1 - \zeta_9^2)$$

is a uniformizing parameter of $\mathbb{Z}_3[\zeta_9]$, and has $\text{ord}_3 = 1/6 < 1/2$.

It remains only to treat the case $n \geq 5, p = 3$. Suppose first $n \geq 9$ and $p = 3$. Pick any super-even primitive Λ. we consider the lisse sheaf $\mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\Lambda(1-tX)} \otimes \mathcal{L}_{\psi_1(at+bt^5+ct^7)}$ on $\mathbb{G}_m \times \mathbb{A}^3$ (parameters (t, a, b, c)), and its cohomology along the fibers

$$\mathcal{G}_\Lambda := R^1(pr_2)_!(\mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\Lambda(1-tX)} \otimes \mathcal{L}_{\psi_1(at+bt^5+ct^7)}).$$

This $\mathcal{G}_\Lambda$ is the restriction of $\mathcal{F}_{\text{univ,odd},\chi_2}$ to an $\mathbb{A}^3$ in $\text{Prim}_{n,\text{odd}}$. The usual moment calculation, now using [10, 3.11.6A], shows that $\mathcal{G}_\Lambda$ has fourth moment 3. As we have the a priori inclusion $G_{geom,\mathcal{G}_\Lambda} \subset O(n)$, Larsen's Alternative [6, p. 113] shows that either $G_{geom,\mathcal{G}_\Lambda}$ is finite, or it is $\text{SO}(n)$ or $\text{O}(n)$. If $G_{geom,\mathcal{G}_\Lambda}$ is not finite, then the larger group G_{geom} contains $\text{SO}(n)$, so by Lemma 6.1 must be $\text{O}(n)$. If $G_{geom,\mathcal{G}_\Lambda}$ were finite for all super-even primitive Λ, then by the diophantine criterion G_{geom} would be finite. Because $\mathcal{F}_{\text{univ,odd},\chi_2}$ is a sheaf of perverse origin, restricting to the subspace of super-even characters of conductor 3, we would find that G_{geom} is finite in the $n = 3 = p$ case, contradiction.

If $n = 7$ and $p = 3$, we repeat the above argument with one important modification. For a given choice of super-even primitive Λ, there is exactly one value c_0 of c for which $\mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\Lambda(1-tX)} \otimes \mathcal{L}_{\psi_1(at+bt^5+ct^7)}$ has lower conductor. So we must work with this sheaf on the product of $\mathbb{G}_m$ with the open set of $\mathbb{A}^3$ where $c - c_0$ is invertible, and form its $R^1(pr_2)_!$, which is the restriction of $\mathcal{F}_{\text{univ,odd},\chi_2}$ to $\mathbb{A}^3[1/(c - c_0)]$. On the entire $\mathbb{A}^3$, the moment calculation would give fourth moment 3. One checks that the fact of omitting the hyperplane $c = c_0$ only changes the calculation by lower order terms, the point being that in $\mathbb{A}^4/\overline{\mathbb{F}_3}$ with coordinates (x, y, z, w), the subscheme defined by the two equations

$$x^5 + y^5 = z^5 + w^5, \ \ x^7 + y^7 = z^7 + w^7,$$

has codimension 2. Now repeat the argument of the previous paragraph.

Here is an alternate proof for the case $n = 7, p = 3$. Over $\mathbb{F}_3$,we first use the "low ordinal" argument. We have the character $\Lambda := \psi_1(t^7 - t^5)\psi_2(t, 0)$, whose sum

$$\begin{aligned} -\sum_{t \in \mathbb{F}_3^\times} \chi_2(t)\psi_1(t^7 - t^5)\psi_2(t, 0) &= -\psi_2(1, 0) + \psi_2(-1, 0) \\ &= -\zeta_9 + \zeta_9^{-1} = \zeta_9^{-1}(1 - \zeta_9^2) \end{aligned}$$

is a uniformizing parameter for $\mathbb{Z}_3[\zeta_9]$, whose $\text{ord}_3 = 1/6 < 1/2$. This shows that $\mathcal{G} := \text{NFT}(\mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\Lambda(1-tX)})$ has a $G_{geom,\mathcal{G}_\Lambda}$ which is not finite. Because the rank $n = 7$ is prime, its $G_{geom,\mathcal{G}}$ must therefore be Lie irreducible, cf. [9, 3.5].

Now consider the three-parameter (a,b,c) family of characters $\Lambda_{a,b,c} := \psi_1(t^7 + at^5 + bt) \otimes \psi_2(ct, 0)$. On $\mathbb{G}_m \times \mathbb{A}^3$ with coordinate (t,a,b,c) we have the lisse sheaf $\mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\Lambda_{a,b,c}(1-tX)}$, its $R^1(pr_2)_! := \mathcal{H}$ is the restriction of $\mathcal{F}_{\mathrm{univ,odd},\chi_2}$ to an $\mathbb{A}^3$ in $\mathrm{Prim}_{n,\mathrm{odd}}$, and its further restriction to the $\mathbb{A}^1$ defined by $a = -1, c = 1$ with parameter b is the sheaf $\mathcal{G}$ above. Therefore, the larger group $G_{geom,\mathcal{H}}$ must be Lie irreducible. By Gabber's theorem [4, 1.6] on prime-dimensional representations, the only possibilities for $G^0_{geom,\mathcal{H}}$ are SO(7) itself or G_2 or the image of $SL(2)$ in $\mathrm{Sym}^6(std_2)$, which we will denote $\mathrm{Sym}^6(SL(2))$. If we get SO(7), then G_{geom} contains SO(7), and so by Lemma 6.1 must be O(7).

We will show that $G^0_{geom,\mathcal{H}}$ is not $\mathrm{Sym}^6(SL(2))$ or G_2. We argue by contradiction. Our $\mathcal{H}$ is a lisse sheaf on $\mathbb{A}^3/\mathbb{F}_3$, with a determinant which is geometrically of order dividing 2. Hence, its determinant is geometrically constant. Moreover, the twisted sheaf $\mathcal{H}_{\mathrm{arith}} := \mathcal{H} \otimes (-G(\psi, \chi_2))^{-degree}$ has its $G_{arith,\mathcal{G}}$ in O(7), so its determinant, being geometrically constant, is either trivial or is $(-1)^{degree}$.

