Witt Vectors and a Question of Rudnick and Waxman

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This is Part III of the paper "Witt vectors and a question of Keating and Rudnick" [12]. We prove equidistribution results for the L-functions attached to "super-even" characters of the group of truncated "big" Witt vectors, and for the L-functions attached to the twists of these characters by the quadratic character.

1 Introduction: The Basic Setting

We work over a finite field $k = \mathbb{F}_q$ of characteristic $p$ inside a fixed algebraic closure $\overline{k}$, and fix an odd integer $n \geq 3$. We form the $k$-algebra

$$B := k[X]/(X^{n+1}).$$

Following Rudnick and Waxman, we say that a character

$$\Lambda : B^\times \to \mathbb{C}^\times$$

is "super-even" if it is trivial on the subgroup $B_{\text{even}}^\times := (k[X^2]/(X^{n+1}))^\times$ of $B^\times$.

If $\Lambda$ is nontrivial and super-even, one defines its L-function $L(\mathbb{A}^1/k, \Lambda, T)$, a priori a formal power series, by

$$L(\mathbb{A}^1/k, \Lambda, T) := (1 - T)^{-1} \prod_{P \text{ monic irreducible}} (1 - \Lambda(P)T^{\deg P})^{-1},$$

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where the product is over all monic irreducible polynomials \( P \in k[X] \) other than \( X \). In fact it is a polynomial. For \( \Lambda \) primitive (see Section 2), it is a polynomial of degree \( n - 1 \), and there is a unique conjugacy class \( \theta_{k, \Lambda} \) in the compact symplectic group \( \text{USp}(n - 1) \) such that

\[
\det(1 - T \theta_{k, \Lambda}) = L(k^1/k, \Lambda, T/\sqrt{q}).
\]

The question of the distribution of the symplectic conjugacy classes \( \theta_{k, \Lambda} \) attached to variable super-even characters arises in the work of Rudnick and Waxman on (the variance in) a function field analogue of Hecke’s theorem that Gaussian primes are equidistributed in angular sectors.

We will show (Theorem 5.1) that for odd \( n \geq 7 \), in any sequence of finite fields \( k_i \) of cardinalities tending to \( \infty \), the collections of conjugacy classes

\[
\{\theta_{k_i, \Lambda}\}_{\Lambda \text{ primitive super-even}}
\]

become equidistributed in the space \( \text{USp}(n - 1)^\# \) of conjugacy classes of \( \text{USp}(n - 1) \) for its induced Haar measure. For \( n = 3, 5 \) we need to exclude certain small characteristics, see Section 5.

Our second set of results deals with equidistribution in orthogonal groups. When the field \( k \) has odd characteristic, there is a quadratic character \( \chi_2 \) of \( k^\times \), which induces a quadratic character \( \chi_2 \) of \( B^\times \) given by \( f \mapsto \chi_2(f(0)) \). Given a super-even primitive character \( \Lambda \mod \chi^{n+1} \) as above, we form the L-function \( L(\mathbb{G}_m/k, \chi_2 \Lambda, T) \) and get an associated conjugacy class \( \theta_{k, \chi_2 \Lambda} \) in the compact orthogonal group \( O(n, \mathbb{R}) \). A natural question, although one which does not (yet) have applications to function field analogues of classical number-theoretic results, is whether these orthogonal conjugacy classes are suitably equidistributed in the compact orthogonal group.

We show (Theorem 7.1) that for a fixed odd integer \( n \geq 5 \), in any sequence \( k_i \) of finite fields of odd cardinalities tending to infinity, the conjugacy classes

\[
\{\theta_{k_i, \chi_2 \Lambda}\}_{\Lambda \text{ primitive super-even}}
\]

become equidistributed in the space \( O(n, \mathbb{R})^\# \) of conjugacy classes of \( O(n, \mathbb{R}) \). The same result holds for \( n = 3 \) if we restrict the characteristics of the finite fields to be different from 5.

With these two results about symplectic and orthogonal equidistribution established, a natural question is what one can say about the joint distribution.
We also show (Theorem 8.1) that the classes \( \theta_{k,\Lambda} \) and \( \theta_{k,\chi_2 \Lambda} \) are independent, in the following sense. Fix an odd integer \( n \geq 5 \). In any sequence \( k_i \) of finite fields of odd cardinalities tending to infinity, the collections of pairs of conjugacy classes

\[
\left\{ (\theta_{k_i,\Lambda}, \theta_{k_i,\chi_2 \Lambda}) \right\}_\Lambda \text{ primitive super–even}
\]

become equidistributed in the space \( \text{USp}(n - 1)^{\#} \times \text{O}(n, \mathbb{R})^{\#} \) of conjugacy classes of the product \( \text{USp}(n - 1) \times \text{O}(n, \mathbb{R}) \). The same result holds for \( n = 3 \) if we restrict the characteristics of the finite fields to be different from 5.

This last result does not yet have applications to function field analogues of classical number-theoretic results, but is an instance of a natural question having a nice answer.

2 The Situation in Odd Characteristic

Throughout this section, we suppose that \( k \) has odd characteristic \( p \). Then \( B_{\text{even}}^\times \) is the subgroup of \( B^\times \) consisting of those elements which are invariant under \( X \mapsto -X \).

Let us denote by \( B_{\text{odd}}^\times \subset B^\times \) the subgroup of elements \( f(X) \in B^\times \) with constant term 1 which satisfy \( f(-X) = 1/f(X) \) in \( B^\times \).

**Lemma 2.1.** (p odd) The product \( B_{\text{even}}^\times \times B_{\text{odd}}^\times \) maps isomorphically to \( B^\times \) by the map \((f, g) \mapsto fg\). \( \square \)

**Proof.** We first note that this map is injective. For if \( g = 1/f \), then \( g \) is both even and odd and hence \( g(-X) \) is both \( g(X) \) and \( 1/g(X) \). Thus \( g^2 = 1 \) in \( B^\times \). But the subgroup of elements of \( B^\times \) with constant term 1 is a \( p \)-group. By assumption \( p \) is odd, hence \( g = 1 \). To see that the map is surjective, note first that \( B_{\text{even}}^\times \) contains the constants \( k^\times \). So it suffices to show that the image contains every element of \( B^\times \) with constant term 1. This last group being a \( p \)-group, it suffices that the image contains the square of every such element. This results from writing

\[
h(X)^2 = [h(X)h(-X)][h(X)/h(-X)].
\]

Recall from [12, & 2] that the quotient group \( B^\times /k^\times \) is, via the Artin–Hasse exponential, isomorphic to the product

\[
\prod_{m \geq 1 \text{ prime to } p, \ m \leq n} W_{\ell(m,n)}(A),
\]
with \( \ell(m, n) \) the integer defined by

\[
\ell(m, n) = 1 + \text{the largest integer } k \text{ such that } mp^k \leq n.
\]

Via this isomorphism, the quotient \( B^\times / B^\times_{\text{even}} \cong B^\times_{\text{odd}} \) becomes the sub-product

\[
\prod_{m \geq 1 \text{ prime to } p, \, m \leq n, \, m \text{ odd}} W_{\ell(m, n)}(A).
\]

Under these isomorphisms, the map from \( \Lambda^1(k) \) to \( B^\times /k^\times \), \( t \mapsto 1 - tX \), becomes the map

\[
1 - tX \mapsto \prod_{m \geq 1 \text{ prime to } p, \, m \leq n} (t^m, 0, \ldots, 0) \in W_{\ell(m, n)}(A),
\]

and its projection to \( B^\times_{\text{odd}} \) becomes the map

\[
1 - tX \mapsto \prod_{m \geq 1 \text{ prime to } p, \, m \leq n, \, m \text{ odd}} (t^m, 0, \ldots, 0) \in W_{\ell(m, n)}(A).
\]

Any super-even character takes values in the subfield \( \mathbb{Q}(\mu_{p^\infty}) \subset \mathbb{C} \). We choose a prime number \( \ell \neq p \), and an embedding of \( \mathbb{Q}(\mu_{p^\infty}) \subset \overline{\mathbb{Q}_\ell} \). This allows us to view \( \Lambda \) as taking values in \( \overline{\mathbb{Q}_\ell}^\times \), and will allow us to invoke \( \ell \)-adic cohomology.

**Corollary 2.2.** (p odd) For \( \Lambda \) a super-even character of \( B^\times \), and \( L_{\Lambda(1-tX)} \) the associated lisse rank one \( \overline{\mathbb{Q}_\ell}^\times \)-sheaf on \( \Lambda^1/k \), we have

\[
L_{\Lambda^2(1-tX)} \cong L_{\Lambda((1-tX)/(1+tX))}.
\]

**Proof.** Indeed, we have

\[
\Lambda^2(1 - tX) = \Lambda((1 - tX)^2)
\]

\[
= \Lambda([(1-tX)(1+tX)]((1-tX)/(1+tX)))
\]

\[
= \Lambda((1-tX)/(1+tX)),
\]

the last equality because \( \Lambda \) is super-even. \qed

Recall that a character \( \Lambda \) of \( B^\times \) is called primitive if it is nontrivial on the subgroup \( 1 + kX^n \). The Swan conductor \( \text{Swan}(\Lambda) \) of \( \Lambda \) is the largest integer \( d \leq n \) such that \( \Lambda \) is nontrivial on the subgroup \( 1 + kX^d \). One knows [12, Lemma 1.1] that the Swan
conductor of $\Lambda$ is equal to the Swan conductor at $\infty$ of the lisse, rank one sheaf $\mathcal{L}_{\Lambda(1-tX)}$ on the affine $t$-line.

When $\Lambda$ is a nontrivial super-even character, its Swan conductor is an odd integer $1 \leq d \leq n$. Its $L$-function on $\mathbb{A}^1/k$ is given by

$$\det(1 - T\text{Frob}_k|H^1_c(\mathbb{A}^1 \otimes_k \overline{k}, \mathcal{L}_{\Lambda(1-tX)})),$$

a polynomial of degree $d - 1$, which is “pure of weight one cf. [16].” In other words, it is of the form $\prod_{i=1}^{\text{Swan}(\Lambda)-1} (1 - \beta_i T)$ with each $\beta_i$ an algebraic integer all of whose complex absolute values are $\sqrt{q}$.

**Lemma 2.3.** (p odd) Suppose $\Lambda$ is a nontrivial super-even character.

1. The lisse sheaf $\mathcal{L}_{\Lambda(1-tX)}$ is isomorphic to its dual sheaf $\mathcal{L}_{\overline{\Lambda}(1-tX)}$; indeed it is the pullback $[t \mapsto -t]^*(\mathcal{L}_{\overline{\Lambda}(1-tX)})$ of its dual.
2. The resulting cup product pairing

$$H^1_c(\mathbb{A}^1 \otimes_k \overline{k}, \mathcal{L}_{\Lambda(1-tX)}) \times H^1_c(\mathbb{A}^1 \otimes_k \overline{k}, \mathcal{L}_{\Lambda(1-tX)})$$

$$\to H^2_c(\mathbb{A}^1 \otimes_k \overline{k}, \mathbb{Q}_l) \cong \mathbb{Q}_l(-1)$$

given by

$$(\alpha, \beta) \mapsto \alpha \cup [t \mapsto -t]^*(\beta)$$

is a symplectic autoduality. □

**Proof.** As the group $B_{\text{odd}}^\times$ is a $p$-group, its character group is a $p$-group, so every super-even character has a unique square root. So for [1] it suffices to treat the case of $\Lambda^2$, in which case the assertion is obvious from Corollary 2.2 above. For [2], we note first that both our $\mathcal{L}$’s are totally wildly ramified at $\infty$, so for each the forget supports map $H^1_c \to H^1$ is an isomorphism. Thus, the cup product pairing is an autoduality. Viewed inside the $H^1$ of $\mathcal{C}$, the cohomology group in question is the $\Lambda$-isotypical component of the $H^1$ of $\mathcal{C}$. The fact that the pairing is symplectic then results from the fact that cup-product is alternating on $H^1$ of $\mathcal{C}$; cf. [10, 3.10.1–2] for an argument of this type. □

For $\Lambda$ primitive and super-even, we define a conjugacy class $\theta_{k,\Lambda}$ in the compact symplectic group $\text{USp}(n-1)$ in terms of its reversed characteristic polynomial
by the formula

\[ \det(1 - T_{\theta,\Lambda}) = L(\mathbb{A}^1 \otimes_k \bar{k}, \mathcal{L}_{\Lambda(1-t\chi)})(T/\sqrt{q}). \]

We next recall how to realize these conjugacy classes in an algebro-geometric way. For each integer \( r \geq 1 \), choose a faithful character \( \psi_r : W_r(\mathbb{F}_p) \cong \mathbb{Z}/p^r\mathbb{Z} \to \mu_{p^r} \).

For convenience, choose these characters so that under the maps \( x \mapsto px \) of \( \mathbb{Z}/p^r\mathbb{Z} \) to \( \mathbb{Z}/p^{r+1}\mathbb{Z} \), we have

\[ \psi_r(x) = \psi_{r+1}(px). \]

[For example, take \( \psi_r(x) := \exp(2\pi i x/p^r) \).]

