Inequalities related to Lefschetz pencils and integrals of Chern classes

Nicholas M. Katz and Rahul Pandharipande

Introduction We work over an algebraically closed field k, in which a prime number ℓ is invertible. We fix a projective, smooth, connected k-scheme X/k, of dimension $n \geq 1$. We also fix a projective embedding $i : X \subset \mathbb{P}$. This allows us to speak of smooth hyperplane sections $X \cap L$ of X, or more generally of smooth hypersurface sections $X \cap H_d$ of X of any degree $d \geq 1$ (i.e., H_d is a degree d hypersurface in the ambient $\mathbb{P}$, and the scheme-theoretic intersection $X \cap H_d$ is smooth over k, and of codimension one in X).

The paper [Ka-LAM] applied results of Larsen to the problem of determining the monodromy of the universal family of smooth hypersurface sections $X \cap H_d$ of X of fixed degree d. Consider the lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}_{\ell,d}$ on the parameter space, given by

$$H_d \mapsto H^{n-1}(X \cap H_d, \overline{\mathbb{Q}}_\ell)/H^{n-1}(X, \overline{\mathbb{Q}}_\ell).$$

Denote by N_d the rank of $\mathcal{F}_{\ell,d}$. One knows that $\mathcal{F}_{\ell,d}$ is orthogonally self dual if n-1 is even, and symplectically self-dual if n-1 is odd. We denote by

$$G_{geom} \subset O(N_d), \text{ if } n-1 \text{ is even},$$
$$G_{geom} \subset Sp(N_d), \text{ if } n-1 \text{ is odd},$$

the Zariski closure of the image of the geometric fundamental group of the parameter space in the $\overline{\mathbb{Q}}_\ell$-representation which $\mathcal{F}_{\ell,d}$ "is".

Deligne showed in [De-Weil II, 4.4.1 and 4.4.2^a] that for any $d \geq 2$ (and also for d=1 if X admits a Lefschetz pencil of hyperplane sections, which it always does in characteristic zero, cf. [SGA 7 Exp. XVII, 2.5.2]), one has

$$G_{geom} = Sp(N_d), \text{ if } n-1 \text{ is odd}.$$

When n-1 is even, he showed [De-Weil II, 4.4.1, 4.4.2^s, and 4.4.9] that either G_{geom} is the full orthogonal group $O(N_d)$, or G_{geom} is a finite reflection group.

The finite reflection case does occur. For instance, if $n-1 = 0$, G_{geom} is the symmetric group S_{1+N_d} in its N_d-dimensional "deleted permutation" representation. And for $X = \mathbb{P}^3$ embedded linearly in

$\mathbb{P}$, and d=3, we get the universal family of smooth cubic surfaces in $\mathbb{P}^3$, which has finite monodromy group equal to the Weyl group of E_6 in its reflection representation. However, one has the following result.

Theorem 0 [Ka-LAM, 2.2.4] Suppose $n-1 \geq 2$ is even and $d \geq 3$. If $N_d > 8$, then $G_{geom} = O(N_d)$.

One knows [SGA 7 Exp. XVIII, 6.4.2.1] that for fixed (X, i), N_d as a function of d is a polynomial of degree n, of the form

$$N_d = \deg(X)d^n + \text{lower terms}.$$

This makes it clear that for $d >> 0$, we will indeed have $N_d > 8$. But how large does d really need to be to insure that $N_d > 8$? Is $d \mapsto N_d$ strictly increasing for $d \geq 2$?

Consider first the case when X is $\mathbb{P}^n$, embedded linearly in $\mathbb{P}$. In this case, N_d is given by the well-known formula

$$N_d = ((d-1)/d)((d-1)^n - (-1)^n).$$

Armed with this explicit formula, one checks easily that $d \mapsto N_d$ is strictly increasing for $d \geq 2$, and that one has:

if $n = 3$ and $d \geq 4$, then $N_d > 8$,

if $n \geq 4$ and $d \geq 3$, then $N_d > 8$.