So over any even degree extension of $\mathbb{F}_3$, in particular over $\mathbb{F}_9$, our twisted sheaf $\mathcal{H}_{\mathrm{arith}}$ has $G_{arith,\mathcal{H}} \subset SO(7)$. If $G^0_{geom,\mathcal{H}}$ is one of the groups $\mathrm{Sym}^6(SL(2))$ or G_2, then $G_{arith,\mathcal{H}}$ lies in the normalizer of $\mathrm{Sym}^6(SL(2))$, respectively of G_2, in SO(7). But each of these groups is its own normalizer in SO(7). Therefore $G_{arith,\mathcal{H}}$ is either the group $\mathrm{Sym}^6(SL(2))$ or G_2. One knows that $\mathrm{Sym}^6(SL(2)) \subset G_2$, so we find an inclusion $G_{arith,\mathcal{G}} \subset G_2$. One knows that the traces of elements of the compact form UG_2 of G_2 lie in the interval $[-2, 7]$. So the traces of Frobenius on $\mathcal{H}_{\mathrm{arith}}$ at $\mathbb{F}_9$-points will all lie in the interval $[-2, 7]$. Concretely, these are the sums

$$(1/3) \sum_{t \in \mathbb{F}_9^\times} \chi_2(\mathrm{Norm}_{\mathbb{F}_9/\mathbb{F}_3}(t)) \psi_1(\mathrm{Trace}_{\mathbb{F}_9/\mathbb{F}_3}(t^7 + at^5 + bt)) \psi_2(\mathrm{Trace}_{W_2(\mathbb{F}_9)/W_2(\mathbb{F}_3)}(ct, 0)).$$

A machine calculation shows that at the point $(a = -1, b = 0, c = 1 + i)$, (i being either primitive fourth root of unity in $\mathbb{F}_9$), this trace is $-6.10607/3 = -2.03536$, contradiction. [Machine calculation also shows that at the point $(a = i, b = -1 - i, c = 1 + i)$ this trace is $-7.29086/3 = -2.43029$.]

If $n = 5$ and $p = 3$, the argument is quite similar. Over $\mathbb{F}_3$,we first use the "low ordinal" argument. We have the character $\Lambda := \psi_1(t^5)\psi_2(t, 0)$, whose sum

$$- \sum_{t \in \mathbb{F}_3^\times} \chi_2(t)\psi_1(t^5)\psi_2(t, 0) = -\psi_1(1)\psi_2(1\,0) + \psi_1(-1)\psi_2(-1, 0)$$
$$= -\zeta_3\zeta_9 + \zeta_3^{-1}\zeta_9^{-1} = \zeta_9^{-4} - \zeta_9^4 = \zeta_9^{-4}(1 - \zeta_9^8)$$

is a uniformizing parameter for $\mathbb{Z}_3[\zeta_9]$, whose $\mathrm{ord}_3 = 1/6 < 1/2$. This shows that $\mathcal{G} := \mathrm{NFT}(\mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\Lambda(1-tX)})$ has a $G_{geom,\mathcal{G}_\Lambda}$ which is not finite. Because the rank $n = 5$ is prime, its $G_{geom,\mathcal{G}}$ must therefore be Lie irreducible, cf. [9, 3.5]. Thus, $G^0_{geom,\mathcal{G}}$ is a connected semisimple group in an irreducible five-dimensional representation. By Gabber's theorem [4, 1.6] on prime-dimensional representations, the only possibilities for $G^0_{geom,\mathcal{G}}$ are SO(5) itself or the image of $SL(2)$ in $\mathrm{Sym}^4(std_2)$, which we will denote $\mathrm{Sym}^4(SL(2))$. If we get SO(5) for $\mathcal{G}_\Lambda$, then G_{geom} contains SO(5), and so by Lemma 6.1 must be O(5).

So it suffices to show that $G^0_{geom,\mathcal{G}}$ is not $\mathrm{Sym}^4(SL(2))$. We argue by contradiction. Our $\mathcal{G}$ is a lisse sheaf on $\mathbb{A}^1/\mathbb{F}_3$, with a determinant which is geometrically of order dividing 2. Hence its determinant is geometrically constant. Moreover, the twisted sheaf $\mathcal{G}_{\mathrm{arith}} := \mathcal{G} \otimes (-G(\psi, \chi_2))^{-degree}$ has its $G_{arith,\mathcal{G}}$ in O(5), so its determinant, being geometrically constant, is either trivial or is $(-1)^{degree}$.

So over any even degree extension of $\mathbb{F}_3$, in particular over $\mathbb{F}_9$, our twisted sheaf $\mathcal{G}_{\mathrm{arith}}$ has $G_{arith,\mathcal{G}} \subset SO(5)$. Therefore, $G_{arith,\mathcal{G}}$ lies in the normalizer of $\mathrm{Sym}^4(SL(2))$ in SO(5). But this normalizer is just $\mathrm{Sym}^4(SL(2))$ itself, and hence $G_{arith,\mathcal{G}}$ is the group $\mathrm{Sym}^4(SL(2))$. Therefore, the traces of Frobenius on $\mathcal{G}_{\mathrm{arith}}$ at $\mathbb{F}_9$-rational points are among the traces of elements of $SU(2)$ in $\mathrm{Sym}^4(std_2)$. For an element γ of $SU(2)$ with $\mathrm{Trace}(\gamma) = T$, its trace in $\mathrm{Sym}^4(std_2)$ is $1 - 3T^2 + T^4$. The minimum of this polynomial on the interval $[-2, 2]$ is $-5/4$.

