Sums of Betti numbers in arbitrary characteristic

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Introduction In [Mil], Milnor gave an explicit upper bound for the sum of the Betti numbers of a complex affine algebraic variety V. If V is defined in $\mathbb{C}^N$, $N \geq 1$, by $r \geq 1$ equations F_i, i =1 to r, all of degree ≤ d, Milnor showed

$$\Sigma_i h^i(V, \mathbb{Q}) \leq d(2d-1)^{2N-1}.$$

Oleinik [Ol] and Thom [Th] gave similar results.

It is standard (cf. the proof below that Theorem 2 implies Theorem 1) to infer from Milnor's result an explicit upper bound for the sum of the compact Betti numbers: one finds

$$\Sigma_i h^i{}_c(V, \mathbb{Q}) \leq 2^r(1 + rd)(1 + 2rd)^{2N+1}.$$

What happens if we work over an algebraically closed field k of arbitrary characteristic? Let $V \subset \mathbb{A}^N$, $N \geq 1$, be a closed subscheme, defined by the vanishing of r polynomials $F_1, ..., F_r$ in $k[X_1, ..., X_N]$, all of degree ≤ d. Bombieri [Bom], combining Dwork's p−adic methods [Dw] and Deligne's results [De−Weil II, 3.3.2−8], gave an explicit upper bound for the Euler characteristic of V. For any prime ℓ invertible in k, Bombieri proved

$$|\chi_c(V, \mathbb{Q}_\ell)| \leq (4(1+d) + 5)^{N+r}.$$

[Recall Laumon's result [Lau] that $\chi_c(V, \mathbb{Q}_\ell) = \chi(V, \mathbb{Q}_\ell)$.] Strictly speaking, Bombieri proves his result when k is the algebraic closure of a finite field, but by a standard "descend to a finitely generated ground field, spread out, and then specialize to a sufficiently general closed point" argument, this case implies the general case.

Adolphson and Sperber [A−S] later improved on Bombieri's bound. To state their result, we define $D_{N,r}$ to be the homogeneous form of degree N in r+1 variables over $\mathbb{Z}$ all of whose coefficients are 1:

$$D_{N,r}(X_0, X_1, ..., X_r) := \Sigma_{|W|=N} X^W.$$

Their general result [A−S, 5.2.7] gives

$$|\chi_c(V, \mathbb{Q}_\ell)| \leq 2^r D_{N,r}(1, 1+d, 1+d, ..., 1+d).$$

Because $D_{N,r}$ is dominated coefficientwise by $(\Sigma_{i=0 \text{ to } r} X_i)^N$, this in turn gives

$$|\chi_c(V, \mathbb{Q}_\ell)| \leq 2^r \times (r+1+rd)^N.$$

In Part I of this note, we will use the affine weak Lefschetz theorem to deduce from the Bombieri bound, or from the Adolphson−Sperber bound, or indeed from **any** universal bound

$$|\chi_c(V, \mathbb{Q}_\ell)| \leq E(N, r, d),$$

on the Euler characteristic, a corresponding universal bound on the sum of the compact Betti numbers.

In Part II, we will use the same technique to get universal (but non−explict) bounds for the sum of the compact Betti numbers with coefficients in a general compatible system. We then give explicit bounds for some particularly simple compatible systems, as consequences of the explicit bounds of Bombieri and Adolphson−Sperber for the corresponding Euler characteristics.

Part I : Sums of Betti numbers with $\mathbb{Q}_\ell$ coefficients

The Axiomatic Set−up

Given a univeral bound

$$|\chi_c(V, \mathbb{Q}_\ell)| \le E(N, r, d),$$

we define new constants

$$A(N, r, d) := E(N, r, d) + 2 + 2\Sigma_{n=1 \text{ to } N-1} E(n, r, d),$$

$$B(N, r, d) := 1 + \Sigma_{\text{subsets } S \neq \emptyset \text{ of } \{1,2,..., r\}} A(N+1, 1, 1+ d(\#S)).$$

Theorem 1 Let k be an algebraically closed field. Let $V \subset \mathbb{A}^N$, $N > 1$, be a closed subscheme, defined by the simultaneous vanishing of r polynomials $F_1, ..., F_r$ in $k[X_1, ..., X_N]$, each of degree $\le d$. Fix a prime number ℓ invertible in k. Then we have

$$\Sigma_i h^i{}_c(V, \mathbb{Q}_\ell) \le B(N, r, d).$$

Theorem 2 Hypotheses as in Theorem 1, suppose in addition that either dim(V) = 0, or that V/k is smooth and connected of dimension $n \ge 1$. Fix a prime number ℓ invertible in k. Then we have

$$\Sigma_i h^i{}_c(V, \mathbb{Q}_\ell) \le A(N, r, d).$$

Proofs of Theorems 1 and 2 We first admit the truth of Theorem 2, and show how it implies Theorem 1. Let us define

$$\sigma_c(V, \mathbb{Q}_\ell) := \Sigma_i h^i{}_c(V, \mathbb{Q}_\ell),$$

$$\sigma(V, \mathbb{Q}_\ell) := \Sigma_i h^i(V, \mathbb{Q}_\ell).$$

When V is smooth, Poincare duality for each connected component of V shows that we have $\sigma_c(V, \mathbb{Q}_\ell) = \sigma(V, \mathbb{Q}_\ell)$.