Every character of \( W_r(k) \) is of the form

\[ w \mapsto \psi_r(\text{Trace}_{W_r(k)/W_r(\mathbb{F}_p)}(aw)) \]

for a unique \( a \in W_r(k) \). We denote this character \( \psi_{r,a} \).

A super-even character \( \Lambda \) of \( B^\times \), under the isomorphism

\[ B_{\text{odd}}^\times \cong \prod_{m \geq 1 \text{ prime to } p, \ m \leq n, \ m \text{ odd}} W_{\ell(m,n)}(k), \]

becomes a character of \( \prod_{m \geq 1 \text{ prime to } p, \ m \leq n, \ m \text{ odd}} W_{\ell(m,n)}(k) \), where it is of the form

\[ (w(m))_m \mapsto \prod_{m} \psi_{\ell(m,n),a(m)}(w(m)) \]

for uniquely defined elements \( a(m) \in W_{\ell(m,n)}(k) \).

The lisse sheaf \( \mathcal{L}_{\Lambda(1-t\chi)} \) on \( \mathbb{A}^1/k \) thus becomes the tensor product

\[ \mathcal{L}_{\Lambda(1-t\chi)} \cong \otimes_m \mathcal{L}_{\psi_{\ell(m,n),a(m)}(t^m,0')}, \]

over the \( m \geq 1 \) prime to \( p, \ m \leq n, \ m \text{ odd} \).

Recall from [12, Lemma 3.2] the following lemma, which will be applied here to super-even characters \( \Lambda \).

**Lemma 2.4.** (p odd) Write the odd integer \( n = n_0p^{r-1} \) with \( n_0 \) prime to \( p \) and \( r \geq 1 \). Then, we have the following results about a super-even character \( \Lambda \) of \( B^\times \).

1. We have \( \text{Swan}_\infty(\otimes_m \mathcal{L}_{\psi_{\ell(m,n),a(m)}(t^m,0')}) = n \) if and only if the Witt vector \( a(n_0) \in W_{\ell(n_0,n)}(k) = W_r(k) \) has its initial component \( a(n_0)_0 \in k^\times \).
(2) We have $\text{Swan}_\infty(\mathcal{L}_{\Lambda(1-tX)}) = n$ if and only if $\Lambda$ is a primitive super-even character of $B^\times$.

We continue with our odd $n \geq 3$ written as $n = n_0p^{r-1}$ with $n_0$ prime to $p$ and $r \geq 1$. As explained above, the sheaves $\mathcal{L}_{\Lambda(1-tX)}$ with $\Lambda$ primitive are exactly the sheaves

$$\bigotimes_m \mathcal{L}_{\phi(m,n)}(a(m)(t^m,0',s))$$

for which the Witt vector $a(n_0) \in W_{\ell(n_0,n)}(k) = W_r(k)$ has its initial component $a(n_0)_0 \in k^\times$. Let us denote by

$$W_r^\times \subset W_r$$

the open subscheme of $W_r$ defined by the condition that the initial component $a_0$ be invertible.

Let us denote by $\mathbb{W}$ the product space $\prod_{m \geq 1 \text{ prime to } p, m \leq n, m \text{ odd}} W_{\ell(m,n)}$. Thus $\mathbb{W}$ is a unipotent group over $\mathbb{F}_p$, with $\mathbb{W}(k) = B_{\text{odd}}^\times$, whose $k$-valued points are the super-even characters of $B^\times$.

On the space $\mathbb{A}^1 \times_k \mathbb{W}$, with coordinates $(t, (a(m)_m))$, we have the lisse rank one $\mathbb{Q}_\ell$-sheaf

$$\mathcal{L}_{\text{univ, odd}} := \bigotimes_m \mathcal{L}_{\phi(m,n)}(a(m)(t^m,0',s)).$$

Denoting by

$$pr_2 : \mathbb{A}^1 \times_k \mathbb{W} \to \mathbb{W}$$

the projection on the second factor, we form the sheaf

$$\mathcal{F}_{\text{univ, odd}} := R^1(pr_2)_!(\mathcal{L}_{\text{univ, odd}})$$

on $\mathbb{W}$. This is a sheaf of perverse origin in the sense of [8].

For $E/k$ a finite extension, and $\Lambda_{(a(m))_m}$ a super-even nontrivial character of $(E[X]/(X^{n+1}))^\times$ given by a non-zero point $a = (a(m))_m \in \mathbb{W}(E)$, we have

$$\det(1 - TFrob_E, \mathcal{F}_{\text{univ, odd}}) \cdot \mathcal{F}_{\text{univ, odd}}$$

$$\quad = \det(1 - TFrob_E, H^1_c(\mathbb{A}^1 \otimes_k \overline{k}, \mathcal{L}_{\Lambda_{(a(m))_m}(1-tX)}))$$

$$\quad = L(\mathbb{A}^1/E, \Lambda_{(a(m))_m})(T).$$
Let us denote by 

\[
\text{Prim}_{n,\text{odd}} \subset \prod_{m \geq 1 \text{ prime to } p, \, m \leq n, \, m \text{ odd}} W_{(m,n)}
\]

the open set defined by the condition that the \(n_0\) component lie in \(W_r^\times\). Exactly as in [12], we see that the restriction of \(\mathcal{F}_{\text{univ, odd}}\) to \(\text{Prim}_{n,\text{odd}}\) is lisse of rank \(n - 1\), pure of weight one. By Lemma 2.3 above, it is symplectically self-dual toward \(\mathbb{Q}_\ell(-1)\). Moreover, the Tate-twisted sheaf \(\mathcal{F}_{\text{univ, odd}}(1/2)\), restricted to \(\text{Prim}_{n,\text{odd}}\), is pure of weight zero and symplectically self-dual.

We now state an equicharacteristic version of our equidistribution theorem in odd characteristic.

**Theorem 2.5.** Suppose either

1. \(n \geq 3\) and \(p \geq 7\)
   or
2. \(n \geq 7\) and \(p \geq 3\)
   or
3. \(n = 3\) and \(p = 3\)
   or
4. \(n = 5\) and \(p = 3\) or \(p = 5\).

The geometric and arithmetic monodromy groups of the lisse sheaf \(\mathcal{F}_{\text{univ, odd}}(1/2)|_{\text{Prim}_{n,\text{odd}}}\) are given by \(G_{\text{geom}} = G_{\text{arith}} = \text{Sp}(n - 1)\). \(\square\)

3 **Analysis of the Situation in Characteristic 2 and a Variant Situation in Arbitrary Characteristic**

We work over a finite field \(k = \mathbb{F}_q\) of arbitrary characteristic \(p\) inside a fixed algebraic closure \(\overline{k}\), and fix an integer \(n \geq 3\) which is prime to \(p\). We choose a prime number \(\ell \neq p\), and an embedding of \(\mathbb{Q}(\mu_{p^n}) \subset \mathbb{Q}_\ell\). We form the \(k\)-algebra

\[
B := k[X]/(X^{n+1})
\]

Inside \(B^\times\), we have the subgroup \(B_{p^\ell}^\times\) consisting of \(p^\ell\)th powers of elements of \(B^\times\). Concretely, \(B_{p^\ell}^\times\) is the image of \(k[[X^p]]^\times\) in \(B^\times\). When \(p = 2\), \(B_{p^\ell}^\times\) is the subgroup \(B_{\text{even}}^\times\)
A character

\[ \Lambda : B^\times \to \mathbb{C}^\times \]

is trivial on the subgroup \( B^\times_{p^\prime \text{th powers}} \) of \( B^\times \) if and only if \( \Lambda^p = 1 \).

**Lemma 3.1.** Via the Artin–Hasse exponential, the quotient group \( B^\times / B^\times_{p^\prime \text{th powers}} \) is isomorphic to the additive group consisting of all polynomials \( f(X) = \sum a_mX^m \) in \( k[X] \) such that

\[
\text{degree}(f) \leq n, \ a_0 = 0, a_m = 0 \text{ if } p \mid m. \]

**Proof.** The Artin–Hasse is the formal series, a priori in \( 1 + X \mathbb{Q}[[X]] \), defined by

\[
AH(X) := \exp(-\sum_{n \geq 0} X^{pn}/p^n) = 1 - X + \cdots.
\]

The “miracle” is that in fact \( AH(X) \) has \( p \)-integral coefficients, that is, it lies in \( 1 + X \mathbb{Z}_{(p)}[[X]] \).

For \( R \) any \( \mathbb{Z}_{(p)} \) algebra, that is, any ring in which every prime number other than \( p \) is invertible, in particular for \( k \), one knows that every element of the multiplicative group \( 1 + XR[[X]] \) has a unique representation as an infinite product

\[
\prod_{m \geq 1 \text{ prime to } p, \ a \geq 0} AH(a_{mp^a}X^{mp^a})^{1/m}
\]

with coefficients \( a_{mp^a} \in R \).

In the quotient group \( (1 + XR[[X]])/(1 + X^pR[[X^p]]) \), the factors with \( a \geq 1 \) die, so every element in this quotient group is of the form

\[
\prod_{m \geq 1 \text{ prime to } p} AH(a_mX^m)^{1/m}
\]

for some choice of coefficients \( a_m \in R \). The key observation is that for any two elements \( a, b \in R \), we have

\[
AH(aX)AH(bX)/AH((a + b)X) \in 1 + X^pR[[X^p]]).
\]

To see this, we argue as follows. The quotient lies in \( 1 + XR[[X]] \). By reduction to the universal case (when \( R \) is the polynomial ring \( \mathbb{Z}_{(p)}[a, b] \) in two variables \( a, b \), it suffices
to treat the case when $R$ lies in a $\mathbb{Q}$-algebra, where we must show that only powers of $X^p$ occur. It suffices to check this after extension of scalars from $R$ to the $\mathbb{Q}$-algebra $R \otimes_{\mathbb{Z}} \mathbb{Q}$. So we reduce to the case when $R$ is a $\mathbb{Q}$-algebra, in which case the assertion is obvious, as

$$AH(aX)AH(bX)/AH((a+b)X) = \exp(-\sum_{n \geq 1} (a^p/n + b^p/n - (a+b)^p/n)X^p/n/p^n)$$

is visibly a series in $X^p$.

Thus, the map

$$\prod_{m \geq 1 \text{ prime to } p} R \rightarrow (1 + XR[[X]])/(1 + X^pR[[X]])$$

given by

$$(a_m)_m \mapsto \prod_{m \geq 1 \text{ prime to } p} AH(a_mX^m)^{1/m} \mod 1 + X^pR[[X]]$$

is a surjective group homomorphism with source the additive group $\prod_{m \geq 1 \text{ prime to } p} R$. Truncating mod $X^{n+1}$, and taking $R = k$, we get a surjective homomorphism from the additive group consisting of all polynomials $f(X) = \sum_i a_mX^m$ in $k[X]$ such that

$$\text{degree}(f) \leq n, \ a_0 = 0, a_m = 0 \text{ if } p|m,$$

to $B^\times/B^\times_{p\text{'th powers}}$. This map is an isomorphism, because source and target have the same cardinality.

Let us denote by $\mathbb{W}[p]$ the additive groupscheme over $\mathbb{F}_p$ whose $R$-valued points are the Artin–Schreier reduced polynomials of degree $\leq n$ over $R$ which are strongly odd $[10, 3.10.4]$, that is, those polynomials $f(X) = \sum_i a_mX^m$ in $R[X]$ such that

$$\text{degree}(f) \leq n, \ a_0 = 0, a_m = 0 \text{ if either } p|m \text{ or } 2|m.$$

Let us denote by $B^\times_{\text{even}, p\text{'th powers}}$ the subgroup of $B^\times$ generated by both $B^\times_{\text{even}}$ and $B^\times_{p\text{'th powers}}$.

**Corollary 3.2.** The quotient $B^\times/B^\times_{\text{even}, p\text{'th powers}}$ is isomorphic to the additive group $\mathbb{W}[p](k)$. □
The group $\mathbb{W}p(k)$ is its own Pontryagin dual, by the pairing

$$(f, g) \mapsto \psi_1(\text{constant term of } f(X)g(1/X)).$$

For a character of $B^\times/B^\times_{\text{even}, p}$, the corresponding lisse, rank one sheaf $L_\Lambda(1-tX)$ on $\mathbb{A}^1$ is of the form $L_{\psi_1(f(t))}$ for a unique $f(t) \in k[t]$ which is strongly odd and Artin–Schreier reduced of degree $\leq n$. This $\Lambda$ is primitive if and only if $f$ has degree $n$. For such $\Lambda$, we define a conjugacy class $\theta_{k, \Lambda}$ in the compact symplectic group $\text{USp}(n-1)$ in terms of its reversed characteristic polynomial by the formula

$$\det(1-T\theta_{k, \Lambda}) = L(\mathbb{A}^1 \otimes_k \overline{k}, L_{\Lambda(1-tX)})(T/\sqrt{q}).$$

When $p = 2$, these are precisely the conjugacy classes attached to the super-even characters which are primitive.

