Gnomes in the Fog: The Reception of Brouwer's Intuitionism in the 1920s, by Dennis E. Hesseling, Science Networks – Historical Studies, vol. 28, Birkhäuser, Basel, 2003, xxviii+447 pp., ISBN 3-7643-6536-6

A classical mathematician *C* and an intuitionist *I* use the same logical operators, $\wedge\ \forall\ \rightarrow\ \neg\ \exists\ \vee$, but with different semantics and different deductive procedures. When *C* asserts a closed formula A, she is making an ontological statement: A is true in the structure under consideration. Then $A \vee \neg A$ always holds: either A is true or it is not, there is no third possibility. But when *I* asserts A, he is making an epistemological statement: I know A. Thus to him, $A \vee \neg A$ means: I know A or I know that A is absurd—but now there is clearly a third possibility. He rejects the principle of the excluded middle because to him it is a principle of omniscience.

I overheard two conversations between *C* and *I*. She spoke first:

> I have just proved $\exists x A$.
> Congratulations! What is it?
> I don't know. I assumed $\forall x \neg A$ and derived a contradiction.
> Oh. You proved $\neg\forall x \neg A$.
> That's what I said.

> I have proved $A \vee B$.
> Good. Which did you prove?
> What?
> You said you proved A or B; which did you prove?
> Neither; I assumed $\neg A \wedge \neg B$ and derived a contradiction.
> Oh, you proved $\neg[\neg A \wedge \neg B]$.
> That's right. It's another way of saying the same thing.

He does not agree with her last statement, of course, but at least *C* can explain to him what she has proved in a way that he can understand. This elimination of $\exists$ and $\vee$ must continue all the way, in the component formulas A and B, until atomic formulas are reached. Here *C* and *I* agree if they are discussing arithmetic, but in analysis *C* must explain an atomic formula C as meaning $\neg\neg$ C to *I*.

Now that *C* has explained her theorem, will *I* accept her proof? Yes, Gödel showed this in a remarkable five page paper published in 1933 [Gö]. At least, this is so for arithmetic. In set theory, *I* will admit the correctness of the deduction but the axioms of set theory will be gibberish to him.

The moral of Gödel's result is that Brouwer did not require a *restriction* of classical mathematics, as both he and his opponents believed, but rather provided an *extension* of it: he introduced two new logical operators, the constructive $\exists$ and the constructive $\vee$ different from their classical counterparts. The intuitionistic semantics and proof syntax for $\exists x A$ and $A \vee B$ are quite different from those for $\neg\forall x \neg A$ and $\neg[\neg A \wedge \neg B]$. Stricter standards of proof are required and richer information is obtained.

To *C*, a closed formula C contains one bit of information: true or false. To *I*, C is an incomplete communication. For example, if C is $\exists x A$ and *I* asserts that he knows C,

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then he is saying that he knows a number n (let us stick to arithmetic) such that he knows $A_x[n]$, and similarly for $A \vee B$ he asserts that he knows A or B and is prepared to say which.

Kleene, not himself an intuitionist, saw deeply into the nature of intuitionism and formalized intuitionistic knowledge in terms of *realizability*. He encoded intuitionistic knowledge of C in a code c, far richer than the classical true or false; see [Kl45] and [Kl52, §82].

The intuitionistic knowledge of a closed formula C of arithmetic can be explained in terms of an interactive program:

If C is:	and the code c is:	the program:
atomic	anything	evaluates C as true or false.
$A \wedge B$	$\langle a, b\rangle$	splits into two child processes, with code a for A and b for B.
$\forall xA$	c	prompts for a number input n and runs $c\{n\}$ on $A_x[n]$.
$A \rightarrow B$	c	prompts for code input a and forms a pipe: it runs a on A, and if this terminates, runs $c\{a\}$ on B.
$\exists xA$	$\langle n, a\rangle$	prints the output number n and runs a on $A_x[n]$.
$A \vee B$	$\langle 1, a\rangle$	prints "first" and runs a on A.
$A \vee B$	$\langle 2, a\rangle$	prints "second" and runs a on B.

(We can replace $\neg A$ by $A \rightarrow 0 = 1$, so $\neg$ does not require separate discussion.)

Then "c realizes C" means that c is correct code for the interactive program C. As the program runs, it also reduces the code to a form suitable for the subformulas that are encountered. A fuller account of Kleene's realizability in terms of programming is available online [ENe].

The nature of intuitionism was greatly clarified by Heyting when he formalized the intuitionistic predicate calculus and intuitionistic arithmetic in 1930. (Full disclosure: the reviewer is a formalist.) Kleene introduced realizability in the expectation that every theorem of intuitionistic arithmetic S was realizable, and David Nelson established this by an algorithm [DNe]. The high point of this beautiful result is his proof by a single induction that every induction axiom is realizable. Since $0 = 1$ is not realizable, Nelson's theorem entails a consistency proof for S and so a fortiori for Peano Arithmetic P (since by the Gödel interpretation S is an extension of P). The realization predicate cannot itself be expressed in arithmetic, so conflict with Gödel's second theorem, on the impossibility of self-consistency proofs, is avoided.

Call a formula *classical* in case it does not contain the output operators $\exists$ or $\vee$. For a classical closed formula C of arithmetic, "c realizes C" does not depend on the code c and, from a classical perspective, is equivalent to the classical true-false notion. The extra constructive information that a realization code c contains is algorithms for constructing existentially quantified objects and for deciding between alternatives. This is why intuitionistic reasoning is important to programming. Thanks to Kleene, the two constructive tendencies, algorithms and intuitionism, met. The connection between intuitionism and programming is deeply pursued by Per Martin-Löf and his school; see [M-L] and [NoPeSm].