From the excision sequence for $\mathbb{A}^N - V \subset \mathbb{A}^N$ in compact cohomology,

$$\to H^{i-1}{}_c(\mathbb{A}^N, \mathbb{Q}_\ell) \to H^{i-1}{}_c(V, \mathbb{Q}_\ell) \to H^i{}_c(\mathbb{A}^N - V, \mathbb{Q}_\ell) \to ,$$

we get

$$\sigma_c(V, \mathbb{Q}_\ell) \le \sigma_c(\mathbb{A}^N, \mathbb{Q}_\ell) + \sigma_c(\mathbb{A}^N - V, \mathbb{Q}_\ell) = 1 + \sigma(\mathbb{A}^N - V, \mathbb{Q}_\ell),$$

the last equality because $\mathbb{A}^N - V$ is smooth.

The open variety $\mathbb{A}^N - V$ is covered by the affine open sets

$$U_i : = \mathbb{A}^n[1/F_i], \ i = 1 \text{ to } r.$$

From the Mayer−Vietoris spectral sequence of this covering

$$E_1{}^{p,q} := \oplus_{1 \le i_1 < i_2 < ... < i_p \le r} H^q(U_{i_1} \cap U_{i_2} \cap ... \cap U_{i_p}, \mathbb{Q}_\ell) \Rightarrow H^{p+q}(\mathbb{A}^N - V, \mathbb{Q}_\ell)$$

we get

$$\sigma(\mathbb{A}^N - V, \mathbb{Q}_\ell) \le \Sigma_{p=1 \text{ to } r} \Sigma_{1 \le i_1 < i_2 < ... < i_p \le r} \sigma(U_{i_1} \cap U_{i_2} \cap ... \cap U_{i_p}, \mathbb{Q}_\ell).$$

Now each intersection $U_{i_1} \cap U_{i_2} \cap ... \cap U_{i_p}$ is of the form

$$\mathbb{A}^N[1/(\prod_{j=1 \text{ to } p} F_{i_j})],$$

which we may view as ths smooth hypersurface in $\mathbb{A}^{N+1}$ (coordinates $X_1, ..., X_N, Z$) defined by the single equation

$$1 - Z(\prod_{j=1 \text{ to } p} F_{i_j}) = 0,$$

whose degree is at most 1 + pd. So by Theorem 2, we have

$$\sigma(U_{i_1} \cap U_{i_2} \cap ... \cap U_{i_p}, \mathbb{Q}_\ell) \le A(N+1, 1, 1 + pd).$$

Combining this with the inequality above,

$$\sigma(\mathbb{A}^N - V, \mathbb{Q}_\ell) \le \Sigma_{p=1 \text{ to } r} \Sigma_{1 \le i_1 < i_2 < ... < i_p \le r} \sigma(U_{i_1} \cap U_{i_2} \cap ... \cap U_{i_p}, \mathbb{Q}_\ell),$$

we get Theorem 1.

To prove Theorem 2, we argue as follows. If dim(V) = 0, then both $H^a_c(V, \mathbb{Q}_\ell)$ and $H^a(V, \mathbb{Q}_\ell)$ vanish for $a \neq 0$, so we have

$$\sigma_c(V, \mathbb{Q}_\ell) = \sigma(V, \mathbb{Q}_\ell) = \chi(V, \mathbb{Q}_\ell) \le E(N, r, d) \le A(N, r, d),$$

as required.

Suppose now V/k is smooth and connected of dimension $n \ge 1$. We proceed by induction on the dimension n of V. Because V is smooth and connected, $\sigma_c(V, \mathbb{Q}_\ell) = \sigma(V, \mathbb{Q}_\ell)$; we will work below with σ. If n=1, then

$$\sigma(V, \mathbb{Q}_\ell) = h^1(V, \mathbb{Q}_\ell) + 1$$
$$= -\chi(V, \mathbb{Q}_\ell) + 2 \le E(N, r, d) + 2 \le A(N, r, d).$$

as required. Suppose now that $n \ge 2$, and that the theorem holds for n−1. By the affine weak Lefschetz theorem [Ka−ACT, 3.4.1 and 3.4.3], there is a dense open set of affine hyperplanes H in $\mathbb{A}^N$ for which $V \cap H$ is smooth and connected of dimension n−1, and for which the restriction maps on cohomology

$$H^i(V, \mathbb{Q}_\ell) \to H^i(V \cap H, \mathbb{Q}_\ell)$$

are injective for i = n−1, and isomorphisms for $i \le n-2$. The key point is that H is an $\mathbb{A}^{N-1}$, in which $V \cap H$ is defined by r equations, each of degree at most d.