On the product $\mathbb{A}^1 \times \mathbb{W}p$, with coordinates $(t, f)$, we have the lisse, rank one Artin–Schreier sheaf

$$L_{\text{univ,odd,AS}} := L_{\psi_1(f(t))},$$

and the projection

$$pr_2 : \mathbb{A}^1 \times \mathbb{W}p \to \mathbb{W}p.$$ 

We then define the sheaf $\mathcal{F}_{\text{univ,AS}}$ by

$$\mathcal{F}_{\text{univ,odd,AS}} := R^1(pr_2)_!(L_{\text{univ,AS}}).$$

This is a sheaf of perverse origin on $\mathbb{W}p$.

On the open set $\text{Prim}_{n, \text{odd}}[p] \subset \mathbb{W}p$ where the coefficient $a_n$ of $X^n$ is invertible, $\mathcal{F}_{\text{univ,odd,AS}}$ is lisse of rank $n-1$, pure of weight one, and symplectically self-dual.

The following theorem is essentially proven in [10, 3.10.7], cf. the remark below.

**Theorem 3.3.** Fix an odd integer $n \geq 3$ which is prime to $p$. If either $n \geq 7$ or $p \geq 7$, the geometric and arithmetic monodromy groups of $\mathcal{F}_{\text{univ,odd,AS}}(1/2)|_{\text{Prim}_{n, \text{odd}}[p]}$ are given by $G_{\text{geom}} = G_{\text{arith}} = \text{Sp}(n-1).$
Remark 3.4. We say “essentially” because in [10, 3.10.7], the parameter space $D(1,n,\text{odd})$ consists of all strictly odd polynomials of degree $n$; the requirement of being Artin–Schreier reduced is not imposed. When $p = 2$, the Artin–Schreier reducedness is automatic, implied by strict oddness. When $p$ is odd, $D(1,n,\text{odd})$ contains the image of the space of strongly odd polynomials of degree $\leq n/p$ under the map $g \mapsto g - g^p$, and is the product of $\text{Prim}_{n,\text{odd}}[p]$ with this subspace. But one knows that $\mathcal{L}_{\psi_1(f(t)) + g(t) - g(t)p}$ is isomorphic to $\mathcal{L}_{\psi_1(f(t))}$. Thus, the universal $\mathcal{F}$ on $D(1, n, \text{odd})$ is the pullback of $\mathcal{F}_{\text{univ, odd,AS}}|\text{Prim}_{n,\text{odd}}[p]$ by the “Artin–Schrier reduction” map of $D(1, n, \text{odd})$ on to $\text{Prim}_{n,\text{odd}}[p]$. □

4 Proof of Theorem 2.5

We have a priori inclusions $G_{\text{geom}} \subset G_{\text{arith}} \subset \text{Sp}(n-1)$, so it suffices to show that $G_{\text{geom}} = \text{Sp}(n-1)$.

We first treat the case (Cases (1) and (2)) when either $n \geq 7$ or $p \geq 7$. In this case, we exploit the fact that if $n$ is prime to $p$, then $\text{Prim}_{n,\text{odd,AS}}$ lies in $\text{Prim}_{n,\text{odd}}$, and the restriction of $\mathcal{F}_{\text{univ, odd}}|\text{Prim}_{n,\text{odd}}$ to $\text{Prim}_{n,\text{odd,AS}}$ is the sheaf $\mathcal{F}_{\text{univ, odd,AS}}|\text{Prim}_{n,\text{odd,AS}}$.

Thus if $n$ is prime to $p$, already a pullback of $\mathcal{F}_{\text{univ, odd}}|\text{Prim}_{n,\text{odd}}$ has $G_{\text{geom}} = \text{Sp}(n-1)$.

We must now treat the case when $p|n$. Because $n$ is odd, $p \geq 3$. We first apply the “low $p$-adic ordinal” argument of [12, Lemma 7.2.], which, when $n$ and $p$ are both odd, conveniently produces a super-even primitive character $\Lambda$ whose $\mathbb{F}_p$-character sum has low $p$-adic ordinal. This insures that the Fourier Transform $\text{NFT}(\mathcal{L}_\Lambda)$, which is the restriction of $\mathcal{F}_{\text{univ, odd}}|\text{Prim}_{n,\text{odd}}$ to a line in $\text{Prim}_{n,\text{odd}}$, has a $G_{\text{geom}}$ which is not finite. This $\text{NFT}(\mathcal{L}_\Lambda)$ is an irreducible Airy sheaf in the sense of [15, 11.1], according to which it either has finite $G_{\text{geom}}$, or is Artin–Schrier induced, or is Lie irreducible. As $\text{NFT}(\mathcal{L}_\Lambda)$ has rank $n-1$ prime to $p$, it cannot be Artin–Schrier induced. Therefore $\text{NFT}(\mathcal{L}_\Lambda)$ is Lie-irreducible. According to [15, 11.6], its $G_{\text{geom}}^0$ is either $\text{Sp}(n-1)$ or $\text{SL}(n-1)$. As we have an a priori inclusion of its $G_{\text{geom}}$ in $\text{Sp}(n-1)$, $\text{NFT}(\mathcal{L}_\Lambda)$ has $G_{\text{geom}} = \text{Sp}(n-1)$. So also in this case, already a pullback of $\mathcal{F}_{\text{univ, odd}}|\text{Prim}_{n,\text{odd}}$ has $G_{\text{geom}} = \text{Sp}(n-1)$.

Suppose now that $(n, p)$ is either $(3,3)$ or $(5,3)$ or $(5,5)$. In these cases, $n \geq p \geq 3$ and $\ell(1,n) = 2$, so the “low $p$-adic ordinal” argument of [12, Lemma 7.2.] again produces a super-even primitive character $\Lambda$ whose $\mathbb{F}_p$-character sum has low $p$-adic ordinal. Again here $n-1$ is prime to $p$, and we conclude as in the previous paragraph.

This concludes the proof of Theorem 2.5.
5 The Target Theorem

Our goal is to prove the following equidistribution theorem. Endow the space $\text{USp}(n-1)^*$ of conjugacy classes of $\text{USp}(n-1)$ with the direct image of the total mass one Haar measure on $\text{USp}(n-1)$. Equidistribution in the theorem below is with respect to this measure.

**Theorem 5.1.** We have the following results.

1. Fix an odd integer $n \geq 7$. In any sequence of finite fields $k_i$ of (possibly varying) characteristics $p_i$, whose cardinalities $q_i$ are archimedeanly increasing to $\infty$, the collections of conjugacy classes

   \[ \{ \theta_{k_i, \Lambda} \}_{\Lambda \text{ primitive super-even}} \]

   become equidistributed in $\text{USp}(n-1)^*$.

2. For $n = 3$, we have the same result if every $k_i$ has characteristic $p_i = 3$ or $p_i \geq 7$.

3. For $n = 5$, we have the same result if every $k_i$ has characteristic $p_i \geq 3$.  

**Proof.** Whenever $p$ is an allowed characteristic, then by Theorem 3.3 for $p = 2$ and by Theorem 2.5 for odd $p$, the relevant monodromy groups are $G_{\text{geom}} = G_{\text{arith}} = \text{Sp}(n-1)$.

Fix the odd integer $n \geq 3$. By the Weyl criterion, it suffices to show that for each irreducible nontrivial representation $\Xi$ of $\text{USp}(n-1)$, there exists a constant $C(\Xi)$ such that for any allowed characteristic $p$ and any finite field $k$ of characteristic $p$, we have the estimate

\[ |\sum_{\Lambda \text{ super-even and primitive}} \text{Trace}(\Xi(\theta_{k, \Lambda}))| \leq \#\text{Prim}_{n, \text{odd}}(k)C(\Xi)/\sqrt{#k}.\]

For a given allowed characteristic $p$, Deligne’s equidistribution theorem [1, 3.5.3], as explicated in [13, 9.2.6, part (2)], we can take

\[ C(\Xi, p) := \sum_i h^i_c(\text{Prim}_{n, \text{odd}} \otimes_{\text{Sp}} \mathbb{F}_p, \mathbb{F}_p, \Xi(\mathcal{F}_{\text{univ,odd}})). \]

This sum of Betti numbers is uniformly bounded as $p$ varies. In fact, we have the following estimate.

**Lemma 5.2.** Fix an irreducible nontrivial representation $\Xi$ of $\text{USp}(n-1)$. Let $M \geq 1$ be an integer such that $\Xi$ occurs in $\text{std}^\otimes M_{n-1}$. [For example, if the highest weight of $\Xi$ is
\[ \sum_i r_i \omega_i \text{ in Bourbaki numbering, then } \omega_i \text{ occurs in } \Lambda^i(\text{std}_{n-1}) \subset \text{std}_{n-1}^{\otimes i}, \text{ and so we may take } M := \sum_i i r_i. \] In characteristic \( p > n \), we have the estimate

\[ \sum_i h^i_c(\text{Prim}_{n, \text{odd}} \otimes \mathbb{F}_p, \mathcal{F}(\text{univ, odd})) \leq \sum_i h^i_c(\text{Prim}_{n, \text{odd}} \otimes \mathbb{F}_p, \mathcal{F}_{\text{univ, odd}}^{\otimes M}) \leq 3(n + 2)^{M+1+(n+3)/2} \leq 3(n + 2)^{M+1}. \]

**Proof.** The first asserted inequality is obvious, since \( \mathcal{E}(\text{univ, odd}) \) is a direct summand of \( (\mathcal{F}_{\text{univ, odd}}|_{\text{Prim}_{n, odd}})^{\otimes M} \).

When \( p > n \), the space \( \mathcal{W} \) is the space of odd polynomials \( f \) of degree \( \leq n \), the sheaf \( \mathcal{L}_{\text{univ, odd}} \) on \( \mathbb{A}^1 \times \mathcal{W} \) with coordinates \( (t, f) \) is \( \mathcal{L}_{\varphi_1(f(t))} \), and \( \mathcal{F}_{\text{univ, odd}} \) on \( \mathcal{W} \) is \( R^1(pr_2)_!(\mathcal{L}_{\varphi_1(f(t))}) \). The space \( \text{Prim}_{n, \text{odd}} \subset \mathcal{W} \) is the space of odd polynomials of degree \( n \), that is, the open set of \( \mathcal{W} \) where the coefficient \( a_n \) of \( f = \sum_{i \leq n} a_i t^i \) is invertible. The key point is that over \( \text{Prim}_{n, odd} \), the \( R^i(pr_2)_!(\mathcal{L}_{\varphi_1(f(t))}) \) vanish for \( i \neq 1 \) (as one sees looking fiber by fiber). By the Kunneth formula [14, Exp. XVII, 5.4.3], the \( M \)th tensor power of \( \mathcal{F}_{\text{univ, odd}}|_{\text{Prim}_{n, odd}} \) is \( R^M(pr_2)_!(\mathcal{L}_{\varphi_1(f(t_1)+f(t_2)+\cdots+f(t_M))}) \) for the projection of \( \mathbb{A}^M \times \text{Prim}_{n, odd} \) on to \( \text{Prim}_{n, odd} \), and the \( R^i(pr_2)_!(\mathcal{L}_{\varphi_1(f(t_1)+f(t_2)+\cdots+f(t_M))}) \) vanish for \( i \neq M \). [One might note that \( f(t_1) + f(t_2) + \cdots + f(t_M) \) is, for each \( f \), a Deligne polynomial [10, 3.5.8] of degree \( n \) in \( M \) variables.] So the cohomology groups which concern us are

\[ H^i_c(\text{Prim}_{n, odd} \otimes \mathbb{F}_p, \mathcal{F}_{\text{univ, odd}}^{\otimes M}) = H^{i+M}_{C}(\mathbb{A}^M \times \text{Prim}_{n, odd}, \mathcal{L}_{\varphi_1(f(t_1)+f(t_2)+\cdots+f(t_M))}). \]

Here the space \( \mathbb{A}^M \times \text{Prim}_{n, odd} \) is the open set in \( \mathbb{A}^{M+(n+1)/2} \), coordinates \( t_1, \ldots, t_M, a_1, a_3, \ldots, a_n \) where \( a_n \) is invertible, so defined in \( \mathbb{A}^{M+1+(n+1)/2} \), with one new coordinate \( z \), by one equation \( za_n = 1 \). The function \( f(t_1) + f(t_2) + \cdots + f(t_M) \) is a polynomial in the \( M+(n+1)/2 \) variables the \( t_i \) and the \( a_j \) of degree \( n + 1 \). The asserted estimate is then a special case of [7, Theorem 12].