The twisting factor over $\mathbb{F}_9$ is $-1/3$, so the sums, indexed by $a \in \mathbb{F}_9$,

$$(1/3) \sum_{t \in \mathbb{F}_9^\times} \chi_2(\mathrm{Norm}_{\mathbb{F}_9/\mathbb{F}_3}(t))\psi_1(\mathrm{Trace}_{\mathbb{F}_9/\mathbb{F}_3}(t^5 + at))\psi_2(\mathrm{Trace}_{W_2(\mathbb{F}_9)/W_2(\mathbb{F}_3)}(t, 0)),$$

must all lie in the interval $[-5/4, 5]$. We get a contradiction, because for $a = 1 + i$ (for i either primitive fourth root of unity in $\mathbb{F}_9$), machine calculation shows that this sum is $-4.75877048/3 = -1.58626$. ∎

7 Equidistribution for the Twists by the Quadratic Character

Fix an odd integer $n \geq 3$. For each finite field k of odd characteristic, and each primitive super-even character Λ of $(k[X]/(X^{n+1}))^\times$, the reversed characteristic polynomial

$$\det(1 - TFrob_k, H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\Lambda(1-tX)})/(-G(\psi, \chi_2))$$

is the reversed characteristic polynomial $\det(1 - T\theta_{k,\chi_2\Lambda})$ of a unique conjugacy class $\theta_{k,\chi_2\Lambda}$ of the compact orthogonal group $O(n, \mathbb{R})$. Because n is odd, the group $O(n)$ is the product $(\pm 1) \times \mathrm{SO}(n)$, the decomposition being

$$A = \det(A)\tilde{A};\ \tilde{A} := A/\det(A).$$

Conjugacy classes of $\mathrm{O}(n, \mathbb{R})$ have the same product decomposition

$$\theta_{k,\chi_2\Lambda} = \det(\theta_{k,\chi_2\Lambda})\tilde{\theta}_{k,\chi_2\Lambda},$$

with $\tilde{\theta}_{k,\chi_2\Lambda}$ a conjugacy class of $\mathrm{SO}(n, \mathbb{R})$.

Endow the space $\mathrm{O}(n, \mathbb{R})^{\#}$ of conjugacy classes of $\mathrm{O}(n, \mathbb{R})$ with the direct image of the total mass one Haar measure on $\mathrm{O}(n, \mathbb{R})$. Equidistribution in the theorem below is with respect to this measure.

Theorem 7.1. Fix an odd integer $n \geq 5$. In any sequence of finite fields k_i of (possibly varying) odd characteristics p_i, whose cardinalities q_i are archimedeanly increasing to ∞, the collections of conjugacy classes

$$\{\theta_{k_i,\chi_2\Lambda}\}_{\Lambda \text{ primitive super-even}}$$

become equidistributed in $\mathrm{O}(n, \mathbb{R})^{\#}$. We have the same result for $n = 3$ if we require that no p_i is 5. □

Proof. Fix the odd integer $n \geq 3$. Whenever p is an allowed characteristic, then by Theorem 6.2 the relevant monodromy groups are $G_{\mathrm{geom}} = G_{\mathrm{arith}} = \mathrm{O}(n)$.

By the Weyl criterion, it suffices show that for each irreducible nontrivial representation Ξ of $\mathrm{O}(n, \mathbb{R})$, there exists a constant $C(\Xi)$ such that for any allowed characteristic p and any finite field k of characteristic p, we have the estimate

$$\Big|\sum_{\Lambda \text{ super-even and primitive}} \mathrm{Trace}(\Xi(\theta_{k,\chi_2\Lambda}))\Big| \leq \#\mathrm{Prim}_{n,\mathrm{odd}}(k)C(\Xi)/\sqrt{\#k}.$$

The group $\mathrm{O}(n)$ is the product $(\pm 1) \times \mathrm{SO}(n)$, the decomposition being

$$A = (\det(A))(\det(A)A).$$

So the irreducible nontrivial representations Ξ are products $\det^a \times \Xi_0$ with a being 0 or 1 and Ξ_0 an irreducible representation of $\mathrm{SO}(n)$, such that either $a = 1$ or Ξ_0 is irreducible nontrivial. We have seen, in the proof of Lemma 6.1, that over a given finite field $k = \mathbb{F}_q$ of odd characteristic, the $q - 1$ pullbacks $[t \mapsto at]^{\star}(\Lambda(1 - tX))$ of a given super-even primitive character will give rise to the conjugacy class $\theta_{k,\chi_2\Lambda}$ exactly $(q-1)/2$ times, and to the conjugacy class $-\theta_{k,\chi_2\Lambda}$ exactly $(q-1)/2$ times. This shows that when the representation Ξ is of the form $\det \times \Xi_0$, then the sum

$$\sum_{\Lambda \text{ super-even and primitive}} \mathrm{Trace}(\Xi(\theta_{k,\chi_2\Lambda}))$$

vanishes identically. So we need only be concerned with the Weyl sums for irreducible nontrivial representations Ξ_0.

Thus, we have reduced the theorem to the following one.

Theorem 7.2. Fix an odd integer $n \geq 5$. In any sequence of finite fields k_i of (possibly varying) odd characteristics p_i, whose cardinalities q_i are archimedeanly increasing to ∞, the collections of conjugacy classes

$$\{\tilde{\theta}_{k_i,\chi_2\Lambda}\}_{\Lambda \text{ primitive super-even}}$$

become equidistributed in $\mathrm{SO}(n,\mathbb{R})^{\#}$. We have the same result for $n=3$ if we require that no p_i is 5. □

For a given allowed characteristic p, and an irreducible nontrivial representation Ξ of $\mathrm{SO}(n)$, Deligne's equidistribution theorem [1, 3.5.3], as explicated in [13, 9.2.6, part (2)], tells us we can take

$$C(\Xi,p) := \sum_i h_c^i(\mathrm{Prim}_{n,\mathrm{odd}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}, \Xi(\mathcal{F}_{\mathrm{univ,odd},\chi_2})).$$

This sum of Betti numbers is uniformly bounded as p varies. Indeed, we have the following lemma.

