Confessions of an Apostate Mathematician

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Midway between Moses and Jesus there appeared a figure of like charisma: golden-thighed Pythagoras. Son of a Greek mother and a Phoenician father, he spent years in Egypt, endured a period of captivity in Babylon, and founded the Pythagorean Brotherhood in southern Italy. The Brotherhood was something rare for its time, and alas! for ours: a true Brotherhood and Sisterhood. The legends surrounding Pythagoras differ wildly, but all attest to an extraordinary radiance of face and person.

To develop my thesis, I must speak of the mathematics of antiquity. My knowledge in this field is second-hand and superficial. There is no following "but": I am simply admitting at the outset a weakness of the presentation.

There was no mathematics before Pythagoras, just as there was no Christianity before Jesus. The story that the Egyptians discovered geometry because the annual flooding of the Nile washed away boundary marks is in a class with the story of Newton and the apple. The Babylonians were wizards at computation, they had at least one of the two ways of generating what we now call Pythagorean triples (as Otto Neugebauer brilliantly demonstrated from the Plimpton cuneiform tablets), they could approximate the square root of two to as many sexagesimal places as desired, but the question of the rationality or irrationality of the square root of two simply would have made no sense to them, because they had no numbers in their conceptual framework. This is not surprising, because numbers did not yet exist in those days.

Numbers were invented (or revealed, as believers would maintain) by Pythagoras. Numbers are divine, the only true divinity, the source of all that is in the world, holy, to be worshiped and glorified. Such is the Pythagorean religion, and such is the origin of mathematics. This is the religion from which I am apostate.

What an extraordinary religion it is! Had it not been for Pythagoras, would it ever have entered anyone's head that it is a good and normal thing to have communities of people think deeply about numbers and magnitudes, to develop an unimaginably complex web of formal deductions – a web of reasoning covering the globe and spanning millennia?

Eudoxus

Plato traveled all the way to Sicily to obtain a Pythagorean manuscript. Plato was a dreadful fellow, the source of a persistent evil from which the world has not yet been liberated, but he was the teacher of Eudoxus. When I was in the Athenian Agora, thoughts of Socrates, Plato, and Aristotle, of St. Paul, of the dramatists, soon faded in the realization that I was walking where Eudoxus once walked! Eudoxus was the greatest mathematician that ever lived. It was he who invented the method of exhaustion (definite integrals) that Archimedes developed so successfully; it was he who enunciated the subtle principle we call the Archimedean axiom (but which Archimedes himself attributes to Eudoxus); he did extremely intricate work on the apparent motion of the planets and he founded mathematical physics with a work on dynamics. Not one of his manuscripts survives.

We have not yet spoken of the deepest invention of Eudoxus. To their consternation, the Pythagoreans discovered that the diagonal of a square has no ratio, of numbers, to the side: the square root of two is irrational, with all the emotional and intellectual connotations of that word. This was a true crisis in religion. If numbers are the only divinity, the source of all, how can this be? Must we admit magnitudes into the pantheon as well? Seemingly so, for the late Pythagorean description of the quadrivium was this:

arithmetic:	numbers at rest
music:	numbers in motion
geometry:	magnitudes at rest
astronomy:	magnitudes in motion

But Eudoxus eliminated the dualism of number and magnitude. His idea was this: rather than say what the ratio of two magnitudes *is*, it suffices to define a notion of *two* such (possibly nonexistent) ratios *being equal*, and this he did by a subtle quantification over all the Pythagorean numbers. Despite the fact that it was in the *Elements* for all to read, this construction of the real number system was not understood – not even by Galilei. It was just last century that the notion was re-invented by Richard Dedekind. I know of no parallel to this in the history of human thought.

Eudoxus defined a notion of two things being equal in order to construct the things themselves. Here was a triumph of formalism, a victory of syntax over semantics!

The origin of numbers

From the perspective of monotheistic faith, we reject the religious idea of numbers as divine and uncreated. What then are they? Are they created? I have written elsewhere:

> The famous saying by Kronecker that God created the numbers, all else is the work of Man, presumably was not meant to be taken seriously. Nowhere in the book of Genesis do we find the passage: And God said, let there be numbers, and there were numbers; odd and even created he them, and he said unto them, be fruitful and multiply; and he commanded them to keep the laws of induction.

Everything in creation is contingent; every created thing is dependent on the will of the Creator for its being. If numbers are uncreated, they are divine – this we reject. If numbers are created, they are contingent – this we find absurd. What other possibility is there? Simply that numbers do not exist – not until human beings make them. Despite the assertion of William Butler Yeats that "Things out of perfection sail," very few would maintain that the poem of which that is the first line existed before Yeats made it. Why do we mathematicians, makers like poets and musicians, describe what we do as discovery rather than invention? This is Pythagorean religion.

