Stochastic mechanics of relativistic fields

Edward Nelson

Department of Mathematics, Princeton University, Princeton NJ 08544, USA

E-mail: `nelson@math.princeton.edu`

Abstract. After a brief review of stochastic mechanics of nonrelativistic particle systems, the paper discusses in a qualitative way issues concerning the application of stochastic mechanics to relativistic fields and their relation to quantum field theory. It is suggested that gauge theories will be essential to this program.

1. Stochastic mechanics of nonrelativistic particle systems

A *basic Lagrangian* is

$$L = \frac{1}{2} m_{ij} v^i v^j - \varphi + A_i v^i \tag{1}$$

Here the mass tensor m_{ij} is a Riemannian metric, v^i is the velocity, φ is the scalar potential, and A_i is the covector potential; the summation convention is used. This is the general form of a Lagrangian for which the force is a dynamical variable (depending only on position and velocity). For simplicity, we discuss only the case for which the configuration space is $\mathbb{R}^{sN}$ for N particles in s space dimensions. But tensor notation is still useful; the masses of the various particles can be included in the diagonal constant mass tensor m_{ij}. Also for simplicity we assume here that $A_i = 0$ and denote the scalar potential φ by V.

Hamilton's principal function is

$$S(x,t) = -\int_t^{t_1} L(X(s,x,t), \dot{X}(s,x,t), s)\, ds \tag{2}$$

where $X(s,x,t)$ is the configuration at time s starting at x at time t and $\dot{X}$ is its time derivative. The requirement that S be stationary when the flow is perturbed by a time-dependent vector field leads to the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{2} \nabla^i S \nabla_i S + V = 0 \tag{3}$$

Let w be the Wiener process on $\mathbb{R}^{sN}$, the stochastic process of mean 0 characterized by

$$dw^i dw_i = \hbar dt + \mathrm{o}(dt) \tag{4}$$

Stochastic mechanics postulates that the motion of the configuration is a Markov process governed by the stochastic differential equation

$$dX^i = b^i(X(t),t)dt + dw^i \tag{5}$$

where b^i is the mean forward velocity. Call this *basic stochasticization*; this terminology is better than "stochastic quantization" since the physics remains classical. The fluctuations are of order $dt^{\frac{1}{2}}$ in time dt, and with a coefficient larger than $\hbar$ in (4) this postulate could be falsified by experiment, without violating the Heisenberg uncertainty principle.

Now try to substitute the Markov process X into (2) and require that the conditional expectation $\mathbb{E}_t$ of the action, given the configuration at time t, be stationary with respect to variations of the forward velocity b^i. The trajectories of the process X are not differentiable, so replace the derivatives in $\dot{X}$ by difference quotients, and replace the integral by a Riemann sum. There is a singular term whose conditional expectation is 0, so it drops out, and there is a singular term that is a constant, independent of the trajectory, so it drops out from the variation. Then pass to the limit when the Riemann sum becomes an integral. The result is the *stochastic principal function*

$$S(x,t) = -\mathbb{E}_{x,t} \int_t^{t_1} \left(\frac{1}{2} b^i b_i + \frac{\hbar}{2} \nabla_i b^i - V \right) (X(s), s) ds \tag{6}$$

where $\mathbb{E}_{x,t}$ is the expectation conditioned by $X(t) = x$.

Let

$$R = \frac{\hbar}{2} \log \rho \tag{7}$$

where ρ is the time-dependent probability distribution of the configuration. Computation shows that

$$\begin{aligned} &\frac{\partial S}{\partial t} + \frac{1}{2} \nabla^i S \nabla_i S + V - \frac{1}{2} \nabla^i R \nabla_i R - \frac{\hbar}{2} \Delta R = 0 \\ &\frac{\partial R}{\partial t} + \nabla_i R \nabla^i S + \frac{\hbar}{2} \Delta S = 0 \end{aligned} \tag{8}$$

The first equation is the *stochastic Hamilton-Jacobi equation.* There is no deterministic analogue of the second equation since $R = 0$ when $\hbar = 0$. These two coupled nonlinear partial differential equations determine the process X.

With

$$\psi = e^{R+iS} \tag{9}$$

these equations are equivalent to the Schrödinger equation

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} \left(-\frac{1}{2} \Delta + V \right) \psi \tag{10}$$

As I reported to this conference two years ago [1], despite many successes there are problems with stochastic mechanics as a candidate for a physically realistic theory. There can be instantaneous signaling between widely separated correlated but dynamically uncoupled systems. Although the probability distribution of position measurements at a single time are the same for quantum mechanics and stochastic mechanics (because $|\psi|^2 = \rho$ is the probability density of the configuration), measurements at two different times, even when mutually compatible, can differ in the two theories.

The problem is that stochastic mechanics describes a Markov process in multidimensional configuration space, not in physical space.

