European Journal of Mathematics (2024) 10:65
https://doi.org/10.1007/s40879-024-00770-0

RESEARCH ARTICLE

On some Airy sheaves of Laurent type

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To the memory of Nikolai Aleksandrovich Vavilov

Received: 30 April 2024 / Revised: 9 September 2024 / Accepted: 10 September 2024

Abstract

We study certain one-parameter families of exponential sums of Airy–Laurent type. Their general theory was developed in Katz and Tiep (Airy sheaves of Laurent type: an introduction, https://web.math.princeton.edu/~nmk/kt31_11sept.pdf). In the present paper, we make use of that general theory to compute monodromy groups in some particularly simple families (in the sense of “simple to remember"), realizing Weyl groups of type E_6 and E_8.

Keywords Local systems · Airy sheaves · Monodromy groups · Weyl groups

Mathematics Subject Classification 11T23 · 20C15 · 20D06 · 20C33 · 20G40

1 Introduction

In classical analysis, Airy functions are the Fourier transforms of exponentials $e^{g(x)}$ of polynomials, (originally for the polynomial $g(x) := x^3/3$) and Airy differential equations are the linear differential equations

Antonio Rojas-León was partially supported by grant PID2020-114613GB-I00 funded by MICIU/AEI/10.13039/501100011033. Pham Huu Tiep gratefully acknowledges the support of the NSF (grant DMS-2200850) and the Joshua Barlaz Chair in Mathematics.

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Published online: 06 November 2024

$$g'\left(\frac{d}{dt}\right)y + ty = 0$$

they satisfy. These differential equations have an irregular singularity at ∞, and have quite interesting differential Galois groups. In the seminal paper [21] of Such, he introduces their ℓ-adic finite field analogues, the local systems whose trace functions are of the form

$$t \mapsto -\sum_x \psi(g(x) + tx).$$

The local systems we are concerned with here are generalizations of these Airy local systems in several ways. We allow the "t term" tx to be replaced by tx^a, we allow the polynomial $g(x)$ to be replaced by a Laurent polynomial $f(1/x) + g(x)$, and we allow an "outside factor" $\chi(x)$ in the sum. Here is a more detailed discussion.

We work in **odd** characteristic $p > 0$, and denote by $\overline{\mathbb{F}_p}$ an algebraic closure of $\mathbb{F}_p$. We also fix a prime $\ell \neq p$ to be able to speak of $\overline{\mathbb{Q}_\ell}$-adic cohomology. We fix integers

$$A \geqslant 1, \quad a \geqslant 1$$

about which we assume

$$p \nmid Aa.$$

We fix polynomials

$$f(x) \in k[x], \quad \deg(f) = A, \quad k \text{ some finite subfield of } \overline{\mathbb{F}_p},$$
$$g(x) \in k[x], \quad \deg(g) < a, \quad k \text{ some finite subfield of } \overline{\mathbb{F}_p},$$

We make the assumption that $f(x)$ is Artin–Schreier reduced: this means that in the expression $f(x) = \sum_i c_i x^i$, we have $c_i = 0$, if $p \mid i$. We define

$$\mathrm{gcd}_{\mathrm{deg}}(f) := \gcd(\{i \mid c_i \neq 0\})$$

the greatest common divisor of the degrees of the monomials appearing in f. We suppose

$$\gcd(a, \mathrm{gcd}_{\mathrm{deg}}(f)) = 1.$$

We fix χ a (possibly trivial) multiplicative character of a finite extension $k/\mathbb{F}_p$ containing the coefficients of f and g. We denote by ψ a chosen nontrivial additive character of $\mathbb{F}_p$. For L/k a finite extension, we denote by χ_L, respectively by ψ_L, the composition of χ, respectively of ψ, with $\mathrm{Norm}_{L/k}$, respectively with $\mathrm{Trace}_{L/\mathbb{F}_p}$.

We denote by $\mathcal{G}(f, g, a, \chi)$ the lisse sheaf on $\mathbb{G}_m/k$ whose trace function at time $t \in L^\times$, for L/k a finite extension, is

$$t \mapsto -\sum_{x \in L^\times} \psi_L(f(1/x) + g(x) + tx^a)\chi_L(x).$$

We will mostly be concerned with the case when $\chi = \chi_2$, the quadratic character.

2 Basic facts about $\mathcal{G}(f, g, a, \chi)$

The local system $\mathcal{G}(f, g, a, \chi)$ is lisse of rank $D = A + a$ on $\mathbb{G}_m$, and pure of weight one. We view it as being the Fourier transform

$$\mathrm{FT}_\psi\big([a]_\star\big(\mathcal{L}_{\psi(f(1/x)+g(x))} \otimes \mathcal{L}_\chi(x)\big)\big).$$

Lemma 2.1 *Let $A \geqslant 1$, $a \geqslant 1$, $p \nmid Aa$, f be Artin–Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = 1$. Then the $I(\infty)$-representation of $\mathcal{G}(f, g, a, \chi)$ is irreducible. It has rank $A + a$ and all slopes $A/(A + a)$.*

Proof This is a straightforward application of Laumon's results on the local monodromy of FT_ψ. The input sheaf to FT_ψ is lisse on $\mathbb{G}_m$ of rank a, with $I(0)$-slopes A/a. The hypothesis

$$\gcd(a, \gcd_{\deg}(f)) = 1$$

implies that the $I(0)$-representation of the input sheaf is irreducible, cf. the proof of [19, Lemma 2.1].

Then the $I(\infty)$-representation of $\mathcal{G}(f, g, a, \chi)$ is $\mathrm{FTloc}(0, \infty)$(rank a, slopes A/a), which has rank $A + a$ and all slopes $A/(A + a)$, cf. [10, 7.4.4 (4)]. The asserted irreducibility result from the irreducibility of the input and the fact that $\mathrm{FTloc}(0, \infty)$ is a suitable equivalence of categories. □

Lemma 2.2 *Suppose that $A = 1$ and that $p \nmid (a + 1)$. Then the image of $P(\infty)$ in the representation attached to $\mathcal{G}(f, g, a, \chi)$ is isomorphic to the additive group of the field $\mathbb{F}_p(\mu_{a+1})$.*

Proof The $I(\infty)$-representation is irreducible of prime to p rank $a + 1$ and has $\mathsf{Swan} = 1$. By [9, 1.14], the $I(\infty)$-representation is the Kummer direct image $[a+1]_\star\mathcal{L}$ of some $\mathcal{L}$ with $\mathsf{Swan} = 1$. Moreover, as a $P(\infty)$-representation, the $I(\infty)$ representation is the direct sum of the multiplicative translates, by elements of μ_{a+1}, of $\mathcal{L}$. Because $\mathcal{L}$ has $\mathsf{Swan} = 1$, it is of the form $\mathcal{L}_{\psi(ax)}$ for some $a \neq 0$ in $\overline{\mathbb{F}_p}$. Now repeat the (end of) the proof of [14, Lemma 1.2]. □

Lemma 2.3 *If both f, g are odd polynomials and a is odd, the local system $\mathcal{G}(f, g, a, \chi_2)$ is geometrically self-dual. Indeed, its constant field twist by $1/\mathsf{Gauss}(\psi, \chi_2)$ is arithmetically self-dual.*

Proof The local system $\mathcal{G}(f, g, a, \chi_2)$ is geometrically irreducible. The oddness of f, g, a insures that its constant field twist by $1/\mathsf{Gauss}(\psi, \chi_2)$, which is pure of weight zero, has real traces, hence the asserted autoduality. □

Theorem 2.4 *If both f, g are odd polynomials and a is odd, the geometric determinant of $\mathcal{G}(f, g, a, \chi_2)$ is $\mathcal{L}_{\chi_2}$.*

Proof We first explain the idea. For fixed data f, a, the local system $\mathcal{G}(f, g, a, \chi_2)$ makes sense for any *odd* polynomial g of degree $< a$. Such g form an affine space $\mathbb{A}^{(a-1)/2}$, and indeed there is a local system on $\mathbb{A}^{(a-1)/2} \times \mathbb{G}_m$, call it $\mathcal{G}_{\text{univ}}(f, a, \chi)$, whose trace function at a point (g, t) is

$$(g, t) \mapsto -\sum_x \psi(f(1/x) + g(x) + tx^a)\chi_2(x).$$

Because this local system is self-dual, its determinant, call it $\mathcal{L}_{\text{univ}}$, is either trivial or nontrivial of order 2. Viewing μ_2 as $\mathbb{Z}/2\mathbb{Z}$, we view $\mathcal{L}_{\text{univ}}$ as an element of $H^1((\mathbb{A}^{(a-1)/2} \times \mathbb{G}_m)/\overline{\mathbb{F}_p}, \mathbb{Z}/2\mathbb{Z})$. Because we are in odd characteristic p, the groups $H^i(\mathbb{A}^{(a-1)/2}/\overline{\mathbb{F}_p}, \mathbb{Z}/2\mathbb{Z}) = 0$ for all $i > 0$, and $H^0(\mathbb{A}^{(a-1)/2}/\overline{\mathbb{F}_p}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

By the Kunneth formula [3, Corollary 1.11], the map

$$\text{pr}_2 : \mathbb{A}^{(a-1)/2} \times \mathbb{G}_m \to \mathbb{G}_m, \quad (g, t) \mapsto t,$$

induces by pullback an isomorphism

$$H^1(\mathbb{G}_m/\overline{\mathbb{F}_p}, \mathbb{Z}/2\mathbb{Z}) \cong H^1\big((\mathbb{A}^{(a-1)/2} \times \mathbb{G}_m)/\overline{\mathbb{F}_p}, \mathbb{Z}/2\mathbb{Z}\big).$$

For any fixed $g_0 \in \mathbb{A}^{(a-1)/2}$, pullback by the inclusion

$$\text{incl}_{g_0} : \mathbb{G}_m \subset \mathbb{A}^{(a-1)/2} \times \mathbb{G}_m, \quad t \mapsto (g_0, t)$$

induces an isomorphism

$$H^1\big((\mathbb{A}^{(a-1)/2} \times \mathbb{G}_m)/\overline{\mathbb{F}_p}, \mathbb{Z}/2\mathbb{Z}\big) \cong H^1(\mathbb{G}_m/\overline{\mathbb{F}_p}, \mathbb{Z}/2\mathbb{Z}).$$

The composition $\text{pr}_2 \circ \text{incl}_{g_0}$ is the identity map of $\mathbb{G}_m$ to itself. Therefore the composition of their pullbacks, $\text{incl}^\star_{g_0} \circ \text{pr}^\star_2$ is the identity on $H^1(\mathbb{G}_m/\overline{\mathbb{F}_p}, \mathbb{Z}/2\mathbb{Z})$.

On the one hand, if we view $\mathcal{L}_{\text{univ}}$ as the pullback by pr_2 of a class L_0 on $\mathbb{G}_m$, then for any g_0 we have

$$\text{incl}^\star_{g_0}\mathcal{L}_{\text{univ}} = L_0.$$

The key point is that this pullback $\text{incl}^\star_{g_0}\mathcal{L}_{\text{univ}}$ is the determinant of the local system $\mathcal{G}(f, g_0, a, \chi_2)$ on $\mathbb{G}_m/\overline{\mathbb{F}_p}$. What we must show is that L_0 is $\mathcal{L}_{\chi_2}$. For this, it suffices to check at the single point $g_0 = 0$.

The local system $\mathcal{G}(f, g_0 = 0, a, \chi_2)$ is the local system denoted by $\mathcal{G}(f, a, \chi_2)$ in [20]. This local system has all ∞-slopes $A/(A + a) < 1$, and hence its entire G_{geom} is the Zariski closure of all conjugates of the image of $I(0)$. So its determinant is $\mathcal{L}_{\chi_2}$ if and only if the determinant of its $I(0)$-representation is $\mathcal{L}_{\chi_2}$. But its $I(0)$-representation is the direct sum

$$\bigoplus_{\rho\,:\,\rho^a=\chi_2} \mathcal{L}_\rho \oplus (\overline{\mathbb{Q}_\ell})^A.$$

Because a is odd, its determinant is $\mathcal{L}_{\chi_2}$. Indeed, if we fix one a^{th} root ρ_0 of χ_2, then all the a^{th} roots are $\rho_0\Lambda$ as Λ runs over the characters of order dividing a. So we have

$$\prod_{\rho\,:\,\rho^a=\chi_2} \rho = \rho_0^a \times \prod_{\Lambda\in\mathsf{Char}(a)} \Lambda = \chi_2 \times \mathbb{1} = \chi_2.$$

□

Theorem 2.5 *If f is an odd polynomial, a is odd, and $a > 2\deg(g)$, the geometric determinant of $\mathcal{G}(f, g, a, \chi_2)$ is $\mathcal{L}_{\chi_2}$.*

Proof By [19, Corollary 2.2], the geometric determinant is either trivial or is $\mathcal{L}_{\chi_2}$. Using the argument above, it first suffices to check for the specialization $g = 0$, and then to observe that, again by the previous argument, this specialization has the asserted determinant $\mathcal{L}_{\chi_2}$. □

We will now consider the following local systems on $\mathbb{G}_m^3$ with trace functions

$$(r, s, t) \mapsto -\sum_{x\in L^\times} \psi_L(r/x + sx^B + tx^a)\chi_{2,L}(x),$$

where

$$(B, a, p) = (3, 7, 5),\ (5, 7, 3),\ \text{or}\ (1, 7, 3), \tag{2.5.1}$$

or

$$(B, a, p) = (1, 5, 3) \text{ or } (2, 5, 3). \tag{2.5.2}$$

We will deviate from our previous notation, and denote this system as $\mathcal{G}_{r,s}$ as we always let t vary. In particular, $\mathcal{G}_{r_0,s_0}$ is the pullback of $\mathcal{G}_{r,s}$ by $r = r_0$ and $s = s_0$ for any $(r_0, s_0) \in \mathbb{G}_m^2$.