From affine weak Lefschetz, we have the inequality

$$h^n(V, \mathbb{Q}_\ell) \le (-1)^n \chi(V, \mathbb{Q}_\ell) + (-1)^{n-1} \chi(V \cap H, \mathbb{Q}_\ell)$$
$$= h^n(V, \mathbb{Q}_\ell) + [h^{n-1}(V \cap H, \mathbb{Q}_\ell) - h^{n-1}(V, \mathbb{Q}_\ell)].$$

We also have the inequality

$$\sigma(V, \mathbb{Q}_\ell) \le h^n(V, \mathbb{Q}_\ell) + \sigma(V \cap H, \mathbb{Q}_\ell).$$

Putting these inequalities together, we find

$$\sigma(V, \mathbb{Q}_\ell) \le (-1)^n \chi(V, \mathbb{Q}_\ell) + (-1)^{n-1} \chi(V \cap H, \mathbb{Q}_\ell) + \sigma(V \cap H, \mathbb{Q}_\ell).$$

So by induction we find

$$\sigma(V, \mathbb{Q}_\ell) \le E(N, r, d) + E(N-1, r, d) + A(N-1, r, d) := A(N, r, d). \quad \text{QED}$$

Theorem 3 Over an algebraically closed field k, let $X \subset \mathbb{P}^N$, $N \ge 1$, be defined by the vanishing of $r \ge 1$ homogeneous forms F_i, all of degree at most d. Then for any prime ℓ invertible in k, we have

$$\Sigma_i h^i{}_c(X, \mathbb{Q}_\ell) = \Sigma_i h^i(X, \mathbb{Q}_\ell) \le 1 + \Sigma_{n=1 \text{ to } N} B(n, r, d).$$

proof We have $\sigma_c(X, \mathbb{Q}_\ell) = \sigma(X, \mathbb{Q}_\ell)$ because X is projective. Use the $\sigma_c(X, \mathbb{Q}_\ell)$ description. Write X as $(X\cap\mathbb{A}^N)\coprod(X\cap\mathbb{P}^{N-1})$, and proceed by induction on N. QED

Explicit incarnations

If we begin with the Bombieri bound

$$E(N, r, d) = (4(1+d) + 5)^{N+r},$$

the corresponding constants A(N, r, d) and B(N, r, d) are easily checked to satisfy

$$A(N, r, d) \le (5/4)\times(4(1+d)+5)^{N+r},$$

$$B(N, r, d) \le 2^r A(N+1, 1, 1 + rd)$$

$$\le 2^r\times(5/4)\times(4(2+ rd) + 5)^{N+2}.$$

If we begin with the weakening

$$E(N, r, d) = 2^r\times(r+1+rd)^N$$

of the Adolphson−Sperber bound, the corresponding constants are easily checked to satisfy

$$A(N, r, d) \le 3\times2^r\times(r+1+rd)^N,$$

$$B(N, r, d) \le 2^r A(N+1, 1, 1 + rd) \le 2^r\times3\times2\times(1+ 1 + (1 + rd))^{N+1}..$$

Corollary of Theorem 1 Hypotheses and notations as in Theorem 1, we have the inequalities

$$\Sigma_i h^i{}_c(V, \mathbb{Q}_\ell) \le 2^r\times(5/4)\times(4(2+ rd) + 5)^{N+2},$$

and

$$\Sigma_i h^i{}_c(V, \mathbb{Q}_\ell) \le 2^r\times3\times2\times(1+ 1 + (1 + rd))^{N+1}.$$

Corollary of Theorem 2 Hypotheses and notations as in Theorem 2, we have the inequalities

$$\Sigma_i h^i{}_c(V, \mathbb{Q}_\ell) \le (5/4)\times(4(1+d)+5)^{N+r},$$

and

$$\Sigma_i h^i{}_c(V, \mathbb{Q}_\ell) \le 3\times2^r\times(r+1+rd)^N.$$

Corollary of Theorem 3 (cf. [Ka−ESES, pp. 881−882 and page 889]) Hypotheses and notations as in Theorem 3, we have the inequalties

$$\Sigma_i h^i{}_c(X, \mathbb{Q}_\ell) = \Sigma_i h^i(X, \mathbb{Q}_\ell) \le (13/12)\times2^r\times(5/4)\times(4(2+ rd) + 5)^{N+2}.$$

and

$$\Sigma_i h^i{}_c(X, \mathbb{Q}_\ell) = \Sigma_i h^i(X, \mathbb{Q}_\ell) \le (3/2)\ 2^r\times3\times2\times(1+ 1 + (1 + rd))^{N+1}.$$

First application to independence of ℓ

Theorem 4 Let k be an algebraically closed field, V/k a separated k−scheme of finite type. There exists a constant $M_c(V/k)$ such that for every prime ℓ invertible in k, we have

$$\Sigma_i h^i{}_c(V, \mathbb{Q}_\ell) \le M_c(V/k).$$

proof If V is affine, this is a trivial consequence of Theorem 1, where we give an a priori bound for $M_c(V/k)$ in terms of the number and degrees of the equations defining V. Proceed by induction on the minimum number n of affine open sets needed to cover V. Take an affine open cover $\{U_i\}_{i=1 \text{ to } n}$ of V. Define

$$V_{n-1} := \cup_{i \le n-1} U_i.$$

Then V is $V_{n-1} \cup U_n$. By the excision sequence for $U_n \subset V$, we get

$$\sigma_c(V, \mathbb{Q}_\ell) \le \sigma_c(U_n, \mathbb{Q}_\ell) + \sigma_c(V - U_n, \mathbb{Q}_\ell).$$

But $Z := V - U_n$ is closed in V and doesn't meet U_n, so it is the union of its n−1 affine open sets $Z \cup U_i$ for i = 1 to n−1. The theorem holds by induction for $Z := V - U_n$ and for U_n, so we may take

$$M_c(V/k) := M_c(U_n/k) + M_c((V - U_n)/k). \qquad \text{QED}$$

Much deeper is the following result.