Here is another method, which avoids the problem of finding good bounds for the sum of the Betti numbers in large characteristic, but which itself only applies when \( p > 2(n-1)+1 \). As above, the primitive super-even \( \Lambda \)'s give precisely the Artin–Schreier sheaves \( \mathcal{L}_{\varphi_1(f(t))} \) for \( f \) running over the strictly odd polynomials of degree \( n \). Each of these sheaves has its Fourier Transform, call it

\[ \mathcal{G}_f := \text{NFT}(\mathcal{L}_{\varphi_1(f(t))}), \]
lisse of rank $n - 1$ on $\mathbb{A}^1$, with all $\infty$-slopes equal to $n/(n-1)$, and one knows [3, Theorem 19] that its $G_{\text{geom}}$ is $\text{Sp}(n-1)$. [In the reference [3, Theorem 19], the hypothesis is stated as $p > 2n + 1$, but what is used is that $p > 2\text{rank}(\mathcal{G}_f) + 1$.] This $\mathcal{G}_f$ is just the restriction of $\mathcal{F}_{\text{univ,odd}}$ to the line $a \mapsto f(t) + at$, and the restriction of $\Xi(\mathcal{F}_{\text{univ,odd}})$ to this line is $\Xi(\mathcal{G}_f)$. Because $\mathcal{G}_f$ has $G_{\text{geom}} = \text{Sp}(n-1)$, and has all $\infty$-slopes $\leq n/(n-1)$, we have the estimate

$$h^1_c(\mathbb{A}^1 \otimes \overline{\mathbb{F}}_p, \Xi(\mathcal{G}_f)) \leq \dim(\Xi)/(n-1), \text{ other } h^i_c = 0,$$

cf. the proof of [12, 8.2]. Thus, we get

$$|\sum_{a \in k, \Lambda \equiv f(t) + at} \text{Trace}(\Xi(\theta_{k,\Lambda}))| \leq (\dim(\Xi)/(n-1))#k/\sqrt{#k}.$$ 

Summing this estimate over equivalence classes of strictly odd $f$'s of degree $n$ (for the equivalence relation $f \equiv g$ if $\text{deg}(f - g) \leq 1$), we get, in characteristic $p > 2(n-1) + 1$, the estimate

$$|\sum_{\Lambda \text{ super--even and primitive}} \text{Trace}(\Xi(\theta_{k,\Lambda}))| \leq \#\text{Prim}_{n,\text{odd}}(k)(\dim(\Xi)/(n-1))/\sqrt{#k}.$$

Thus, we may take

$$C(\Xi) := \max(\dim(\Xi)/(n-1), \max_{p \leq 2n-1, \text{ allowed}} C(\Xi, p)).$$

\section{Twisting by the Quadratic Character}

In this section, $k = \mathbb{F}_q$ is a finite field of odd characteristic, and $\chi_2 : k^\times \to \pm 1$ denotes the quadratic character, extended to $k$ by $\chi_2(0) := 0$. We can view $\chi_2$ as the character of $B^\times$ given by $f(X) \mapsto \chi_2(f(0))$.

For $\Lambda$ any nontrivial super-even character of $B^\times$, the $L$-function

$$\det(1 - T\text{Frob}_k|H^1_c(\mathbb{G}_m \otimes_k \overline{k}, \mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\Lambda(1-tX)}))$$

is a polynomial of degree $\text{Swan}(\Lambda)$, which is pure of weight one. For any nontrivial additive character $\psi$ of $k$, with Gauss sum

$$G(\psi, \chi_2) := \sum_{t \in k^\times} \psi(t)\chi_2(t),$$
the product

\[-1/G(\psi, \chi_2)(- \sum_{t \in k^\times} \chi_2(t) \Lambda(1-tX))\]

is easily checked to be real.

On the space $G_m \times_k W$, with coordinates $(t, (a(m)_m))$, we have the lisse rank one $\mathbb{Q}_l$-sheaf

$$L_{\chi_2(t)} \otimes L_{\text{univ, odd}} := L_{\chi_2(t)} \otimes L_{\psi((m)(m)(tm,0))}. $$

Denoting by

$$pr_2 : G_m \times_k W \to W$$

the projection on the second factor, we form the sheaf

$$\mathcal{F}_{\text{univ, odd}, \chi_2} := R^1(pr_2)_!(L_{\chi_2(t)} \otimes L_{\text{univ, odd}})$$

on $W$. This is a sheaf of perverse origin in the sense of [8].

For $E/k$ a finite extension, and $\Lambda_{(a(m)m)}$ a super-even nontrivial character of $(E[X]/(X^{n+1}))^\times$ given by a non-zero point $a = (a(m)m) \in W(E)$, we have

$$\det(1 - T Frob_{E,((a(m)m))} | \mathcal{F}_{\text{univ, odd}, \chi_2})$$

$$= \det(1 - T Frob_{E}, H^1(G_m \otimes k, L_{\chi_2(t)} \otimes L_{\Lambda((a(m)m)(1-\tauX))}).$$

The restriction of $\mathcal{F}_{\text{univ, odd}, \chi_2}$ to $\text{Prim}_{n, odd}$ is lisse of rank $n$, pure of weight one. It is geometrically irreducible, because for any super-even primitive $\Lambda$, its restriction to a suitable line is NFT($L_{\chi_2(t)} \otimes L_{\Lambda(1-tX)})$. The sheaf

$$\mathcal{F}_{\text{univ, odd}, \chi_2}(-G(\psi, \chi_2))^{\text{degree}} |_{\text{Prim}_{n, odd}}$$

is thus geometrically irreducible, and pure of weight zero. Its trace function is $\mathbb{R}$-valued, so this sheaf is isomorphic to its dual. Since its rank is the odd integer $n$, the resulting autoduality must be orthogonal. Thus, the $G_{\text{geom}}$ and $G_{\text{arith}}$ of $\mathcal{F}_{\text{univ, odd}, \chi_2}(-G(\psi, \chi_2))^{\text{degree}} |_{\text{Prim}_{n, odd}}$ have

$$G_{\text{geom}} \subset G_{\text{arith}} \subset O(n).$$

**Lemma 6.1.** $G_{\text{geom}} \not\subset SO(n).$
**Proof.** If $G_{\text{geom}}$ were contained in $\text{SO}(n)$, then $\det(F_{\text{univ,odd},n} | \text{Prim}_{n,\text{odd}})$ would be geometrically constant. In particular, for any two primitive super-even characters $\Lambda_0$ and $\Lambda_1$ of $B^\circ$, we would have

$$\det(F_{\text{rob}} | H^1_c(\mathbb{G}_m \otimes_k \kappa, L_{\chi_2(t)} \otimes L_{\Lambda_0(1-tX)})) = \det(F_{\text{rob}} | H^1_c(\mathbb{G}_m \otimes_k \kappa, L_{\chi_2(t)} \otimes L_{\Lambda_1(1-tX)})).$$

Fix a primitive super-even $\Lambda_0$. Choose a nonsquare $a \in k^\times$, and take

$$\Lambda_1(1-tX) = \Lambda_0(1-atX).$$

[Concretely, if $\Lambda_0$ has "coordinates" $a(m)$, with

$$L_{\Lambda_01-tX} \cong \otimes m L_{\psi(m,n,a(m))(tm,0)}(m) ,$$

then $\Lambda_1$ has coordinates $\text{Teich}(a^n)a(m)$.)

We will show that the two determinants have opposite signs. The sums

$$-\sum_{t \in k^\times} \chi_2(t) \Lambda_0(1-atX)$$

and

$$-\sum_{t \in k^\times} \chi_2(t) \Lambda_0(1-tX)$$

have opposite signs; make the change of variable $t \mapsto t/a$ in the first sum, and remember that $\chi_2(a) = -1$. These sums over odd degree extensions of $k$ continue to have opposite signs, while these sums over even degree extensions coincide. In terms of the eigenvalues $\alpha_i, i = 1, \ldots, n$ and $\beta, i = 1, \ldots, n$ of $F_{\text{rob}}$ on the cohomology groups in question, this means precisely that for the Newton symmetric functions, we have

$$N_i(\alpha's) = (-1)^i N_i(\beta's)$$

for all $i \geq 1$. But

$$(-1)^i N_i(\beta's) = N_i(-\beta's).$$

Thus, the $\alpha$'s and the $-\beta$'s have the same Newton symmetric functions. As we are in $\overline{\mathbb{Q}}$, a field of characteristic zero, the $\alpha$'s and the $-\beta$'s have the same elementary symmetric
functions, hence agree as sets with multiplicity. Since \( n \) is odd,
\[
\prod_{j=1}^{n} \alpha_j = \prod_{j=1}^{n} (-\beta_j) = -\prod_{j=1}^{n} \beta_j.
\]
Thus, the two determinants in question have opposite signs.

\[\square\]

**Theorem 6.2.** Suppose either

1. \( n \geq 5 \) and \( p \geq 5 \), or
2. \( n \geq 3 \) and \( p \geq 7 \), or
3. \( n = 3 \) and \( p = 3 \), or
4. \( n \geq 5 \) and \( p \geq 3 \).

In short, \( n \geq 3 \) and \( p \) are odd, and \((n,p) \neq (3,5)\).

Then \( F_{\text{univ,odd},\chi_2}(-G(\psi, \chi_2))^{\text{degree}}|_{\text{Prim}_n,\text{odd}} \) has
\[
G_{\text{geom}} = G_{\text{arith}} = O(n).
\]

**Proof.** From the inclusions
\[
G_{\text{geom}} \subset G_{\text{arith}} \subset O(n),
\]
it suffices to prove that \( G_{\text{geom}} = O(n) \).

Suppose first that \( p \geq 5 \) and \( n \geq 5 \). For any super-even primitive \( \Lambda \), we consider the lisse sheaf \( L_{\chi_2(t)} \otimes L_{\Lambda(1-tx)} \otimes L_{\psi_1(at+bt^3)} \) on \( \mathbb{G}_m \times \mathbb{A}^2 \) (parameters \((t,a,b)\)), and its cohomology along the fibers
\[
G_{\Lambda} := R^1(pr_2)_! (L_{\chi_2(t)} \otimes L_{\Lambda(1-tx)} \otimes L_{\psi_1(at+bt^3)}).
\]
This \( G_{\Lambda} \) is the restriction of \( F_{\text{univ,odd},\chi_2} \) to an \( \mathbb{A}^2 \) in \( \text{Prim}_{n,\text{odd}} \). The moment calculation of [6, pp. 115–119] or [10, 3.11.4] shows that \( G_{\Lambda} \) has fourth moment 3. As we have the a priori inclusion \( G_{\text{geom},\mathcal{G}_{\Lambda}} \subset O(n) \), Larsen’s Alternative [6, p. 113] shows that either \( G_{\text{geom},\mathcal{G}_{\Lambda}} \) is finite, or it is \( \text{SO}(n) \) or \( O(n) \).

The group \( G_{\text{geom},\mathcal{G}_{\Lambda}} \) is a subgroup of \( G_{\text{geom}} \). Thus if \( G_{\text{geom},\mathcal{G}_{\Lambda}} \) is not finite, then \( G_{\text{geom},\mathcal{G}_{\Lambda}} \) contains \( \text{SO}(n) \), and hence \( G_{\text{geom}} \) contains \( \text{SO}(n) \). By the previous lemma, we must have \( G_{\text{geom}} = O(n) \).

It remains to show that there exists at least one super-even primitive \( \Lambda \) for which \( G_{\text{geom},\mathcal{G}_{\Lambda}} \) is not finite. If \( G_{\text{geom},\mathcal{G}_{\Lambda}} \) is always finite, then by the diophantine criterion [4, 8.14.6] for finiteness, for every finite extension \( E/k \) and for every super-even primitive
character $\Lambda$ of $(B \otimes_k E)^*$, the sum

$$- \sum_{t \in E^*} \chi_2(t) \Lambda(1 - tX)$$

is divisible by $\sqrt{\#E}$ as an algebraic integer. If this holds for all $\Lambda$, then the diophantine criterion, applied to $\mathcal{F}_{\text{univ,odd},x_2}|\text{Prim}_{n,\text{odd}}$, shows that $G_{\text{geom}}$ is finite. However, $\mathcal{F}_{\text{univ,odd},x_2}$ is a sheaf of perverse origin. Restricting it to the subspace of super-even characters of conductor 5, it would result from [8] that we have finite $G_{\text{geom}}$ in the $n = 5$ case.

For $p \geq 7$, one knows [9, 3.12] that $G_{\text{geom},n=5}$ is not finite, indeed it contains $\text{SO}(5)$. For $p = 5 = n$, we show that $G_{\text{geom},n=5}$ is not finite by the “low ordinal” method. Take the character of conductor 5 given by $t \mapsto \psi_2(t,0)$ (concretely, the character $t \mapsto \exp(2\pi it^p/p^2)$ of the Heilbronn sum in the case $p = 5$). Then, the sum

$$- \sum_{t \in \mathbb{F}_5^*} \chi_2(t) \psi_2(t,0)$$

has ord$_p = 1/10 < 1/2$. Indeed, the Teichmuller representatives of $1, 2, 3, 4 \mod 25$ are $1, 7, -7, -1$. Denote by $\zeta_{25}$ the primitive 25th root of unity which is the value $\psi_2(1,0)$. Then minus our sum is

$$\zeta_{25} - \zeta_{25}^7 - \zeta_{25}^{-7} + \zeta_{25}^{-1} = \zeta_{25}(1 - \zeta_6^2) - \zeta_{25}^{-7}(1 - \zeta_6^6)$$

$$= (\zeta_{25} - \zeta_{25}^{-7})(1 - \zeta_6^2) = -\zeta_{25}^{-7}(1 - \zeta_6^8)(1 - \zeta_6^6)$$

is the product of two uniformizing parameters in $\mathbb{Z}_p[\zeta]$, each with ord$_p = 1/20$.