Recall the conditions (**S+**) and (**S-**) defined in [17, Section 1.1]. A basic fact about $\mathcal{G}_{r,s}$ is the following

Proposition 2.6 *For the three choices of (B, a, p) as in (2.5.1), and for any $(r_0, s_0) \in \mathbb{G}_m^2$, $\mathcal{G}_{r_0,s_0}$, and hence $\mathcal{G}_{r,s}$, satisfies* (**S+**).

Proof We already proved in Lemma 2.1 that the underlying representation V is $I(\infty)$-irreducible. Next, primitivity of $\mathcal{G}_{r_0,s_0}$ is proved for $(B, a, p) = (3, 7, 5), (5, 7, 3)$ in [19, Theorem 2.10(d)], and for $(B, a, p) = (1, 7, 3)$ in [19, Theorem 2.11]. An application of [19, Proposition 2.8] shows that V is tensor indecomposable over $I(0)$. Finally, for the first two choices of (B, a, p), V is not tensor induced by [19, Proposition 2.9(d)]. The same conclusion holds for the third choice by [19, Lemma 3.9], for in this case, $D = 8 = 2^3$, so the only possible n is 3, which is not prime to $p = 3$. □

Proposition 2.7 *For the two choices of* (B, a, p) *as in* (2.5.2)*, and for any* $(r_0, s_0) \in \mathbb{G}_m^2$, $\mathcal{G}_{r_0,s_0}$*, and hence* $\mathcal{G}_{r,s}$*, satisfies* (**S+**).

Proof This is [19, Corollary 2.13]. □

A natural question is what we can say about these local systems after specializing $s = 0$.

Proposition 2.8 *In any odd characteristic* p*, for any integer* $a \geqslant 4$ *with* $p \nmid a$*, and any multiplicative character* Λ *of order* N *prime to* p*, the local system on* $(\mathbb{G}_m \times \mathbb{G}_m)/\mathbb{F}_p(\mu_{aN})$ *whose trace function is*

$$(r, t) \mapsto -\sum_{x \neq 0} \psi(r/x + tx^a)\Lambda(x)$$

satisfies (**S+**)*. In fact, its specialization* $r = 1$ *satisfies* (**S+**) *and has* G_{geom} *containing a scalar multiple of a complex reflection if* $\Lambda \neq \mathbb{1}$.

Proof Choose an a^{th} root χ of Λ, i.e., a character χ of $\mathbb{F}_p(\mu_{aN})^\times$ with $\chi^a = \Lambda$. [Concretely, in terms of a generator ω of $\mathbb{F}_p(\mu_{aN})^\times$, $\Lambda(\omega)$ has order N, and the choice of χ is the choice of an a^{th} root of $\Lambda(\omega)$ in μ_{aN}. One then takes $\chi(\omega)$ to be this choice of a^{th} root.] We first prove that the $r = 1$ specialization, namely the local system on $\mathbb{G}_m/\mathbb{F}_p(\mu_{aN})$ whose trace function is

$$t \mapsto -\sum_{x \neq 0} \psi(1/x + tx^a)\chi^a(x)$$

is geometrically isomorphic to the Kloosterman sheaf $\mathcal{K}\ell(\mathbb{1}, \{\overline{\chi}\rho\}_{\rho \in \mathsf{Char}(a)})$. Indeed, this $r = 1, s = 0$ local system is geometrically calculated in terms of hypergeometric sheaves $\mathcal{H}$ as being

$$\begin{aligned} \mathrm{FT}_\psi\big([a]_\star(\mathcal{L}_{\psi(1/x)} \otimes \mathcal{L}_{\chi^a(x)})\big) &= \mathrm{FT}_\psi(\mathcal{L}_\chi \otimes [a]_\star \mathcal{H}(\varnothing; \mathbb{1})) \\ &\cong \mathrm{FT}_\psi(\mathcal{L}_\chi \otimes \mathcal{H}(\varnothing; \mathsf{Char}(a)) \\ &= \mathrm{FT}_\psi(\mathcal{H}(\varnothing; \chi\mathsf{Char}(a))) = \mathcal{K}\ell_\psi(\mathbb{1}, \overline{\chi}\mathsf{Char}(a)), \end{aligned}$$

cf. [10, 8.1.12 & 8.4.2]. Now, if Λ has order $e > 1$ then the a^{th} power of a generator g_0 of $I(0)$ of this Kloosterman sheaf has spectrum $\{1, \zeta_e, \ldots, \zeta_e\}$ and thus it is the ζ_e-multiple of a complex reflection.

We next check that this Kloosterman sheaf is primitive. It suffices to show it is not Kummer induced, by Pink's result [8, Lemmas 11, 12]. We argue by contradiction. Suppose it is Kummer induced of some degree $d \geqslant 2$, $d \mid (a + 1)$. Choose a prime divisor r of d. Then it is Kummer induced of degree r, and its characters are cosets of $\mathsf{Char}(r)$. If $r = a + 1$, then its characters are precisely $\mathsf{Char}(r)$. But if we take two distinct elements of $\overline{\chi}\mathsf{Char}(a)$, their ratio is a nontrivial element of $\mathsf{Char}(a)$, so has order dividing a, so cannot be an element of $\mathsf{Char}(r)$ (simply because $\gcd(r, a) = 1$).

If $r < a+1$, then $r \leqslant (a+1)/2$, hence outside the $\mathsf{Char}(r)$ coset of $\mathbb{1}$, there are two distinct elements of $\overline{\chi}\mathsf{Char}(a)$, and their ratio again gives a contradiction.

Because this Kloosterman sheaf is primitive, it has (**S+**) by [16, 1.7]. Once this $r = 1$ pullback local system has (**S+**), then so does the (r, t) local system. □

Corollary 2.9 *For any odd prime p, any integer $a \geqslant 4$ with $p \nmid a$, any integer $1 \leqslant B < a$, and any multiplicative character Λ of order N prime to p, the local system on $(\mathbb{G}_m \times \mathbb{A}^1 \times \mathbb{G}_m)/\mathbb{F}_p(\mu_{aN})$ whose trace function is*

$$(r, s, t) \mapsto -\sum_{x \neq 0} \psi(r/x + sx^B + tx^a)\,\Lambda(x)$$

satisfies (**S+**).

Proof Indeed, its $s = 0$ pullback satisfies (**S+**). □

Next we study the moment $M_{2,2}$ of $\mathcal{G}(1/x, x^B, a, \chi)$ for $B = 1, 2$. See [11, 1.16, 1.17] or [17, Section 2.6] for a discussion of the moments of pure local systems, and for their computation as a limsup.

Lemma 2.10 *For any prime p, any integer $a \geqslant 2$ with $p \nmid a$, and any multiplicative character χ of order prime to p, consider the local system $\mathcal{G}(1/x, x, a, \chi)$ on $\mathbb{G}_m^3/\mathbb{F}_p(\text{values of } \chi)$ whose trace function at points of $\mathbb{G}_m(L)^3$, for $L/\mathbb{F}_p(\text{values of } \chi)$ a finite extension, is*

$$(r, s, t) \mapsto \frac{-1}{\sqrt{\# L}} \sum_{x \in L^\times} \psi_L(r/x + sx + tx^a)\,\chi_L(x).$$

Then $M_{2,2} \leqslant 3$, with equality precisely when a is odd and χ is either $\mathbb{1}$ or χ_2, this second case allowed only for p odd.

Proof Since $M_{2,2}$ only decreases as G_{geom} grows, it suffices to prove $M_{2,2} \leqslant 3$ when we freeze $t = 1$, and consider the two parameter local system on $\mathbb{G}_m \times \mathbb{G}_m$ whose trace function is

$$(r, s) \mapsto \frac{-1}{\sqrt{\# L}} \sum_{x \in L^\times} \psi_L(r/x + sx + x^a)\,\chi_L(x).$$

By [18, 2.1], we may calculate its $M_{2,2}$ as the limsup, over finite extensions $\mathbb{F}_q/\mathbb{F}_p(\text{values of } \chi)$ of

$$\frac{1}{q^2(q-1)^2} \sum_{r,s \in \mathbb{F}_q^\times} \sum_{x,y,z,w \in \mathbb{F}_q^\times} \psi_{\mathbb{F}_q}\Bigg(r\left(\frac{1}{x} + \frac{1}{y} - \frac{1}{z} - \frac{1}{w}\right) + s(x + y - z - w)$$
$$+ x^a + y^a - z^a - w^a\Bigg)\chi(xy)\overline{\chi}(zw).$$

We first show that this limsup does not change if, instead of summing over $(r,s) \in (\mathbb{F}_q^\times)^2$, we sum over $(r,s) \in (\mathbb{F}_q)^2$. An individual summand with $r=0$, any s, is

$$\sum_{x,y,z,w \in \mathbb{F}_q^\times} \psi_{\mathbb{F}_q}\big(s(x+y-z-w)+x^a+y^a-z^a-w^a\big)\chi(xy)\overline{\chi}(zw) = \left|\sum_{x \in \mathbb{F}_q^\times} \psi_{\mathbb{F}_q}(sx+x^a)\chi(x)\right|^4,$$

which is $\leqslant (a\sqrt{q})^4$, hence is $O(q^2)$. Similarly, an individual summand with $s=0$, any r, is $\leqslant (a\sqrt{q})^4$, hence is $O(q^2)$. The total number of $(r,s) \in (\mathbb{F}_q)^2$ with $rs=0$ is $2q-1$, so we are only changing the inner sum by $O(q^3)$, while we are dividing by $1/q^2(q-1)^2$.

We now examine

$$\frac{1}{q^2(q-1)^2}\sum_{r,s \in \mathbb{F}_q}\sum_{x,y,z,w \in \mathbb{F}_q^\times} \psi_{\mathbb{F}_q}\left(r\left(\frac{1}{x}+\frac{1}{y}-\frac{1}{z}-\frac{1}{w}\right)+s(x+y-z-w)\right.$$
$$\left.+x^a+y^a-z^a-w^a\right)\chi(xy)\overline{\chi}(zw)$$
$$= \frac{1}{q^2(q-1)^2}\sum_{x,y,z,w \in \mathbb{F}_q^\times} \psi_{\mathbb{F}_q}(x^a+y^a-z^a-w^a)\chi(xy)\overline{\chi}(zw)$$
$$\times \sum_{r,s \in \mathbb{F}_q} \psi_{\mathbb{F}_q}\left(r\left(\frac{1}{x}+\frac{1}{y}-\frac{1}{z}-\frac{1}{w}\right)+s(x+y-z-w)\right)$$
$$= \frac{1}{(q-1)^2}\sum_{\substack{x,y,z,w \in \mathbb{F}_q^\times \\ 1/x+1/y=1/z+1/w,\ x+y=z+w}} \psi_{\mathbb{F}_q}(x^a+y^a-z^a-w^a)\chi(xy)\overline{\chi}(zw).$$

We now examine the two equations in x, y, z, w with $xyzw \neq 0$ given by

$$1/x+1/y = 1/z+1/w, \quad x+y = z+w,$$

which we rewrite as

$$(x+y)/xy = (z+w)/zw, \quad x+y = z+w.$$

If $x+y=z+w=0$, we have the plane $y=-x, z=-w$. If $x+y=z+w \neq 0$, then we divide by them and get $1/xy = 1/zw$. Thus we have $x+y=z+w$ and $xy=zw$, and hence the two sets $\{x,y\}$ and $\{z,w\}$ coincide. So we have the two planes $x=z, y=w$ and $x=w, y=z$. On each of these last two planes, the function $x^a+y^a-z^a-w^a$ vanishes, so $\psi(x^a+y^a-z^a-w^a)=1$, and $\chi(xy)\overline{\chi}(zw)=1$.

On the first plane $y=-x, z=-w$, the sum $x^a+y^a-z^a-w^a$ vanishes precisely when a is odd, and $\chi(xy)\overline{\chi}(zw) = \chi(x^2/w^2)$.

Thus if both a is odd and $\chi^2 = \mathbb{1}$, then the first plane also contributes 1 to $M_{2,2}$. In general its contribution is

$$\frac{1}{(q-1)^2} \sum_{x,w \in \mathbb{F}_q^\times} \psi_{\mathbb{F}_q}\big(x^a + (-x)^a - w^a - (-w)^a\big)\chi(-x^2)\overline{\chi}(-w^2)$$
$$= \frac{1}{(q-1)^2}\left|\sum_{x \in \mathbb{F}_q^\times} \psi_{\mathbb{F}_q}(x^a + (-x)^a)\chi(-x^2)\right|^2,$$

which is $O(1/q)$ unless both a is odd and $\chi^2 = \mathbb{1}$.

