MATHEMATICS 217: HONORS LINEAR ALGEBRA

Spring Term, 2002

The textbook for the course is *Linear Algebra* by Hoffman and Kunze (second edition, Prentice-Hall, 1971). The course probably will cover most of the first seven chapters, and a bit more if time permits. Weekly assignments will consist of reading and problem sets; assignments will be distributed in class on Monday and due the following Monday. There will be in-class mid-term and final examinations, or take-home problem sets in lieu thereof, and possibly some shorter quizzes during the term.

The teacher is Edward Nelson, `nelson@math.princeton.edu`, Fine 1208, 8-4206. If you wish to see me, let me know in class or email me and we will get together as soon as is mutually convenient. The grader is Peter Keevash, `keevash@math.princeton.edu`, Fine 606, 8-4186.

The Web page for the course is

`http://www.math.princeton.edu/~nelson/217.html`.

Here are the assignments. **They are from a previous year, and with high probability changes will be made.**

First Week, February 4–8

READING:

As a preliminary it may be advisable to read the first two sections of the Appendix (pages 386–391), which describe some basic notions and notation that will be used freely in the course; this may be quite familiar material, but a quick review could still be helpful. The first topic covered in the course is the study of systems of linear equations through the use of matrices; you may have seen much of this material already, but it is important to become familiar with computations using matrices, as background for the more general discussion of linear spaces to follow. The first reading assignment is Chapter 1, pages 1–26.

PROBLEMS:

page 5, problem 6;
page 15, problems 1, 8;
page 21, problems 1, 3;
page 26, problem 3, 6, 8, 9.

Second Week, February 11–15

READING:

The next topic is a survey of some general properties of vector spaces: subspaces, bases, dimension, and coordinates. The reading assignment is Chapter 2, pages 28–66.

PROBLEMS:

page 39, problems 4, 7, 9;
page 48, problems 6, 7, 8;
page 54, problems 3, 7;

Third Week, February 18–22

READING:

The principal object of study in linear algebra is the set of linear transformations between two vector spaces; the fundamentals are discssed in Chapter 3, but most of the rest of the course will treat various aspects of linear transformations. The reading assignment is Chapter 3, pages 67–116.

PROBLEMS:

page 74, problem 12;
page 83, problems 3, 4, 8;
page 95, problem 5;
page 105, problems 3, 9;
page 116, problem 5.
and the following:

For a fixed real $n \times n$ matrix A consider the mapping T from the set of $n \times n$ real matrices X to itself defined by $T(X) = AX - XA$. Is this a linear transformation? Why? What can you say about the kernel (null space) and range of this mapping? (This is a difficult problem, since the range and null space depend in a somewhat subtle way on the nature of the matrix A. I don't expect a complete discussion of all possibilties; but say something that you find interesting.)

Fourth Week, February 25–March 1

READING:

Some knowledge of properties of polynomials over a field F is important in the more detailed study of normal forms for matrices, particularly a familiarity with the notions of the greatest common divisor of a set of polynomials and the prime factorization of polynomials. These topics are discussed in Chapter 4. The formal treatment of polynomials in sections 4.1 and 4.2 is unnecessarily abstract for our purposes; if you view polynomials as formal expressions $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ with coefficients $a_i \in F$ and with the expected rules for addition, multiplication, and differentiation of polynomials, that is really enough as background for the more technical and important results in this chapter. The reading assignment is part of section 4.3 (just page 124) and all of sections 4.4 and 4.5 (pages 127-138).

PROBLEMS:

page 126, problems 1, 3 (the Lagrange polynomials are the polynomials (4-12));
page 134, problems 2(b), 4;
page 139, problem 2.

Fifth Week, March 4–8

READING:

Determinants are a basic tool in linear algebra, and are the topic for this week. The reading assignment is Chapter 5, sections 5.1 through 5.4, pages 140 to 165. The more general topics discussed in the last part of Chapter 5 will not be needed, but are interesting if you are inclined to explore more.

PROBLEMS:

page 148, problems 10, 12;
page 155, problems 2, 6, 7;
page 162, problems 4, 10, 11, 13.

Sixth Week, March 11–15

READING:

With the general background for linear algebra out of the way, we turn next to the study of canonical forms for linear transformations, the main subject for the remainder of the term. The reading assignment for this week is the first part of Chapter 6, sections 6.1 through 6.3, pages 181–198.

There will be an in-class midterm or a take-home assignment instead. *Here is a sample from a previous year—this is not assigned.*

(1.) Consider the matrix

$$A = \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix}.$$

(a) Find the row-reduced echelon form for this matrix.
(b) Find the simplest normal form for this matrix under the set of all transformations that take A to the matrix XAY where X is any nonsingular 4×4 real matrix and Y is any nonsingular 5×5 matrix.
(c) Find a basis for the space of linear functionals on the vector space $\mathbb{R}^5$ that vanish on the linear subspace of $\mathbb{R}^5$ spanned by the rows of the matrix A.
(d) Determine a basis for the kernel (or null space) of the linear transformation $A : \mathbb{R}^5 \longrightarrow \mathbb{R}^4$ defined by the matrix A.
(e) Describe the set of vectors $Y \in \mathbb{R}^4$ for which there are no solutions of the linear equation $AX = Y$?
(f) Find $\det A_i$ where A_i is the 4×4 matrix obtained by deleting column i of the matrix A. How do these determinants compare with the corresponding determinants for the row-reduced echelon form of the matrix A?