Suppose now $n = 3$ and $p \geq 7$. In this case, it is shown in [9, 3.7] that $G_{\text{geom}}$ contains $\text{SO}(3)$. In view of Lemma 6.1, we have $G_{\text{geom}} = \text{O}(3)$.

Suppose that $n = 3 = p$. It suffices to show that $G_{\text{geom}}$ is not finite. For then the identity component $G_{\text{geom}}^0$ is a nontrivial semisimple (because $\mathcal{F}_{\text{univ,odd},x_2}|\text{Prim}_{3,\text{odd}}$ is pure) connected subgroup of $\text{SO}(3)$. The only such subgroup is $\text{SO}(3)$ itself. Indeed, such a subgroup is the image of $\text{SL}(2)$ in a three-dimensional orthogonal representation, and the only such representation is $\text{Sym}^2(\text{std}_2)$, whose image is $\text{SO}(3)$. We show that $G_{\text{geom}}$ is not finite by the “low ordinal” argument. For $\zeta_9$ the primitive ninth root of unity $\zeta_9 := \psi_2(1,0)$, the sum

$$- \sum_{t \in \mathbb{F}_3^*} \chi_2(t) \psi_2(t,0) = -(\zeta_9 - \zeta_9^{-1}) = \zeta_9^{-1}(1 - \zeta_9^2)$$

is a uniformizing parameter of $\mathbb{Z}_3[\zeta_9]$, and has ord$_3 = 1/6 < 1/2$. 
It remains only to treat the case \( n \geq 5, p = 3 \). Suppose first \( n \geq 9 \) and \( p = 3 \). Pick any super-even primitive \( \Lambda \). we consider the lisse sheaf \( \mathcal{L} \mathcal{X}_2(t) \otimes \mathcal{L}_{\Lambda(1-\iota x)} \otimes \mathcal{L}_{\psi_1(\alpha t + b \theta^5 + c \theta^7)} \) on \( \mathbb{G}_m \times \mathbb{A}^3 \) (parameters \( (t, a, b, c) \)), and its cohomology along the fibers

\[
\mathcal{G}_\Lambda := R^1(pr_2)_!(\mathcal{L}_{\mathcal{X}_2(t)} \otimes \mathcal{L}_{\Lambda(1-\iota x)} \otimes \mathcal{L}_{\psi_1(\alpha t + b \theta^5 + c \theta^7)}).
\]

This \( \mathcal{G}_\Lambda \) is the restriction of \( \mathcal{F}_{\text{univ,odd},x_2} \) to an \( \mathbb{A}^3 \) in \( \text{Prim}_{n,\text{odd}} \). The usual moment calculation, now using [10, 3.11.6A], shows that \( \mathcal{G}_\Lambda \) has fourth moment 3. As we have the a priori inclusion \( G_{\text{geom},\mathcal{G}_\Lambda} \subset O(n) \), Larsen’s Alternative [6, p. 113] shows that either \( G_{\text{geom},\mathcal{G}_\Lambda} \) is finite, or it is \( \text{SO}(n) \) or \( \text{SO}(n) \). If \( G_{\text{geom},\mathcal{G}_\Lambda} \) is not finite, then the larger group \( G_{\text{geom}} \) contains \( \text{SO}(n) \), so by Lemma 6.1 must be \( O(n) \). If \( G_{\text{geom},\mathcal{G}_\Lambda} \) were finite for all super-even primitive \( \Lambda \), then by the diophantine criterion \( G_{\text{geom}} \) would be finite. Because \( \mathcal{F}_{\text{univ,odd},x_2} \) is a sheaf of perverse origin, restricting to the subspace of super-even characters of conductor 3, we would find that \( G_{\text{geom}} \) is finite in the \( n = 3 = p \) case, contradiction.

If \( n = 7 \) and \( p = 3 \), we repeat the above argument with one important modification. For a given choice of super-even primitive \( \Lambda \), there is exactly one value \( c_0 \) of \( c \) for which \( \mathcal{L}_{\mathcal{X}_2(t)} \otimes \mathcal{L}_{\Lambda(1-\iota x)} \otimes \mathcal{L}_{\psi_1(\alpha t + b \theta^5 + c \theta^7)} \) has lower conductor. So we must work with this sheaf on the product of \( \mathbb{G} \mathcal{G}_m \) with the open set of \( \mathbb{A}^3 \) where \( c - c_0 \) is invertible, and form its \( R^1(pr_2)_! \), which is the restriction of \( \mathcal{F}_{\text{univ,odd},x_2} \) to \( \mathbb{A}^3[1/(c - c_0)] \). On the entire \( \mathbb{A}^3 \), the moment calculation would give fourth moment 3. One checks that the fact of omitting the hyperplane \( c = c_0 \) only changes the calculation by lower order terms, the point being that in \( \mathbb{A}^4/\mathbb{F}_3 \) with coordinates \( (x, y, z, w) \), the subscheme defined by the two equations

\[
x^5 + y^5 = z^5 + w^5, \quad x^7 + y^7 = z^7 + w^7,
\]

has codimension 2. Now repeat the argument of the previous paragraph.

Here is an alternate proof for the case \( n = 7, p = 3 \). Over \( \mathbb{F}_3 \), we first use the “low ordinal” argument. We have the character \( \Lambda := \psi_1(t^7 - t^5) \psi_2(t, 0) \), whose sum

\[
- \sum_{t \notin \mathbb{Z}^3} \chi_2(t) \psi_1(t^7 - t^5) \psi_2(t, 0) = -\psi_2(1, 0) + \psi_2(-1, 0)
\]

\[
= -\zeta + \zeta^{-1} = \zeta^{-1}(1 - \zeta^2)
\]

is a uniformizing parameter for \( \mathbb{Z}^2[\zeta] \), whose ord_3 = 1/6 < 1/2. This shows that \( \mathcal{G} := \text{NFT}(\mathcal{L}_{\mathcal{X}_2(t)} \otimes \mathcal{L}_{\Lambda(1-\iota x)}) \) has a \( G_{\text{geom},\mathcal{G}_\Lambda} \) which is not finite. Because the rank \( n = 7 \) is prime, its \( G_{\text{geom},\mathcal{G}} \) must therefore be Lie irreducible, cf. [9, 3.5].
Now consider the three-parameter \((a, b, c)\) family of characters \(\Lambda_{a,b,c} := \psi_1(t^7 + at^5 + bt) \otimes \psi_2(ct, 0)\). On \(G_m \times A^3\) with coordinate \((t, a, b, c)\) we have the lisse sheaf \(\mathcal{L}_{\mathcal{X}_2(t)} \otimes \mathcal{L}_{\Lambda_{a,b,c}(1-tx)}\), its \(R^1(pr_2) := \mathcal{H}\) is the restriction of \(\mathcal{F}_{univ,odd,\mathcal{X}_2}\) to an \(A^3\) in \(\text{Prim}_{n,odd}\), and its further restriction to the \(A^1\) defined by \(a = -1, c = 1\) with parameter \(b\) is the sheaf \(\mathcal{G}\) above. Therefore, the larger group \(G_{geo,\mathcal{G}}\) must be Lie irreducible. By Gabber’s theorem \([4, 1.6]\) on prime-dimensional representations, the only possibilities for \(G_{geo,\mathcal{G}}^0\) are \(SO(7)\) itself or \(G_2\) or the image of \(SL(2)\) in \(Sym^6(\text{std}_2)\), which we will denote \(Sym^6(SL(2))\). If we get \(SO(7)\), then \(G_{geo}\) contains \(SO(7)\), and so by Lemma 6.1 must be \(O(7)\).

We will show that \(G_{geo,\mathcal{G}}^0\) is not \(Sym^6(SL(2))\) or \(G_2\). We argue by contradiction. Our \(\mathcal{H}\) is a lisse sheaf on \(A^3/\mathbb{F}_3\), with a determinant which is geometrically of order dividing 2. Hence, its determinant is geometrically constant. Moreover, the twisted sheaf \(\mathcal{H}_{arith} := \mathcal{H} \otimes (-G(\psi, \chi_2))^{-\text{degree}}\) has its \(G_{arith,\mathcal{G}}\) in \(O(7)\), so its determinant, being geometrically constant, is either trivial or is \((-1)^{\text{degree}}\).

So over any even degree extension of \(\mathbb{F}_3\), in particular over \(\mathbb{F}_9\), our twisted sheaf \(\mathcal{H}_{arith}\) has \(G_{arith,\mathcal{G}} \subset SO(7)\). If \(G_{geo,\mathcal{G}}^0\) is one of the groups \(Sym^6(SL(2))\) or \(G_2\), then \(G_{arith,\mathcal{G}}\) lies in the normalizer of \(Sym^6(SL(2))\), respectively of \(G_2\), in \(SO(7)\). But each of these groups is its own normalizer in \(SO(7)\). Therefore \(G_{arith,\mathcal{G}}\) is either the group \(Sym^6(SL(2))\) or \(G_2\). One knows that \(Sym^6(SL(2)) \subset G_2\), so we find an inclusion \(G_{arith,\mathcal{G}} \subset G_2\). One knows that the traces of elements of the compact form \(UG_2\) of \(G_2\) lie in the interval \([-2, 7]\). So the traces of Frobenius on \(\mathcal{H}_{arith}\) at \(\mathbb{F}_9\)-points will all lie in the interval \([-2, 7]\). Concretely, these are the sums

\[
(1/3) \sum_{t \in \mathbb{F}_9} \chi_2(\text{Norm}_{\mathbb{F}_9/\mathbb{F}_3}(t)) \psi_1(\text{Trace}_{\mathbb{F}_9/\mathbb{F}_3}(t^7 + at^5 + bt)) \psi_2(\text{Trace}_{\mathbb{F}_2(\mathbb{F}_9)/\mathbb{F}_2(\mathbb{F}_3)}(ct, 0)).
\]

A machine calculation shows that at the point \((a = -1, b = 0, c = 1 + i)\), \((i\) being either primitive fourth root of unity in \(\mathbb{F}_9)\), this trace is \(-6.10607/3 = -2.03536\), contradiction. [Machine calculation also shows that at the point \((a = i, b = -1 - i, c = 1 + i)\) this trace is \(-7.29086/3 = -2.43029\).]

If \(n = 5\) and \(p = 3\), the argument is quite similar. Over \(\mathbb{F}_3\), we first use the “low ordinal” argument. We have the character \(\Lambda := \psi_1(t^5)\psi_2(t, 0)\), whose sum

\[
- \sum_{t \in \mathbb{F}_3} \chi_2(t) \psi_1(t^5) \psi_2(t, 0) = -\psi_1(1)\psi_2(1) + \psi_1(-1)\psi_2(-1, 0)
\]

\[
= -\zeta_3\zeta_9 + \zeta_3^{-1}\zeta_9^{-1} = \zeta_9^{-4} - \zeta_9^{-4}(1 - \zeta_9^8)
\]
is a uniformizing parameter for \( \mathbb{Z}_3[\zeta_9] \), whose \( \text{ord}_3 = 1/6 < 1/2 \). This shows that 
\( G := \text{NFT}(L_{x_2(t)} \otimes L_{\Lambda(1-t \chi)}) \) has a \( G_{\text{geom}, G} \) which is not finite. Because the rank \( n = 5 \) is prime, its \( G_{\text{geom}, G} \) must therefore be Lie irreducible, cf. [9, 3.5]. Thus, \( G_{\text{geom}, G}^0 \) is a connected semisimple group in an irreducible five-dimensional representation. By Gabber's theorem [4, 1.6] on prime-dimensional representations, the only possibilities for \( G_{\text{geom}, G}^0 \) are \( \text{SO}(5) \) itself or the image of \( \text{SL}(2) \) in \( \text{Sym}^4(\text{std}_2) \), which we will denote \( \text{Sym}^4(\text{SL}(2)) \). If we get \( \text{SO}(5) \) for \( G_{\Lambda} \), then \( G_{\text{geom}} \) contains \( \text{SO}(5) \), and so by Lemma 6.1 must be \( \text{O}(5) \).

So it suffices to show that \( G_{\text{geom}, G}^0 \) is not \( \text{Sym}^4(\text{SL}(2)) \). We argue by contradiction. Our \( G \) is a lisse sheaf on \( \mathbb{A}^1/\mathbb{F}_3 \), with a determinant which is geometrically of order dividing 2. Hence its determinant is geometrically constant. Moreover, the twisted sheaf \( G_{\text{arith}} := G \otimes (-G(\psi, \chi_2))^{-\text{degree}} \) has its \( G_{\text{arith}, G} \) in \( \text{O}(5) \), so its determinant, being geometrically constant, is either trivial or is \((-1)^{\text{degree}}\).