Kleene's explanation of intuitionistic knowledge is semantic and does not refer to proofs at all. But many intuitionists attempt to explain the intuitionistic meaning of the input operators $\forall$, $\rightarrow$, and $\neg$ syntactically in terms of proofs, as in "$A \rightarrow B$ means that

I know how from a proof of A to find a proof of B". This viewpoint has been vigorously questioned by Dummett [Du]. Taken literally, it would mean that we could dispense with modus ponens, which is the rule of inference: from A and A $\rightarrow$ B infer B. But if A $\rightarrow$ B means that I know how from a proof of A to find a proof of B, why not just do it rather than bother with A $\rightarrow$ B? Alternatively, if one attempts to construct a syntactical predicate "c verifies C" along the lines of Kleene's semantic predicate "c realizes C", it runs afoul of Gödel's second theorem and $\neg\neg\, 0 = 0$ turns out to be "unverifiable"; see [ENe]. No one has given an account of the usual intuitionistic description of the meaning of $\rightarrow$ that manages, in Dummett's words, "to escape, not merely circularity, but total vacuousness". Curiously, the question of the intuitionistic meaning of $\rightarrow$ does not seem to have played any role in the debate of the 1920s, though the meaning of $\exists$ and $\vee$ was central.

And this brings us to the book under review, whose main focus is on the debate from the appearance of Weyl's *Grundlagenkrise* in 1921 to about 1928. It is a fascinating story. If you have a strong interest in foundations, you will begin at the beginning, continue until you reach the end, then stop. Those with a more peripheral interest in the subject will enjoy reading the conclusions to each chapter and to the book, browsing in the author index to see the roles in the debate of Kolmogorov, Lévy, Zariski, and many more, and skimming here and there. I found Chapter 1 on Brouwer's predecessors gripping and Chapter 6 on the cultural context less so.

I have said nothing in this article about Brouwer's "second act of intuitionism" (§2.6) and free choice sequences, but it makes interesting reading. There seems to be a consensus that Bishop's non-intuitionistic constructive analysis [Bi] is preferable to Brouwer's approach to analysis.

The author, Dennis Hesseling, is a mathematician with an additional background in history and philosophy, and he speaks with confidence in all three areas. A book as fine as this deserves more care from its publisher. In a number of places, particularly in footnotes and the glossary, formulas are garbled, with $\overline{}$ printed instead of the intended symbol. The author's English is fluent and a pleasure to read, but there is an occasional unidiomatic usage which careful editing would have corrected.

In strong contrast to the other great scientific debate of the twentieth century, that between Niels Bohr and Einstein, the debate between Brouwer and Hilbert was acerbic, with uncollegial words and acts, primarily on the part of Hilbert (see especially §2.8). It strikes me as a curious historical fact that Bohr persuaded physicists, who used to study the real world, to give up their belief in the objective reality of the physical world whereas mathematicians, who study an abstract world that we ourselves create, followed Hilbert (who was a Platonist at heart) in refusing to abandon our belief in the objective reality of mathematical entities.

Hilbert's program to secure the foundations of mathematics by finitary means is often called a failure. But it achieved several things of fundamental value and lasting importance. Before the Formalist Enlightenment, even mathematicians who like Peano and Zermelo were striving to be formal did not make an absolute distinction between syntax and semantics. The distinction is essential; an axiom system in which one is required to understand the meaning of some of the axioms is not an axiomatization at all. This distinction may be due more to Hilbert's assistant Bernays than to Hilbert himself (§5.2.2).

Thanks to this enlightenment, it became possible for the first time to study theories with the same precision and clarity with which one studies groups or fields. Mathematics consists of reasoning and computation, and of the two computation is the more fundamental. Aristotle, Leibniz, Boole, and Hilbert each took a big step towards realizing the vision of reducing reasoning to computation.

One of the most interesting themes of "Gnomes in the Fog" is the strong influence that Brouwer had on Hilbert's program. As Hesseling documents, Brouwer's opposition was not to formalism, which hardly existed at the time he founded intuitionism, but to classical mathematics regarded as contentual. (This is a useful word that I learned from the book. It is the English equivalent of the German *inhaltlich*, and the OED defines it as "Belonging to, or dealing with, content" with the first citation from 1909.) Gradually, step by step and without acknowledging his debt to Brouwer, Hilbert retreated from a contentual view of classical mathematics and refined the formalist position to the point where consistency was to be established essentially by intuitionistic methods. As Fraenkel summarized the debate (§3.2.2), "As I see it, Brouwer has reached the biggest success for his point of view by winning as an advocate of his starting position—Hilbert! ... Hilbert indeed has taken over the demand for constructivity and the rejection of a contentual ground for the application of Aristotelian logic on infinite totalities."

In evaluating the work of those like Brouwer who have made truly fundamental contributions to human knowledge—Columbus and Freud also come to mind—we are in the position that our world outlook has been completely changed by their work and we cannot picture what it was like beforehand. Before Brouwer, were mathematicians aware that the usual proofs of the fundamental theorem of algebra are not constructive (§4.3)?

Let us honor those with the courage foolishly to set sail into unknown seas and the endurance to reach land, though the land discovered differ from the land of the vision. What is the fog and who are the gnomes? The answer is found in the quotations with which this beautiful book begins and ends.

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