Theorem 5 (de Jong, Berthelot) Let k be an algebraically closed field, V/k a separated k−scheme of finite type. There exists a constant M(V/k) such that for every prime ℓ invertible in k, we have

$$\Sigma_i h^i(V, \mathbb{Q}_\ell) \le M(V/k).$$

proof First reduce to the case when V is afine by the Mayer−Vietoris spectral sequence, then to the case where V is reduced and irreducible. Then use de Jong's theorem [de J, Thm. 3.1] on alterations to produce a Deligne−style proper hypercovering $X_. \to V$ in which each X_n is a smooth affine k−scheme, cf. [Ber, page 34]. We get a spectral sequence

$$E_1{}^{p,q} = H^q(X_p, \mathbb{Q}_\ell) \to H^{p+q}(V, \mathbb{Q}_\ell).$$

So for each integer a ≥ 0, we have

$$h^a(V, \mathbb{Q}_\ell) \le \Sigma_{p+q=a} h^q(X_p, \mathbb{Q}_\ell).$$

Since $h^a(V, \mathbb{Q}_\ell) = 0$ for a > dim(V) by the Lefschetz affine theorem, we get

$$\sigma(V, \mathbb{Q}_\ell) \le \Sigma_{p \le \dim(V)} \sigma(X_p, \mathbb{Q}_\ell)$$
$$= \Sigma_{p \le \dim(V)} \sigma_c(X_p, \mathbb{Q}_\ell) \text{ (because each } X_p \text{ is smooth)}$$
$$\le \Sigma_{p \le \dim(V)} M_c(X_p/k). \qquad \text{QED}$$

Remark/Question If V is affine, defined in $\mathbb{A}^N$ by r equations all of degree ≤ d, then Theorem 1 gives an explicit value for $M_c(V/k)$. So if in addition V/k is smooth and connected, then M(V/k) = $M_c(V/k)$ is explicitly bounded. If V/k is not smooth and connected, is there an equally explicit bound for M(V/k)?

Part II : Sums of Betti numbers with coefficients in a compatible system

Througthout this part, we work over a finite field F. Let V/F be a separated F–scheme of finite type. For any prime ℓ, we have the notion of a lisse $\overline{\mathbb{Q}}_\ell$–sheaf $\mathcal{F}$ on V. If V is connected, then once we fix a geometric point $\overline{v}$ in V, such an $\mathcal{F}$ is a finite–dimensional continuous $\overline{\mathbb{Q}}_\ell$–representation of $\pi_1(V, \overline{v})$. In the general case, V is a finite disjoint union of its connected components V_i, and giving $\mathcal{F}$ on V is the same as giving a lisse $\overline{\mathbb{Q}}_\ell$–sheaf $\mathcal{F}|V_i$ on V_i for each i.

We will work with a fairly general notion of compatible system on V/F. Let K be a field of characteristic zero, and Λ a nonempty collection of pairs

(a prime ℓ_λ invertible in F, an embedding ι_λ of K into $\overline{\mathbb{Q}}_{\ell_\lambda}$).

A (Λ, K)–compatible system of lisse sheaves on V/F is a collection, one for each λ in Λ, of a lisse $\overline{\mathbb{Q}}_{\ell_\lambda}$–sheaf $\mathcal{F}_\lambda$ on V, with the following compatibility property: For every finite extension E/F, for every E–valued point v in V(E), and for every λ in Λ, the characteristic polynomial

$$\det(1 - T\mathrm{Frob}_{E,v} \mid \mathcal{F}_\lambda)$$

lies in K[T] (when we use ι_λ to view K as a subfield of $\overline{\mathbb{Q}}_{\ell_\lambda}$) and in K[T] it is independent of of the auxiliary choice of λ.

We will prove the following theorem.

Theorem 6 Given a (Λ, K)–compatible system of lisse sheaves $\{\mathcal{F}_\lambda\}_\lambda$ on V/F, there exists a constant $M_c(V/F, \{\mathcal{F}_\lambda\})$ such that for every λ in Λ, we have

$$\sum_i h^i_c(V\otimes_F\overline{F}, \mathcal{F}_\lambda) \le M_c(V/F, \{\mathcal{F}_\lambda\}).$$

Proof of Theorem 6 The fundamental observation is that the Euler characteristic $\chi_c(V\otimes_F\overline{F}, \mathcal{F}_\lambda)$ (= $\chi(V\otimes_F\overline{F}, \mathcal{F}_\lambda)$ by Laumon [Lau]) is independent of λ, because it is minus the degree of the L–function $L(V/F, \mathcal{F}_\lambda)(T)$ as a rational function, thanks to Grothendieck's Lefschetz Trace Formula:

$$L(V/F, \mathcal{F}_\lambda)(T) = \prod_i \det(1 - T\mathrm{Frob}_F \mid H^i_c(V\otimes_F\overline{F}, \mathcal{F}_\lambda)^{(-1)^{i+1}}.$$

But in a compatible system, the L–functions are all equal in 1+TK[[T]], so, being rational functions in overfields, they all lie in K(T) and are equal there, so in particular all have the same degree.