Now we return to $\mathcal{G}(1/x, x, a, \chi)$ on $\mathbb{G}_m^3$. We have proven that for its $t = 1$ specialization, we have $M_{2,2} \leqslant 3$, with equality precisely when both a is odd and $\chi^2 = \mathbb{1}$. When we do not have equality, we have $M_{2,2} = 2$. As $M_{2,2}$ can only decrease for a bigger group, we certainly have $M_{2,2} \leqslant 2$, and hence $M_{2,2} = 2$, for $\mathcal{G}(1, x, x, a, \chi)$ unless both a is add and $\chi^2 = \mathbb{1}$. In the case when a is odd and $\chi^2 = \mathbb{1}$, then $\mathcal{G}(1, x, x, a, \chi)$ is self dual, hence has $M_{2,2} \geqslant 3$, and thus has the asserted $M_{2,2} = 3$. □

Lemma 2.11 *For any odd prime p, any integer $a \geqslant 3$ with $p \nmid a$, and any multiplicative character χ of order prime to p, consider the local system $\mathcal{G}(1/x, x^2, a, \chi)$ on $\mathbb{G}_m^3/\mathbb{F}_p(\text{values of } \chi)$ whose trace function at points of $\mathbb{G}_m(L)^3$, for $L/\mathbb{F}_p(\text{values of } \chi)$ a finite extension, is*

$$(r, s, t) \mapsto \frac{-1}{\sqrt{\#L}} \sum_{x \in L^\times} \psi_L(r/x + sx^2 + tx^a)\chi_L(x).$$

Then $M_{2,2} = 2$.

Proof By [18, 2.1], we may calculate the $M_{2,2}$ as the limsup, over finite extensions $\mathbb{F}_q/\mathbb{F}_p(\text{values of } \chi)$ of

$$\frac{1}{q^2(q-1)^3} \sum_{r,s,t \in \mathbb{F}_q^\times} \sum_{x,y,z,w \in \mathbb{F}_q^\times} \psi_{\mathbb{F}_q}\left(r\left(\frac{1}{x} + \frac{1}{y} - \frac{1}{z} - \frac{1}{w}\right) + s(x^2 + y^2 - z^2 - w^2)\right.$$
$$\left. + t(x^a + y^a - z^a - w^a)\right)\chi(xy)\overline{\chi}(zw).$$

We will refer to

$$\sum_{r,s,t \in \mathbb{F}_q^\times} \sum_{x,y,z,w \in \mathbb{F}_q^\times} \psi_{\mathbb{F}_q}\left(r\left(\frac{1}{x} + \frac{1}{y} - \frac{1}{z} - \frac{1}{w}\right) + s(x^2 + y^2 - z^2 - w^2)\right.$$
$$\left. + t(x^a + y^a - z^a - w^a)\right)\chi(xy)\overline{\chi}(zw)$$

as “a summand".

We first show that this limsup does not change if, instead of summing over $(r, s, t) \in (\mathbb{F}_q^\times)^3$, we sum over $(r, s, t) \in (\mathbb{F}_q)^3$. The summand with $(r, s, t) = (0, 0, 0)$ is

$$\sum_{x,y,z,w\in\mathbb{F}_q^\times} \chi(xy)\overline{\chi}(zw) = \left|\sum_{x\in\mathbb{F}_q^\times} \chi(x)\right|^4,$$

which vanishes if $\chi \neq \mathbb{1}$, and is $(q-1)^4$ if $\chi = \mathbb{1}$.

An individual summand with $r = 0$, any $(s, t) \neq (0, 0)$, is

$$\sum_{x,y,z,w\in\mathbb{F}_q^\times} \psi_{\mathbb{F}_q}\big(s(x^2+y^2-z^2-w^2)+t(x^a+y^a-z^a-w^a)\big)\chi(xy)\overline{\chi}(zw)$$
$$= \left|\sum_{x\in\mathbb{F}_q^\times} \psi_{\mathbb{F}_q}(sx^2+tx^a)\chi(x)\right|^4,$$

which is $\leqslant ((a+1)\sqrt{q})^4$, hence is $O(q^2)$. Similarly, an individual summand with $s = 0$, any $(r, t) \neq (0, 0)$, is $\leqslant ((a+1)\sqrt{q})^4$, hence is $O(q^2)$. Finally, an individual summand with $t = 0$, any $(r, s) \neq (0, 0)$, is $\leqslant (3\sqrt{q})^4$, hence is $O(q^2)$. The total number of $(r, s, t) \in (\mathbb{F}_q)^3$ with $(r, s, t) \neq (0, 0, 0)$ but $rst = 0$ is $O(q^2)$, so those terms are only changing the inner sum by $O(q^4)$, and the $(0, 0, 0)$ term is either 0 or $(q-1)^4$, while we are dividing by $q^2(q-1)^3$.

We now examine

$$\frac{1}{q^2(q-1)^3}\sum_{r,s,t\in\mathbb{F}_q,\ x,y,z,w\in\mathbb{F}_q^\times} \psi_{\mathbb{F}_q}\left(r\left(\frac{1}{x}+\frac{1}{y}-\frac{1}{z}-\frac{1}{w}\right)+s(x^2+y^2-z^2-w^2)\right.$$
$$\left.+\,t(x^a+y^a-z^a-w^a)\right)\chi(xy)\overline{\chi}(zw)$$
$$= \frac{1}{q^2(q-1)^3}\sum_{x,y,z,w\in\mathbb{F}_q^\times} \chi(xy)\overline{\chi}(zw)\sum_{t\in\mathbb{F}_q}\psi_{\mathbb{F}_q}\big(t(x^a+y^a-z^a-w^a)\big)$$
$$\times\sum_{r\in\mathbb{F}_q}\psi_{\mathbb{F}_q}\left(r\left(\frac{1}{x}+\frac{1}{y}-\frac{1}{z}-\frac{1}{w}\right)\right)\sum_{s\in\mathbb{F}_q}\psi_{\mathbb{F}_q}\big(s(x^2+y^2-z^2-w^2)\big)$$
$$= \frac{q}{(q-1)^3}\sum_{\substack{x,y,z,w\in\mathbb{F}_q^\times\\(x,y,z,w)\in\Sigma}} \chi(xy)\overline{\chi}(zw),$$

where the locus Σ is defined by the equations

$$1/x + 1/y = 1/z + 1/w, \quad x^2+y^2 = z^2+w^2, \quad x^a+y^a = z^a+w^a.$$

We first look at the intersection of Σ with $x + y = z + w$. If $x + y = z + w = 0$, then as $p \neq 2$ the second equation $x^2 + y^2 = z^2 + w^2$ gives the two lines $y = -x$,

$z = -w = \pm x$. If $x + y = z + w \neq 0$, then we divide the first equation by them and get $1/xy = 1/zw$. Thus we have $x + y = z + w$ and $xy = zw$, and hence the two sets $\{x, y\}$ and $\{z, w\}$ coincide. So we have the two planes $x = z$, $y = w$ and $x = w$, $y = z$. On each of these two planes, $x^a + y^a - z^a - w^a = 0$ and $\chi(xy)\overline{\chi}(zw) = 1$. So the contribution of this intersection to the limsup is $2 + O(1/q)$.

It remains to show that the number of points $(x, y, z, w) \in \Sigma(\mathbb{F}_q^\times)$ with $x + y \neq z + w$ is $O(q)$. Note that if $x + y = 0$ then $1/z + 1/w = 1/x + 1/y = 0$ and hence $z + w = 0 = x + y$. So we may assume $x + y \neq 0$, $z + w \neq 0$, and introduce new variables

$$u = x + y, \quad v = z + w, \quad t = xy/(x + y),$$

so that

$$s := xy = tu, \quad zw = tv, \quad \text{but } u \neq v.$$

Now

$$u^2 - 2tu = x^2 + y^2 = z^2 + w^2 = v^2 - 2tv,$$

and so

$$u + v = 2t, \quad \text{i.e. } v = 2t - u.$$

An easy induction on odd $a \geqslant 3$ shows that there are some integers $c_0 = 1, c_1, \ldots, c_{(a-1)/2}$ such that $x^a + y^a = \sum_{i=0}^{(a-1)/2} c_i u^{n-2i} s^i$; in fact, $c_i = \frac{n}{n-i}\binom{n-i}{i}$. It follows that $x^a + y^a = \sum_{i=0}^{(a-1)/2} c_i u^{n-i} t^i$, and so the condition $x^a + y^a - z^a - w^a = 0$ is equivalent to the vanishing of

$$\sum_{i=0}^{(a-1)/2} c_i u^{n-i} t^i - \sum_{i=0}^{(a-1)/2} c_i (2t - u)^{n-i} t^i,$$

a homogeneous polynomial in u, t of degree a with the coefficient for u^a being $1 - (-1)^a = 2$. As $p \neq 2$, given any $t \in \mathbb{F}_q$ there are at most a values for u that satisfy this last condition. For each (u, t), there are at most two pairs (x, y) with $x + y = u$, $xy = tu$, and there are at most two pairs (z, w) with $z + w = 2t - u$, $xy = t(2t - u)$. It follows that the number of points $(x, y, z, w) \in \Sigma(\mathbb{F}_q^\times)$ with $x + y \neq z + w$ is at most $4aq$, as stated. ◻

3 Preliminaries on specializations of G_{geom}

We first quote verbatim from [18, Section 11, Theorem 11.1], cf. [10, 8.17, 8.18].

"The situation we consider is the following. We are given a normal connected affine noetherian scheme $S = \mathrm{Spec}(A)$ with A a noetherian normal integral domain with fraction field K, and a chosen algebraic closure $\overline{K}$ of K. Thus $\mathrm{Spec}(K)$ is a generic point η of S, and $\mathrm{Spec}(\overline{K})$ is a geometric point $\overline{\eta}$ of S. We are given X/S a smooth S-scheme of relative dimension D, with geometrically connected fibres, and $\phi \in X(S)$ a section of X/S. Then $\phi(\overline{\eta})$ is a geometric point of X. We are given a finite group G and a surjective homomorphism

$$\pi_1(X, \phi(\overline{\eta})) \twoheadrightarrow G.$$

For each geometric point s of S, $\phi(s)$ is a geometric point of X_s (and also of X). We have a continuous group homomorphism

$$\pi_1(X_s, \phi(s)) \to \pi_1(X, \phi(s)) \cong \pi_1(X, \phi(\overline{\eta})).$$

This last isomorphism is only canonical up to inner automorphism of the target group $\pi_1(X, \phi(\overline{\eta}))$. By composition, we get a group homomorphism

$$\pi_1(X_s, \phi(s)) \to G$$

which is well defined up to inner automorphism of G. This applies in particular with s taken to be $\overline{\eta}$. We are interested in how the image of $\pi_1(X_s, \phi(s))$ in G compares with the image of $\pi_1(X_{\overline{\eta}}, \phi(\overline{\eta}))$ in G: when are these two subgroups of G conjugate in G? Let us denote these image groups G_s and $G_{\overline{\eta}}$.

Theorem 3.1 ([18, 11.1]) *There exists a dense open set $U \subset S$ such that for any geometric point $s \in U$, G_s and $G_{\overline{\eta}}$ are conjugate subgroups of G. Moreover, for any geometric point $s \in S$, G_s is conjugate to a subgroup of $G_{\overline{\eta}}$."*

Because G is a *finite* group, this theorem has the following more precise corollary.

Corollary 3.2 *The set of points $s \in S$ at which G_s is conjugate to $G_{\overline{\eta}}$ in G is open in S.*

Proof The group $G_{\overline{\eta}}$ is finite (because it is a subgroup of the finite group G). Therefore it has only finitely many subgroups, say $G_{\overline{\eta}} = H_0, H_1, \ldots, H_r$. The proof of [10, 8.17, 8.18] shows that each of the sets

$$Z_i := \{s \in S \mid G_s \text{ is conjugate to } H_i\}$$

and each of the sets

$$W_i := \{s \in S \mid G_s \text{ is conjugate to a subgroup of } H_i\}$$

is constructible (a finite union of sets of the form (open set) $\cap$ (closed set)). But each set W_i is stable by specialization, hence is closed, cf. [5, Chapter II, 3.18]. Therefore $\bigcup_{i \geqslant 1} W_i$ is closed. Then its open complement is precisely the set Z_0. □

4 Finiteness theorems

We denote by q the cardinality of k, and by k_r the unique extension of k of degree r in $\overline{\mathbb{F}_q}$. $V : (\mathbb{Q}/\mathbb{Z})_{\text{prime to } p} \to [0, 1)$ will denote Kubert's V-function for the prime p (cf. [14]).