In particular, for $d \geq 3$ and $n-1 \geq 2$ even, we always have $N_d > 8$, except in the one exceptional case ($n = 3$, $d = 3$) of cubic surfaces in $\mathbb{P}^3$, for which $N_3 = 6$.

This observation led us to wonder how N_d for a general (X, i), which we will denote

$$N_d(X, i)$$

to emphasize its dependence on both X and i, compared with N_d for the special case ($\mathbb{P}^n$, linear embedding), which we will denote

$$N_d(\mathbb{P}^n, \text{lin}).$$

Theorem 1 Let X/k be projective, smooth, and geometrically connected, of dimension $n \geq 1$. Let $i : X \subset \mathbb{P}$ be a projective

embedding. Then we have the following results.
1) For every integer $d \geq 1$, we have the inequality

$$N_d(X, i) - N_1(X, i) \geq \deg(X)N_d(\mathbb{P}^n, \mathrm{lin}).$$

2 For every integer $d \geq 1$, we have the inequality

$$N_d(X, i) \geq \deg(X)N_d(\mathbb{P}^n, \mathrm{lin}).$$

3) The function $d \mapsto N_d(X, i)$ is strictly increasing for $d \geq 2$.

Corollary 2 Hypotheses as in Theorem 1, suppose $n \geq 3$ and $d \geq 3$. Then we have $N_d(X, i) > 8$ except for the one exceptional case

$$n = 3,\ d = 3, \text{ and } (X, i) \text{ is } (\mathbb{P}^3, \mathrm{lin}),$$

of cubic surfaces in $\mathbb{P}^3$.

proof of Corollary 2 As noted above, we have $N_d(\mathbb{P}^n, \mathrm{lin}) > 8$ except in the one exceptional case. Since $\deg(X) \geq 1$, it follows from part 2) of the theorem that $N_d(X, i) > 8$ except in the case

$$n = 3,\ d = 3,\ \deg(X) = 1.$$

But the only projective smooth connected 3-fold X in $\mathbb{P}$ with $\deg(X) = 1$ is $\mathbb{P}^3$, linearly embedded, cf. [Hart, Ch. I, Ex. 7.6, page 55]. QED

Corollary 3 Notations as in Theorem 1, suppose $n \geq 3$ is odd, $d \geq 3$, and (X, i, d) is not $(\mathbb{P}^3, \mathrm{lin}, 3)$. The monodromy of the universal family of smooth, degree d hypersurface sections of X has $G_{geom} = O(N_d(X, i))$.

proof of Corollary 3 This is immediate from Corollary 2 and the cited Theorem 0. QED

In proving Theorem 1, we stumbled across a striking inequality, relating the number

$$\#\mathrm{VanCycles}_d(X, i)$$

of vanishing cycles in any Lefschetz pencil of degree d hypersurface sections of (X, i) to the dimension $N_d(X, i)$ of the space they span, cf. [SGA 7, Expose XVIII, 3.2.10, 6.6 and 6.6.1].

Theorem 4 Hypotheses as in Theorem 1, suppose in addition that either the fibre dimension n-1 is odd, or that char(k) $\neq$ 2. Then for every integer $d \geq 2$ (and also for d=1 if X admits a Lefschetz pencil of hyperplane sections), we have the inequality

$$\#\mathrm{VanCycles}_d(X, i) \geq 2N_d(X, i).$$

The proofs of these results are based on positivity properties of integrals involving Chern classes, see Section II. These positivities, although elementary, do not seem to have been noticed before, and seem to be useful in other contexts as well. To illustrate this, we apply these positivities in the final section IV of the paper to Chern integral expressions for "dimensions" of exponential sums.

Section I: Standard facts about Chern classes

We continue to work over an algebraically closed field k in which a prime number ℓ is invertible. We fix an integer $n \geq 0$ and a projective, smooth (but not necessarily connected) equidimensional k-scheme X/k of dimension n, given with a projective embedding i : $X \subset \mathbb{P}$. We denote by $A^\bullet(X)$ the Chow ring of X, cf. [Gro-Ch] or [Ful, Ch. 8]. If X is the disjoint union of connected components X_α, then $A^\bullet(X)$ is the direct sum ring $\oplus_\alpha A^\bullet(X_\alpha)$. The theory is usually stated for connected X, but it is convenient to allow the more general case, so that one can inductively take hyperplane sections as many times as one likes.

