THE MYSTERY OF STOCHASTIC MECHANICS

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Classical Hamilton-Jacobi theory

N particles of various masses on a Euclidean space.

Incorporate the masses in the flat Riemannian metric m_{ij}, the *mass tensor*. Then if v^i is a velocity, $v_i = m_{ij}v^j$ is a momentum.

Kinetic energy: $\frac{1}{2}v^i v_i$.

Potential energy: V.

Lagrangian: $L = \frac{1}{2}v^i v_i - V$.

Position at time t of the configuration: $\xi(t)$.

Initial time: t.

Final time: t_1.

Hamilton's principal function:

$$S(x,t) = -\int_t^{t_1} L\big(\xi(s)\big)\,ds.$$

Substantial derivative (derivative along trajectories):

$$D = \frac{\partial}{\partial t} + v^i \nabla_i.$$

Then $DS = L$.

Vector field: v with $\dfrac{d\xi}{dt} = v$.

Principle of least action in Hamilton-Jacobi theory: v is a critical point for S, for unconstrained variations.

That is, let v' be another vector field, let $\delta v = v' - v$, and denote by a prime quantities with v replaced by v'. Then

$$\begin{aligned} D(S' - S) &= D'S' - DS + (D - D')S' \\ &= L' - L - \delta v^i \nabla_i S' \\ &= L' - L - \delta v^i \nabla_i S + \mathrm{o}(\delta v). \end{aligned}$$

Now

$$\begin{aligned} L' - L &= v_i \delta v^i + \mathrm{o}(\delta v), \\ S' - S &= -\int_t^{t_1} (v_i - \nabla_i S) \delta v^i \, ds + \mathrm{o}(\delta v). \end{aligned}$$

Since this is true for all variations, we have the Hamilton-Jacobi condition:

$$v_i = \nabla_i S.$$

Together with $DS = L$; i.e.,

$$\frac{\partial s}{\partial t} + v^i \nabla_i S = \frac{1}{2} v^i v_i - V,$$

this gives the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{2} \nabla^i \nabla_i S + V = 0.$$

If we take the gradient we obtain Newton's equation $F = ma$.

Stochastic Hamilton-Jacobi theory

Following Guerra and Morato, construct a conservative Markovian dynamics for a Markov process ξ.

Wiener process (Brownian motion) w with

$$\mathrm{E} dw^i(t) dw_i(t) = \hbar dt.$$

Kinematics:

$$d\xi(t) = b\big(\xi(t), t\big) dt + dw(t).$$

Dynamics: $\delta \mathrm{E} \int L \, dt = 0$ (heuristically).

The trajectories are non-differentiable, so what is the meaning of

$$\frac{1}{2} \frac{d\xi^i}{dt} \frac{d\xi_i}{dt}$$

in the Lagrangian L?

Let $dt > 0$ and $d\xi(t) = \xi(t + dt) - \xi(t)$, so $\dfrac{d\xi}{dt}$ is a quotient, not a derivative.

Compute $\mathrm{E}\dfrac{1}{2}\dfrac{d\xi^i}{dt}\dfrac{d\xi_i}{dt}$ up to $\mathrm{o}(1)$.

$$d\xi^i = \int_t^{t+dt} b^i\big(\xi(r), r\big)\, dr + dw^i.$$

Note: dw is of order $dt^{1/2}$.

$$\begin{aligned} d\xi^i &= \int_t^{t+dt} b^i\Big(\xi(t) + \int_t^r b\big(\xi(s), s)ds + w(r) - w(t), r\Big)dr \\ &\quad + dw^i \\ &= b^i dt + \nabla_k b^i W^k + dw^i + \mathrm{O}(dt^2) \end{aligned}$$

where

$$W^k = \int_t^{t+dt} [w^k(r) - w^k(t)]\, dr.$$

Therefore

$$\frac{1}{2}d\xi^i d\xi_i = \frac{1}{2}b^i b_i dt + b^i dw_i dt + \nabla_k b^i W^k dw_i \\ + \frac{1}{2}dw^i dw_i + \mathrm{o}(dt^2).$$

Now

$$\mathrm{E}W^k dw_i = \hbar\delta_i^k \int_t^{t+dt} (r-t)dt = \frac{\hbar}{2}\delta_i^k,$$

so $\nabla_k b^i W^k = \frac{\hbar}{2}\nabla_i b^i$.