Theorem 4.1 *Let $d_1 > d_2 > \cdots > d_n > 0$ be prime to p integers, $\mathbf{d} = (d_1, \ldots, d_n)$ and $\mathcal{F}$ the local system on $\mathbb{G}_{m,k}^{n+1}$ whose trace function is given by*

$$F(k_r; s, t_1, \ldots, t_n) = -\frac{1}{\sqrt{|k_r|}} \sum_{x \in k_r^\times} \psi_{k_r}\left(s/x + t_1 x^{d_1} + \cdots + t_n x^{d_n}\right) \chi_{2,k_r}(x).$$

Then $\mathcal{F}$ has finite (geometric and arithmetic) monodromy group if and only if

$$V\left(d_1 x_1 + \cdots + d_n x_n + \frac{1}{2}\right) + V(x_1) + \cdots + V(x_n) \geqslant \frac{1}{2}$$

for every $(x_1, \ldots, x_n) \in (\mathbb{Q}/\mathbb{Z})^n_{\text{prime to } p}$.

Proof By [14, Proposition 2.1], we need to show that $F(k_r; s, t_1, \ldots, t_n)$ is an algebraic integer or, equivalently, that

$$\operatorname{ord}_{q^r}\left(\sum_{x \in k_r^\times} \psi_{k_r}\left(s/x + t_1 x^{d_1} + \cdots + t_n x^{d_n}\right) \chi_{2,k_r}(x)\right) \geqslant \frac{1}{2}$$

for every $r \geqslant 1$ and $(s, t_1, \ldots, t_n) \in (k_r^\times)^{n+1}$. Taking the Mellin transform on $\mathbb{G}_m^{n+1}$, this is equivalent to

$$\begin{aligned}
&\sum_{s, t_1, \ldots, t_n \in k_r^\times} \eta(s)\xi_1(t_1)\cdots\xi_n(t_n) \sum_{x \in k_r^\times} \psi_{k_r}\left(s/x + t_1 x^{d_1} + \cdots + t_n x^{d_n}\right) \chi_{2,k_r}(x) \\
&\quad = \sum_{x \in k_r^\times} \chi_{2,k_r}(x) \left(\sum_{s \in k_r^\times} \psi_{k_r}(s/x)\eta(s)\right)\left(\sum_{t_1 \in k_r^\times} \psi_{k_r}(t_1 x^{d_1})\xi_1(t_1))\right) \\
&\qquad\qquad \cdots \left(\sum_{t_n \in k_r^\times} \psi_{k_r}(t_n x^{d_n})\xi_n(t_n)\right) \\
&\quad = \sum_{x \in k_r^\times} \chi_{2,k_r}(x)\eta(x) G_r(\eta) \bar{\xi}_1^{d_1}(x) G_r(\xi_1) \cdots \bar{\xi}_n^{d_n}(x) G_r(\xi_n) \\
&\quad = G_r(\eta) G_r(\xi_1) \cdots G_r(\xi_n) \sum_{x \in k_r^\times} \left(\chi_{2,k_r} \eta \bar{\xi}_1^{d_1} \cdots \bar{\xi}_n^{d_n}\right)(x) \\
&\quad = \begin{cases} 0 & \text{if } \chi_{2,k_r} \eta \bar{\xi}_1^{d_1} \cdots \bar{\xi}_n^{d_n} \neq \mathbf{1}, \\ (q^r - 1) G_r(\eta) G_r(\xi_1) \cdots G_r(\xi_n) & \text{if } \chi_{2,k_r} \eta \bar{\xi}_1^{d_1} \cdots \bar{\xi}_n^{d_n} = \mathbf{1} \end{cases}
\end{aligned}$$

has $\mathrm{ord}_{q^r} \geqslant 1/2$ for every $\eta, \xi_1, \ldots, \xi_n \in \widehat{k_r^\times}$, where $G_r(\chi)$ denotes the Gauss sum associated to the multiplicative character χ on k_r. This reduces to

$$\mathrm{ord}_{q^r}\left(G_r\left(\chi_{2,k_r}\xi_1^{d_1}\cdots\xi_n^{d_n}\right)G_r(\xi_1)\cdots G_r(\xi_n)\right) \geqslant \frac{1}{2}$$

for every $\xi_1, \ldots, \xi_n \in \widehat{k_r^\times}$ which, by Stickelberger, is equivalent to the given condition. □

If $d_1x_1 + \cdots + d_nx_n + 1/2 \neq 0$, using that $V(y) + V(-y) = 1$ for $y \neq 0$, we can rewrite the condition as

$$V\left(d_1x_1 + \cdots + d_nx_n + \frac{1}{2}\right) \leqslant V(-x_1) + \cdots + V(-x_n) + \frac{1}{2},$$

which trivially holds for $d_1x_1 + \cdots + d_nx_n + 1/2 = 0$. So we have

Corollary 4.2 *The local system $\mathcal{F}$ has finite monodromy if and only if the following two conditions hold:*

(i) $V(d_1x_1 + \cdots + d_nx_n + 1/2) \leqslant V(-x_1) + \cdots + V(-x_n) + 1/2$ *for every* $(x_1, \ldots, x_n) \in (\mathbb{Q}/\mathbb{Z})^n_{\text{prime to } p}$.
(ii) $V(x_1) + V(x_2) + \cdots + V(x_n) \geqslant 1/2$ *for every* $(x_1, \ldots, x_n) \in (\mathbb{Q}/\mathbb{Z})^n_{\text{prime to } p}$ *with* $\sum_{i=1}^n d_ix_i = 1/2$.

Note that the second condition is the criterion for the local system on $\mathbb{G}_m^n$ with trace function

$$F(k_r; t_1, \ldots, t_n) = -\frac{1}{\sqrt{|k_r|}}\sum_{x\in k_r^\times}\psi_{k_r}\left(t_1x^{d_1} + \cdots + t_nx^{d_n}\right)\chi_{2,k_r}(x)$$

to have finite monodromy. In terms of the sum-of-digits function $[-]_{p,r,-}$ defined in [13, Appendix], the first condition becomes

$$\left[d_1x_1 + \cdots + d_nx_n + \frac{p^r - 1}{2}\right]_{p,r,-} \leqslant [p^r - 1 - x_1]_{p,r,-} + \cdots + [p^r - 1 - x_n]_{p,r,-} + \frac{r(p-1)}{2}$$

for every $r \geqslant 1$ and every $0 < x_1, \ldots, x_n < p^r$ such that $p^r - 1$ does not divide $d_1x_1 + \cdots + d_nx_n + (p^r - 1)/2$. An argument similar to [14, Theorem 2.12] then shows

Proposition 4.3 *Suppose that there exists some real $A \geqslant 0$ such that*

$$\left[d_1x_1 + \cdots + d_nx_n + \frac{p^r - 1}{2}\right]_p \leqslant [p^r - 1 - x_1]_p + \cdots + [p^r - 1 - x_n]_p + \frac{r(p-1)}{2} + A$$

for every $r \geqslant 1$ and every $0 \leqslant x_1, \dots, x_n \leqslant p^r - 1$, where $[x]_p$ denotes the sum of the p-adic digits of x. Then condition (i) *in Corollary 4.2 holds.*

For $r \geqslant 1$ and an n-tuple $(x_1, \dots, x_n)$ with $0 \leqslant x_i \leqslant p^r - 1$, let

$$C(r; x_1, \dots, x_n) = \left[\sum_{i=1}^n d_i x_i + \frac{p^r - 1}{2}\right]_p - \sum_{i=1}^n [p^r - 1 - x_i]_p - \frac{r(p-1)}{2}.$$

For $s \geqslant 1$ we say that the n-tuple $(z_1, \dots, z_n)$ with $0 \leqslant z_1, \dots, z_n \leqslant p^s - 1$ is *s-good* if one of these conditions holds:

(a) $C(s; z_1, \dots, z_n) \leqslant 0$.
(b) There exist an $s' < s$ and an n-tuple $(z'_1, \dots, z'_n)$ with $0 \leqslant z'_i \leqslant p^{s'} - 1$ such that $C(s'; z'_1, \dots, z'_n) \geqslant C(s; z_1, \dots, z_n)$ and for every $j > 0$ the $(s+j)$-th digit in the p-adic expansion of $\sum_{i=1}^n d_i z_i + (p^s - 1)/2$ is greater than or equal to the $(s'+j)$-th digit in the p-adic expansion of $\sum_{i=1}^n d_i z'_i + (p^{s'} - 1)/2$ (counting the digits from right to left).

We say that the n-tuple $(x_1, \dots, x_n)$ with $0 \leqslant x_1, \dots, x_n \leqslant p^r - 1$ has *good termination* if, for some $1 \leqslant s < r$, the n-tuple $(z_1, \dots, z_n)$ whose i-th coordinate is the number formed by the last s p-adic digits of x_i (i.e. the remainder of the division of x_i by p^s) is s-good.

Proposition 4.4 *Suppose that there exists some $r_0 \geqslant 1$ such that all n-tuples $(x_1, \dots, x_n)$ with $0 \leqslant x_1, \dots, x_n \leqslant p^{r_0} - 1$ have good termination. Then the hypothesis of Proposition 4.3 holds.*

Proof Let

$$A = \max_{1 \leqslant r \leqslant r_0} \max_{1 \leqslant x_1, \dots, x_n \leqslant p^r - 1} C(r; x_1, \dots, x_n).$$

We will prove by induction on r that

$$C(r; x_1, \dots, x_n) \leqslant A$$

for every $r \geqslant 1$ and every $0 \leqslant x_1, \dots, x_n \leqslant p^r - 1$. For $r \leqslant r_0$ this is obvious by definition of A. Fix $r > r_0$ and assume that the inequality holds for all smaller r. Let $(x_1, \dots, x_n)$ be an n-tuple with $0 \leqslant x_1, \dots, x_n \leqslant p^r - 1$. By hypothesis it has good termination, since every $s < r_0$ is also $< r$. Hence there is some $1 \leqslant s < r_0$ such that the n-tuple $(z_1, \dots, z_n)$ whose i-th coordinate is the number formed by the last s p-adic digits of x_i is s-good. Let $y_i = p^{-s}(x_i - z_i)$, that is, the number obtained from x_i by removing its last s p-adic digits. Then $0 \leqslant y_i \leqslant p^{r-s} - 1$, $[p^r - 1 - x_i] = [p^{r-s} - 1 - y_i] + [p^s - 1 - z_i]$ and

$$\left[\sum_{i=1}^n d_i x_i + \frac{p^r - 1}{2}\right] = \left[\sum_{i=1}^n d_i y_i + \frac{p^{r-s} - 1}{2}\right] + \left[\sum_{i=1}^n d_i z_i + \frac{p^s - 1}{2}\right] - \delta(p-1)$$

where δ is the number of digit carries in the (outer) sum

$$\sum_{i=1}^{n} d_i x_i + \frac{p^r - 1}{2} = p^s\left(\sum_{i=1}^{n} d_i y_i + \frac{p^{r-s} - 1}{2}\right) + \left(\sum_{i=1}^{n} d_i z_i + \frac{p^s - 1}{2}\right).$$

In particular, $C(r; x_1, \dots, x_n) = C(r-s; y_1, \dots, y_n) + C(s; z_1, \dots, z_n) - \delta(p-1)$. We now have two options according to the definition of s-good.

(a) $C(s; z_1, \dots, z_n) \leqslant 0$. Then

$$\begin{aligned} C(r; x_1, \dots, x_n) &\leqslant C(r-s; y_1, \dots, y_n) + C(s; z_1, \dots, z_n) \\ &\leqslant C(r-s; y_1, \dots, y_n) \leqslant A \end{aligned}$$

by induction.

(b) There exist an $s' < s$ and an n-tuple $(z'_1, \dots, z'_n)$ with $0 \leqslant z'_i \leqslant p^{s'} - 1$ such that $C(s'; z'_1, \dots, z'_n) \geqslant C(s; z_1, \dots, z_n)$ and for every $j > 0$ the $(s+j)$-th digit in the p-adic expansion of $\sum_{i=1}^{n} d_i z_i + (p^s - 1)/2$ is greater than or equal to the $(s'+j)$-th digit in the p-adic expansion of $\sum_{i=1}^{n} d_i z'_i + (p^{s'} - 1)/2$ (counting from the right).

Let $x'_i = p^{s'} y_i + z'_i$ for $i = 1, \dots, n$. Then

$$C(r - s + s'; x'_1, \dots, x'_n) = C(r-s; y_1, \dots, y_n) + C(s'; z'_1, \dots, z'_n) - \epsilon(p-1),$$

where ϵ is the number of digit carries in the (outer) sum

$$\sum_{i=1}^{n} d_i x'_i + \frac{p^{r-s+s'} - 1}{2} = p^{s'}\left(\sum_{i=1}^{n} d_i y_i + \frac{p^{r-s} - 1}{2}\right) + \left(\sum_{i=1}^{n} d_i z'_i + \frac{p^{s'} - 1}{2}\right).$$

The hypothesis on the digits of $\sum_{i=1}^{n} d_i z_i + (p^s - 1)/2$ and $\sum_{i=1}^{n} d_i z'_i + (p^{s'} - 1)/2$ implies that $\epsilon \leqslant \delta$. Therefore

$$\begin{aligned} C(r; x_1, \dots, x_n) &= C(r-s; y_1, \dots, y_n) + C(s; z_1, \dots, z_n) - \delta(p-1) \\ &\leqslant C(r-s; y_1, \dots, y_n) + C(s'; z'_1, \dots, z'_n) - \epsilon(p-1) \\ &= C(r-s+s'; x'_1, \dots, x'_n) \leqslant A \end{aligned}$$

by induction. □

Theorem 4.5 *The local system on $\mathbb{G}^3_{m,\mathbb{F}_3}$ whose trace function is given by*

$$(\mathbb{F}_{3^r}; s, t, u) \mapsto -\frac{1}{3^{r/2}} \sum_{x \in \mathbb{F}_{3^r}^\times} \psi_{\mathbb{F}_{3^r}}(s/x + tx + ux^7)\chi_{2,\mathbb{F}_{3^r}}(x)$$

has finite monodromy.