We next use the ideas of Part I to pass from bounds for Euler characteristics to bounds for sums of Betti numbers. Of course, if V has dimension zero, the Euler characteristic is the sum of the Betti numbers, so in this case we are done.

Because we are dealing with compact cohomology, we may cut V into pieces. So we reduce to the case when V is both affine and smooth/F. We may make extend scalars from F to any finite extension E/F, so we may further reduce to the case when V/F is smooth and geometrically connected of some dimension n. We have already treated the zero–dimensional case, so we may assume $n \ge 1$.

Suppose, then, that V/F is smooth and geometrically connected, of some dimension $n \ge 1$.

Given a (Λ, K)–compatible system $\{\mathcal{F}_\lambda\}$ on V/F, their contragredients $\{\mathcal{F}_\lambda^\vee\}$ also form a (Λ, K)–compatible system on V/F. By Poincare duality, we have

$$\Sigma_i h^i_c(V\otimes_F\overline{F}, \mathcal{F}_\lambda) = \Sigma_i h^i(V\otimes_F\overline{F}, \mathcal{F}_\lambda^\vee).$$

So interchanging $\{\mathcal{F}_\lambda\}$ and $\{\mathcal{F}_\lambda^\vee\}$, it suffices to prove

Theorem 7 Hypotheses and notations as in Theorem 6, suppose in addition that V/F is smooth and geometrically connected, of some dimension $n \geq 1$. Given a (Λ, K)–compatible system of lisse sheaves $\{\mathcal{F}_\lambda\}_\lambda$ on V/F, there exists a constant $M(V/F, \{\mathcal{F}_\lambda\})$ such that for every λ in Λ, we have

$$\Sigma_i h^i(V\otimes_F\overline{F}, \mathcal{F}_\lambda) \leq M(V/F, \{\mathcal{F}_\lambda\}).$$

Proof of Theorem 7 We first treat the case n=1. Then V/F is a smooth, geometrically connected affine curve, so $H^a(V\otimes_F\overline{F}, \mathcal{F}_\lambda)$ vanishes except possibly for a = 0 or a = 1. So we have

$$\Sigma_i h^i(V\otimes_F\overline{F}, \mathcal{F}_\lambda) = -\chi(V\otimes_F\overline{F}, \mathcal{F}_\lambda) + 2h^0(V\otimes_F\overline{F}, \mathcal{F}_\lambda)$$
$$\leq -\chi(V\otimes_F\overline{F}, \mathcal{F}_\lambda) + 2\mathrm{rank}(\mathcal{F}_\lambda).$$

The rank of $\mathcal{F}_\lambda$ is independent of L, being the degree of the characteristic polynomial of $\mathrm{Frob}_{E,v}$ at any chosen point v of V with values in some finite extension E/F.

Suppose now that $n \geq 2$. The idea is to apply repeatedly the affine weak Lefschetz theorem. To fix ideas, suppose that V is given as a locally closed subscheme of $\mathbb{A}^N$, with coordinates $x_1, ..., x_N$. Denote by $\mathcal{H}$ the space of equations

$$H : \Sigma_i a_i x_i + b = 0$$

of possibly degenerate affine hyperplanes H in $\mathbb{A}^N$ (i.e, $\mathcal{H}$ is the space $\mathrm{AffMaps}(\mathbb{A}^N, \mathbb{A}^1)$ of [Ka–ACT, 1.3]). For any $m \geq 1$, the m–fold product $\mathcal{H}^m$ is the space $\mathrm{AffMaps}(\mathrm{AffMaps}(\mathbb{A}^N, \mathbb{A}^m)$. Here is the result we need.

Theorem 8 Let k be an algebraically closed field, V/k a smooth connected affine variety of dimension $n \geq 2$, given as a locally closed subscheme of $\mathbb{A}^N$. Let ℓ be a prime invertible in k, and $\mathcal{F}$ a lisse $\overline{\mathbb{Q}}_\ell$–sheaf on V. Then there exists a dense open set U in $\mathcal{H}^{n-1}$ with the following properties:

1) For any k–valued point $(H_1, H_2, ..., H_{n-1})$ in U(k), and any i with $1 \leq i \leq n-1$, the intersection $V\cap H_1\cap H_2\cap...\cap H_i$ is smooth and connected of dimension n–i. For i=1, the restriction map on cohomology

$$H^a(V, \mathcal{F}) \to H^a(V\cap H_1, \mathcal{F})$$

is injective for a = n–1, and bijective for $a \leq n-2$. For $i \geq 2$, the restriction map on cohomology

$$H^a(V\cap H_1\cap H_2\cap...\cap H_i-1, \mathcal{F}) \to H^a(V\cap H_1\cap H_2\cap...\cap H_i, \mathcal{F})$$

is injective for $a = n - i$, and bijective for $a \leq n - i - 1$.