Lemma 7.3. Fix an irreducible nontrivial representation Ξ of $\mathrm{SO}(n)$. Choose an integer $M \geq 1$ such that Ξ occurs in $std_n^{\otimes M}$. For $p > n$, we have the estimate

$$\begin{aligned}
&\sum_i h_c^i(\mathrm{Prim}_{n,\mathrm{odd}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}, \Xi(\mathcal{F}_{\mathrm{univ,odd},\chi_2})) \\
&\quad \leq \sum_i h_c^i(\mathrm{Prim}_{n,\mathrm{odd}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}, \mathcal{F}_{\mathrm{univ,odd},\chi_2}^{\otimes M}) \\
&\quad \leq 3(n+3+M)^{(n+3)/2+M+1} \leq 3(n+3+M)^{n+M+1}.
\end{aligned}$$

□

Proof. The proof is similar to that of Lemma 5.2. For $p > n$, we again invoke the Kunneth formula and end up with isomorphisms

$$H_c^i(\mathrm{Prim}_{n,\mathrm{odd}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}, \mathcal{F}_{\mathrm{univ,odd},\chi_2}^{\otimes M}) \\ = H_c^{i+M}((\mathbb{A}^M \times \mathrm{Prim}_{n,\mathrm{odd}}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}, \mathcal{L}_{\chi_2(t_1 t_2 \ldots t_M)} \mathcal{L}_{\psi_1(f(t_1)+\ldots f(t_m))}).$$

The asserted estimate is then a special case of [7, Theorem 12]. ∎

We can also use the Fourier transform method in large characteristic, for any $n \neq 7$. If $p > n$, the primitive super-even Λ's give precisely the Artin–Schreier sheaves $\mathcal{L}_{\psi_1(f(t))}$ for f running over the strictly odd polynomials of degree n. For each of these, the Fourier transform

$$\mathcal{G}_f := \mathrm{NFT}(\mathcal{L}_{\chi_2(t)} \otimes sL_{\psi_1(f(t))})$$

is lisse of rank n and geometrically irreducible, hence Lie irreducible by [3, Proposition 5]. Its G_{geom} lies in $\mathrm{SO}(n)$. Its ∞-slopes are

$$\{0, n-1 \text{ slopes } n/(n-1)\}.$$

By [4, 7.1.1 and 7.2.7 (2)] there are (effective) non-zero integers $N_1(n-1)$ and $N_2(n-1)$ such that if p, in addition to being $> 2n+1$, does not divide the integer $2nN_1(n-1)N_2(n-1)$, then $G_{geom,\mathcal{G}_f}$ is either $\mathrm{SO}(n)$,or, if $n = 7$, possibly G_2. [It is this ambiguity which rules out the case $n = 7$.]

Because $\mathcal{G}_f$ has $G_{geom,\mathcal{G}_f} = \mathrm{SO}(n)$, and has all ∞-slopes $\leq n/(n-1)$, we have the estimate

$$h_c^1(\mathbb{A}^1 \otimes \overline{\mathbb{F}_p}, \Xi(\mathcal{G}_f)) \leq \dim(\Xi)/(n-1), \text{ other } h_c^i = 0,$$

cf. the proof of [12, 8.2]. Thus, we get

$$\Big|\sum_{a \in k, \Lambda \cong f(t)+at} \mathrm{Trace}(\Xi(\tilde{\theta}_{k,\Lambda}))\Big| \leq (\dim(\Xi)/(n-1))\#k/\sqrt{\#k}.$$

Summing this estimate over equivalence classes of strictly odd f's of degree n (for the equivalence relation $f \cong g$ if $\deg(f-g) \leq 1$), we get, in characteristic $p > 2n+1$, p not dividing $2nN_1(n-1)N_2(n-1)$, the estimate

$$|\sum_{\Lambda\ \text{super-even and primitive}} \mathrm{Trace}(\Xi(\tilde{\theta}_{k,\Lambda}))|$$
$$\leq \#\mathrm{Prim}_{n,\mathrm{odd}}(k)(\dim(\Xi)/(n-1))/\sqrt{\#k}.$$

Denote by Excep(n) the finite set of odd primes p either which are $\leq 2n+1$ or which divide $2nN_1(n-1)N_2(n-1)$. We may take

$$C(\Xi) := \mathrm{Max}\ (\dim(\Xi)/(n-1), \mathrm{Max}_{\ p\in\mathrm{Excep}(n)} C(\Xi, p)).$$ ■

Remark 7.4. In the case $n=7$ and $p \geq 17$, it is proven in [4, 9.1.1] that for any $a \neq 0$ and for $f = ax^7$, the sheaf $\mathcal{G}_f$ has $G_{geom,\mathcal{G}_f} = G_2$. We will show elsewhere that for p sufficiently large, we also have $G_{geom,\mathcal{G}_f} = G_2$ for any f of the form $ax^7 + abx^5 + ab^2(25/84)x^3$. It is plausible that these are the only such f. If that were the case, then the exceptions would be uniformly small enough (over $\mathbb{F}_q$, $q^2(q-1)$ out of $q^3(q-1)$ $\tilde{\theta}$'s in all) that we would get the same result for $n=7$ as for the other odd n, with all odd primes allowed. □

8 A Theorem of Joint Equidistribution

Theorem 8.1. Fix an odd integer $n \geq 5$. In any sequence of finite fields k_i of (possibly varying) odd characteristics p_i, whose cardinalities q_i are archimedeanly increasing to ∞, the collections of pairs of conjugacy classes

$$\{(\theta_{k_i,\Lambda}, \theta_{k_i,\chi_2\Lambda})\}_{\Lambda\ \text{primitive super-even}}$$

become equidistributed in the space $\mathrm{USp}(n-1)^{\#} \times \mathrm{O}(n,\mathbb{R})^{\#}$ of conjugacy classes in the product group $\mathrm{USp}(n-1) \times \mathrm{O}(n,\mathbb{R})$. We have the same result for $n=3$ if we require that no p_i is 5. □

Proof. We consider the direct sum sheaf

$$(\mathcal{F}_{\mathrm{univ,odd}} \oplus \mathcal{F}_{\mathrm{univ,odd},\chi_2})|\mathrm{Prim}_{n,\mathrm{odd}}.$$

The two factors have, respectively,

$$G_{\mathrm{geom}} = G_{\mathrm{arith}} = \mathrm{Sp}(n-1),\ \ G_{\mathrm{geom}} = G_{\mathrm{arith}} = \mathrm{O}(n)$$

in any odd characteristic p. So G_{geom} (respectively G_{arith}) for the direct sum is a subgroup of the product $\text{Sp}(n-1) \times \text{O}(n)$ which maps on to each factor.

Suppose first that n is neither 3 nor 5. Then, these two factors have no nontrivial quotients which are isomorphic. So by Goursat's lemma, the direct sum sheaf has

$$G_{\text{geom}} = G_{\text{arith}} = \text{Sp}(n-1) \times \text{O}(n)$$

in any odd characteristic p.