2. Stochastic mechanics of fields

There are two motivations for applying stochastic mechanics to fields. One is the hope that since fields live on physical spacetime nonlocality problems may be avoided. The other is that it may provide useful technical tools in constructive quantum field theory. The strategy is to apply basic stochasticization to a basic field Lagrangian.

Consider a real scalar field φ on d-dimensional spacetime. I shall discuss the free field of mass $\mu > 0$ and the field with a φ^4 interaction. Choose a spacelike hyperplane $\mathbb{R}^s$, where s, the number of space dimensions, is $d-1$. The configuration space is a set of scalar functions φ on $\mathbb{R}^s$. Impose a spatial cutoff and a momentum cutoff. That is, put the system in a box of side λ and represent the free field as a set of harmonic oscillators (a device going back to Jeans) with momentum bounded by κ. Then we have a system with finitely many degrees of freedom.

For the free field, of course, the limit $\lambda \to \infty, \kappa \to \infty$ presents no difficulty. But it is unwise to attempt to formulate an interacting field on the same space since by Haag's theorem it must have an inequivalent representation of the canonical commutation relations.

Formally, the φ^4 theory has the interaction Hamiltonian

$$\int_{\mathbb{R}^s} \varphi^4(x)\,dx \tag{11}$$

With our cutoffs, the integral becomes a sum. But this expression is hopelessly singular as $\kappa \to \infty$, and the interaction Hamiltonian is replaced by

$$\int_{\mathbb{R}^s} :\varphi^4(x):\,dx \tag{12}$$

The colons denote Wick ordering, which can be expressed simply by summing over quadruples of *distinct* oscillators.

For $d = 2$ it was shown that the Hamiltonian is bounded below by a constant independent of κ, giving the limit $\kappa \to \infty$. This was the origin of hypercontractivity, which has grown into a rich topic in analysis. The construction of the limit $\lambda \to \infty$ followed—in fact, with φ^4 replaced by any polynomial of even degree.

For $d = 3$ the φ^4 theory was rigorously established by Glimm and Jaffe, with mass and coupling constant renormalization, by heroic work with cluster expansions.

For $d > 4$ Aizenman established a no-go theorem: the limiting theory with cutoffs removed is non-interacting.

For $d = 4$ there are partial no-go results, and most workers in constructive quantum field theory believe that the limiting theory with cutoffs removed is non-interacting also for $d = 4$, but there is no complete proof—the problem is open.

In this work in constructive quantum field theory, Minkowski space was replaced by Euclidean space, giving a problem in probability theory: the Euclidean fields commute. Then the relativistic theory was reconstructed from the Euclidean theory. This is a powerful analytic technique, but it does not give a candidate for quantum theory as emergent from an underlying classical theory.

What does stochastic mechanics have to say? We have a basic Lagrangian. But the separation of the energy for the free field into kinetic energy and potential energy is not a relativistically invariant procedure. Probability theory and relativity do not mix well, due to the indefinite nature of the Minkowski metric.

For the ground state of the free field, the corresponding random field of stochastic mechanics was constructed by Guerra and Ruggiero [2]. It is a Gaussian field with Euclidean-invariant covariance function. But it plays a different role than in Euclidean field theory; relativistic effects persist. Suppose that the field is coupled to an external potential localized in a bounded spacetime region Ω, bounded in the past by t_1 and in the future by t_2. Consider the Markov process, not necessarily for the ground state, of the free field at time t_1. It interacts with the external potential and has a certain value at t_2. By the basic theorem of stochastic mechanics, the probability distribution of the process at time t_2 is the same as that given by quantum mechanics. Since the quantum field is relativistic, the influence of the external potential affects only the points at time t_2 that lie in the future cone of some point in Ω.

But we cannot conclude a similar result for the Markov process itself. Supraluminal influences persist, alas, and the hope that stochastic mechanics applied to field theory gives a reasonable candidate for quantum field theory as emergent from a classical theory does not seem to be valid.

3. Epiphany at the airport

This was the discouraged note on which I ended my talk at the conference. But returning home I had a long wait at the Zurich airport, and a ray of hope appeared.

Consider the electromagnetic field and represent the vector potentials A_μ as harmonic oscillators with basic stochasticization. Then, although they are subject to supraluminal signaling, the physically real electric field and magnetic field should not be, and they transform covariantly under Lorentz transformations. Gauge theories can be expected to play the role of harmonizing the requirements of probability theory and of relativity theory. If spinor fields can be treated stochastically, with their associated currents represented as random fields, and if the divergence problems endemic to relativistic field theory can be handled (two big ifs), the way may be open to a theory treating the phenomena at present described by quantum filed theory as emerging from classical physics with intrinsic randomness.

References

[1] Nelson E 2012 *J. Phys.: Conf. Series* **361**
[2] Guerra F and Ruggiero P 1973 *Phys. Rev. Lett.* **31**