So over any even degree extension of \( \mathbb{F}_3 \), in particular over \( \mathbb{F}_9 \), our twisted sheaf \( G_{\text{arith}} \) has \( G_{\text{arith},G} \subset \text{SO}(5) \). Therefore, \( G_{\text{arith}, G} \) lies in the normalizer of \( \text{Sym}^4(\text{SL}(2)) \) in \( \text{SO}(5) \).

But this normalizer is just \( \text{Sym}^4(\text{SL}(2)) \) itself, and hence \( G_{\text{arith}, G} \) is the group \( \text{Sym}^4(\text{SL}(2)) \). Therefore, the traces of Frobenius on \( G_{\text{arith}} \) at \( \mathbb{F}_9 \)-rational points are among the traces of elements of \( \text{SU}(2) \) in \( \text{Sym}^4(\text{std}_2) \). For an element \( \gamma \) of \( \text{SU}(2) \) with \( \text{Trace}(\gamma) = T \), its trace in \( \text{Sym}^4(\text{std}_2) \) is \( 1 - 3T^2 + T^4 \). The minimum of this polynomial on the interval \([-2, 2]\) is \(-5/4\).

The twisting factor over \( \mathbb{F}_9 \) is \(-1/3\), so the sums, indexed by \( a \in \mathbb{F}_9 \),

\[
(1/3) \sum_{t \in \mathbb{F}_9} \chi_2(\text{Norm}_{\mathbb{F}_9/\mathbb{F}_3}(t)) \psi_1(\text{Trace}_{\mathbb{F}_9/\mathbb{F}_3}(t^5 + at)) \psi_2(\text{Trace}_{\mathbb{F}_9/\mathbb{F}_3}(t, 0)),
\]

must all lie in the interval \([-5/4, 5]\). We get a contradiction, because for \( a = 1 + i \) (for \( i \) either primitive fourth root of unity in \( \mathbb{F}_9 \)), machine calculation shows that this sum is \(-4.75877048/3 = -1.58626\).

\[\blacksquare\]

7 Equidistribution for the Twists by the Quadratic Character

Fix an odd integer \( n \geq 3 \). For each finite field \( k \) of odd characteristic, and each primitive super-even character \( \Lambda \) of \( (k[X]/(X^{n+1}))^\times \), the reversed characteristic polynomial

\[
\det(1 - T\text{Prob}_k, H^1_c(G_m \otimes_k k, L_{x_{2}(t)} \otimes L_{\Lambda(1-t \chi)})/(-G(\psi, \chi_2)))
\]

is the reversed characteristic polynomial \( \det(1 - T\theta_{k,x_2\Lambda}) \) of a unique conjugacy class \( \theta_{k,x_2\Lambda} \) of the compact orthogonal group \( \text{O}(n, \mathbb{R}) \). Because \( n \) is odd, the group \( \text{O}(n) \) is the
product \((\pm 1) \times \text{SO}(n)\), the decomposition being

\[ A = \det(A)\tilde{A}; \quad \tilde{A} := A/\det(A). \]

Conjugacy classes of \(\text{O}(n, \mathbb{R})\) have the same product decomposition

\[ \theta_{k,\chi_2\Lambda} = \det(\theta_{k,\chi_2\Lambda})\tilde{\theta}_{k,\chi_2\Lambda}, \]

with \(\tilde{\theta}_{k,\chi_2\Lambda}\) a conjugacy class of \(\text{SO}(n, \mathbb{R})\).

Endow the space \(\text{O}(n, \mathbb{R})^\#\) of conjugacy classes of \(\text{O}(n, \mathbb{R})\) with the direct image of the total mass one Haar measure on \(\text{O}(n, \mathbb{R})\). Equidistribution in the theorem below is with respect to this measure.

**Theorem 7.1.** Fix an odd integer \(n \geq 5\). In any sequence of finite fields \(k_i\) of (possibly varying) odd characteristics \(p_i\), whose cardinalities \(q_i\) are archimedeanly increasing to \(\infty\), the collections of conjugacy classes

\[ \{\theta_{k_i,\chi_2\Lambda}\}_\Lambda \text{ primitive super–even} \]

become equidistributed in \(\text{O}(n, \mathbb{R})^\#\). We have the same result for \(n = 3\) if we require that no \(p_i\) is 5.

**Proof.** Fix the odd integer \(n \geq 3\). Whenever \(p\) is an allowed characteristic, then by Theorem 6.2 the relevant monodromy groups are \(G_{\text{geom}} = G_{\text{arith}} = \text{O}(n)\).

By the Weyl criterion, it suffices show that for each irreducible nontrivial representation \(\Xi\) of \(\text{O}(n, \mathbb{R})\), there exists a constant \(C(\Xi)\) such that for any allowed characteristic \(p\) and any finite field \(k\) of characteristic \(p\), we have the estimate

\[ |\sum_{\Lambda \text{ super–even and primitive}} \text{Trace}(\Xi(\theta_{k,\chi_2\Lambda}))| \leq \#\text{Prim}_{n,\text{odd}}(k)C(\Xi)/\sqrt{k}. \]

The group \(\text{O}(n)\) is the product \((\pm 1) \times \text{SO}(n)\), the decomposition being

\[ A = (\det(A))(\det(A)A). \]

So the irreducible nontrivial representations \(\Xi\) are products \(\det^a \times \Xi_0\) with \(a\) being 0 or 1 and \(\Xi_0\) an irreducible representation of \(\text{SO}(n)\), such that either \(a = 1\) or \(\Xi_0\) is irreducible nontrivial. We have seen, in the proof of Lemma 6.1, that over a given finite field \(k = \mathbb{F}_q\) of odd characteristic, the \(q - 1\) pullbacks \([t \mapsto at]^*(\Lambda(1 - tX))\) of a given super–even
primitive character will give rise to the conjugacy class $\theta_{k,\chi_2}^\Lambda$ exactly $(q - 1)/2$ times, and to the conjugacy class $-\theta_{k,\chi_2}^\Lambda$ exactly $(q - 1)/2$ times. This shows that when the representation $\Xi$ is of the form $\det \times \Xi_0$, then the sum

$$\sum_{\Lambda \text{ super--even and primitive}} \text{Trace}(\Xi(\theta_{k,\chi_2}^\Lambda))$$

vanishes identically. So we need only be concerned with the Weyl sums for irreducible nontrivial representations $\Xi_0$.

Thus, we have reduced the theorem to the following one.

**Theorem 7.2.** Fix an odd integer $n \geq 5$. In any sequence of finite fields $k_i$ of (possibly varying) odd characteristics $p_i$, whose cardinalities $q_i$ are archimedeanly increasing to $\infty$, the collections of conjugacy classes

$$\{\tilde{\theta}_{k_i,\chi_2}^\Lambda\}_\Lambda \text{ primitive super--even}$$

become equidistributed in $\text{SO}(n, \mathbb{R})'$. We have the same result for $n = 3$ if we require that no $p_i$ is 5. □

For a given allowed characteristic $p$, and an irreducible nontrivial representation $\Xi$ of $\text{SO}(n)$, Deligne’s equidistribution theorem [1, 3.5.3], as explicated in [13, 9.2.6, part (2)], tells us we can take

$$C(\Xi, p) := \sum_i h^i_c(\text{Prim}_{n,\text{odd}} \otimes_{\mathbb{F}_p} \mathbb{F}_p, \Xi(\mathcal{F}_{\text{univ,odd},\chi_2})).$$

This sum of Betti numbers is uniformly bounded as $p$ varies. Indeed, we have the following lemma.

**Lemma 7.3.** Fix an irreducible nontrivial representation $\Xi$ of $\text{SO}(n)$. Choose an integer $M \geq 1$ such that $\Xi$ occurs in $\text{std}_n^\otimes M$. For $p > n$, we have the estimate

$$\sum_i h^i_c(\text{Prim}_{n,\text{odd}} \otimes_{\mathbb{F}_p} \mathbb{F}_p, \Xi(\mathcal{F}_{\text{univ,odd},\chi_2}))$$

$$\leq \sum_i h^i_c(\text{Prim}_{n,\text{odd}} \otimes_{\mathbb{F}_p} \mathbb{F}_p, \mathcal{F}_{\text{univ,odd},\chi_2}^\otimes M)$$

$$\leq 3(n + 3 + M)^{(n+3)/2+M+1} \leq 3(n + 3 + M)^{n+M+1}. \quad \square$$
**Proof.** The proof is similar to that of Lemma 5.2. For \( p > n \), we again invoke the Kunneth formula and end up with isomorphisms

\[
H^i_c(\text{Prim}_{n,\text{odd}} \otimes_{\mathbb{F}_p} \mathcal{F}_{\text{univ,odd},x2}^M) = H^{i+M}_c((\mathbb{A}^M \times \text{Prim}_{n,\text{odd}}) \otimes_{\mathbb{F}_p} \mathcal{L}_{x2}(t_1 t_2 \ldots t_M) \mathcal{L}_{\psi_1(f(t_1) + \ldots + f(t_M)))}.
\]

The asserted estimate is then a special case of [7, Theorem 12].

We can also use the Fourier transform method in large characteristic, for any \( n \neq 7 \). If \( p > n \), the primitive super-even \( \Lambda \)'s give precisely the Artin–Schreier sheaves \( \mathcal{L}_{\psi_1(f(t))} \) for \( f \) running over the strictly odd polynomials of degree \( n \). For each of these, the Fourier transform

\[
G_f := \text{NFT}(\mathcal{L}_{x2}(t) \otimes sL_{\psi_1(f(t)))}
\]

is lisse of rank \( n \) and geometrically irreducible, hence Lie irreducible by [3, Proposition 5]. Its \( G_{\text{geom}} \) lies in \( \text{SO}(n) \). Its \( \infty \)-slopes are

\[
\{0, n - 1 \text{ slopes } n/(n - 1)\}.
\]

By [4, 7.1.1 and 7.2.7 (2)] there are (effective) non-zero integers \( N_1(n - 1) \) and \( N_2(n - 1) \) such that if \( p \), in addition to being \( > 2n + 1 \), does not divide the integer \( 2nN_1(n - 1)N_2(n - 1) \), then \( G_{\text{geom},G_f} \) is either \( \text{SO}(n) \), or, if \( n = 7 \), possibly \( G_2 \). [It is this ambiguity which rules out the case \( n = 7 \).]

Because \( G_f \) has \( G_{\text{geom},G_f} = \text{SO}(n) \), and has all \( \infty \)-slopes \( \leq n/(n - 1) \), we have the estimate

\[
h^1_c(\mathbb{A}^1 \otimes \mathcal{F}_{\mathbb{F}_p}, \Xi(G_f)) \leq \dim(\Xi)/(n - 1), \text{ other } h^i_c = 0,
\]

cf. the proof of [12, 8.2]. Thus, we get

\[
\sum_{a \in k, \Lambda \equiv f(t) + at} \text{Trace}(\Xi(\tilde{\theta}_{k,\Lambda})) \leq (\dim(\Xi)/(n - 1))\#k/\sqrt{\#k}.
\]

Summing this estimate over equivalence classes of strictly odd \( f \)'s of degree \( n \) (for the equivalence relation \( f \cong g \) if \( \deg(f - g) \leq 1 \)), we get, in characteristic \( p > 2n + 1 \), \( p \) not
dividing $2nN_1(n - 1)N_2(n - 1)$, the estimate

$$\left| \sum_{\Lambda \text{ super-even and primitive}} \text{Trace}(\mathbb{E}(\tilde{\theta}_{k,\Lambda})) \right| \leq \#\text{Prim}_{n,\text{odd}}(k)(\dim(\mathbb{E})/(n - 1))/\sqrt{k}.$$ 

Denote by $\text{Excep}(n)$ the finite set of odd primes $p$ either which are $\leq 2n + 1$ or which divide $2nN_1(n - 1)N_2(n - 1)$. We may take

$$C(\mathbb{E}) := \text{Max} \left( \frac{\dim(\mathbb{E})}{(n - 1)}, \text{Max}_{p \in \text{Excep}(n)} \frac{C(\mathbb{E}, p)}{C(\mathbb{E}, p)} \right).$$

**Remark 7.4.** In the case $n = 7$ and $p \geq 17$, it is proven in [4, 9.1.1] that for any $a \neq 0$ and for $f = ax^7$, the sheaf $G_f$ has $G_{\text{geom},G_f} = G_2$. We will show elsewhere that for $p$ sufficiently large, we also have $G_{\text{geom},G_f} = G_2$ for any $f$ of the form $ax^7 + abx^5 + ab^2(25/84)x^3$. It is plausible that these are the only such $f$. If that were the case, then the exceptions would be uniformly small enough (over $\mathbb{F}_q$, $q^2(q - 1)$ out of $q^3(q - 1)$ $\tilde{\theta}$'s in all) that we would get the same result for $n = 7$ as for the other odd $n$, with all odd primes allowed. 