Let us temporarily admit that for $n = 5$, the direct sum sheaf also has

$$G_{\text{geom}} = G_{\text{arith}} = \text{Sp}(n-1) \times \text{O}(n)$$

in any odd characteristic p. Let us also admit that for $n = 3$ the direct sum sheaf has

$$G_{\text{geom}} = G_{\text{arith}} = \text{Sp}(n-1) \times \text{O}(n)$$

in any odd characteristic $p \neq 5$.

By the Weyl criterion, it suffices to show that for each irreducible nontrivial representation $\Pi \otimes \Xi$ of $\text{USp}(n-1) \times \text{O}(n, \mathbb{R})$, there exists a constant $C(\Pi \otimes \Xi)$ such that for any odd characteristic p and any finite field k of characteristic p, we have the estimate

$$\begin{aligned} &|\sum_{\Lambda \text{ super-even and primitive}} \text{Trace}(\Pi(\theta_{k,\Lambda}))\text{Trace}(\Xi(\theta_{k,\chi_2\Lambda}))| \\ &\quad \leq \#\text{Prim}_{n,\text{odd}}(k) C(\Pi \otimes \Xi)/\sqrt{\#k}. \end{aligned}$$

For $a \in k^\times$ a nonsquare, the effect of $\Lambda \mapsto [t \mapsto at]^*\Lambda$ is to leave $\theta_{k_i,\Lambda}$ unchanged, but to replace $\theta_{k_i,\chi_2\Lambda}$ by minus itself. So exactly as in the proof of Theorem 7.2 above, the Weyl sums vanish identically when the Ξ factor is of the form $\det \otimes \Xi_0$ for Ξ_0 a representation of $\text{SO}(n)$. So we need only be concerned with the Weyl sums for irreducible nontrivial representations of the form $\Pi \otimes \Xi_0$.

Thus, we have reduced the theorem to the following one.

Theorem 8.2. Fix an odd integer $n \geq 5$. In any sequence of finite fields k_i of (possibly varying) odd characteristics p_i, whose cardinalities q_i are archimedeanly increasing to ∞, the collections of pairs of conjugacy classes

$$\{(\theta_{k_i,\Lambda}, \tilde{\theta}_{k_i,\chi_2\Lambda})\}_{\Lambda \text{ primitive super-even}}$$

become equidistributed in the space $\mathrm{USp}(n-1)^\# \times \mathrm{SO}(n,\mathbb{R})^\#$ of conjugacy classes in the product group $\mathrm{USp}(n-1) \times \mathrm{SO}(n,\mathbb{R})$. We have the same result for $n = 3$ if we require that no p_i is 5. □

For a given odd characteristic p, and an irreducible nontrivial representation $\Pi \otimes \Xi$ of $\mathrm{Sp}(n-1) \times \mathrm{SO}(n)$, Deligne's equidistribution theorem [1, 3.5.3], as explicated in [13, 9.2.6, part (2)], we can take

$$C(\Pi \otimes \Xi, p) := \sum_i h_c^i(\mathrm{Prim}_{n,\mathrm{odd}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}, \Pi(\mathcal{F}_{\mathrm{univ,odd}}) \otimes \Xi(\mathcal{F}_{\mathrm{univ,odd},\chi_2})).$$

This sum of Betti numbers is uniformly bounded as p varies. Notice that if either Ξ, respectively Π, is trivial, then its partner Π, respectively Ξ, must be nontrivial, and the result is given by Lemma 5.2, respectively Lemma 7.3. So it suffices to prove the following lemma.

Lemma 8.3. Fix irreducible nontrivial representations Π of $\mathrm{USp}(n-1)$ and Ξ of $\mathrm{SO}(n,\mathbb{R})$. Choose integers $M_1 \geq 1$ and $M_2 \geq 1$ such that Π occurs in $std_{n-1}^{\otimes M_1}$ and such that Ξ occurs in $std_n^{\otimes M_2}$. Then, we have the estimate

$$\begin{aligned}
&\sum_i H_c^i(\mathrm{Prim}_{n,\mathrm{odd}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}, \Pi(\mathcal{F}_{\mathrm{univ,odd}}) \otimes \Xi(\mathcal{F}_{\mathrm{univ,odd},\chi_2})) \\
&\quad \leq \sum_i h_c^i(\mathrm{Prim}_{n,\mathrm{odd}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}, \mathcal{F}_{\mathrm{univ,odd}}^{\otimes M_1} \otimes \mathcal{F}_{\mathrm{univ,odd},\chi_2}^{\otimes M_2}) \\
&\quad \leq 3(n+3+M_2)^{(n+3)/2+1+M_1+M_2} \leq 3(n+3+M_2)^{n+1+M_1+M_2}.
\end{aligned}$$

□

Proof. The proof is similar to the proofs of Lemmas 5.2 and 7.3. For $p > n$, we invoke the Kunneth formula to obtain isomorphisms

$$\begin{aligned}
&H_c^i(\mathrm{Prim}_{n,\mathrm{odd}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}, \mathcal{F}_{\mathrm{univ,odd}}^{\otimes M_1} \otimes \mathcal{F}_{\mathrm{univ,odd},\chi_2}^{\otimes M_2}) \\
&\quad = H_c^{i+M_1+M_2}((\mathbb{A}^{M_1} \times \mathbb{A}^{M_2} \times \mathrm{Prim}_{n,\mathrm{odd}}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}, \mathcal{H})
\end{aligned}$$

for $\mathcal{H}$ the sheaf

$$\mathcal{L}_{\chi_2(s_1 \ldots s_{M_2})} \otimes \mathcal{L}_{\psi_1(f(t_1)+\cdots+f(t_{M_1})+f(s_1)+\ldots f(s_{M_2}))}.$$