(2) Consider the polynomials $p(x) = x^3 - 2$ and $q(x) = x + 1$.
(a) Find the greatest common divisor of these two polynomials when they are viewed as rational polynomials (that is, as polynomials in $\mathbb{Q}[x]$), and express that greatest common divisor as a linear combination of $p(x)$ and $q(x)$.
(b) Find the greatest common divisor of these two polynomials when they are viewed as polynomials over the field $\mathbb{F}_3$ of the integers modulo 3 (that is, as polynomials in $\mathbb{F}_3[x]$), and express that greatest common divisor as a linear combination of $p(x)$ and $q(x)$.

(3) Suppose that T is a linear functional on the vector space $F[x]$ of polynomials over a field F, that T is not identically zero, and that $T(p_1 p_2) = T(p_1)T(p_2)$ for any two polynomials $p_1, p_2 \in F[x]$. Show that there is some element $t \in F$ such that $T(p) = p(t)$ for every polynomial $p \in F[x]$.

(4) Suppose that T is a linear transformation from a finite-dimensional vector space V over a field F to itself, and that $T^2 = T$. Show that the vector space V has a basis such that the matrix representing the linear transformation T in terms of this basis is a diagonal matrix with diagonal entries either 0 or 1.

Seventh Week, March 25–29

READING:

The reading this week continues the study of linear transformations on a vector space. The reading assignment is Chapter 6, section 6.4, the first part of section 6.6 (pages 209–210), and section 6.8. A more direct treatment of some of these topics will be discussed in class, to avoid the rather abstract approach the text uses in the parts of Chapter 6 not assigned.

PROBLEMS:

page 190, problems 6, 8, 9, 10.
page 197, problems 2, 6, 9, 10.

Eighth Week, April 1–5

READING:

The next topic is the Jordan normal form for a linear transformation for which the characteristic polynomial factors completely, hence in particular for arbitrary complex linear transformations. The reading assignment is primarily the separate notes distributed in class. The notes give a somewhat different presentation of the material from that in the textbook, and among other things do not treat the rational normal form. A further although quite trivial notational difference is that the Jordan form in the notes is upper triangular while that in the textbook is lower triangular. However in addition to the notes it is interesting to read the application of the normal form for complex matrices to solutions of ordinary differential equations, given in Example 14, page 223 and Example 8, page 248, in the textbook.

PROBLEMS:

page 205, problems 1, 8, 10, 11.
page 218, problem 3.
page 225, problems 1, 2, 6

Ninth and Tenth Weeks, April 8–19

READING:

The discussion during the remainder of the term will focus on another role for matrices, as defining inner products on vector spaces. The basic properties of inner product spaces are discussed in Chapter 8, sections 8.1 and 8.2, pages 270–290.

PROBLEMS:

page 230, problems 1, 4, 7.
page 249, problems 3, 4, 5, 6, 8, 10.

Eleventh Week, April 22-26

READING:

This week continues the discussion of inner product spaces, with the reading assignment the remainder of Chapter 8, sections 8.3 through 8.5, pages 290–317.

PROBLEMS:

page 275, problems 8, 9.
page 288, problems 1, 2, 9, 15, 17.

There will be an in-class final or take-home instead. *Here is a sample from a previous year—this is not assigned.*

1. Let A be the linear transformation from $\mathbb{R}^4$ to itself defined by the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 3 \\ -1 & 2 & 1 & -1 \\ -1 & 1 & 0 & -3 \\ 1 & -2 & -1 & 1 \end{pmatrix}.$$

Find the row-reduced echelon form of this matrix.
Find a basis for the image and the kernel of this linear transformation.
Find $\det A$.

2. Let A be the matrix

$$A = \begin{pmatrix} 1 & 2 & -4 & 4 \\ 2 & -1 & 4 & -8 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 3 \end{pmatrix}.$$

Find the characteristic and minimal polynomials of this matrix.
Find the Jordan normal form J for this matrix, and find an explicit matrix B such that $BAB^{-1} = J$.

3. Determine all possible Jordan canonical forms for a matrix with the characteristic polynomial $(x-2)^3(x-3)^2$.

4. Show that a 3×3 complex matrix satisfying $A^3 = A$ can be diagonalized.
Is this true for a 4×4 matrix? Why?
Is it true for a real matrix? Why?

5. Suppose that A is an $n \times n$ Jordan block with the eigenvalue a; find the decomposition of the matrix A^2 into Jordan blocks. [Caution: this depends on the eigenvalue a.]

6. Show that a real symmetric matrix A has a real symmetric cube root B, a real symmetric matrix such that $B^3 = A$.

7. Let V be the real vector space of real polynomials of degree at most equal to 3, with the inner product

$$(f, g) = \int_0^1 f(x)g(x)dx.$$

Find a polynomial $p_t \in V$ such that $(f, p_t) = f(t)$ for any real number t.
What is the adjoint T^* of the operation T of differentiation on V?

8. If T is a normal linear operator on a finite-dimensional complex vector space show that there is a complex polynomial $f(x)$ such that $T^* = f(T)$. [Suggestion:

represent T by a diagonal matrix.]

9. Find a real orthogonal matrix O such that $OA\,{}^tO$ is diagonal where

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

10. Show that a complex linear transformation A is normal if and only if $A = A_1 + i\,A_2$ where A_1, A_2 are self-adjoint operators that commute.