**8 A Theorem of Joint Equidistribution**

**Theorem 8.1.** Fix an odd integer $n \geq 5$. In any sequence of finite fields $k_i$ of (possibly varying) odd characteristics $p_i$, whose cardinalities $q_i$ are archimedeanly increasing to $\infty$, the collections of pairs of conjugacy classes

$$\{(\theta_{k_i,\Lambda}, \theta_{k_i,x2,\Lambda})\}_{\Lambda \text{ primitive super-even}}$$

become equidistributed in the space $\text{USp}(n - 1)^\# \times O(n, \mathbb{R})^\#$ of conjugacy classes in the product group $\text{USp}(n - 1) \times O(n, \mathbb{R})$. We have the same result for $n = 3$ if we require that no $p_i$ is 5.

**Proof.** We consider the direct sum sheaf

$$(\mathcal{F}_{\text{univ,odd}} \oplus \mathcal{F}_{\text{univ,odd},x2})|\text{Prim}_{n,\text{odd}}.$$ 

The two factors have, respectively,

$$G_{\text{geom}} = G_{\text{arith}} = \text{Sp}(n - 1), \ G_{\text{geom}} = G_{\text{arith}} = O(n).$$
in any odd characteristic $p$. So $G_{\text{geom}}$ (respectively $G_{\text{arith}}$) for the direct sum is a subgroup of the product $\text{Sp}(n - 1) \times \text{O}(n)$ which maps on to each factor.

Suppose first that $n$ is neither 3 nor 5. Then, these two factors have no nontrivial quotients which are isomorphic. So by Goursat’s lemma, the direct sum sheaf has

$$G_{\text{geom}} = G_{\text{arith}} = \text{Sp}(n - 1) \times \text{O}(n)$$

in any odd characteristic $p$.

Let us temporarily admit that for $n = 5$, the direct sum sheaf also has

$$G_{\text{geom}} = G_{\text{arith}} = \text{Sp}(n - 1) \times \text{O}(n)$$

in any odd characteristic $p$. Let us also admit that for $n = 3$ the direct sum sheaf has

$$G_{\text{geom}} = G_{\text{arith}} = \text{Sp}(n - 1) \times \text{O}(n)$$

in any odd characteristic $p \neq 5$.

By the Weyl criterion, it suffices to show that for each irreducible nontrivial representation $\Pi \otimes \Xi$ of $\text{USp}(n - 1) \times \text{O}(n, \mathbb{R})$, there exists a constant $C(\Pi \otimes \Xi)$ such that for any odd characteristic $p$ and any finite field $k$ of characteristic $p$, we have the estimate

$$\left| \sum_{\Lambda \text{ super--even and primitive}} \text{Trace}(\Pi(\theta_{k,\Lambda})) \text{Trace}(\Xi(\theta_{k,\Xi}(\Lambda))) \right| \leq \#\text{Prim}_{n,\text{odd}}(k) C(\Pi \otimes \Xi) / \sqrt{\#k}.$$ 

For $a \in k^\times$ a nonsquare, the effect of $\Lambda \mapsto [t \mapsto at]^\Lambda \Lambda$ is to leave $\theta_{k,\Lambda}$ unchanged, but to replace $\theta_{k,\Xi}(\Lambda)$ by minus itself. So exactly as in the proof of Theorem 7.2 above, the Weyl sums vanish identically when the $\Xi$ factor is of the form $\det \otimes \Xi_0$ for $\Xi_0$ a representation of $\text{SO}(n)$. So we need only be concerned with the Weyl sums for irreducible nontrivial representations of the form $\Pi \otimes \Xi_0$.

Thus, we have reduced the theorem to the following one.

**Theorem 8.2.** Fix an odd integer $n \geq 5$. In any sequence of finite fields $k_i$ of (possibly varying) odd characteristics $p_i$, whose cardinalities $q_i$ are archimedeanly increasing to $\infty$, the collections of pairs of conjugacy classes

$$\{((\theta_{k_i,\Lambda}, \tilde{\theta}_{k_i,\Xi}(\Lambda)))_{\Lambda \text{ primitive super--even}}$$
become equidistributed in the space $\text{USp}(n - 1)^* \times \text{SO}(n, \mathbb{R})^*$ of conjugacy classes in the product group $\text{USp}(n - 1) \times \text{SO}(n, \mathbb{R})$. We have the same result for $n = 3$ if we require that no $p_i$ is 5.

For a given odd characteristic $p$, and an irreducible nontrivial representation $\Pi \otimes \Xi$ of $\text{Sp}(n - 1) \times \text{SO}(n)$, Deligne’s equidistribution theorem [1, 3.5.3], as explicated in [13, 9.2.6, part (2)], we can take $C(\Pi \otimes \Xi, p) := \sum_i h^i_c(\text{Prim}_{n, \text{odd}} \otimes_{\mathbb{F}_p} \mathbb{F}_p, \Pi(\mathcal{F}_{\text{univ, odd}}) \otimes \Xi(\mathcal{F}_{\text{univ, odd}, x_2})).$

This sum of Betti numbers is uniformly bounded as $p$ varies. Notice that if either $\Xi$, respectively $\Pi$, is trivial, then its partner $\Pi$, respectively $\Xi$, must be nontrivial, and the result is given by Lemma 5.2, respectively Lemma 7.3. So it suffices to prove the following lemma.

**Lemma 8.3.** Fix irreducible nontrivial representations $\Pi$ of $\text{USp}(n - 1)$ and $\Xi$ of $\text{SO}(n, \mathbb{R})$. Choose integers $M_1 \geq 1$ and $M_2 \geq 1$ such that $\Pi$ occurs in $\text{std}^\otimes_{M_1}$ and such that $\Xi$ occurs in $\text{std}^\otimes_{M_2}$. Then, we have the estimate

$$\sum_i H^i_c(\text{Prim}_{n, \text{odd}} \otimes_{\mathbb{F}_p} \mathbb{F}_p, \Pi(\mathcal{F}_{\text{univ, odd}}) \otimes \Xi(\mathcal{F}_{\text{univ, odd}, x_2})) \leq \sum_i h^i_c(\text{Prim}_{n, \text{odd}} \otimes_{\mathbb{F}_p} \mathbb{F}_p, \mathcal{F}_{\text{univ, odd}}^{\otimes M_1} \otimes \mathcal{F}_{\text{univ, odd}, x_2}^{\otimes M_2}) \leq 3(n + 3 + M_2)^{(n+3)/2+1+M_1+M_2} \leq 3(n + 3 + M_2)^{n+1+M_1+M_2}.$$

**Proof.** The proof is similar to the proofs of Lemmas 5.2 and 7.3. For $p > n$, we invoke the Kunneth formula to obtain isomorphisms

$$H^i_c(\text{Prim}_{n, \text{odd}} \otimes_{\mathbb{F}_p} \mathbb{F}_p, \mathcal{F}_{\text{univ, odd}}^{\otimes M_1} \otimes \mathcal{F}_{\text{univ, odd}, x_2}^{\otimes M_2}) = H^{i+M_1+M_2}_c((\Lambda^{M_1} \times A^{M_2} \times \text{Prim}_{n, \text{odd}}) \otimes_{\mathbb{F}_p} \mathbb{F}_p, \mathcal{H})$$

for $\mathcal{H}$ the sheaf

$$\mathcal{L}_{x_2(s_1\ldots s_{M_2})} \otimes \mathcal{L}_{\psi_1(f(t_1)+\ldots+f(t_{M_1})+f(s_1)+\ldots+f(s_{M_2}))}.$$

The asserted estimate is a special case of [7, Theorem 12].

For $n$ not 5 or 7, we can also use the Fourier transform method. For $p > 2n + 1$ and not dividing $2nN_1(n - 1)N_2(n - 1)$, we know that for $\Lambda$ super-even primitive, $\mathcal{L}_{\Lambda(1-ix)}$
is precisely of the form $L_{\psi_1(f(t))}$ for an odd polynomial $f$ of degree $n$. We have seen above that the Fourier transforms

$$G_f := \text{NFT}(L_{\psi_1(f(t))} \otimes (\sqrt{q})^{-\text{degree}},$$

$$G_{f,\chi_2} := \text{NFT}(L_{\chi_2(t)} \otimes L_{\psi_1(f(t))} \otimes (-G(\psi, \chi_2))^{-\text{degree}} \otimes \det$$

have

$$G_{\text{geom},G_f} = G_{\text{arith},G_f} = \text{Sp}(n - 1),$$

and

$$G_{\text{geom},G_{f,\chi_2}} = G_{\text{arith},G_{f,\chi_2}} = \text{SO}(n).$$

Their direct sum $G_f \oplus G_{f,\chi_2}$ has $G_{\text{geom}} = G_{\text{arith}} = \text{Sp}(n - 1) \times \text{SO}(n)$. Both $G_f$ and $G_{f,\chi_2}$ have all $\infty$-slopes $\leq n/(n - 1)$, hence so does any tensor product

$$\Pi(G_f) \otimes \Xi(G_{f,\chi_2}).$$

So for any nontrivial irreducible representation $\Pi \otimes \Xi$ of $\text{Sp}(n - 1) \times \text{SO}(n)$ we have the estimate

$$h^1_c(\mathbb{A}^1 \otimes F_p, \Pi(G_f) \otimes \Xi(G_{f,\chi_2})) \leq \dim(\Pi) \dim(\Xi)/(n - 1), \text{ other } h^i_c = 0,$$

cf. the proof of [12, 8.2].

So we get the estimate

$$\sum_{a \in k, \Lambda \equiv f(t) + at} \text{Trace}(\Pi(\theta_{k,\Lambda,\chi_2})) \text{Trace}(\Xi(\tilde{\theta}_{k,\Lambda,\chi_2})) \leq (\dim(\Pi) \dim(\Xi)/(n - 1))#k/\sqrt{#k}. $$

Summing this estimate over equivalence classes of strictly odd $f$’s of degree $n$ (for the equivalence relation $f \equiv g$ if $\text{deg}(f - g) \leq 1$), we get, in characteristic $p > 2n + 1$, $p$ not dividing $2nN_1(n - 1)N_2(n - 1)$, the estimate

$$\sum_{\Lambda \text{ super--even and primitive}} \text{Trace}(\Pi(\theta_{k,\Lambda})) \text{Trace}(\Xi(\tilde{\theta}_{k,\chi_2\Lambda})) \leq \#\text{Prim}_{n,\text{odd}}(k)C(\Pi \otimes \Xi)/\sqrt{#k}. $$
Thus for \( n \geq 9 \) we may take

\[
C(\Pi \otimes \Xi) := \max (\dim(\Pi) \dim(\Xi)/(n-1), \max_{p \in \text{Excep}(n)} C(\Xi, p)).
\]

[For \( n \) either 5 or 7, we do not know that every individual Fourier transform has the correct \( G_{\text{geom}} \), hence their exclusion.]

9 Joint Equidistribution in the Case \( n = 3 \)

The problem we must deal with in the \( n = 3 \) case is that the quotient \( SL(2)/\pm 1 \) is isomorphic to the quotient \( O(3)/\pm 1 \cong SO(3) \), namely \( SO(3) \) is the image of the representation \( \text{Sym}^2(\text{std}_2) \) of \( SL(2) \). We must rule out the possibility that the conjugacy classes

\[
\{(\theta_k, \lambda, \tilde{\theta}_k, \lambda) \}_\lambda \text{ primitive super--even}
\]

are related by

\[
\tilde{\theta}_k, \lambda = \text{Sym}^2(\theta_k, \lambda).
\]

We begin with the case of characteristic \( p = 3 \). In this case, up to tensoring with an \( L_{\psi_1(x)} \), the super-even primitive characters of conductor 3 correspond to the Artin–Schreier–Witt sheaves \( L_{\psi_2(ax,0)} \) for some invertible scalar \( a \). By the obvious change of variable \( x \mapsto x/a \), this reduces us to considering the Fourier transforms

\[
\mathcal{F} := \text{NFT}(L_{\psi_2(x,0)}) \otimes (\sqrt{q})^{-\text{degree}},
\]

\[
\mathcal{G} := \text{NFT}(L_{\chi_2(x)} \otimes L_{\psi_2(x,0)}) \otimes (-G(\psi, \chi_2))^{-\text{degree}} \otimes \det.
\]

What we must show is that there is no geometric isomorphism between \( \text{Sym}^2(\mathcal{F}) \) and \( \mathcal{G} \). For then by Goursat’s lemma, the \( G_{\text{geom}} \) for \( \mathcal{F} \oplus \mathcal{G} \) will be the full product \( SL(2) \times SO(3) \), and \( a \text{ fortiori} \) the \( G_{\text{arith}} \) will also be the full product.