Thus $A^\bullet(X)$ is the $\mathbb{Z}_{\geq 0}$-graded commutative ring with unit, whose underlying abelian group is the group of algebraic cycles on X, modulo rational equivalence. The grading is by codimension of support, and the multiplication is by intersection product. The group $A^n(X)$ is the group of 0-cycles on X, modulo rational equivalence. There is a canonical surjective group homomorphism

$$\deg : A^n(X) \to \mathbb{Z},$$

which sends a 0-cycle $\Sigma_i n_i P_i$ to its degree $\Sigma_i n_i$. Using it, one defines a canonical surjective group homomorphism

$$\int_X : A^\bullet(X) \to \mathbb{Z}$$

as follows. Given an element $\xi = \Sigma_i \xi_i$ in $A^\bullet(X)$, ξ_i in $A^i(X)$, one defines

$$\int_X \xi := \deg(\xi_n).$$

The ring $A^\bullet(X)$ receives a theory of Chern classes of coherent sheaves on X. The total chern class of the tangent bundle of X/k is called the total chern class of X, denoted c(X). Thus

$$c(X) = 1 + c_1(X) + ... + c_n(X).$$

The first fundamental fact we need is the integration formula

$$\int_X c(X) = \chi(X) := \Sigma_i (-1)^i \dim H^i(X, \overline{\mathbb{Q}}_\ell).$$

For the ambient $\mathbb{P} = \mathbb{P}^N$, the graded ring $A^\bullet(\mathbb{P}^N)$ is the truncated polynomial ring

$$A^\bullet(\mathbb{P}^N) = \mathbb{Z}[L]/(L^{N+1}),$$

where L in $A^1(\mathbb{P}^N)$ is the class of a hyperplane. For an integer $d \geq 1$, and for H_d a degree d hypersurface in $\mathbb{P}$, the class of H_d in $A^1(\mathbb{P}^N)$ is dL.

Via the pullback ring homomorphism

$$i^* : A^\bullet(\mathbb{P}^N) \to A^\bullet(X),$$

we obtain an element $i^*(L)$ in $A^1(X)$. When no confusion is likely, we will denote this element of $A^1(X)$ simply as L. We have $L^{n+1} = 0$ in $A^\bullet(X)$, and

$$\int_X L^n = \deg(X).$$

If H_d in $\mathbb{P} = \mathbb{P}^N$ is a degree d hypersurface such that $X \cap H_d$ is smooth and of codimension one in X, then the class of $X \cap H_d$ in $A^1(X)$ is dL.

The second basic fact we need is this. For $X \cap H_d$ a smooth, degree d hypersurface section of a projective smooth equidimensional X with $\dim X \geq 1$, denote by

$$\alpha : X \cap H_d \mapsto X$$

the inclusion. We get an induced homomorphism of groups

$$\alpha_* : A^\bullet(X \cap H_d) \to A^{\bullet+2}(X),$$

as well as a ring homomorphism

$$\alpha^* : A^\bullet(X) \to A^\bullet(X \cap H_d).$$

The chern class of $X \cap H_d$ is given by

$$c(X \cap H_d) = \alpha^*(c(X)/(1 + dL)).$$

For any element ξ in $A^\bullet(X)$, we have the projection formula

$$\alpha_*(\alpha^* \xi) = dL\xi,$$

and the integration formula

$$\int_{X\cap H_d} \alpha^*\xi = \int_X \alpha_*(\alpha^*\xi) = \int_X dL\xi.$$

Thus for any power series f(L) in L with $\mathbb{Z}$-coefficients, we have

$$\int_{X\cap H_d} c(X\cap H_d)f(L) = \int_X c(X)(dL/(1 + dL))f(L).$$

The third basic fact we need is this. Suppose we are given an integer $r \geq 1$, a sequence of $r \geq 1$ integers $d_1, d_2, \ldots, d_r$, each $d_i \geq 1$, and for each d_i a hypersurface H_{d_i} in $\mathbb{P}$ of degree d_i such that the following two transversality conditions hold:

1) For any integer j with $1 \leq j \leq r$ and with $j \leq \dim X$,

$$X\cap H_{d_1}\cap\ldots\cap H_{d_j} \text{ is smooth and of codimension } j \text{ in } X,$$

2) For any integer j with $\dim X < j \leq r$,

$$X\cap H_{d_1}\cap\ldots\cap H_{d_j} \text{ is empty.}$$

Then we have the integration formula

$$\int_X c(X)\prod_{i=1 \text{ to } r} (d_iL/(1 + d_iL)) = \chi(X\cap H_{d_1}\cap\ldots\cap H_{d_r}).$$

[If $r > \dim X$, both sides vanish. If $r \leq \dim X$, use the previous integration formula r times.]

Section II: Statements of the basic positivities

We now introduce the element t in $A^\bullet(X)$ defined by

$$t := -L/(1 + L).$$

Since L is nilpotent, with $L^{n+1} = 0$, we have

$$t = -L + L^2 - L^3 + L^4\ldots = \Sigma_{i=1 \text{ to } n} (-1)^iL^i.$$

Thus t is also nilpotent, $t^{n+1} = 0$, and

$$L = -t/(1 + t).$$

Positivity Lemma 5 Let k be an algebraically closed field k, X/k a projective, smooth connected k-scheme of dimension $n \geq 1$, given with a projective embedding $i : X \subset \mathbb{P}$. We have the following positivity results.

1) For all integers $k \geq 0$, we have

$$\int_X (-1)^nc(X)(1 + t)^2t^k \geq 0.$$

2) For $k > n$, we have

$$\int_X (-1)^nc(X)(1 + t)^2t^k = 0.$$

3) For $k = n$, we have

$$\int_X (-1)^n c(X)(1 + t)^2 t^n = \deg(X).$$

4) If $n \geq 1$, we have

$$\int_X (-1)^n c(X)(1 + t)^2 t^{n-1} \geq 2\deg(X) - 2.$$

5) If $n \geq 2$, we have

$$\int_X (-1)^n c(X)(1 + t)^2 t^{n-2} \geq \deg(X) - 1.$$

Normalization Lemma 6 If (X, i) is $(\mathbb{P}^n, \text{lin})$, then for $k \geq 0$ we have

$$\int_X (-1)^n c(X)(1 + t)^2 t^k = 0, \text{ if } k \neq n,$$
$$= 1, \text{ if } k = n.$$

Section III: Proofs

We choose a prime number ℓ invertible in k, and use $\overline{\mathbb{Q}}_\ell$-cohomology. We write

$$H^j(X) := H^j(X, \overline{\mathbb{Q}}_\ell).$$

We fix an isomorphism $\overline{\mathbb{Q}}_\ell(1) \cong \overline{\mathbb{Q}}_\ell$, so that we can view the cycle class map as a ring homomorphism

$$A^\bullet(X) \to H^{2\bullet}(X)$$

from the Chow ring to the even part of the cohomology ring. We will also denote by L in $H^2(X)$ the image of the class L in $A^1(X)$. For X as in the Positivity Lemma, the strong Lefschetz theorem tells us that for each integer $j \geq 0$, cupping j times with L is an isomorphism

$$L^j : H^{n-j}(X) \cong H^{n+j}(X).$$

In particular, cupping once with L defines an injective map

$$L : H^{n-2}(X) \to H^n(X).$$

The weak Lefschetz theorem tells us that if $X \cap L$ is a smooth hyperplane section of X, then the restriction map

$$H^j(X) \to H^j(X \cap L)$$

is an isomorphism for $j \leq n-2$, and is injective for $j=n-1$.