Proof By Corollary 4.2, we need to show

(i) $V(x_1 + 7x_2 + 1/2) \leqslant V(-x_1) + V(-x_2) + 1/2$ for every $(x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^2_{\text{prime to } 3}$.

(ii) $V(x) + V(-7x + 1/2) \geqslant 1/2$ for every $x \in (\mathbb{Q}/\mathbb{Z})_{\text{prime to } 3}$.

Condition (ii) is the criterion for finite monodromy of the local system with trace function

$$F(\mathbb{F}_{3^r}; t, u) \mapsto -\frac{1}{3^{r/2}} \sum_{x \in \mathbb{F}_{3^r}^\times} \psi_{\mathbb{F}_{3^r}}(tx + ux^7)\chi_{2,\mathbb{F}_{3^r}}(x)$$

which holds by [12, Theorem 4.3] since $7 = (3^3+1)/(3+1)$. For the first condition, following Proposition 4.4, we check by a computer search that all pairs (x_1, x_2) with $0 \leqslant x_1, x_2 \leqslant 3^4 - 1$ have good termination.

For each $s = 1, 2, 3$, the following tables show the list of all pairs (z_1, z_2) with $0 \leqslant z_1, z_2 \leqslant 3^s - 1$ such that

(a) $C(s; z_1, z_2) > 0$, and

(b) do not have good termination (i.e. all their last-digits truncations appear in the previous tables marked with •).

If condition (b) for being s-good can be applied to them, we show the possible values of s', z_1', z_2' on the table, otherwise we mark it with • and move it on to the next s. All values of z_i and z_i' are shown as their 3-adic expansion. The columns D and D' show the result of removing the last s (respectively s') digits of $z_1 + 7z_2 + (3^s - 1)/2$ (resp. of $z_1' + 7z_2' + (3^{s'} - 1)/2$). Each digit of the number in column D must be greater than or equal to the corresponding digit of the number in column D'.

$s = 1$

z_1	z_2	$C(s; z_1, z_2)$	D	
1	2	2	12	•
2	2	4	12	•

$s = 2$

z_1	z_2	$C(s; z_1, z_2)$	D	s'	z_1'	z_2'	$C(s'; z_1', z_2')$	D'
21	22	2	21	•				
02	22	2	20	•				
12	12	2	11	•				
12	22	2	21	•				
22	02	2	2	•				
22	12	2	12	1	1	2	2	12
22	22	4	21	•				

$s=3$								
z_1	z_2	$C(s; z_1, z_2)$	D	s'	z_1'	z_2'	$C(s'; z_1', z_2')$	D'
121	222	2	21	2	21	22	2	21
221	122	2	12	1	1	2	2	12
221	222	2	22	1	1	2	2	12
102	222	2	21	2	21	22	2	21
202	122	2	12	1	1	2	2	12
202	222	4	21	2	22	22	4	21
112	212	2	20	2	02	22	2	20
212	112	2	11	2	12	12	2	11
212	212	2	21	2	12	22	2	21
112	222	2	11	2	12	12	2	11
212	122	2	12	1	1	2	2	12
212	222	2	22	1	1	2	2	12
022	202	2	12	1	1	2	2	12
222	202	2	20	2	02	22	2	20
222	212	2	21	2	21	22	2	21
022	222	2	21	2	21	22	2	21
122	122	2	12	1	1	2	2	12
122	222	4	21	2	22	22	4	21
222	122	4	12	1	2	2	4	12
222	222	4	22	1	2	2	4	12

In the last table (for $s = 3$) there are no remaining cases left with •, so this finishes the proof. □

Theorem 4.6 *The local system on $\mathbb{G}^3_{m,\mathbb{F}_3}$ whose trace function is given by*

$$(\mathbb{F}_{3^r}; s, t, u) \mapsto -\frac{1}{3^{r/2}} \sum_{x \in \mathbb{F}_{3^r}^\times} \psi_{\mathbb{F}_{3^r}}(s/x + tx^5 + ux^7)\chi_{2,\mathbb{F}_{3^r}}(x)$$

has finite monodromy.

Proof By Corollary 4.2, we need to show
(i) $V(5x_1+7x_2+1/2) \leqslant V(-x_1)+V(-x_2)+1/2$ for every $(x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^2_{\text{prime to }3}$.
(ii) $V(x_1) + V(x_2) \geqslant 1/2$ for every $(x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^2_{\text{prime to }3}$ such that $5x_1 + 7x_2 = 1/2$.

Condition (ii) is the criterion for finite monodromy of the local system with trace function

$$F(\mathbb{F}_{3^r}; t, u) \mapsto -\frac{1}{3^{r/2}} \sum_{x \in \mathbb{F}_{3^r}^\times} \psi_{\mathbb{F}_{3^r}}(tx^5 + ux^7)\chi_{2,\mathbb{F}_{3^r}}(x)$$

which holds by [17, Theorem 10.3.13 (vi)]. For the first condition, following Proposition 4.4, we check by a computer search that all pairs (x_1, x_2) with $0 \leqslant x_1, x_2 \leqslant 3^6 - 1$ have good termination. For each $s = 1, 2, 3, 4, 5$, the following tables show the list of all pairs (z_1, z_2) with $0 \leqslant z_1, z_2 \leqslant 3^s - 1$ such that

(a) $C(s; z_1, z_2) > 0$, and
(b) do not have good termination (i.e. all their last-digits truncations appear in the previous tables marked with •).

If condition (b) for being s-good can be applied to them, we show the possible values of s', z_1', z_2' on the table, otherwise we mark it with • and move it on to the next s. All values of z_i and z_i' are shown as their 3-adic expansion. The columns D and D' show the result of removing the last s (respectively s') digits of $5z_1 + 7z_2 + (3^s - 1)/2$ (resp. of $5z_1' + 7z_2' + (3^{s'} - 1)/2$). Each digit of the number in column D must be greater than or equal to the corresponding digit of the number in column D'.

$s = 1$

z_1	z_2	$C(s; z_1, z_2)$	D	
1	2	2	20	•
2	2	4	22	•

$s = 2$

z_1	z_2	$C(s; z_1, z_2)$	D	s'	z_1'	z_2'	$C(s'; z_1', z_2')$	D'
11	22	4	22	1	2	2	4	22
21	02	2	12	•				
21	12	2	22	1	2	2	4	22
21	22	2	101	•				
02	22	2	21	1	1	2	2	20
22	12	4	22	1	2	2	4	22
22	22	2	102	•				

$s = 3$

z_1	z_2	$C(s; z_1, z_2)$	D	s'	z_1'	z_2'	$C(s'; z_1', z_2')$	D'
121	202	2	22	1	2	2	4	22
221	102	2	21	1	1	2	2	20
021	222	2	22	1	2	2	4	22
121	122	2	21	1	1	2	2	20
221	222	4	102	•				
022	222	2	22	1	2	2	4	22

$s = 4$								
z_1	z_2	$C(s; z_1, z_2)$	D	s'	z'_1	z'_2	$C(s'; z'_1, z'_2)$	D'
0221	2222	4	22	1	2	2	4	22
1221	0222	2	12	2	21	02	2	12
1221	1222	2	22	1	2	2	4	22
1221	2222	2	101	2	21	22	2	101
2221	0222	2	21	1	1	2	2	20
2221	1222	2	100	•				
2221	2222	2	110	•				

$s = 5$								
z_1	z_2	$C(s; z_1, z_2)$	D	s'	z'_1	z'_2	$C(s'; z'_1, z'_2)$	D'
02221	11222	2	12	2	21	02	2	12
02221	21222	2	22	1	2	2	4	22
12221	11222	2	21	1	1	2	2	20
12221	21222	2	100	4	2221	1222	2	100
22221	01222	2	20	1	1	2	2	20
22221	21222	4	102	3	221	222	4	102
12221	12222	2	22	1	2	2	4	22
12221	22222	2	101	2	21	22	2	101
22221	02222	2	21	1	1	2	2	20
22221	22222	2	110	4	2221	2222	2	110

In the last table (for $s = 5$) there are no remaining cases left with •, so this finishes the proof. □

Theorem 4.7 *The local system on $\mathbb{G}^3_{m,\mathbb{F}_5}$ whose trace function is given by*

$$F(\mathbb{F}_{5^r}; s, t, u) \mapsto -\frac{1}{5^{r/2}} \sum_{x \in \mathbb{F}_{5^r}^\times} \psi_{\mathbb{F}_{5^r}}(s/x + tx^3 + ux^7)\chi_{2,\mathbb{F}_{5^r}}(x)$$

has finite monodromy.

Proof By Corollary 4.2, we need to show
(i) $V(3x_1+7x_2+1/2) \leqslant V(-x_1)+V(-x_2)+1/2$ for every $(x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^2_{\text{prime to }5}$.
(ii) $V(x_1) + V(x_2) \geqslant 1/2$ for every $(x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^2_{\text{prime to }5}$ such that $3x_1 + 7x_2 = 1/2$.

Condition (ii) is the criterion for finite monodromy of the local system with trace function

$$F(\mathbb{F}_{5^r}; t, u) \mapsto -\frac{1}{5^{r/2}} \sum_{x \in \mathbb{F}_{5^r}^\times} \psi_{\mathbb{F}_{5^r}}(tx^3 + ux^7)\chi_{2,\mathbb{F}_{5^r}}(x)$$

which holds by [17, Theorem 10.3.13 (ix)]. For the first condition, following Proposition 4.4, we check by a computer search that all pairs (x_1, x_2) with $0 \leqslant x_1, x_2 \leqslant 5^6-1$

have good termination. For each $s = 1, 2, 3, 4, 5$, the following tables show the list of all pairs (z_1, z_2) with

$$0 \leqslant z_1, z_2 \leqslant 5^s - 1$$

such that

(a) $C(s; z_1, z_2) > 0$, and
(b) do not have good termination (i.e. all their last-digits truncations appear in the previous tables marked with •).

If condition (b) for being s-good can be applied to them, we show the possible values of s', z'_1, z'_2 on the table, otherwise we mark it with • and move it on to the next s. All values of z_i and z'_i are shown as their 5-adic expansion. The columns D and D' show the result of removing the last s (respectively s') digits of $3z_1 + 7z_2 + (5^s - 1)/2$ (resp. of $3z'_1 + 7z'_2 + (5^{s'} - 1)/2$). Each digit of the number in column D must be greater than or equal to the corresponding digit of the number in column D'.

$s = 1$

z_1	z_2	$C(s; z_1, z_2)$	D	
3	4	4	12	•
4	4	4	13	•

$s = 2$

z_1	z_2	$C(s; z_1, z_2)$	D	s'	z'_1	z'_2	$C(s'; z'_1, z'_2)$	D'
23	44	4	13	1	4	4	4	13
33	34	4	12	1	3	4	4	12
33	44	4	14	1	4	4	4	13
43	34	4	13	1	4	4	4	13
43	44	8	14	•				
24	44	4	13	1	4	4	4	13
34	44	4	14	1	4	4	4	13
44	34	4	13	1	4	4	4	13

$s = 3$

z_1	z_2	$C(s; z_1, z_2)$	D	s'	z'_1	z'_2	$C(s'; z'_1, z'_2)$	D'
043	444	4	12	1	3	4	4	12
143	444	4	13	1	4	4	4	13
243	144	4	4	•				

243	344	4	12	1	3	4	4	12
243	444	4	14	1	4	4	4	13
343	244	4	11	•				
343	344	4	13	1	4	4	4	13
343	444	8	14	2	43	44	8	14
443	044	4	4	•				
443	244	4	12	1	3	4	4	12
443	344	8	13	•				
443	444	4	20	•				

$s = 4$

z_1	z_2	$C(s; z_1, z_2)$	D	s'	z'_1	z'_2	$C(s'; z'_1, z'_2)$	D'
3243	4144	4	13	1	4	4	4	13
4243	3144	4	12	1	3	4	4	12
4243	4144	4	14	1	4	4	4	13
1343	4244	4	12	1	3	4	4	12
2343	4244	4	13	1	4	4	4	13
3343	1244	4	4	3	443	044	4	4
3343	3244	4	12	1	3	4	4	12
3343	4244	4	14	1	4	4	4	13
4343	2244	4	11	3	343	244	4	11
4343	3244	4	13	1	4	4	4	13
4343	4244	8	14	2	43	44	8	14
3443	4044	4	13	1	4	4	4	13
4443	3044	4	12	1	3	4	4	12
4443	4044	4	14	1	4	4	4	13
0443	2344	4	4	3	443	044	4	4
0443	4344	4	12	1	3	4	4	12
1443	3344	4	11	3	343	244	4	11
1443	4344	4	13	1	4	4	4	13
2443	1344	4	4	3	443	044	4	4
2443	3344	4	12	1	3	4	4	12
2443	4344	8	13	3	443	344	8	13
3443	0344	4	3	•				
3443	2344	4	11	3	343	244	4	11
3443	3344	4	13	1	4	4	4	13
3443	4344	8	14	2	43	44	8	14
4443	0344	4	4	3	443	044	4	4
4443	1344	4	10	•				
4443	2344	4	12	1	3	4	4	12
4443	3344	8	13	3	443	344	8	13
4443	4344	4	20	3	443	444	4	20
1443	4444	4	13	1	4	4	4	13
2443	3444	4	12	1	3	4	4	12
2443	4444	4	14	1	4	4	4	13
3443	3444	4	13	1	4	4	4	13
3443	4444	8	14	2	43	44	8	14
4443	0444	4	4	3	443	044	4	4
4443	2444	4	12	1	3	4	4	12
4443	3444	4	14	1	4	4	4	13
4443	4444	4	20	3	443	444	4	20

$s=5$								
z_1	z_2	$C(s;z_1,z_2)$	D	s'	z_1'	z_2'	$C(s';z_1',z_2')$	D'
23443	40344	4	12	1	3	4	4	12
33443	40344	4	13	1	4	4	4	13
43443	10344	4	4	3	443	044	4	4
43443	30344	4	12	1	3	4	4	12
43443	40344	4	14	1	4	4	4	13
14443	21344	4	4	3	443	044	4	4
14443	41344	4	12	1	3	4	4	12
24443	31344	4	11	3	343	244	4	11
24443	41344	4	13	1	4	4	4	13
34443	11344	4	4	3	443	044	4	4
34443	31344	4	12	1	3	4	4	12
34443	41344	8	13	3	443	344	8	13
44443	01344	4	3	4	3443	0344	4	3
44443	21344	4	11	3	343	244	4	11
44443	31344	4	13	1	4	4	4	13
44443	41344	8	14	2	43	44	8	14

In the last table (for $s=5$) there are no remaining cases left with $\bullet$, so this finishes the proof.