2) For any i with $1 \leq i \leq n-1$, consider the affine $\mathcal{H}^{n-1}$–scheme

$$\pi_i : \mathcal{X}_i \to \mathcal{H}^{n-1}$$

where $\mathcal{X}_i$ is the closed subscheme of $V\times\mathcal{H}^{n-1}$ whose fibre over an S−valued point $(H_1, H_2, ..., H_{n-1})$ of $\mathcal{H}^{n-1}$ is $V\cap H_1\cap H_2\cap...\cap H_i$. The sheaves $(R^j(\pi_i)_*\mathcal{F}) \mid U$ are lisse on U, and their formation is compatible with arbitrary change of base on U.

proof of Theorem 8 Property 2) always holds on some dense open set, by Deligne's generic base change theorem [De−TF, Thm. 1.9]. To see that property 1) holds on a dense open set, apply [Ka−ACT, 3.9.6 and 3.9.8] with d there taken to be our n, n there our N, π there the inclusion of V into $\mathbb{A}^N$, and f there the constant map "zero", and take m successively to be 1, 2, ..., n−1. This successively produces dense open sets U_i in $\mathcal{H}^i$ such that at a point $(H_1, H_2, ..., H_i)$ in U_i, the affine variety $(V\otimes_F\bar{F})\cap H_1\cap H_2\cap...\cap H_i$ is smooth of dimension n−i and connected, and the restriction map on cohomology

$$H^a(V\otimes_F\bar{F}, \mathcal{F}) \to H^a((V\otimes_F\bar{F})\cap H_1\cap H_2\cap...\cap H_i, \mathcal{F})$$

is injective for $a = n - i$, and bijective for $a \le n - i - 1$. View $\mathcal{H}^i$ as the image of $\mathcal{H}_{n-1}$ under the map "projection on the first i factors", and take $\tilde{U}_i \subset \mathcal{H}^{n-1}$ to be the inverse image of U_i. Then over the dense open set $\tilde{U}_1\cap\tilde{U}_2\cap...\cap\tilde{U}_{n-1}$, property 1) holds. QED

We now resume the proof of Theorem 7. For each λ, we apply the previous theorem to $\mathcal{F}_\lambda$ on $V\otimes_F\bar{F}$. We get a dense open set $U(\lambda)$ in $\mathcal{H}^{n-1}\otimes_F\bar{F}$ over which properties 1) and 2) hold for $\mathcal{F}_\lambda$ on $V\otimes_F\bar{F}$.

By property 2), there exist non−negative integers

$$h^i(\text{codim } j, \lambda), \text{ indexed by } i \ge 0, 0 \le j \le n-i$$

such that

$$h^i(\text{codim } 0, \lambda) = h^i(V\otimes_F\bar{F}, \mathcal{F}_\lambda),$$

and such that for every point $(H_1, H_2, ..., H_{n-1})$ in $U(\lambda)(\bar{F})$, and every j with $1 \le j \le n-1$, we have

$$h^i(\text{codim } j, \lambda) = h^i((V\otimes_F\bar{F})\cap H_1\cap H_2\cap...\cap H_j, \mathcal{F}_\lambda).$$

We define

$$\chi(\text{codim } j, \lambda) := \Sigma_i (-1)^i h^i(\text{codim } j, \lambda),$$

and we define

$$\sigma(\text{codim } j, \lambda) := \Sigma_i h^i(\text{codim } j, \lambda),$$

Exactly as in the proof of Theorem 2, picking any point $(H_1, H_2, ..., H_{n-1})$ in $U(\lambda)(\bar{F})$ and successively applying affine weak Lefschetz shows that we have the following inequality:

(σ−χ inequality)

$$\sigma(\text{codim } 0, \lambda) \le |\chi(\text{codim. } 0, \lambda)| + 2\text{rank}(\mathcal{F}_\lambda) + 2\Sigma_{j=1 \text{ to } n-1} |\chi(\text{codim. } j, \lambda)|.$$

We have already remarked that $\text{rank}(\mathcal{F}_\lambda)$ is independent of λ, and that $\chi(\text{codim. } 0, \lambda) :=$

$\chi(V\otimes_F\bar{F}, \mathcal{F}_\lambda)$ is independent of λ (being minus the degree of the L–function). It remains only to show that for each j with $1 \le j \le n-1$, $\chi(\text{codim. } j, \lambda)$ is independent of λ. Given two points λ_1 and λ_2 in Λ, the dense open sets $U(\lambda_1)$ and $U(\lambda_2)$ have a nonempty intersection, so there exists a point $(H_1, H_2, ..., H_{n-1})$ in $U(\lambda_1)(\bar{F})\cap U(\lambda_2)(\bar{F})$. Such a point is defined over some finite extension E/F. Comparing the degrees of the L–functions

$$L((V\otimes_F E)\cap H_1\cap H_1\cap...\cap H_j/E, \mathcal{F}_{\lambda_1})(T) = L((V\otimes_F E)\cap H_1\cap H_1\cap...\cap H_j/E, \mathcal{F}_{\lambda_2})(T)$$

(equality in K(T)), we get

$$\chi(\text{codim. } j, \lambda_1) = \chi(\text{codim. } j, \lambda_2),$$

as required. QED

Theorem 9 (Berthelot, de Jong) Given a (Λ, K)–compatible system of lisse sheaves $\{\mathcal{F}_\lambda\}_\lambda$ on V/F, there exists a constant $M(V/F, \{\mathcal{F}_\lambda\})$ such that for every λ in Λ, we have

$$\sum_i h^i(V\otimes_F\bar{F}, \mathcal{F}_\lambda) \le M(V/F, \{\mathcal{F}_\lambda\}).$$

proof Repeat the proof of Theorem 5. QED

Some Diophantine applications

In this section, we will combine the $\sigma-\chi$ inequality above with the explicit bounds of Bombieri and Adolphson–Sperber.