The asserted estimate is a special case of [7, Theorem 12]. ■

For n not 5 or 7, we can also use the Fourier transform method. For $p > 2n+1$ and not dividing $2nN_1(n-1)N_2(n-1)$, we know that for Λ super-even primitive, $\mathcal{L}_{\Lambda(1-tX)}$

is precisely of the form $\mathcal{L}_{\psi_1(f(t))}$ for an odd polynomial f of degree n. We have seen above that the Fourier transforms

$$\mathcal{G}_f := \mathrm{NFT}(\mathcal{L}_{\psi_1(f(t))}) \otimes (\sqrt{q})^{-degree},$$
$$\mathcal{G}_{f,\chi_2} := \mathrm{NFT}(\mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\psi_1(f(t))}) \otimes (-G(\psi,\chi_2))^{-degree} \otimes \det$$

have

$$G_{geom,\mathbb{G}_f} = G_{arith,\mathbb{G}_f} = \mathrm{Sp}(n-1),$$

and

$$G_{geom,\mathcal{G}_{f,\chi_2}} = G_{arith,\mathcal{G}_{f,\chi_2}} = \mathrm{SO}(n).$$

Their direct sum $\mathcal{G}_f \oplus \mathcal{G}_{f,\chi_2}$ has $G_{\mathrm{geom}} = G_{\mathrm{arith}} = \mathrm{Sp}(n-1) \times \mathrm{SO}(n)$. Both $\mathcal{G}_f$ and $\mathcal{G}_{f,\chi_2}$ have all ∞-slopes $\leq n/(n-1)$, hence so does any tensor product

$$\Pi(\mathcal{G}_f) \otimes \Xi(\mathcal{G}_{f,\chi_2}).$$

So for any nontrivial irreducible representation $\Pi \otimes \Xi$ of $\mathrm{Sp}(n-1) \times \mathrm{SO}(n)$ we have the estimate

$$h^1_c(\mathbb{A}^1 \otimes \overline{\mathbb{F}_p}, \Pi(\mathbb{G}_f) \otimes \Xi(\mathcal{G}_{f,\chi_2})) \leq \dim(\Pi)\dim(\Xi)/(n-1), \text{ other } h^i_c = 0,$$

cf. the proof of [12, 8.2].

So we get the estimate

$$|\sum_{a\in k, \Lambda \cong f(t)+at} \mathrm{Trace}(\Pi(\tilde{\theta}_{k,\Lambda,\chi_2}))\mathrm{Trace}(\Xi(\tilde{\theta}_{k,\Lambda,\chi_2}))|$$
$$\leq (\dim(\Pi)\dim(\Xi)/(n-1))\#k/\sqrt{\#k}.$$

Summing this estimate over equivalence classes of strictly odd f's of degree n (for the equivalence relation $f \cong g$ if $\deg(f-g) \leq 1$), we get, in characteristic $p > 2n+1$, p not dividing $2nN_1(n-1)N_2(n-1)$, the estimate

$$|\sum_{\Lambda \text{ super-even and primitive}} \mathrm{Trace}(\Pi(\theta_{k,\Lambda}))\mathrm{Trace}(\Xi(\tilde{\theta}_{k,\chi_2\Lambda}))|$$
$$\leq \#\mathrm{Prim}_{n,\mathrm{odd}}(k)C(\Pi \otimes \Xi)/\sqrt{\#k}.$$

Thus for $n \geq 9$ we may take

$$C(\Pi \otimes \Xi) := \mathrm{Max}\ (\dim(\Pi)\dim(\Xi)/(n-1), \mathrm{Max}_{\ p \in Excep(n)} C(\Xi, p)).$$

[For n either 5 or 7, we do not know that every individual Fourier transform has the correct G_{geom}, hence their exclusion.] ■

9 Joint Equidistribution in the Case $n = 3$

The problem we must deal with in the $n = 3$ case is that the quotient $SL(2)/\pm 1$ is isomorphic to the quotient $\mathrm{O}(3)/\pm 1 \cong \mathrm{SO}(3)$, namely $\mathrm{SO}(3)$ is the image of the representation $\mathrm{Sym}^2(std_2)$ of $SL(2)$. We must rule out the possibility that the conjugacy classes

$$\{(\theta_{k,\Lambda}, \tilde{\theta}_{k,\chi_2\Lambda})\}_{\Lambda \text{ primitive super-even}}$$

are related by

$$\tilde{\theta}_{k,\chi_2\Lambda} = \mathrm{Sym}^2(\theta_{k,\Lambda}).$$

We begin with the case of characteristic $p = 3$. In this case, up to tensoring with an $\mathcal{L}_{\psi_1(tx)}$, the super-even primitive characters of conductor 3 correspond to the Artin–Schreier–Witt sheaves $\mathcal{L}_{\psi_2(ax,0)}$ for some invertible scalar a. By the obvious change of variable $x \mapsto x/a$, this reduces us to considering the Fourier transforms

$$\mathcal{F} := \mathrm{NFT}(\mathcal{L}_{\psi_2(x,0)}) \otimes (\sqrt{q})^{-degree},$$
$$\mathcal{G} := \mathrm{NFT}(\mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi_2(x,0)}) \otimes (-G(\psi, \chi_2))^{-degree} \otimes \det.$$

What we must show is that there is no geometric isomorphism between $\mathrm{Sym}^2(\mathcal{F})$ and $\mathcal{G}$. For then by Goursat's lemma, the G_{geom} for $\mathcal{F} \oplus \mathcal{G}$ will be the full product $SL(2) \times \mathrm{SO}(3)$, and *a fortiori* the G_{arith} will also be the full product.

If there were a geometric isomorphism between $\mathrm{Sym}^2(\mathcal{F})$ and $\mathcal{G}$, then $\mathrm{Hom}_{\pi_{(\mathbb{A}^1)}geom}(\mathrm{Sym}^2(\mathcal{F}), \mathcal{G})$ would be a one-dimensional (both objects are geometrically irreducible) representation of $\pi_1^{arith}/\pi_1^{geom} = \mathrm{Gal}(\overline{\mathbb{F}_3}/\mathbb{F}_3)$. In other words, for some scalar A, we would have an arithmetic isomorphism

$$\mathrm{Sym}^2(\mathcal{F}) \cong \mathcal{G} \otimes A^{degree}.$$

The scalar A would necessarily have $|A| = 1$ for any complex embedding $\overline{\mathbb{Q}_\ell} \subset \mathbb{C}$, because both $\mathcal{F}$ and $\mathcal{G}$ are pure of weight zero. In particular, for any finite extension $E/\mathbb{F}_3$ and any $t \in E$, and any complex embedding, we would have an equality of absolute values

$$|\mathrm{Trace}(Frob_{E,t}|\mathrm{Sym}^2(\mathcal{F}))| = |\mathrm{Trace}(Frob_{E,t}|\mathcal{G})|.$$