Theorem 4.8 *The local system on $\mathbb{G}^3_{m,\mathbb{F}_3}$ whose trace function is given by*

$$(\mathbb{F}_{3^r};s,t,u)\mapsto -\frac{1}{3^{r/2}}\sum_{x\in\mathbb{F}^\times_{3^r}}\psi_{\mathbb{F}_{3^r}}(s/x+tx+ux^5)\chi_{2,\mathbb{F}_{3^r}}(x)$$

has finite monodromy.

Proof By Corollary 4.2, we need to show
(i) $V(x_1+5x_2+1/2)\leqslant V(-x_1)+V(-x_2)+1/2$ for every $(x_1,x_2)\in(\mathbb{Q}/\mathbb{Z})^2_{\text{prime to }3}$.
(ii) $V(x)+V(-5x+1/2)\geqslant 1/2$ for every $x\in(\mathbb{Q}/\mathbb{Z})_{\text{prime to }3}$.

Condition (ii) is the criterion for finite monodromy of the local system with trace function

$$F(\mathbb{F}_{3^r};t,u)\mapsto -\frac{1}{3^{r/2}}\sum_{x\in\mathbb{F}^\times_{3^r}}\psi_{\mathbb{F}_{3^r}}(tx+ux^5)\chi_{2,\mathbb{F}_{3^r}}(x)$$

which holds by [12, Theorem 4.2] since $5=(3^2+1)/2$. For the first condition, following Proposition 4.4, we check by a computer search that all pairs (x_1,x_2) with $0\leqslant x_1,x_2\leqslant 3^5-1$ have good termination. For each $s=1,2,3,4$, the following tables show the list of all pairs (z_1,z_2) with $0\leqslant z_1,z_2\leqslant 3^s-1$ such that

(a) $C(s;z_1,z_2)>0$, and
(b) do not have good termination (i.e. all their last-digits truncations appear in the previous tables marked with $\bullet$).

If condition (b) for being s-good can be applied to them, we show the possible values of s', z_1', z_2' on the table, otherwise we mark it with • and move it on to the next s. All values of z_i and z_i' are shown as their 3-adic expansion. The columns D and D' show the result of removing the last s (respectively s') digits of $z_1 + 5z_2 + (3^s - 1)/2$ (resp. of $z_1' + 5z_2' + (3^{s'} - 1)/2$). Each digit of the number in column D must be greater than or equal to the corresponding digit of the number in column D'.

$s = 1$

z_1	z_2	$C(s; z_1, z_2)$	D	
2	1	2	2	•
2	2	2	11	•

$s = 2$

z_1	z_2	$C(s; z_1, z_2)$	D	s'	z_1'	z_2'	$C(s'; z_1', z_2')$	D'
12	21	2	11	1	2	2	2	11
22	21	2	12	1	2	2	2	11
12	22	2	12	1	2	2	2	11
22	22	4	12	•				

$s = 3$

z_1	z_2	$C(s; z_1, z_2)$	D	s'	z_1'	z_2'	$C(s'; z_1', z_2')$	D'
022	222	2	12	1	2	2	2	11
122	222	4	12	2	22	22	4	12
222	022	2	2	1	2	1	2	2
222	122	2	11	1	2	2	2	11
222	222	2	20	•				

$s = 4$

z_1	z_2	$C(s; z_1, z_2)$	D	s'	z_1'	z_2'	$C(s'; z_1', z_2')$	D'
0222	2222	2	12	1	2	2	2	11
2222	1222	2	11	1	2	2	2	11
2222	2222	2	20	3	222	222	2	20

In the last table (for $s = 4$) there are no remaining cases left with •, so this finishes the proof. □

Theorem 4.9 *The local system on $\mathbb{G}^3_{m,\mathbb{F}_3}$ whose trace function is given by*

$$(\mathbb{F}_{3^r}; s, t, u) \mapsto -\frac{1}{3^{r/2}} \sum_{x \in \mathbb{F}_{3^r}^\times} \psi_{\mathbb{F}_{3^r}}(s/x + tx^2 + ux^5)\chi_{2,\mathbb{F}_{3^r}}(x)$$

has finite monodromy.

Proof By Corollary 4.2, we need to show
(i) $V(2x_1+5x_2+1/2) \leqslant V(-x_1)+V(-x_2)+1/2$ for every $(x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^2_{\text{prime to 3}}$.
(ii) $V(x_1) + V(x_2) \geqslant 1/2$ for every $(x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^2_{\text{prime to 3}}$ such that $2x_1 + 5x_2 = 1/2$.

Condition (ii) is the criterion for finite monodromy of the local system with trace function

$$F(\mathbb{F}_{3^r}; t, u) \mapsto -\frac{1}{3^{r/2}} \sum_{x \in \mathbb{F}_{3^r}^\times} \psi_{\mathbb{F}_{3^r}}(tx^2 + ux^5)\chi_{2,\mathbb{F}_{3^r}}(x)$$

which holds by [17, Theorem 10.3.13 (i)]. For the first condition, following Proposition 4.4, we check by a computer search that all pairs (x_1, x_2) with $0 \leqslant x_1, x_2 \leqslant 3^6 - 1$ have good termination. For each $s = 1, 2, 3, 4, 5$, the following tables show the list of all pairs (z_1, z_2) with $0 \leqslant z_1, z_2 \leqslant 3^s - 1$ such that

(a) $C(s; z_1, z_2) > 0$ and
(b) do not have good termination (i.e. all their last-digits truncations appear in the previous tables marked with •).

If condition (b) for being s-good can be applied to them, we show the possible values of s', z'_1, z'_2 on the table, otherwise we mark it with • and move it on to the next s. All values of z_i and z'_i are shown as their 3-adic expansion. The columns D and D' show the result of removing the last s (respectively s') digits of $2z_1+5z_2+(3^s - 1)/2$ (resp. of $2z'_1 + 5z'_2 + (3^{s'} - 1)/2$). Each digit of the number in column D must be greater than or equal to the corresponding digit of the number in column D'.

$s = 1$				
z_1	z_2	$C(s; z_1, z_2)$	D	
1	1	1	2	•
1	2	1	11	•
2	2	2	12	•

In the last table (for $s = 5$) there are no remaining cases left with •, so this finishes the proof. □

$s = 2$

z_1	z_2	$C(s; z_1, z_2)$	D	s'	z'_1	z'_2	$C(s'; z'_1, z'_2)$	D'
21	21	3	12	•				
11	22	2	12	1	2	2	2	12
21	12	1	11	1	1	2	1	11
21	22	1	20	•				
22	22	2	20	•				

$s = 3$

z_1	z_2	$C(s; z_1, z_2)$	D	s'	z'_1	z'_2	$C(s'; z'_1, z'_2)$	D'
021	221	1	12	1	2	2	2	12
221	121	1	12	1	2	2	2	12
221	221	3	20	•				
021	222	1	12	1	2	2	2	12
221	122	1	12	1	2	2	2	12
221	222	1	21	1	1	2	1	11
022	222	2	12	1	2	2	2	12
122	122	1	11	1	1	2	1	11
122	222	1	20	2	21	22	1	20
222	122	2	12	1	2	2	2	12
222	222	2	21	2	22	22	2	20

$s = 4$

z_1	z_2	$C(s; z_1, z_2)$	D	s'	z'_1	z'_2	$C(s'; z'_1, z'_2)$	D'
0221	2221	3	12	2	21	21	3	12
1221	1221	2	11	•				
1221	2221	2	20	3	221	221	3	20
2221	0221	1	10	•				
2221	1221	3	12	2	21	21	3	12
2221	2221	3	21	3	221	221	3	20

$s = 5$

z_1	z_2	$C(s; z_1, z_2)$	D	s'	z'_1	z'_2	$C(s'; z'_1, z'_2)$	D'
11221	21221	3	12	2	21	21	3	12
21221	11221	2	11	4	1221	1221	2	11
21221	21221	2	20	3	221	221	3	20
02221	20221	1	11	4	2221	0221	1	10
12221	20221	2	12	1	2	2	2	12
22221	00221	1	2	1	1	1	1	2
22221	10221	1	11	4	2221	0221	1	10
22221	20221	1	20	3	221	221	3	20

5 G_{geom} for local systems of rank 8

In this section, we will determine $G_{\text{geom},r,s}$, the geometric monodromy group of $\mathcal{G}_{r,s}$, and G_{geom,r_0,s_0}, the geometric monodromy group of $\mathcal{G}_{r_0,s_0}$ for any $(r_0, s_0) \in \mathbb{G}_m^2$, of rank 8. With (B, a, p) as in (2.5.1), it follows from Theorems 4.5, 4.7, and 4.6 that $G_{\text{geom},r,s}$ is finite. [Those results show that the restriction of $\mathcal{G}_{r,s}$ to the open dense subset $\mathbb{G}_m^3$ of $\mathbb{G}_m^1 \times \mathbb{A}^1 \times \mathbb{G}_m$ has finite G_{geom}. In general, for a local system $\mathcal{F}$ on some smooth, geometrically connected variety X, and $U \subset X$ a dense open set, $\pi_1(U)$ maps onto $\pi_1(X)$, so $\mathcal{F}$ on X and $\mathcal{F}|_U$ on U have the same G_{geom} (indeed have the same image of π_1).]

Theorem 5.1 *Let $(B, a, p) = (3, 7, 5)$. Then both $\mathcal{G}_{r,s}$, and $\mathcal{G}_{r_0,s_0}$ for any $(r_0, s_0) \in \mathbb{G}_m^2$, have $G_{\text{geom}} = W(E_8)$, the Weyl group of type E_8.*

Proof By Theorem 4.7, $G := G_{\text{geom},r,s}$ is a finite subgroup of $\mathrm{GL}_8(\mathbb{C})$, whence the same holds for its subgroup $H := G_{\text{geom},r_0,s_0}$. Next, H satisfies (**S+**) by Proposition 2.6, whence the same holds for G.

Let φ denote the G-character afforded by the underlying representation. By Lemma 2.3, φ takes real values for any specializations of (r, s), and hence φ is real-valued. This implies that

$$\mathbf{Z}(H) \leqslant \mathbf{Z}(G) \leqslant C_2. \tag{5.1.1}$$

Next, by Lemma 2.2 the image of $P(\infty)$ in H is isomorphic to the additive group of $\mathbb{F}_5(\mu_8) = \mathbb{F}_{5^2}$, which is elementary abelian of order 5^2, whence (5.1.1) implies that

$$5^2 \text{ divides } |H/\mathbf{Z}(H)|. \tag{5.1.2}$$

Now we can apply [16, Lemma 1.1] to both H and G. If either of them is an extraspecial normalizer, then there is some $\epsilon = \pm$ such that

$$H \leqslant 2_\epsilon^{1+6} \cdot \mathrm{O}_6^\epsilon(2),$$

which violates (5.1.2). So both H and G are almost quasisimple; in particular, $L := H^{(\infty)}$ is a quasisimple group with

$$S := L/\mathbf{Z}(L)$$

being the unique non-abelian composition factor of H. The condition (**S+**) implies that $\varphi|_L$ is irreducible, and so $\mathbf{C}_H(L) = \mathbf{Z}(H)$ by Schur's lemma. It follows that

$$H/\mathbf{Z}(H) \hookrightarrow \mathrm{Aut}(L),$$

and so $5^2 \mid |\mathrm{Aut}(L)|$ by (5.1.2). Now we can inspect Table 2 in [6] to see that $2 \cdot \Omega_8^+(2)$ is the unique possibility for L. Note that $\varphi|_L$ is of type $+$, so

$$2 \cdot \Omega_8^+(2) = L \lhd H \leqslant \mathbf{N}_{\mathrm{O}_8(\mathbb{C})}(L) = W(E_8) = L \cdot 2.$$

Also note from Theorem 2.4 that $\mathcal{G}_{r_0,s_0}$ has geometric determinant χ_2, which shows that H cannot be perfect. Hence $H = W(E_8)$.