The term $b^i dw_i dt$ is singular, of order $dt^{3/2}$, but its expectation is 0.

Finally, $\mathrm{E}dw^i dw_i = \frac{\hbar}{2}ndt$. Hence we have the sought-for result:

$$\mathrm{E}\frac{1}{2}\frac{d\xi^i}{dt}\frac{d\xi_i}{dt} = \frac{1}{2}b^i b_i + \frac{\hbar}{2}\nabla_i b^i + \frac{\hbar}{2}\frac{n}{dt} + \mathrm{o}(1).$$

The singular term $\dfrac{\hbar}{2}\dfrac{n}{dt}$ is a constant, not depending on the trajectory, and it drops out of the variation.

Let

$$L_{+} = \frac{1}{2}b^i b_i + \frac{\hbar}{2}\nabla_i b^i - V$$

and

$$I = \mathrm{E}\int_t^{t_1} L_{+}\bigl(\xi(s)\bigr)\,ds.$$

Let δb be a vector field, $b' = b + \delta b$, and as before denote by primes quantities with b replaced by b'. Let ξ' satisfy

$$d\xi'(t) = b'\bigl(\xi'(t),t\bigr)dt + dw(t)$$

with $\xi'(t) = \xi(t)$ for the initial time t.

Definition: ξ is critical for L in case

$$I' - I = \mathrm{o}(\delta b).$$

Stochastic Hamilton's principal function:

$$S(x,t) = -\mathrm{E}_{x,t} \int_t^{t_1} L_+\big(\xi(s), s\big)\, ds.$$

Then

$$DS = L_+$$

where now D is the mean forward derivative:

$$Df = \lim_{dt \to 0+} \mathrm{E}_t \left(\frac{df(t)}{dt} \right).$$

As before,

$$\begin{aligned} D(S' - S) &= D'S' - DS + (D - D')S' \\ &= L'_+ - L_+ - \delta b^i \nabla_i S' \\ &= L'_+ - L_+ - \delta b^i \nabla_i S + \mathrm{o}(\delta b). \end{aligned}$$

Now

$$L'_+ - L_+ = b_i \delta b^i + \frac{\hbar}{2} \nabla_i \delta b^i + \mathrm{o}(\delta b)$$

and

(∗)

$$I' - I = \mathrm{E} \int_t^{t_1} \left(b_i - \nabla_i S + \frac{\hbar}{2} \nabla_i \right) \delta b^i\, ds + \mathrm{o}(\delta b).$$

Digression on Markovian kinematics:

Markov process: given the present, past and future are independent.

The two directions of time are on an equal footing.

In addition to the forward drift b there is the backward drift b_*.

$$v = \frac{b + b_*}{2} \qquad \text{current velocity,}$$
$$u = \frac{b - b_*}{2} \qquad \text{osmotic velocity.}$$

Let ρ be the probability density of ξ. Then

$$\frac{\partial v}{\partial t} = -\nabla(v\rho) \qquad \text{current equation,}$$
$$\frac{\partial u}{\partial t} = \frac{\hbar}{2}\frac{\nabla\rho}{\rho} \qquad \text{osmotic equation.}$$

End of digression.

The equation (*), namely

$$I' - I = \mathrm{E}\int_t^{t_1} \left(b_i - \nabla_i S + \frac{\hbar}{2}\nabla_i\right)\delta b^i\,ds + \mathrm{o}(\delta b),$$

is awkward because it involves $\nabla_i \delta b^i$. Integrate by parts:

$$\begin{aligned}\mathrm{E}\frac{\hbar}{2}\nabla_i\delta b^i\bigl(\xi(s),s\bigr) &= \int_{\mathbb{R}^n}\frac{\hbar}{2}\left(\nabla_i\delta b^i\right)\rho\,d\mathbf{x}\\ &= -\int_{\mathbb{R}^n}\delta b^i u_i\rho\,d\mathbf{x}.\end{aligned}$$

Since $b - u = v$,

$$I' - I = \mathrm{E}\int_t^{t_1}(v_i - \nabla_i S)\delta b^i\,ds + \mathrm{o}(\delta b).$$