But already for $E = \mathbb{F}_3$ and $t = 0$, these absolute values are different. Write ζ_9 for $e^{2\pi i/9}$. The first is

$$|(1/3)\left(\sum_{x\in\mathbb{F}_3}\psi_2(x,0)\right)^2 - 1| = |(1/3)(1+\zeta_9+\zeta_9^{-1})^2 - 1| = 1.1371\ldots$$

The second, remembering that the Gauss sum has absolute value $\sqrt{3}$, is

$$\left|(1/\sqrt{3})\sum_{x\in\mathbb{F}_3^\times}\chi_2(x)\psi_2(x,0)\right| = \left|(1/\sqrt{3})(\zeta_9-\zeta_9^{-1})\right| = 0.74222\ldots$$

Suppose now that $p \geq 7$. In this case, the super-even primitive Λ's give precisely the Artin–Schreier sheaves $\mathcal{L}_{\psi_1(ax^3+bx)}$ with $(a,t) \in \mathbb{G}_m \times \mathbb{A}^1$. What we must show is that for any $a \neq 0$, the two lisse sheaves on $\mathbb{A}^1$ given by

$$\mathrm{Sym}^2(\mathrm{NFT}(\mathcal{L}_{\psi_1(ax^3)}))(1),$$
$$\mathrm{NFT}(\mathcal{L}_{\chi_2(x)}\otimes\mathcal{L}_{\psi_1(ax^3)})\otimes(-G(\psi_1,\chi_2))^{-\mathrm{degree}}\otimes\det,$$

are not geometrically isomorphic.

Because the question is geometric, we may assume that a is a cube, say $a = 1/\alpha^3$. Making the change of variable $x \mapsto \alpha x$, we reduce to treating the case when $a = 1$. Thus, we must show that

$$\mathcal{F} := \mathrm{Sym}^2(\mathrm{NFT}(\mathcal{L}_{\psi_1(x^3)}))$$

and

$$\mathcal{G} := \mathrm{NFT}(\mathcal{L}_{\chi_2(x)}\otimes\mathcal{L}_{\psi_1(x^3)})$$

are not geometrically isomorphic. For this, we will make use of information about Kloosterman sheaves and hypergeometric sheaves, especially [4, 9.3.2] and [11, 3.7].

We will denote by [3] the cubing map $x \mapsto x^3$. Notice that

$$\mathcal{L}_{\psi_1(x^3)} \cong [3]^\star(\mathcal{L}_{\psi_1(x)}),$$
$$\mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi_1(x^3)} \cong [3]^\star(\mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi_1(x)}).$$

According to [4, 9.3.2], applied with $d = 3$, we have geometric isomorphisms

$$\mathrm{NFT}([3]^\star(\mathcal{L}_{\psi_1(x)})) \cong [3]^\star([x \mapsto -x/27]^\star(\mathcal{K}l(!, \psi_1, \chi_3, \overline{\chi_3}))),$$
$$\mathrm{NFT}([3]^\star(\mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi_1(x)})) \cong [3]^\star([x \mapsto -x/27]^\star(\mathcal{H}yp(!, \psi_1, \mathbb{1}, \chi_3, \overline{\chi_3}; \chi_2))).$$

Here, χ_3 and $\overline{\chi_3}$ are the two Kummer characters of order 3. Thus

$$\mathcal{F} \cong\cong [3]^\star([x \mapsto -x/27]^\star(\mathrm{Sym}^2(\mathcal{K}l(!, \psi_1, \chi_3, \overline{\chi_3}))).$$

According to [11, 3.7], applied with $\rho = \chi_3$ we have a geometric isomorphism

$$\mathrm{Sym}^2(\mathcal{K}l(!, \psi_1, \chi_3, \overline{\chi_3})) \cong [x \mapsto 4x]^\star(\mathcal{H}yp(!, \psi_1, \mathbb{1}, \chi_3, \overline{\chi_3}; \chi_2)).$$

Thus we find

$$\mathcal{F} \cong [3]^\star([x \mapsto -x/27]^\star([x \mapsto 4x]^\star(\mathcal{H}yp(!, \psi_1, \mathbb{1}, \chi_3, \overline{\chi_3}; \chi_2)))),$$

that is,

$$\mathcal{F} \cong [3]^\star[x \mapsto -4x/27]^\star\mathcal{H}yp(!, \psi_1, \mathbb{1}, \chi_3, \overline{\chi_3}; \chi_2),$$

whereas

$$\mathcal{G} \cong [3]^\star[x \mapsto -x/27]^\star\mathcal{H}yp(!, \psi_1, \mathbb{1}, \chi_3, \overline{\chi_3}; \chi_2).$$

To show that $\mathcal{F}$ and $\mathcal{G}$ are not geometrically isomorphic, we argue by contradiction. If $\mathcal{F} \cong \mathcal{G}$, then we have a geometric isomorphism on $\mathbb{G}_m$,

$$[x \mapsto -4x/27]^\star\mathcal{H}yp(!, \psi_1, \mathbb{1}, \chi_3, \overline{\chi_3}; \chi_2) \cong [x \mapsto -x/27]^\star\mathcal{H}yp(!, \psi_1, \mathbb{1}, \chi_3, \overline{\chi_3}; \chi_2).$$

Indeed, if two geometrically irreducible lisse sheaves on $\mathbb{G}_m$ have isomorphic pullbacks by [3], then one is the tensor product of the other with either $\overline{\mathbb{Q}_\ell}$ or $\mathcal{L}_{\chi_3}$ or $\mathcal{L}_{\overline{\chi_3}}$. Of the three candidates, only tensoring with the constant sheaf preserves χ_2 as the tame part of local monodromy at ∞, cf. [4, 8.2.5]. Thus we have the asserted geometric isomorphism, whence a geometric isomorphism

$$[x \mapsto 4x]^\star \mathcal{H}yp(!, \psi_1, \mathbb{1}, \chi_3, \overline{\chi_3}; \chi_2) \cong \mathcal{H}yp(!, \psi_1, \mathbb{1}, \chi_3, \overline{\chi_3}; \chi_2).$$

By [4, 8.5.4], a hypergeometric sheaf is not isomorphic to any nontrivial multiplicative translate of itself. This is the desired contradiction.