The preceding arguments can also be repeated to show that $G \cong W(E_8)$. As $H \leqslant G$, we conclude that $G = H$. □

Theorem 5.2 *Let* $(B, a, p) = (5, 7, 3)$. *Then* $\mathcal{G}_{r,s}$ *has* $G_{\mathrm{geom}} = W(E_8)$, *the Weyl group of type* E_8. *Furthermore, there is a dense open set* $U \subseteq \mathbb{G}_m^2$ *that contains* $(1, 1)$, *such that* $\mathcal{G}_{r_0,s_0}$ *has* G_{geom} *equal to* $W(E_8)$ *when* $(r_0, s_0) \in U$, *and to* S_9 *when* $(r_0, s_0) \notin U$.

Proof By Theorem 4.6, $G := G_{\mathrm{geom},r,s}$ is a finite subgroup of $\mathrm{GL}_8(\mathbb{C})$, whence the same holds for its subgroup $H := G_{\mathrm{geom},r_0,s_0}$. Next, H satisfies (**S+**) by Proposition 2.6, whence the same holds for G.

(a) Let φ denote the G-character afforded by the underlying representation V. By Lemma 2.3, φ takes real values for any specializations of (r, s), and hence φ is real-valued. This implies that

$$\mathbf{Z}(H) \leqslant \mathbf{Z}(G) \leqslant C_2. \tag{5.2.1}$$

Also note from Theorem 2.4 that $\mathcal{G}_{r_0,s_0}$ has geometric determinant χ_2, which shows that $\varphi|_H$ cannot be of symplectic type, and

$$H \leqslant \mathrm{O}(V) \quad \text{but} \quad H \not\leqslant \mathrm{SO}(V). \tag{5.2.2}$$

Next, the wild part of the $I(0)$-representation V has rank 2 and slopes $5/2$, so by [9, 1.14] it is the Kummer induction $[2]_\star\mathcal{L}$ of some $\mathcal{L}$ with $\mathsf{Swan} = 5$, and the $P(0)$-representation is the direct sum $\mathcal{L} \oplus [x \mapsto -x]^\star\mathcal{L}$. Moreover, these two pieces are permuted by any element $g_0 \in I(0)$ which is a generator of $I(0)$ modulo $P(0)$. Thus g_0 acts on V with spectrum $(\alpha, -\alpha, 1, 1, \ldots, 1)$ for some $\alpha \in \mathbb{C}^\times$. Since the image Q of $P(0)$ is a 3-group, it is contained in $\mathrm{SO}(V)$. Now, if the image of g_0 is contained in $\mathrm{SO}(V)$, then so is the image J of $I(0)$. By [16, 4.2], the fact that all ∞-slopes are < 1 implies that H is the normal closure of J, so we get $H \leqslant \mathrm{SO}(V)$, contrary to (5.2.2). Thus $-\alpha^2 = -1$, i.e. $\alpha = \pm 1$ and g_0 acts on V as a reflection.

The version of Mitchell's theorem given in the proof of [17, Theorem 4.2.3] now shows that $H = \mathbf{Z}(H)H_0$, where $H_0 = W(E_8)$, or H_0 is S_9 acting in the deleted natural permutation representation. In the former case, (5.2.1) implies that

$$H = W(E_8).$$

Suppose we are in the latter case. First we consider the case $\mathbf{Z}(H) = C_2$. Then note that $L := H^{(\infty)} \cong \mathsf{A}_9$ and $H/L \cong C_2^2$. In particular, $Q \leqslant L$, and J is contained in $\langle L, g_0\rangle$, a subgroup of index 2, whence normal, in H. Hence the normal closure of J in H is contained in $\langle L, g_0\rangle$, and so cannot be equal to H, a contradiction. We have shown that, in the latter case, $H = \mathsf{S}_9$ in its deleted natural permutation representation.

Now we apply the above consideration to $(r_0, s_0) = (1, 1)$, and assume that $H = \mathsf{S}_9$. We consider the weight zero twist of $\mathcal{G}$ by $1/\mathsf{Gauss}(\psi, \chi_2)$, which is orthogonally self-dual with integer Frobenius traces. Let us denote by H_{arith} its arithmetic monodromy group. Then H_{arith} normalizes H, and hence we have $H \leqslant H_{\mathrm{arith}} \leqslant \mathbf{N}_{\mathrm{O}(V)}(H) =$

$C_2 \times H$. In either case, over any even degree extension $k/\mathbb{F}_3$, we have $H_{\mathrm{arith},k} = H = \mathsf{S}_9$. So over any such k, all Frobenius traces lie in $[-1, 8]$. But a Magma calculation shows that over $\mathbb{F}_{3^4}$, both -3 and -2 (as well as 2 and 3) occur as Frobenius traces. That $H = W(E_8)$ at $(r_0, s_0) = (1, 1)$.

(b) The preceding arguments can also be repeated to show that either $G \cong W(E_8)$ or $G \leqslant C_2 \times \mathsf{S}_9$. Since $W(E_8) = G_{\mathrm{geom},1,1} \leqslant G$ and $|C_2 \times \mathsf{S}_9| < |W(E_8)|$, we conclude that $G \cong W(E_8)$.

By Corollary 3.2, there is a dense open subset U of $\mathbb{G}_m^2$ containing $(1, 1)$ such that $G_{\mathrm{geom},r_0,s_0}$ equals $W(E_8)$ for $(r_0, s_0) \in U$ and $G_{\mathrm{geom},r_0,s_0} = \mathsf{S}_9$ for $(r_0, s_0) \notin U$. □

Theorem 5.3 *Let $(B, a, p) = (1, 7, 3)$. Then both $\mathcal{G}_{r,s}$, and $\mathcal{G}_{r_0,s_0}$ for any $(r_0, s_0) \in \mathbb{G}_m^2$, have $G_{geom} = W(E_8)$, the Weyl group of type E_8.*

Proof (a) By Theorem 4.5, $G := G_{\mathrm{geom},r,s}$ is a finite subgroup of $\mathrm{GL}_8(\mathbb{C})$, whence the same holds for its subgroup $H := G_{\mathrm{geom},r_0,s_0}$. Let φ denote the G-character afforded by the underlying representation V. By Lemma 2.3, φ takes real values for any specializations of (r, s), and hence φ is real-valued. This implies that

$$\mathbf{Z}(H) \leqslant \mathbf{Z}(G) \leqslant C_2. \tag{5.3.1}$$

Also note from Theorem 2.4 that $\mathcal{G}_{r_0,s_0}$ has geometric determinant χ_2, which shows that $\varphi|_H$ cannot be of symplectic type, and

$$H \leqslant G \leqslant \mathrm{O}(V) \quad \text{but} \quad G, H \not\leqslant \mathrm{SO}(V). \tag{5.3.2}$$

Next, the wild part of the $I(0)$-representation V has rank 6, and so the image Q of $P(0)$ is non-abelian, and hence is a 3-group of order at least 3^3. It follows that 3^3 divides $|H|$ and $|G|$. On the other hand, G has $M_{2,2} = 3$ by Lemma 2.10. It follows from [4, Theorem 1.5] and (5.3.2) that either

$$E = 2_+^{1+6} \lhd G \leqslant \mathbf{N}_{\mathrm{O}(V)}(E) = E \cdot \mathrm{O}_6^+(2),$$

or $G = 2 \cdot \mathsf{A}_9$, or $2 \cdot \Omega_8^+(2) \leqslant G \leqslant W(E_8)$. The first possibility is ruled out since 3^3 divides $|G|$. Next, G is not perfect by (5.3.2), ruling out the groups $2 \cdot \mathsf{A}_9$ and $2 \cdot \Omega_8^+(2)$. Hence we conclude that $G = W(E_8)$.

(b) It remains to determine $H = G_{\mathrm{geom},r_0,s_0}$ which is a subgroup of $G = W(E_8)$. By Proposition 2.6, H satisfies condition (**S+**). Hence, by [16, Lemma 1.1], one of the following two cases holds.

(b1) H is an extraspecial normalizer, i.e. R contains a normal 2-subgroup $R = \mathbf{Z}(R)E$, with $E = 2_\epsilon^{1+6}$ acting irreducibly on $V = \mathbb{C}^8$, $\epsilon = \pm$, and $\mathbf{Z}(R) = \mathbf{Z}(E)$ or $\mathbf{Z}(R) = C_4$. Now (5.3.2) implies that $\epsilon = +$ and $\mathbf{Z}(R) = \mathbf{Z}(E)$. Thus $R = E = 2_+^{1+6}$ and

$$H \leqslant \mathbf{N}_{\mathrm{O}(V)}(E) = E \cdot \mathrm{O}_6^+(2).$$

This is however impossible since 3^3 divides $|H|$.

(b2) H is almost quasisimple, i.e. $S \lhd H/\mathbf{Z}(H) \leqslant \mathrm{Aut}(S)$ for a non-abelian simple group S, and the quasisimple group $L = E(H)$, with $S = L/\mathbf{Z}(L)$, acts irreducibly on V. Furthermore, (5.3.1) implies that 3^3 divides $|\mathrm{Aut}(S)|$. Now we analyze the possibilities for (S, L) as listed in [6].

- $S = L = \mathrm{SL}_2(8)$. In this case, $H/\mathbf{Z}(H) \leqslant \mathrm{Aut}(S) = \mathrm{SL}_2(8)\cdot 3$ contains no element of order 4, whereas a generator g_∞ of $I(\infty)$ modulo $P(\infty)$ has order 8 in $H/\mathbf{Z}(H)$, a contradiction.
- $(S, L) = (\mathrm{Sp}_6(2), 2\cdot\mathrm{Sp}_6(2))$. Since $\mathrm{Aut}(S) \cong S$, by (5.3.1) we have $H = L$. Let J and Q denote the image of $I(\infty)$, respectively of $P(\infty)$ in H. Then Q is elementary abelian of order 9 by Lemma 2.2. As J acts irreducibly on V, $J\mathbf{Z}(H)$ transitively permutes the eight nontrivial irreducible characters of Q which all occur in V. Identifying Q with $Q\mathbf{Z}(H)/\mathbf{Z}(H)$, we see that the subgroup $J\mathbf{Z}(H)/\mathbf{Z}(H)$ of S also permutes the eight nontrivial irreducible characters of Q transitively. Note that $S = \mathrm{Sp}_6(2)$ admits an irreducible complex character θ of degree 7, and certainly $\theta|_Q$ contains some irreducible constituent $\lambda \neq 1_Q$. But then all eight nontrivial irreducible characters of Q must occur in $\theta|_Q$ of degree 7, a contradiction.
- $(S, L) = (\mathsf{A}_9, 2\cdot\mathsf{A}_9)$. Since $2\cdot\mathsf{S}_9$ does not act on $\mathbb{C}^8$, we must have $H = L$. In particular, $H/\mathbf{Z}(H)$ contains no element of order 8, whereas g_∞ has order 8 in $H/\mathbf{Z}(H)$, a contradiction.
- $S = L = \mathsf{A}_9$. As in the previous case, the fact that g_∞ has central order 8 implies that $H/\mathbf{Z}(H) \cong \mathsf{S}_9$. It follows from (5.3.1) that H/L is a group of order 2 or 4, whence the image P of $P(0)$ in H is contained in $L = S$. Note that the restriction of the character φ to L is just the character of the deleted permutation module of A_9. Now, $\varphi|_P = 2\cdot 1_P + \alpha + \overline{\alpha}$ for some irreducible character α of P of degree 3, using the fact that φ is real-valued. It follows that the 3-subgroup P of A_9 acts on $\{1, 2, \ldots, 9\}$ with exactly three orbits. The length of any of these orbits is a power of 3. So we conclude that each of them has length 3; say they are $\{1, 2, 3\}$, $\{4, 5, 6\}$, and $\{7, 8, 9\}$. Now, as P fixes each of these three subsets, we see that $P \leqslant \mathsf{A}_3^3$ and hence abelian, contrary to the fact that α has degree 3.
- $(S, L) = (\Omega_8^+(2), 2\cdot\Omega_8^+(2))$. Here we have $\mathbf{Z}(H) = \mathbf{Z}(L) = \mathbf{Z}(G) = C_2$, and $S \leqslant H/\mathbf{Z}(H) \leqslant S\cdot 2$. Again using the fact that $\mathcal{G}_{r_0,s_0}$ having nontrivial determinant we see that $H > L$. Hence we conclude that $|H/\mathbf{Z}(H)| = |S\cdot 2| = |G/\mathbf{Z}(H)|$ and thus $H = G$. □

Remark 5.4 As we mentioned at the beginning of the section, the extension to $\mathcal{G}_{r,s}$ to $\mathbb{G}_m \times \mathbb{A}^1 \times \mathbb{G}_m$ does not change its G_{geom}. Now, the specialization $\mathcal{G}_{1,0}$ of $\mathcal{G}_{r,s}$ in both Theorems 5.2 and 5.3 yield the Kloosterman sheaf $\mathcal{K}\ell(\chi_2, \mathsf{Char}(7)) \otimes \mathcal{L}_{\chi_2}$ by the proof of Proposition 2.8. In particular, its geometric monodromy group $G_{1,0}$ contains minus a reflection. Applying [16, Theorem 9.3 (b)], we see that $G_{1,0} = \mathsf{S}_9$, acting on the tensor product of the deleted permutation representation with the sign representation, i.e. on the non-reflection representation.