Key Lemma 7 Let X/k be as in the Positivity Lemma. Let L_1 and L_2 be hyperplanes in $\mathbb{P}$ such that both $X \cap L_1$ and $X \cap L_2$ are smooth of codimension 1 in X, and such that $X \cap L_1 \cap L_2$ is smooth of codimension 2 (:= empty, if n=1) in X. Then we have the integration formulas

$$\int_X (-1)^n c(X)(1 + t)^2$$
$$= (-1)^n \chi(X) + 2(-1)^{n-1}\chi(X\cap L_1) + (-1)^{n-2}\chi(X\cap L_1\cap L_2).$$
$$= \dim(H^n(X)/LH^{n-2}(X)) + 2\dim(H^{n-1}(X\cap L_1)/H^{n-1}(X))$$
$$+ \dim(H^{n-2}(X\cap L_1\cap L_2)/(H^{n-2}(X)).$$

proof We readily compute

$$(1 + t)^2 = (1 - L/(1 + L))^2 = 1 -2L/(1 + L) + (-L/(1 + L))^2.$$

Thus we have

$$\int_X (-1)^n c(X)(1 + t)^2$$
$$= (-1)^n \chi(X) + 2(-1)^{n-1}\chi(X\cap L_1) + (-1)^{n-2}\chi(X\cap L_1\cap L_2).$$

Now expand out the individual Euler characteristics, using Poincare duality.

$$(-1)^n\chi(X) = h^n(X) \quad - \quad 2h^{n-1}(X) \quad + \quad 2h^{n-2}(X) - ...,$$
$$2(-1)^{n-1}\chi(X\cap L_1) = 2h^{n-1}(X\cap L_1) - 4h^{n-2}(X\cap L_1) + ...$$

(using weak Lefschetz) $= 2h^{n-1}(X\cap L_1) - 4h^{n-2}(X) + ...,$

$$(-1)^{n-2}\chi(X\cap L_1\cap L_2) = \qquad h^{n-2}(X\cap L_1\cap L_2) - ....$$

By weak Lefschetz, the terms indicated by ... cancel out in each degree when we add up. Then use strong Lefschetz to write

$$h^n(X) - h^{n-2}(X) = \dim(H^n(X)/LH^{n-2}(X)). \quad \text{QED}$$

Corollary 8 Fix an integer $d \geq 1$. Let H_d and H'_d be degree d hypersurfaces in $\mathbb{P}$ such that both $X\cap H_d$ and $X\cap H'_d$ are smooth of codimension 1 in X, and such that $X\cap H_d \cap H'_d$ is smooth of codimension 2 (:= empty, if n=1) in X. Then we have the identity

$$(-1)^n\chi(X) + 2(-1)^{n-1}\chi(X\cap H_d) + (-1)^{n-2}\chi(X\cap H_d\cap H'_d).$$
$$= \dim(H^n(X)/LH^{n-2}(X)) + 2\dim(H^{n-1}(X\cap H_d)/H^{n-1}(X))$$
$$+ \dim(H^{n-2}(X\cap H_d\cap H'_d)/(H^{n-2}(X)).$$

proof This is just the second equality of Lemma 7 above, applied to the d-fold Segre embedding built out of the given embedding i. QED

Corollary 9 1) If n := dimX = 1, then

$$\int_X (-1)^n c(X)(1+t)^2 \geq 2\deg(X) - 2.$$

2) If n := dimX = 2, then

$$\int_X (-1)^n c(X)(1+t)^2 \geq \deg(X) - 1.$$

proof For n= 1, 2deg(X) - 2 is the middle term

$$2\dim(H^{n-1}(X\cap L_1)/H^{n-1}(X))$$

in the expression for this integral. For n = 2, deg(X) -1 is the last term

$$\dim(H^{n-2}(X\cap L_1\cap L_2)/(H^{n-2}(X))$$

in the expression for this integral. QED

Proof of Theorem 4 If either n-1 is odd or if char(k) ≠ 2, one knows [SGA 7, Expose XVIII, 3.2.10, 6.6 and 6.6.1] that the number of vanishing cycles in a Lefschetz pencil of degree d hypersurface sections of X is given by

$$\#\mathrm{VanCycles}_d(X, i)$$

$$= (-1)^n\chi(X) + 2(-1)^{n-1}\chi(X\cap H_d) + (-1)^{n-2}\chi(X\cap H_d\cap H'_d).$$