If there were a geometric isomorphism between \( \text{Sym}^2(\mathcal{F}) \) and \( \mathcal{G} \), then \( \text{Hom}_{\pi(A)}(\text{Sym}^2(\mathcal{F}), \mathcal{G}) \) would be a one-dimensional (both objects are geometrically irreducible) representation of \( \pi_{\text{arith}}^{\pi(A)} / \pi_{\text{geom}}^{\pi(A)} = \text{Gal}(\overline{F}_3/F_3) \). In other words, for some scalar \( A \), we would have an arithmetic isomorphism

\[
\text{Sym}^2(\mathcal{F}) \cong \mathcal{G} \otimes A^{\text{degree}}.
\]
The scalar $A$ would necessarily have $|A| = 1$ for any complex embedding $\overline{\mathbb{Q}}_l \subset \mathbb{C}$, because both $\mathcal{F}$ and $\mathcal{G}$ are pure of weight zero. In particular, for any finite extension $E/\mathbb{F}_3$ and any $t \in E$, and any complex embedding, we would have an equality of absolute values

$$|\text{Trace}(\text{Frob}_{E,t}|\text{Sym}^2(\mathcal{F}))| = |\text{Trace}(\text{Frob}_{E,t}|\mathcal{G})|.$$ 

But already for $E = \mathbb{F}_3$ and $t = 0$, these absolute values are different. Write $\zeta_9$ for $e^{2\pi i/9}$. The first is

$$|(1/3) \left( \sum_{x \in \mathbb{F}_3} \psi_2(x,0) \right)^2 - 1| = |(1/3)(1 + \zeta_9 + \zeta_9^{-1})^2 - 1| = 1.1371...$$

The second, remembering that the Gauss sum has absolute value $\sqrt{3}$, is

$$\left| (1/\sqrt{3}) \sum_{x \in \mathbb{F}_3^\times} \chi_2(x) \psi_2(x,0) \right| = \left| (1/\sqrt{3})(\zeta_9 - \zeta_9^{-1}) \right| = 0.74222...$$

Suppose now that $p \geq 7$. In this case, the super-even primitive $\Lambda$’s give precisely the Artin–Schreier sheaves $\mathcal{L}_{\psi_1(ax^3+bx)}$ with $(a, t) \in \mathbb{G}_m \times \mathbb{A}^1$. What we must show is that for any $a \neq 0$, the two lisse sheaves on $\mathbb{A}^1$ given by

- $\text{Sym}^2(\text{NFT}(\mathcal{L}_{\psi_1(ax^3)}))(1)$,
- $\text{NFT}(\mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi_1(ax^3)}) \otimes (-G(\psi_1, \chi_2))^{-\text{degree}} \otimes \det$,

are not geometrically isomorphic.

Because the question is geometric, we may assume that $a$ is a cube, say $a = 1/\alpha^3$. Making the change of variable $x \mapsto \alpha x$, we reduce to treating the case when $a = 1$. Thus, we must show that

$$\mathcal{F} := \text{Sym}^2(\text{NFT}(\mathcal{L}_{\psi_1(x^3)}))$$

and

$$\mathcal{G} := \text{NFT}(\mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi_1(x^3)})$$

are not geometrically isomorphic. For this, we will make use of information about Kloosterman sheaves and hypergeometric sheaves, especially [4, 9.3.2] and [11, 3.7].
We will denote by \([3]\) the cubing map \(x \mapsto x^3\). Notice that
\[
\mathcal{L}_{\psi_1(x^3)} \cong [3]^* (\mathcal{L}_{\psi_1(x)}),
\]
\[
\mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi_1(x^3)} \cong [3]^* (\mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi_1(x)}).
\]

According to \([4, 9.3.2]\), applied with \(d = 3\), we have geometric isomorphisms
\[
\text{NFT}([3]^* (\mathcal{L}_{\psi_1(x)})) \cong [3]^* ([x \mapsto -x/27]^* (\mathcal{Kl}(!, \psi_1, \chi_3, \chi_3))),
\]
\[
\text{NFT}([3]^* (\mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi_1(x^3)})) \cong [3]^* ([x \mapsto -x/27]^* (\mathcal{Hyp}(!, \psi_1, \mathbb{1}, \chi_3, \overline{\chi_3}; \chi_2))).
\]

Here, \(\chi_3\) and \(\overline{\chi_3}\) are the two Kummer characters of order 3. Thus
\[
\mathcal{F} \cong [3]^* ([x \mapsto -x/27]^* (\text{Sym}^2 (\mathcal{Kl}(!, \psi_1, \chi_3, \overline{\chi_3}))).
\]

According to \([11, 3.7]\), applied with \(\rho = \chi_3\) we have a geometric isomorphism
\[
\text{Sym}^2 (\mathcal{Kl}(!, \psi_1, \chi_3, \overline{\chi_3})) \cong [x \mapsto 4x]^* (\mathcal{Hyp}(!, \psi_1, \mathbb{1}, \chi_3, \overline{\chi_3}; \chi_2)).
\]

Thus we find
\[
\mathcal{F} \cong [3]^* ([x \mapsto -x/27]^* ([x \mapsto 4x]^* (\mathcal{Hyp}(!, \psi_1, \mathbb{1}, \chi_3, \overline{\chi_3}; \chi_2)))),
\]
that is,
\[
\mathcal{F} \cong [3]^* [x \mapsto -4x/27]^* \mathcal{Hyp}(!, \psi_1, \mathbb{1}, \chi_3, \overline{\chi_3}; \chi_2),
\]
whereas
\[
\mathcal{G} \cong [3]^* [x \mapsto -x/27]^* \mathcal{Hyp}(!, \psi_1, \mathbb{1}, \chi_3, \overline{\chi_3}; \chi_2).
\]

To show that \(\mathcal{F}\) and \(\mathcal{G}\) are not geometrically isomorphic, we argue by contradiction. If \(\mathcal{F} \cong \mathcal{G}\), then we have a geometric isomorphism on \(\mathbb{G}_m\),
\[
[x \mapsto -4x/27]^* \mathcal{Hyp}(!, \psi_1, \mathbb{1}, \chi_3, \overline{\chi_3}; \chi_2) \cong [x \mapsto -x/27]^* \mathcal{Hyp}(!, \psi_1, \mathbb{1}, \chi_3, \overline{\chi_3}; \chi_2).
\]
Indeed, if two geometrically irreducible lisse sheaves on \(\mathbb{G}_m\) have isomorphic pullbacks by \([3]\), then one is the tensor product of the other with either \(\mathcal{O}_d\) or \(\mathcal{L}_{\chi_3}\) or \(\mathcal{L}_{\overline{\chi_3}}\). Of the three candidates, only tensoring with the constant sheaf preserves \(\chi_2\) as the tame part of
local monodromy at ∞, cf. [4, 8.2.5]. Thus we have the asserted geometric isomorphism, whence a geometric isomorphism

\[ [x \mapsto 4x] \ast \mathcal{Hyp}(!, \psi_1, \mathbb{P}_1, \chi_3, \chi_2) \cong \mathcal{Hyp}(!, \psi_1, \mathbb{P}_1, \chi_3, \chi_2). \]

By [4, 8.5.4], a hypergeometric sheaf is not isomorphic to any nontrivial multiplicative translate of itself. This is the desired contradiction.

Notice that in this \( n = 3 \) case, the "Fourier transform by Fourier transform" method works in every allowed characteristic \( p \neq 5 \), giving the constant

\[ C(\Pi \otimes \Xi) := \dim(\Pi) \dim(\Xi)/2. \]

10 Joint Equidistribution in the Case \( n = 5 \)

Here, the problem is that \( \text{Sp}(4)/\pm \cong \text{SO}(5) \). Indeed, \( \text{SO}(5) \) is the image of \( \text{Sp}(4) \) in its second fundamental representation \( \Lambda^2(\text{std}_4)/\mathbb{P}_1 \). What we must show is that for \( n = 5 \),

\[ \Lambda^2(\mathcal{F}_{\text{univ}, \text{odd}}(1/2))/\mathbb{P}_1 \]

and

\[ \mathcal{F}_{\text{univ}, \text{odd}, \chi_2} \otimes (-1/G(\psi_1, \chi_2))^{\text{degree}} \otimes \det \]

are not geometrically isomorphic in any odd characteristic \( p \). The proof goes along the same lines as did the \( n = 3 \) case.

Notice first that both sides have \( G_{\text{geom}} = G_{\text{arith}} = \text{SO}(5) \), so if they are geometrically isomorphic then they are arithmetically isomorphic.

We first treat the case \( p = 5 \). Because \( p \) is 1 mod 4, the Gauss sum \( G(\psi_1, \chi_2) \) is some square root of 5. So it suffices to show that for the particular super-even character corresponding to \( \mathcal{L}_{\psi_2(t,0)} \),

\[ \text{Trace}(\text{Frob}_{\mathbb{F}_5} | \Lambda^2(H^1(\mathbb{A}^1 \otimes_{\mathbb{F}_5} \overline{\mathbb{F}_5}, \mathcal{L}_{\psi_2(t,0)})(1/\sqrt{5})))) - 1 \]

is not equal to either of

\[ \pm \text{Trace}(\text{Frob}_{\mathbb{F}_5} | H^1(\mathbb{G}_m \otimes_{\mathbb{F}_5} \overline{\mathbb{F}_5}, \mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\psi_2(t,0)})/\sqrt{5}). \]

Computer calculation shows that the first is \(-1.123807...\), while the second is \( \pm 1.033926...\).
Suppose now that \( p \) is an odd prime other than 5. It suffices to show that the restrictions of the two sides to some subvariety of \( \text{Prim}_{n, \text{odd}} \) are not geometrically isomorphic. We will show that the two lisse sheaves

\[
\Lambda^2(\text{NFT}(L_{\psi_1(t^5)})) / \mathbb{I}
\]

and

\[
\text{NFT}(L_{\chi_2(t)} \otimes L_{\psi_1(t^5)})
\]
on \( \mathbb{A}^1 \) are not geometrically isomorphic when restricted to \( \mathbb{G}_m \).

By [4, 9.3.2], we have a geometric isomorphism

\[
\text{NFT}(L_{\psi_1(t^5)}) \cong [x \mapsto x^5][x \mapsto -x/5^5]^{*Kl(!, \psi_1; \rho_1, \rho_2, \rho_3, \rho_4)},
\]

for \( \rho_1, \rho_2, \rho_3, \rho_4 \) the four nontrivial multiplicative characters of order 5.

By [11, 8.6], we have a geometric isomorphism

\[
\Lambda^2(Kl(!, \psi_1; \rho_1, \rho_2, \rho_3, \rho_4))/\mathbb{I} \cong [x \mapsto -4x]^{*\text{Hyp}(!, \psi_1; \mathbb{I}, \rho_1, \rho_2, \rho_3, \rho_4; \chi_2)}.
\]

Thus,

\[
\Lambda^2(\text{NFT}(L_{\psi_1(t^5)})) / \mathbb{I}
\]

\[
\cong [x \mapsto x^5][x \mapsto -x/5^5][x \mapsto -4x]^{*\text{Hyp}(!, \psi_1; \mathbb{I}, \rho_1, \rho_2, \rho_3, \rho_4; \chi_2)}.
\]

At the same time, by [4, 9.3.2], we have a geometric isomorphism

\[
\text{NFT}(L_{\chi_2(t)} \otimes L_{\psi_1(t^5)})
\]

\[
\cong [x \mapsto x^5][x \mapsto -x/5^5]^{*\text{Hyp}(!, \psi_1; \mathbb{I}, \rho_1, \rho_2, \rho_3, \rho_4; \chi_2)}.
\]

So it suffices to show that the two lisse sheaves on \( \mathbb{G}_m \) given by

\[
[x \mapsto -x/5^5][x \mapsto -4x]^{*\text{Hyp}(!, \psi_1; \mathbb{I}, \rho_1, \rho_2, \rho_3, \rho_4; \chi_2)}
\]

and

\[
[x \mapsto -x/5^5]^{*\text{Hyp}(!, \psi_1; \mathbb{I}, \rho_1, \rho_2, \rho_3, \rho_4; \chi_2)}
\]
do not become isomorphic after pullback by the fifth power map $x \mapsto x^5$. We argue by contradiction. As the sheaves are each geometrically irreducible, if their $x \mapsto x^5$ pullbacks are isomorphic, then one is obtained from the other by tensoring with an $L_{\rho}$ for some character $\rho$ of order dividing 5. As both sides have $\chi_2$ as the tame part of their $I(\infty)$-representations, this $\rho$ must be trivial. So we would find that the hypergeometric sheaf

$$[x \mapsto -x/5^5]^* \text{Hyp}(!, \psi_1; \bar{1}, \rho_1, \rho_2, \rho_3, \rho_4; \chi_2)$$

is geometrically isomorphic to its multiplicative translate by $-4$. Because $p \neq 5$, this is a nontrivial multiplicative translation. This contradicts [4, 8.5.4], according to which a geometrically irreducible hypergeometric sheaf is not isomorphic to any nontrivial multiplicative translate of itself.

In this $n = 5$ case, we cannot (at present) apply the "Fourier transform by Fourier transform" method, because we have only analyzed the Fourier transform situation for the single input $L_{\psi_1(t^5)}$, but not for other super-even primitive $\Lambda$'s. Nor do we know for which such $\Lambda$'s, if any, we will in fact have the exceptional isomorphism we ruled out for $L_{\psi_1(t^5)}$.

References