What would the world be like if God had not created numbers? Just what it is like now. There is not a shred of evidence that the numbers have been created. If I give you an addition problem like

$$\begin{array}{r} 3746022518224410025373452134562345711560442783 3 \\ +\ 5232876351453023841215432154322543014325406110 5 \\ \hline \end{array}$$

and you are the first to solve it, you will have created a number that did not exist previously.

But this invention will not create a stir in the mathematical community; it is not the kind of number that we are primarily interested in. More interesting numbers are the n, a, b, c, and d in Lagrange's theorem:

$$\forall n\, \exists a\, \exists b\, \exists c\, \exists d\, [\, n = a^2 + b^2 + c^2 + d^2\,].$$

This may be read: for all n there exist a, b, c, and d such that n is equal to a times a plus b times b plus c times c plus d times d – that is, every number is the sum of four squares. What does this mean?

Various meanings have been ascribed to it. My father, whose work at the Young Men's Christian Association (YMCA) in Piazza Indipendenza first brought me to Rome in 1938 (when I went to first grade in a school near the Piazza Pitagora), used while driving to look at the number on the license plate of the car ahead of him and mentally find a, b, c, and d. So this is one meaning: a pattern of computational challenges. But the formula asserts the existence for *all* n.

A first view of mathematics: realism

Traditionally, from the time of the Pythagorean brotherhood to the present, it meant that for every natural number

$$n = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ \ldots$$

there exist natural numbers a, b, c, and d such that n is equal to the sum of their squares. Until the beginning of this century, with the incisive intuitionistic criticism of L. E. J. Brouwer, this was accepted by almost everyone as having a clear and definite meaning. Mathematics based on this concept is called *classical*. Many proofs in classical mathematics demonstrate that an object having a certain property exists without offering any means to construct such an object. One demonstrates that the assumption that all objects fail to have the property is untenable – it leads to a contradiction – and thereby concludes that there must exist an object having the property. Thus classical mathematics is founded on the picture of mathematical objects eternally existing in Platonic – or better, Pythagorean – reality.

A second view of mathematics: intuitionism

This picture was vigorously attacked by Brouwer in the early part of this century, as being devoid of meaning. Intuitionism is a form of constructivism; to say that a mathematical object exists, for Brouwer, means that one knows how to construct it.

For the four square formula, there was no problem for Brouwer: given n, the a, b, c, and d must be smaller than n, so there is only a finite set to search through. This arrogant phrase, "only a finite set," is widely used by mathematicians, and I have been amused at mathematics parties to hear spouses of mathematicians employ it too. At the end of this century, with the advent of digital computers – which are having a profound influence, whether we like it or not (I do), on mathematics – one may object that this is not a feasible search: one can write down a number n on a sheet of paper such that no computer, from now until the Big Crunch, or whatever other end is in store for the universe, will ever be able to search through all a, b, c, and d smaller than n. The question of elucidating mathematically the nature of a feasible computation is at the forefront of computer science and mathematics, involving deep and exciting unresolved problems, and it is a question before which both realism and intuitionism are helpless.

Brouwer's position was vigorously attacked by Hilbert, defending classical mathematics under the guise of a formalist. An unpleasant, acrimonious, and unnecessary debate ensued. (May I take this opportunity to thank the the organizers of this event for the opportunity to debate, in a totally different spirit, with Professor De Giorgi – it is a great pleasure after so many years to meet him in person to share and dispute our ideas on the foundations of mathematics.) The Brouwer-Hilbert debate was unnecessary because both parties shared a common misconception: that Brouwer's intuitionism was a *restriction* of classical mathematics. But Gödel showed in a short paper, published two years after his epoch-making incompleteness theorem of 1931, that it is actually an *extension* of classical mathematics. At least, this is true for arithmetic (or number theory), but the less said about intuitionistic analysis the better.

Here is a slightly – but only slightly – simplified account of Gödel's argument. To an intuitionist, $\exists x\, \mathrm{A}(x)$ means "I know how to construct an x with the property $\mathrm{A}(x)$." This certainly implies $\neg\forall x \neg\, \mathrm{A}(x)$: "it is not the case that all x fail to have the property $\mathrm{A}(x)$"; the converse implication certainly does not hold. But a realist interprets $\exists x\, \mathrm{A}(x)$ as "in the pre-existing abstract universe of mathematical objects, there is at least one of them, x, with the property $\mathrm{A}(x)$." And to the realist, this is equivalent to $\neg\,\forall x \neg\, \mathrm{A}(x)$, which means the same to the realist and the intuitionist $\big($provided they agree about the meaning of $\mathrm{A}(x)\big)$. Similarly, to an intuitionist $\mathrm{A}(x) \vee \mathrm{B}(x)$ means "x has the property $\mathrm{A}(x)$ or x has the property $\mathrm{B}(x)$, and I can say which." This is stronger than $\neg[\neg\, \mathrm{A}(x)\ \&\ \neg\, \mathrm{B}(x)]$ – "it is not the case that x fails to have the property $\mathrm{A}(x)$ and x fails to have the property $\mathrm{B}(x)$." But to a realist, both mean the same in the Pythagorean heaven. Gödel says in effect: replace each formula $\exists x\, \mathrm{A}(x)$ in a classical proof by $\neg\forall x \neg\, \mathrm{A}(x)$ and each $\mathrm{A}(x) \vee \mathrm{B}(\mathrm{x})$ by $\neg[\neg\, \mathrm{A}(x)\ \&\ \neg\, \mathrm{B}(x)]$. The realist, regarding them as equivalent, cannot object; the intuitionist accepts this as a reinterpretation of what the realist is saying. And then it turns out (and this is a fact, not depending on any view of mathematics) that the classical proof transformed in this way gives an intuitionistic proof of the reinterpreted theorem!