Theorem 10 Let F be finite field of characteristic p, ψ a non–trivial additive character ψ of (F, +) with values in $\mathbb{Q}(\zeta_p)^\times$, K the field $\mathbb{Q}(\zeta_p)$, Λ the set of all embeddings of K into all $\bar{\mathbb{Q}}_\ell$'s with $\ell \neq$ p. Fix integers $N \ge 1$ and $d \ge 0$. Let f in $F[X_1,..., X_N]$ be a polynomial of degree at most d. On $\mathbb{A}^N/E$, we have the (Λ, K)–compatible system $\{\mathcal{F}_\lambda\}$ given by the Artin–Schreier sheaf $\mathcal{L}_{\psi(f)}$. For every λ in Λ, we have the estimate

$$\sigma_c(\mathbb{A}^N\otimes_F\bar{F}, \mathcal{L}_{\psi(f)}) = \sigma(\mathbb{A}^N\otimes_F\bar{F}, \mathcal{L}_{\bar{\psi}(f)}) \le 3(d+1)^N.$$

proof Indeed, the Adolphson–Sperber bound [A–S, 5.2.7] in this case is

$$|\chi(\mathbb{A}^N\otimes_F\bar{F}, \mathcal{L}_{\bar{\psi}(f)})| \le (d + 1)^N.$$

So in the notations of the proof of Theorem 6, we have

$$|\chi(\text{codim } i, \lambda)| \le (d + 1)^{N-i}.$$

By the $\sigma-\chi$ inequality, we get

$$\sigma(\text{codim. } 0, \lambda) \le (d+1)^N + 2\sum_{n \le N-1} (d+1)^n.$$

If $d \ge 1$, this implies the asserted bound. If $d = 0$, then $\mathcal{L}_{\psi(f)}$ is geometrically constant, and $\sigma_c(\mathbb{A}^N\otimes_F\bar{F}, \mathcal{L}_{\psi(f)}) = 1$. QED

Theorem 11 Let F be a finite field $\mathbb{F}_q$ of characteristic p. Fix a non–trivial additive character ψ of (F, +) with values in $\mathbb{Q}(\zeta_p)^\times$. Fix an integer $s \ge 1$, and s nontrivial multiplicative characters ρ_j of $F^\times$ with values in $\mathbb{Q}(\zeta_{q-1})^\times$. Take K to be the field $\mathbb{Q}(\zeta_p, \zeta_{q-1})$, and Λ to be the set of all embeddings of K into all $\overline{\mathbb{Q}}_\ell$'s with $\ell \neq p$.

Fix an integer $N \geq 1$, and s+1 non−negative integers

$$d, e_1, ..., e_s.$$

Fix s+1 polynomials

$$f, G_1, ..., G_s,$$

in $F[X_1,..., X_N]$, with $\deg(f) \leq d$ and $\deg(G_j) \leq e_j$ for j = 1 to s. On

$$U := \text{the open set in } \mathbb{A}^N \text{ where } \prod_j G_j \text{ is invertible,}$$

we have (Λ, K)−compatible system $\{\mathcal{F}_\lambda\}$ given by

$$\mathcal{L}_{\psi(f)} \otimes (\otimes_{j=1 \text{ to } s} \mathcal{L}_{\rho_j(G_j)}).$$

For every λ in Λ, we have the estimate

$$\sigma_c(U \otimes_F \overline{F}, \mathcal{L}_{\psi(f)} \otimes (\otimes_{j=1 \text{ to } s} \mathcal{L}_{\rho_j(G_j)}))$$
$$= \sigma(U \otimes_F \overline{F}, \mathcal{L}_{\overline{\psi}(f)} \otimes (\otimes_{j=1 \text{ to } s} \mathcal{L}_{\overline{\rho}_j(G_j)})) \leq 3(s + 1 + d + \textstyle\sum_j e_j)^N.$$

Proof In this case, the Sperber−Adolphson estimate [A−S, 5.2.7] is

$$|\chi((U \otimes_F \overline{F}, \mathcal{L}_{\overline{\psi}(f)} \otimes (\otimes_{j=1 \text{ to } s} \mathcal{L}_{\overline{\rho}_j(G_j)}))|$$
$$\leq D_{N,s}(1 + d, 1 + e_1, 1 + e_2, ..., 1 + e_s)$$
$$\leq (s + 1 + d + \textstyle\sum_j e_j)^N.$$