By the corollary just above, this last expression is equal to

$$= \dim(H^n(X)/LH^{n-2}(X)) + 2\dim(H^{n-1}(X\cap H_d)/H^{n-1}(X))$$

$$+ \dim(H^{n-2}(X\cap H_d\cap H'_d)/(H^{n-2}(X)),$$

the middle term of which is $2N_d(X, i)$. QED

Proof of the Positivity Lemma 6 For k > n, we have

$$\int_X (-1)^n c(X)(1+t)^2 t^k = 0,$$

simply because $t^{n+1} = 0$. For k ≤ n, denote by

$$X(\mathrm{lin, codim}\ k) \subset X$$

the intersection of X with a general linear space of codimension k in the ambient $\mathbb{P}$. Recalling that t is -L/(1 + L), we see by the projection and integration formulas for chern classes that

$$\int_X (-1)^n c(X)(1+t)^2 t^k$$

$$= \int_{X(\mathrm{lin, codim}\ k)} (-1)^{n-k} c(X(\mathrm{lin, codim}\ k))(1+t)^2.$$

For k = n, X(lin, codim n) consists of deg(X) reduced points, and the integral is equal to deg(X). For k < n, we compute this integral by the Key Lemma, which shows that it is non-negative, and Corollary 9, which gives the asserted inequalities for high values of k. Thus for n ≥ 1, we have

$$\int_X (-1)^n c(X)(1 + t)^2 t^{n-1}$$
$$= \int_{X(\text{lin, codim } n-1)} (-1)c(X(\text{lin, codim } n-1))(1 + t)^2$$
$$\geq 2\deg(X(\text{lin, codim } n-1)) - 2 = 2\deg(X) - 2.$$

Similarly, for n ≥ 2 we have

$$\int_X (-1)^n c(X)(1 + t)^2 t^{n-2}$$
$$= \int_{X(\text{lin, codim } n-2)} (-1)^2 c(X(\text{lin, codim } n-2))(1 + t)^2$$
$$\geq \deg(X(\text{lin, codim } n-2)) - 1 = \deg(X) - 1. \quad \text{QED}$$

Proof of the Normalization Lemma 7 Use the identity above,

$$\int_X (-1)^n c(X)(1 + t)^2 t^k$$
$$= \int_{X(\text{lin, codim } k)} (-1)^{n-k} c(X(\text{lin, codim } k))(1 + t)^2,$$

applied with $X = \mathbb{P}^n$ embedded linearly. We get

$$\int_{\mathbb{P}^n} (-1)^n c(\mathbb{P}^n)(1 + t)^2 t^k$$
$$= \int_{\mathbb{P}^{n-k}} (-1)^{n-k} c(\mathbb{P}^{n-k})(1 + t)^2.$$

For n=k, $\mathbb{P}^{n-k}$ is a point, and the integral is equal to 1. To show that it vanishes for k < n, we must show that for any n ≥ 1, we have

$$\int_{\mathbb{P}^n} (-1)^n c(\mathbb{P}^n)(1 + t)^2 = 0.$$

But this integral is equal to

$$= \dim(H^n(\mathbb{P}^n)/LH^{n-2}(\mathbb{P}^n) + 2\dim(H^{n-1}(\mathbb{P}^{n-1})/H^{n-1}(\mathbb{P}^n))$$
$$+ \dim(H^{n-2}(\mathbb{P}^{n-2})/H^{n-2}(\mathbb{P}^n)),$$

From the known cohomological structure of projective space, $H^\bullet(\mathbb{P}^n) \cong \overline{\mathbb{Q}}_\ell[L]/(L^{n+1})$, with L in degree 2, we see that each term vanishes for n ≥ 1. [When n= 0, it is the first term which is one-dimensional instead of vanishing.] QED

