

EULER-MACLAURIN

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1. INTRODUCTION

1.1. Aysmptotic series. Recall that we say that $f(h)$ has an asymptotic series

$$(1) \quad f(h) = a_0 + a_1h + a_2h^2 + a_3h^3 + \cdots$$

as $h \rightarrow 0$ if for any fixed integer $m \geq 0$

$$f(h) = a_0 + a_1h + \cdots + a_mh^m + \mathcal{O}(h^{m+1}),$$

where the constant implicit in the big- $\mathcal{O}$ depends on m .

1.2. Richardson extrapolation. The technique of using tricks to cancel terms in asymptotic series is called Richardson extrapolation. For example, if f has asymptotic series (1) and we set

$$g(h) = 2f(h/2) - f(h),$$

then g has asymptotic series

$$g(h) = a_0 + b_2h^2 + b_3h^3 + \cdots,$$

where $b_k = a_k(2^{-k+1} - 1)$.

1.3. Taylor's Theorem. If f is a smooth real-valued function on $[a, b]$, then Taylor's Theorem says that $f(x+h)$ has an asymptotic series

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \cdots$$

as $h \rightarrow 0$ assuming that $x, x+h \in [a, b]$. Indeed, the fact that $f(x+h)$ has this asymptotic series is justified by the Lagrange remainder formulation of Taylor's Theorem, which we can state as follows. Let f be a real-valued function that is n times continuously differentiable in $[a, b]$ and $(n+1)$ -times differentiable in (a, b) . Then, for all $x, x_0 \in [a, b]$

$$f(x) = f(x_0) + \sum_{k=1}^n \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1},$$

for some $\xi = \xi(x, x_0)$ in the open interval whose end points are x, x_0 .

2. EULER-MACLAURIN FORMULA

2.1. Trapezoid rule. Suppose that f is a smooth real-valued function on $[a, b]$, and let $h = (b - a)/n$ for a fixed integer n . The Trapezoid rule $T(h)$ with step length h is defined by

$$T(h) = h \left(\frac{1}{2}f(a) + f(a+h) + \cdots + f(b-h) + \frac{1}{2}f(b) \right),$$

or put differently,

$$\frac{T(h)}{h} = \sum_{k=0}^n f(a+hk) - \frac{1}{2}(f(a) + f(b)).$$

2.2. Basic statement. Suppose that f is a smooth real-valued function $[a, b]$. Informally speaking, the Euler-Macluarin formula says that trapezoid rule $T(h)$ has the asymptotic series which only has even powers of h whose first few terms are

$$\begin{aligned} T(h) = \int_a^b f(x)dx + \frac{h^2}{12}(f'(b) - f'(a)) - \frac{h^4}{720}(f'''(b) - f'''(a)) \\ + \frac{h^6}{30240}(f^{(5)}(b) - f^{(5)}(a)) + \cdots \end{aligned}$$

There are many ways to use the Euler-Maclaurin formula. For example, we could use the fact that $T(h)$ has an asymptotic series which only has even powers of h to create an $\mathcal{O}(h^4)$ integration scheme by using Richardson Extrapolation

$$\frac{4T(h/2) - T(h)}{3} = \int_a^b f(x)dx + \mathcal{O}(h^4).$$

Alternatively, we could use the formula to perform an end point correction to Trapezoid rule by

$$T(h) - \frac{h^2}{12}(f'(b) - f'(a)) = \int_a^b f(x)dx + \mathcal{O}(h^4).$$

Both of these methods could be used to create higher order schemes for estimating the integral of f over $[a, b]$.

2.3. Precise statement with remainder formula. Let f be a real-valued function that $2r$ times continuously differentiable on (a, b) . Fix an integer $n \geq 1$ and let $h = (b - a)/n$. Then, the Euler-Macluarin formula states that¹

$$\begin{aligned} \sum_{k=0}^n f(a + kh) = \frac{1}{h} \int_a^b f(x)dx + \frac{1}{2}(f(b) + f(a)) \\ + \sum_{k=1}^{r-1} \frac{h^{2k-1}}{(2k)!} B_{2k}(f^{(2k-1)}(b) - f^{(2k-1)}(a)) + \frac{h^{2r}}{(2r)!} B_{2r} \sum_{k=0}^{n-1} f^{(2r)}(a + kh + \xi h), \end{aligned}$$

for some $0 < \xi < 1$, where B_k denotes the k -th Bernoulli number, which have the generating function

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

¹see for example page 806 of Abramowitz and Stegun

The first few Bernoulli numbers are

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}.$$