So in the notations of the proof of Theorem 6, we have

$$|\chi(\text{codim } i, \lambda)| \leq (s + 1 + d + \textstyle\sum_j e_j)^{N-i}.$$

Now apply the $\sigma-\chi$ inequality. QED

Theorem 12 Fix a a finite field $E = \mathbb{F}_q$ of characteristic p. Fix integers $N \geq 1$, $r \geq 0$, $s \geq 0$, and fix r+s+1 non−negative integers

$$\delta, d_1,..., d_r, e_1, ..., e_s.$$

Fix r+s+1 polynomials

$$f, F_1, ..., F_r, G_1, ..., G_s,$$

in $F[X_1,..., X_N]$, whose degrees are bounded by the given integers:

$$\deg(f) \leq \delta,\ \deg(F_i) \leq d_i \text{ for } i = 1 \text{ to } r,\ \deg(G_j) \leq e_j \text{ for } j = 1 \text{ to } s.$$

Fix a non−trivial additive character ψ of (E, +) with values in $\mathbb{Q}(\zeta_p)^\times$, and $s \geq 0$ nontrivial multiplicative characters ρ_j of $E^\times$ with values in $\mathbb{Q}(\zeta_{q-1})^\times$. Take K to be the field $\mathbb{Q}(\zeta_p, \zeta_{q-1})$, and Λ to be the set of all embeddings of K into all $\overline{\mathbb{Q}}_\ell$'s with $\ell \neq p$.

Define

$$V := \text{the closed subscheme of } \mathbb{A}^N \text{ defined by the } F_i.$$
$$U := \text{the open set in V where } \prod_j Gj_i \text{ is invertible.}$$

On U, we have (Λ, K)−compatible system $\{\mathcal{F}_\lambda\}$ given by

$$\mathcal{L}_{\psi(f)}\otimes(\otimes_{j=1 \text{ to } s}\, \mathcal{L}_{\rho_j(G_j)}).$$

For every λ in Λ, we have the estimate

$$\sigma_c(U\otimes_F\overline{F}, \mathcal{L}_{\psi(f)}\otimes(\otimes_{j=1 \text{ to } s}\mathcal{L}_{\rho_j(G_j)})) \le$$
$$3(s + 1 + \mathrm{Sup}(\delta, \mathrm{Sup}_i(1+d_i)) + \Sigma_j\, e_j)^{N+r}.$$

proof On the space

$$W := \mathbb{A}^{N+r}[1/\Pi_j G_j(X)],$$

with coordinates $X_1,\ldots, X_N$, $Y_1,\ldots, Y_r$, consider the (Λ, K)−compatible system $\{\mathcal{G}_\lambda\}$ given by

$$\mathcal{L}_{\psi(f(X) + \Sigma_i\, Y_iF_i(X)}(\otimes_{j=1 \text{ to } s}\, \mathcal{L}_{\rho_j(G_j(X))}).$$

By Theorem 8, we have

$$\sigma_c(W\otimes_F\overline{F}, \mathcal{G}_\lambda) \le 3(s + 1 + \mathrm{Sup}(\delta, \mathrm{Sup}_i(1+d_i)) + \Sigma_j\, e_j)^{N+r}.$$

So Theorem 9 results from the following well known lemma, which is the cohomological incarnation of the standard way of using additive characters to count points.

Lemma 13 Notations and notations as in Theorem 12 above, for every integer i we have a $\mathrm{Gal}(\overline{F}/F)$−equivariant isomorphism

$$H^i{}_c(U\otimes_F\overline{F}, \mathcal{F}_\lambda)(-r) \cong H^{i+2r}{}_c(W\otimes_F\overline{F}, \mathcal{G}_\lambda).$$

proof Rewrite the coefficients $\mathcal{G}_\lambda$ as

$$(\mathcal{L}_{\psi(f(X))}\otimes(\otimes_{j=1 \text{ to } s}\, (\mathcal{L}_{\rho_j(G_j(X))}))\otimes\mathcal{L}_{\psi(\Sigma_i\, Y_iF_i(X))}.$$

When we project $\mathbb{A}^{N+r}[1/\Pi_jG_j]\times\mathbb{A}^r$ onto $\mathbb{A}^N[1/\Pi_jG_j]$, say by π, standard properties of $\mathcal{L}_\psi$ show that $R(\pi)_!(\mathcal{G}_\lambda)$ is supported in U, that $R^i(\pi)_!(\mathcal{G}_\lambda) = 0$ for i ≠ 2r, and that on U we have

$$R^{2r}(\pi)_!(\mathcal{G}_\lambda) \mid U \cong \mathcal{L}_{\psi(f(X))}\otimes(\otimes_{j=1 \text{ to } s}\, (\mathcal{L}_{\rho_j(G_j(X))})(-r).$$

So the Leray spectral sequence degenerates, and yields the asserted isomorphism. QED

Remark The special case s = 0 and f = 0 of Theorem 12 gives another explicit version of the Corollary to Theorem 1. Put $d := \mathrm{Sup}_i(d_i)$. We get

$$\sigma_c(V\otimes_F\overline{F}, \mathbb{Q}_\ell) \le 3\times(2 + d)^{N+r}$$

for every prime ℓ invertible in F.

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