6 G_{geom} for local systems of rank 6

Theorem 6.1 *Let* $(B, a, p) = (1, 5, 3)$*. Then* $\mathcal{G}_{r,s}$ *has* $G_{\text{geom}} = W(E_6) \times 2$*. Moreover, for any* $(r_0, s_0) \in \mathbb{G}_m^2$*, the geometric monodromy group of* $\mathcal{G}_{r_0,s_0}$ *is* $W(E_6)$*, acting on the non-reflection representation of degree* 6.

Proof (a) By Theorem 4.8, $G := G_{\text{geom},r,s}$ is a finite subgroup of $\mathrm{GL}_6(\mathbb{C})$, whence the same holds for its subgroup $H := G_{\text{geom},r_0,s_0}$ for any $(r_0, s_0) \in \mathbb{G}_m^2$. Let φ denote the G-character afforded by the underlying representation V. Then

$$\mathbb{Q}(\varphi) = \mathbb{Q}, \quad \text{and so} \quad \mathbf{Z}(H) \leqslant \mathbf{Z}(G) \leqslant C_2. \tag{6.1.1}$$

Also note from Theorem 2.4 that $\mathcal{G}_{r_0,s_0}$ has geometric determinant χ_2, and so it cannot be of symplectic type, and

$$H \leqslant G \leqslant \mathrm{O}(V) \quad \text{and} \quad H \not\leqslant \mathrm{SO}(V). \tag{6.1.2}$$

Now we apply Lemma 2.10 to get $M_{2,2}(G) = 3$. Applying [4, Theorem 1.5] and using (6.1.1), we see that G is almost quasisimple, and arrive at one of the following cases for $L = G^{(\infty)}$.

- $L = \mathrm{SU}_3(3)$. In this case, $L \lhd G/\mathbf{Z}(G) \leqslant L \cdot 2$. Using [22] one can check that the rational-valued character $\varphi|_L$ does not have rational-valued extensions to $L \cdot 2$. Hence (6.1.1) implies that $G = \mathbf{Z}(G) \times L$. But in this case $G < \mathrm{SO}(V)$, contradicting (6.1.2).
- $L = \mathrm{SU}_4(2)$. In this case, $L \lhd G/\mathbf{Z}(G) \leqslant L \cdot 2$. Since $G \not\leqslant \mathrm{SO}(V)$ by (6.1.2), G must induce an outer automorphism of L, i.e. $G/\mathbf{Z}(G) = L \cdot 2 \cong W(E_6)$. Together with (6.1.1), this implies that $W(E_6) \leqslant G \leqslant W(E_6) \times 2$. The same arguments applied to $G_{\text{arith},\mathbb{F}_3}$ show that $G_{\text{arith},\mathbb{F}_3} \leqslant W(E_6) \times 2$. In particular, $[G_{\text{arith},\mathbb{F}_3} : G] \leqslant 2$ and $G_{\text{arith},\mathbb{F}_9} = G$. Now, a calculation with Magma [1] over $\mathbb{F}_{3^8}$ shows that the Frobenius at the point $(r, s, t) = (1, 1, w^{437})$ for w a primitive element in $\mathbb{F}_{3^8}$ has trace -4. We also note that a change of variable $x \mapsto rx$ in the trace function sends the trace of the Frobenius at $(1, 1, t)$ to $\chi_2(r)$ times the trace of the Frobenius at $(1, r, tr^5)$. Choosing $r \in \mathbb{F}_{3^8}$ with $\chi_2(r) = -1$, we then get a trace 4, namely at $(1, sr, tr^5)$, in addition to trace -4. Since neither of the two 6-dimensional irreducible representations of $W(E_6)$ possesses both traces 4 and -4, we conclude that $G = W(E_6) \times 2$.

(b) It remains to determine $H = G_{\text{geom},r_0,s_0}$ which is a subgroup of $G = W(E_6) \times 2$. By Proposition 2.7, H satisfies condition (**S+**). Hence, by [16, Lemma 1.1], H is almost quasisimple: $S \lhd H/\mathbf{Z}(H) \leqslant \mathrm{Aut}(S)$ for a non-abelian simple group S, and the quasisimple group $K = E(H)$ with $S = K/\mathbf{Z}(K)$, acts irreducibly on V. By Lemma 2.1, the image of $P(\infty)$ in H is a non-abelian 3-group, so (6.1.1) implies that 3^3 divides $|\mathrm{Aut}(S)|$. Since $K = K^{(\infty)} \leqslant G^{(\infty)} = \mathrm{SU}_4(2)$, the list of maximal subgroups of $\mathrm{SU}_4(2)$ in [2] shows that $K = \mathrm{SU}_4(2) = L$. Now $\mathbf{Z}(G)L$ has index 2 in G and $\mathbf{Z}(G)L \leqslant \mathrm{SO}(V)$. Hence (6.1.2) implies that

$$L < H \leqslant G = L \rtimes C_2^2. \tag{6.1.3}$$

Now we look at the image J of $I(0)$ in H. Since H/L is a 2-group, the image Q of $P(0)$, a 3-subgroup, is contained in L. Next, $J = \langle Q, h \rangle$, where h is the image in H of a generator g_0 of $I(0)$ modulo $P(0)$. Since H/L has exponent 2, we have $h^2 \in L$, and hence

$$[LJ : L] \leqslant 2. \tag{6.1.4}$$

Also, since G/L is abelian, LJ is normal in G and hence also in H. But H is the normal closure of J by [16, Proposition 4.2], so $H \leqslant LJ$. Hence $H = LJ$, and now (6.1.3) and (6.1.4) imply that $[H : L] = 2 = [G : H]$. Among the three subgroups of index 2 in G, $\mathbf{Z}(G) \times L$ is contained in $\mathrm{SO}(V)$, and the other two are isomorphic to $W(E_6)$, which act on V via the two irreducible 6-dimensional representations of $W(E_6)$, the reflection and the non-reflection representation. Using (6.1.2), we obtain

$$H \cong W(E_6). \tag{6.1.5}$$

By Theorem 3.1, there is a subgroup $G_{\bar{\eta}}$ of G and an open dense subset U of $\mathbb{G}_m^2$ such that for all $(r_1, s_1) \in U$, $G_{\mathrm{geom},r_1,s_1}$ is conjugate to $G_{\bar{\eta}}$ in G. Now (6.1.5) implies that $|G_{\mathrm{geom},r_1,s_1}| = |G_{\bar{\eta}}|$. It follows that $G_{\bar{\eta}} \cong W(E_6)$ and hence, being of index 2, that $G_{\bar{\eta}} \lhd G$. Also by Theorem 3.1, for any $(r_2, s_2) \in \mathbb{G}_m^2$, $G_{\mathrm{geom},r_2,s_2}$ is conjugate in G to a subgroup of $G_{\bar{\eta}} \lhd G$, hence it is a subgroup $G_{\bar{\eta}}$. Again using (6.1.5), we obtain that $H = G_{\bar{\eta}}$. In particular, $G_{\mathrm{geom},1,1} = H$, and the calculation in (a) shows that H acts on V via the non-reflection representation. □

Theorem 6.2 *Let* $(B, a, p) = (2, 5, 3)$. *Then both* $\mathcal{G}_{r,s}$, *and* $\mathcal{G}_{r_0,s_0}$ *for any* $(r_0, s_0) \in \mathbb{G}_m^2$, *have* $G_{\mathrm{geom}} = 6_1 \cdot \mathrm{PSU}_4(3) \cdot 2_2$, *the Mitchell group.*

Proof (a) By Theorem 4.9, $G := G_{\mathrm{geom},r,s}$ is a finite subgroup of $\mathrm{GL}_6(\mathbb{C})$, whence the same holds for its subgroup $H := G_{\mathrm{geom},r_0,s_0}$ for any $(r_0, s_0) \in \mathbb{G}_m^2$. Let φ denote the G-character afforded by the underlying representation V. Then

$$\mathbb{Q}(\varphi) \subseteq \mathbb{Q}(\zeta_3), \quad \text{and so} \quad \mathbf{Z}(H) \leqslant \mathbf{Z}(G) \leqslant C_6. \tag{6.2.1}$$

Also note from Theorem 2.5 that $\mathcal{G}_{r,s}$ has geometric determinant χ_2, and so

$$G \nleqslant \mathrm{SL}(V), \quad H \nleqslant \mathrm{SL}(V). \tag{6.2.2}$$

Now we apply Lemma 2.11 to get $M_{2,2}(G) = 2$. Applying [4, Theorem 1.5] and using (6.2.1), we see that G is almost quasisimple, and arrive at one of the following cases for $L = G^{(\infty)}$.

- $L = \mathrm{SU}_4(2)$ or $\mathrm{SU}_3(3)$. In this case, $L \lhd G/\mathbf{Z}(G) \leqslant L \cdot 2$. Using [22] one can check that $M_{2,2} = 3$, a contradiction.
- $L = 6 \cdot \mathrm{PSL}_3(4)$. In this case, $L \lhd G \leqslant L \cdot 2_1$ (in the notation of [22]). The condition (6.2.1) now implies that $G = L$ is perfect, which contradicts (6.2.2).
- $L = 6 \cdot \mathrm{PSU}_4(3)$. In this case, $L \lhd G \leqslant L \cdot 2_2$ (in the notation of [22]). Since G is not perfect by (6.2.2), we have that $G = L \cdot 2_2$, the Mitchell group.

(b) It remains to determine $H = G_{\mathrm{geom},r_0,s_0}$ which is a subgroup of G, the Mitchell group. By Proposition 2.7, H satisfies condition (**S+**). Hence, by [16, Lemma 1.1], H is almost quasisimple: $S \lhd H/\mathbf{Z}(H) \leqslant \mathrm{Aut}(S)$ for a non-abelian simple group S, and the quasisimple group $K = E(H)$ with $S = K/\mathbf{Z}(K)$ acts irreducibly on V. We next show that

$$\mathbb{Q}(\varphi|_K) = \mathbb{Q}(\zeta_3). \tag{6.2.3}$$

By (6.2.1), it suffices to show that $V|_K$ is not self-dual. Assume the contrary. Then V and V^* are two extensions of the absolutely irreducible module $V|_K$ to H. By Gallagher's theorem [7, (6.17)], $V^* \cong V \otimes U$ for some one-dimensional H/K-module U. Applying [19, Corollary 2.7], we see that $U \cong \overline{\mathbb{Q}}_\ell$ is trivial, and thus V is self-dual. But this is impossible by [19, Lemma 2.3]. Using [6] and (6.2.3), we arrive at one of the following cases for K.

- $K = 3 \cdot \mathsf{A}_6$. Since the faithful module $V|_K$ is invariant only under the outer automorphisms 2_3 of K (in the notation of [2]), we have $H = K\mathbf{C}_H(K) = K\mathbf{Z}(H)$ or $H \leqslant \mathbf{Z}(H)K \cdot 2_3$. In the former case, K is perfect and $\mathbf{Z}(H) \leqslant C_6$ has determinant 1 on V, and so $H \leqslant \mathrm{SL}(V)$, contrary to (6.2.2). In the latter case, one can check using [2] that $\mathbb{Q}(\varphi|_H)$ contains $\sqrt{2}$ or $\sqrt{-2}$, contradicting (6.2.1). [Note that the Mitchell group contains a subgroup $3 \cdot \mathsf{A}_6 \cdot 2_3$ which however acts reducibly on the faithful irreducible representations of the Mitchell group — one can see it by checking the character values at involutions insider $3 \cdot \mathsf{A}_6$.]
- $K = 3 \cdot \mathsf{A}_7$. Since the faithful module $V|_K$ is not invariant under outer automorphisms of K, we have $H = K\mathbf{C}_H(K) = K\mathbf{Z}(H)$. As K is perfect and $\mathbf{Z}(H) \leqslant C_6$ has determinant 1 on V, we get $H \leqslant \mathrm{SL}(V)$, contradicting (6.2.2).
- $K = 6 \cdot \mathrm{PSL}_3(4)$. As in part (a), this implies $H = K$ is perfect, again contradicting (6.2.2).
- $K = 6 \cdot \mathrm{PSU}_4(3)$. As in part (a), using (6.2.2) we obtain that $H = K \cdot 2_2$. Since $H \leqslant G$ and $|H| = |G|$, it follows that $H = G$. □

Acknowledgements The authors are grateful to the referee for careful reading and helpful comments on the paper.

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