This is true for all variations, so ξ is critical for L if and only if the stochastic Hamilton-Jacobi condition holds:

$$v_i = \nabla_i S.$$

Let

$$R = \frac{\hbar}{2} \log \rho,$$

so

$$\nabla^i R = u^i.$$

If we write out $DS = L_+$ and express everything in terms of R and S, we obtain the stochastic Hamilton-Jacobi equation:

$$(1) \quad \frac{\partial S}{\partial t} + \frac{1}{2}\nabla^i S \nabla_i S + V - \frac{1}{2}\nabla^i R \nabla_i R - \frac{\hbar}{2}\Delta R = 0.$$

Expressing the current equation $\frac{\partial v}{\partial t} = -\nabla(v\rho)$ in terms of R and S we obtain:

$$(2) \qquad \frac{\partial R}{\partial t} + \nabla_i R \nabla^i S + \frac{\hbar}{2}\Delta S = 0.$$

These two equations are a system of coupled nonlinear partial differential equations expressing necessary and sufficient conditions for a Markov process to be critical.

How can we solve them?

Let

$$\psi = e^{\frac{1}{\hbar}(R+iS)}.$$

Then the system is equivalent to the Schrödinger equation

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar}\left[-\frac{\hbar}{2}\Delta + V\right]\psi.$$

magnetic fields

Riemannian manifolds

spin

Bose-Einstein and Fermi-Dirac statistics

existence of finite-energy Markov processes

momentum

interference

Non-locality

Consider two particles on $\mathbb{R}^1$ with initial wave function

$$\psi(0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{4}\sigma^{-1}(0)x\cdot x}$$

where

$$\sigma^{-1}(0) = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

and

$$x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}.$$

The x^1 and x^2 axes are unrelated; the particles may be separated by an arbitrarily large amount a.

At time 0, turn on a linear restoring force (harmonic oscillator with circular frequency ω) for the second particle. Then the particles are dynamically uncoupled.

Since the particles are very widely separated and dynamically uncoupled, we should expect that

$$E\xi^1(t)\xi^1(0)$$

does not depend on ω.

In fact it does not to fourth order in t, but nevertheless the trajectory of the first particle is immediately affected by the choice of ω in a far distant place.

For me this is unphysical, especially since the effect does not depend on the separation a.

A wrong prediction

Consider two harmonic oscillators, about two widely separated points a_1 and a_2, with circular frequency 1, and let X_i be the Heisenberg position operator, in units of distance from a_i, with Heisenberg momentum operator P_i.

Then

$$\begin{aligned} X_i(t) &= \quad \cos t\, X_i(0) + \sin t\, P_i(0) \\ P_i(t) &= -\sin t\, X_i(0) + \cos t\, P_i(0) \end{aligned}$$

for $i = 1, 2$.

Let the Heisenberg state vector ψ_0 be a real Gaussian centered at $(0, 0)$, and write $\langle A \rangle = (\psi_0, A\psi_0)$. Then $\langle X_i(t) \rangle = \langle P_i(t) \rangle = 0$ since this is true for $t = 0$. The operators $X_1(t)$ and $X_2(s)$ commute.

Choose ψ_0 so that the correlation $\langle X_1(0)X_2(0) \rangle$ is 90%. Thus the oscillators are entangled but dynamically uncoupled. The quantum mechanical correlation function $\langle X_1(t)X_2(0) \rangle$ is periodic of period 2π since $X_1(t)$ is.

Hence $\langle X_1(t)X_2(0) \rangle = .9$ whenever t is a multiple of 2π.

But let (ξ_1, ξ_2) be the corresponding Markov process of stochastic mechanics.

This is a diffusion process and it eventually loses all memory of where it started.

Thus

$$\lim_{n \to \infty} \mathrm{E}\big(\xi_1(2\pi n)\xi_2(0)\big) = 0$$

whereas

$$\langle X_1(2\pi n) X_2(0) \rangle = .9.$$

Here we have an empirical difference between the predictions of quantum mechanics and stochastic mechanics. Measurements of the position of the first particle at time t and of the second particle at time 0 do not interfere with each other, and the two theories predict totally different statistics.

Does anyone doubt that quantum mechanics is right and stochastic mechanics is wrong?