Proof of Theorem 1 The idea is to compute $N_d(X, i) - N_1(X, i)$ as a difference of Euler characteristics, to express this difference as a chern class integral, and then to apply the positivity lemma and the normalization lemma to that integral. We have

$$N_d(X, i) = h^{n-1}(X \cap H_d) - h^{n-1}(X),$$

$$N_1(X, i) = h^{n-1}(X \cap L) - h^{n-1}(X).$$

But

$$(-1)^{n-1}\chi(X \cap H_d) = h^{n-1}(X \cap H_d) - 2h^{n-2}(X \cap H_d) + \ldots,$$

$$(-1)^{n-1}\chi(X \cap L) = h^{n-1}(X \cap L) - 2h^{n-2}(X \cap L) + \ldots.$$

In these two formulas, all the terms after the first agree, by weak Lefschetz. So we get

$$N_d(X, i) - N_1(X, i) = (-1)^{n-1}\chi(X \cap H_d) - (-1)^{n-1}\chi(X \cap L)$$

$$= \int_X (-1)^n c(X)[-dL/(1 + dL) - (-L/(1 + L))].$$

Write this in terms of $t = -L/(1 + L)$. We readily calculate

$$-dL/(1 + dL) = dt/(1 - (d-1)t),$$

$$-L/(1 + L) = t,$$

$$-dL/(1 + dL) - (-L/(1 + L)) = dt(1 + t)/(1 - (d-1)t)$$

$$= (1 + t)^2[(d-1)t/((1 + t)(1 - (d-1)t)).$$

So we get

$$N_d(X, i) - N_1(X, i)$$

$$= \int_X (-1)^n c(X)(1 + t)^2[(d-1)t/((1 + t)(1 - (d-1)t))].$$

Now expand the bracketed term as a power series in t. We have

$$(d-1)t/((1 + t)(1 - (d-1)t)) = (d-1)t(\Sigma_i (-1)^i t^i)(\Sigma_j (d-1)^i t^j)$$

$$=(d-1)t\Sigma_k\ t^k\ \Sigma_{i+j=k}\ (-1)^i(d-1)^j$$

$$=(d-1)t\Sigma_k\ t^k\ ((d-1)^{k+1} - (-1)^{k+1})/((d-1) - (-1))$$

$$= ((d-1)/d)\Sigma_{k\geq 1}\ t^k\ ((d-1)^k - (-1)^k).$$

So we get

$$N_d(X, i) - N_1(X, i)$$

$$= \Sigma_{k\geq 1}\ ((d-1)/d)((d-1)^k - (-1)^k)\int_X (-1)^n c(X)(1 + t)^2 t^k.$$

All the integrals $\int_X (-1)^n c(X)(1 + t)^2 t^k$ are non-negative, and at least one is strictly positive, namely

$$\int_X (-1)^n c(X)(1 + t)^2 t^n = \deg(X).$$

For each $k \geq 1$, the coefficient $((d-1)/d)((d-1)^k - (-1)^k)$ is, for $d \geq 2$, non-negative and strictly increasing in d. So for fixed (X, i), the expression $N_d(X, i) - N_1(X, i)$, and hence $N_d(X, i)$ itself, is strictly increasing for $d \geq 2$. Moreover, looking only at the t^n term, we get the inequality

$$N_d(X, i) - N_1(X, i) \geq \deg(X)((d-1)/d)((d-1)^n - (-1)^n).$$

If we repeat the computation with (X, i) taken to be ($\mathbb{P}^n$, lin), we find

$$N_d(\mathbb{P}^n, \mathrm{lin}) - N_1(\mathbb{P}^n, \mathrm{lin})$$

$$= \Sigma_{k\geq 1} ((d-1)/d)((d-1)^k - (-1)^k)\int_{\mathbb{P}^n} (-1)^n c(\mathbb{P}^n)(1 + t)^2 t^k$$

$$= ((d-1)/d)((d-1)^n - (-1)^n),$$ (by the normalization lemma).