Notice that in this $n = 3$ case, the "Fourier transform by Fourier transform" method works in every allowed characteristic $p \neq 5$, giving the constant

$$C(\Pi \otimes \Xi) := \dim(\Pi)\dim(\Xi)/2.$$

10 Joint Equidistribution in the Case $n = 5$

Here, the problem is that $\mathrm{Sp}(4)/\pm 1$ is isomorphic to the group SO(5). Indeed, SO(5) is the image of Sp(4) in its second fundamental representation $\Lambda^2(std_4)/\mathbb{1}$. What we must show is that for $n = 5$,

$$\Lambda^2(\mathcal{F}_{\mathrm{univ,odd}}(1/2))/\mathbb{1}$$

and

$$\mathcal{F}_{\mathrm{univ,odd},\chi_2} \otimes (-1/G(\psi_1, \chi_2))^{degree} \otimes \det$$

are not geometrically isomorphic in any odd characteristic p. The proof goes along the same lines as did the $n = 3$ case.

Notice first that both sides have $G_{\mathrm{geom}} = G_{\mathrm{arith}} = \mathrm{SO}(5)$, so if they are geometrically isomorphic then they are arithmetically isomorphic.

We first treat the case $p = 5$. Because p is 1 mod 4, the Gauss sum $G(\psi_1, \chi_2)$ is some square root of 5. So it suffices to show that for the particular super-even character corresponding to $\mathcal{L}_{\psi_2(t,0)}$,

$$\mathrm{Trace}(Frob_{\mathbb{F}_5} | \Lambda^2(H^1(\mathbb{A}^1 \otimes_{\mathbb{F}_5} \overline{\mathbb{F}_5}, \mathcal{L}_{\psi_2(t,0)})(1/\sqrt{5}))) - 1$$

is not equal to either of

$$\pm\mathrm{Trace}(Frob_{\mathbb{F}_5} | H^1(\mathbb{G}_m \otimes_{\mathbb{F}_5} \overline{\mathbb{F}_5}, \mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\psi_2(t,0)}))/\sqrt{5}.$$

Computer calculation shows that the first is $-1.123807...$, while the second is $\pm 1.033926...$

Suppose now that p is an odd prime other than 5. It suffices to show that the restrictions of the two sides to some subvariety of $\mathrm{Prim}_{n,\mathrm{odd}}$ are not geometrically isomorphic. We will show that the two lisse sheaves

$$\Lambda^2(\mathrm{NFT}(\mathcal{L}_{\psi_1(t^5)}))/\mathbb{1}$$

and

$$\mathrm{NFT}(\mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\psi_1(t^5)})$$

on $\mathbb{A}^1$ are not geometrically isomorphic when restricted to $\mathbb{G}_m$.

By [4, 9.3.2], we have a geometric isomorphism

$$\mathrm{NFT}(\mathcal{L}_{\psi_1(t^5)}) \cong [x \mapsto x^5]^\star[x \mapsto -x/5^5]^\star Kl(!, \psi_1; \rho_1, \rho_2, \rho_3, \rho_4),$$

for $\rho_1, \rho_2, \rho_3, \rho_4$ the four nontrivial multiplicative characters of order 5.

By [11, 8.6], we have a geometric isomorphism

$$\Lambda^2(Kl(!, \psi_1; \rho_1, \rho_2, \rho_3, \rho_4))/\mathbb{1} \cong [x \mapsto -4x]^\star Hyp(!, \psi_1; \mathbb{1}, \rho_1, \rho_2, \rho_3, \rho_4; \chi_2).$$

Thus,

$$\Lambda^2(\mathrm{NFT}(\mathcal{L}_{\psi_1(t^5)}))/\mathbb{1} \cong [x \mapsto x^5]^\star[x \mapsto -x/5^5]^\star[x \mapsto -4x]^\star Hyp(!, \psi_1; \mathbb{1}, \rho_1, \rho_2, \rho_3, \rho_4; \chi_2).$$

At the same time, by [4, 9.3.2], we have a geometric isomorphism

$$\mathrm{NFT}(\mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\psi_1(t^5)}) \cong [x \mapsto x^5]^\star[x \mapsto -x/5^5]^\star Hyp(!, \psi_1; \mathbb{1}, \rho_1, \rho_2, \rho_3, \rho_4; \chi_2).$$

So it suffices to show that the two lisse sheaves on $\mathbb{G}_m$ given by

$$[x \mapsto -x/5^5]^\star[x \mapsto -4x]^\star Hyp(!, \psi_1; \mathbb{1}, \rho_1, \rho_2, \rho_3, \rho_4; \chi_2)$$

and

$$[x \mapsto -x/5^5]^\star Hyp(!, \psi_1; \mathbb{1}, \rho_1, \rho_2, \rho_3, \rho_4; \chi_2)$$

do not become isomorphic after pullback by the fifth power map $x \mapsto x^5$. We argue by contradiction. As the sheaves are each geometrically irreducible, if their $x \mapsto x^5$ pullbacks are isomorphic, then one is obtained from the other by tensoring with an $\mathcal{L}_\rho$ for some character ρ of order dividing 5. As both sides have χ_2 as the tame part of their $I(\infty)$-representations, this ρ must be trivial. So we would find that the hypergeometric sheaf

$$[x \mapsto -x/5^5]^\star Hyp(!, \psi_1; \mathbb{1}, \rho_1, \rho_2, \rho_3, \rho_4; \chi_2)$$

is geometrically isomorphic to its multiplicative translate by -4. Because $p \neq 5$, this is a nontrivial multiplicative translation.This contradicts [4, 8.5.4], according to which a geometrically irreducible hypergeometric sheaf is not isomorphic to any nontrivial multiplicative translate of itself.

In this $n = 5$ case, we cannot (at present) apply the “Fourier transform by Fourier transform" method, because we have only analyzed the Fourier transform situation for the single input $\mathcal{L}_{\psi_1(t^5)}$, but not for other super-even primitive Λ's. Nor do we know for which such Λ's, if any, we will in fact have the exceptional isomorphism we ruled out for $\mathcal{L}_{\psi_1(t^5)}$.

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