In the light of Gödel's result, we can say that what Brouwer really did was extend classical mathematics by the creation of two new logical operators: the *constructive there exists* and the *constructive or*, stronger than their classical counterparts. Unfortunately for clarity and civility, Gödel's paper did not receive the proper attention or interpretation, and the unseemly squabble dragged on.

A third view of mathematics: formalism

Formalism has a simple answer to the question of the meaning of

$$\forall n \, \exists a \, \exists b \, \exists c \, \exists d \, [\, n = a^2 + b^2 + c^2 + d^2 \,] :$$

it means nothing. (I want to clarify a point immediately. Perhaps it would be better to say that the formula does not denote anything. Mathematics, like music, is meaningful, but nowadays one no longer says of passages of music: this denotes a little bird and that denotes a zephyr. But non-formalists insist even today on giving a denotation to every passage of mathematics.) The symbols in the formula are marks on paper, and the work of the mathematician is to devise deep and beautiful concatenations of such marks according to strict rules. In discussing mathematics, one may speak of truth or visual images, but this has as little relation with doing mathematics as art criticism has with doing art.

Hilbert called himself a formalist, and indeed he crowned his mathematical career with deep and brilliant work on the foundations of mathematics from a formalist perspective, but there is strong reason to suspect that Hilbert never renounced his Pythagorean faith, that formalism for him was a tactic to use against Brouwer.

Let me conclude with a brief but passionate apologia for formalism.

As a description of what mathematicians have been doing, and cherishing, for well over two millennia, it is accurate and leaves nothing out. What we devote our lives to is seeking for proofs; if a proof follows the formal rules, it is correct; if it does not, it is not a proof and is worthless unless it suggests a way to find a proof. No other field of human endeavor has maintained such a consensus over such a vast extent of space and time.

Formalism denies the relevance of truth to mathematics. But, one might object, mathematics works – the evidence is all around us. Does this not imply that there is truth in mathematics? Not in the slightest. Suppose we find a primitive people, or an advanced people, but a people with a world-view utterly alien to ours, who have an herb that is quite effective for a certain illness. They explain its efficacy in terms of the divine action of the *shuki* on the body's *okrus*. We find that the herb is equally effective in our society. How much evidence does this provide for belief in the *shuki*? None at all. The syntax is correct; the semantics is irrelevant. So it is with mathematics. It works. But this is no evidence whatsoever that the religion of mathematics has any truth in it.

In mathematics, reality lies in the symbolic expressions themselves, not in any abstract entities they are thought to denote. The symbol $\exists$ is simply a backwards E. If we conclude that a certain entity exists just because we have derived in a certain formal system a formula beginning with $\exists$, we do so at our peril. The dwelling place of meaning is syntax; semantics is the home of illusion.

How can I continue to be a mathematician when I have lost my faith in the semantics of mathematics? Why should I want to continue doing mathematics if I no longer believe that numbers and stochastic processes and Hilbert spaces exist? Well, why should a composer want to compose music that is not program music? Mathematics is the last of the arts to become nonrepresentational.

And mathematics is slowly beginning to become non-representational. Slowly in departments of mathematics, but quickly in computer science departments. Those who do computer science know that they are inventing and not discovering, and they are making beautiful and deep results concerning the nature of feasible computations. If we who are in traditional departments don't want to miss the boat, it behooves us to saddle a formalist horse pronto.

Abstract beliefs affect concrete actions. Despite its complete lack of justification, the semantic view of mathematics – the discovery of properties of entities existing in a Pythagorean world – has served mathematics reasonably well for a very long time. But now it is time to move forward, to reject the semantic view, and concentrate on what is real in mathematics. And what is real in mathematics is the notation, not an imagined denotation.

Let one brief example suffice. Abraham Robinson's creation of nonstandard analysis was a revolutionary simplification and extension of mathematical practice, but the mathematical community has been very slow, or unwilling, to adopt it because it conflicts with the Pythagorean religion.

We are too timid. If we cannot achieve the depth of Eudoxus, we can at least emulate his willingness to break with universally held opinion.