On the other hand, $N_1(\mathbb{P}^n, \mathrm{lin}) = 0$, so we recover the well known formula

$$N_d(\mathbb{P}^n, \mathrm{lin}) = ((d-1)/d)((d-1)^n - (-1)^n).$$ QED

Section IV: Application to inequalities for dimensions of exponential sums

There are a number of cases in which an exponential sum in several variables is given, up to sign, by the trace of Frobenius on a single cohomology group, and where the dimension of that cohomology group is given by an alternating sum of Euler characteristics, or equivalently by an explicit chern class integral. But it often seems miraculous that the integral in question comes out to be non-negative, and its monotonicity in the degrees involved is far from clear, as is its comparison with "what the corresponding integral on $\mathbb{P}^n$ would be". But the method used here, namely to write the integral as

$$\int_X (-1)^n c(X)(1 + t)^2(\text{a power series } \Sigma_k a_k t^k),$$

often gives a series $\Sigma_k a_k t^k$ with coeffficients which are visibly positive, and which are visibly monotonic in the various degrees that enter. We then apply to it the following theorem.

Theorem 10 Let k be an algebraically closed field, and let X/k be projective, smooth, and equidimensional, of dimension $n \geq 0$, given with a projective embedding $i : X \subset \mathbb{P}$. Denote by L in $A^1(X)$ for the class of a hyperplane section, and put $t := -L/(1 + L)$ in $A^{\bullet}(X)$. For any series $\Sigma_k a_k t^k$ with non-negative integer coefficients, we have the inequalities

$$\int_X (-1)^n c(X)(1+t)^2(\Sigma_k a_k t^k)$$
$$\geq \deg(X)a_n, \text{ if } n \geq 0,$$
$$\geq \deg(X)a_n + (2\deg(X)-2)a_{n-1}, \text{ if } n \geq 1,$$
$$\deg(X)a_n + (2\deg(X)-2)a_{n-1} + (\deg(X)-1)a_{n-2}, \text{ if } n \geq 2,$$

and the equality

$$\int_{\mathbb{P}^n} (-1)^n c(\mathbb{P}^n)(1+t)^2(\Sigma_k a_k t^k) = a_n.$$

proof Immediate reduction to the case when X is connected. The case $n = 0$ is trivial. For $n \geq 1$, apply the Positivity Lemma 6 to get the inequalities, and the Normalization Lemma 7 to get the equality. QED

We now list three particular families of integrals, each of which occurs, at least for certain choices of the parameters, as the "dimension" of an exponential sum. We leave to the reader the pleasant exercise of calculating, in each family, the corresponding series $\Sigma_k a_k t^k$ as above.

Example 1 $\int_X (-1)^n c(X)/((1+L)(1+dL))$, parameter $d \geq 1$, [Ka-SE, 5.1.1] and [Ka-MCS, Theorem 3].

Example 2 $\int_X (-1)^n c(X)(1+b(1-a)\delta L)/((1+a\delta L)(1+b\delta L))$, parameters a, b, δ all ≥ 1, [Ka-SE, 5.1.2]

Example 3 $\int_X (-1)^n c(X)/((1+dL)\Pi_{i=1 \text{ to } r}(1+d_i L))$, parameters d_i and d all ≥ 1, [Ka-SE, 5.4.1] and [Ka-MCS, Theorem 5].

References

[De-Weil II] Deligne, P., La conjecture de Weil II, Pub. Math. I.H.E.S. 52 (1981), 313-428.

{Ful] Fulton, W., *Intersection Theory*, Springer-Verlag, 1984.

[Gro-Ch] Grothendieck, A., La theorie des classes des Chern, Bull. S.M.F. 86 (1958), 137-154.

[Hart] Hartshorne, R., *Algebraic Geometry*, Springer-Verlag, Fourth Printing, 1987.

[Ka-LAM] Katz, N., Larsen's alternative, moments, and the monodromy of Lefschetz pencils, special volume in honor of J. Shalika's 60'th birthday, Johns Hopkins University Press, to appear.

[Ka-MCS] Katz, N., Estimates for "nonsingular" multiplicative character sums, I.M.R.N. no. 7 (2002), 333-349.

[Ka-SE] *Sommes Exponentielles*, Asterisque 79, 1980.

[SGA] A. Grothendieck et al - *Séminaire de Géométrie Algébrique du Bois-Marie*, SGA 1, SGA 4 Parts I, II, and III, SGA 4½, SGA 5, SGA 7 Parts I and II, Springer Lecture Notes in Math. 224, 269-270-305, 569, 589, 288-340, 1971 to 1977.