Mathematics 217 – Spring 2002

Problems in lieu of midterm exam

You may consult any written or online source, but no other person. References other than the text should be explicitly acknowledged. Please write and sign the following statement on your problem set: **"This paper represents my own work in accordance with University regulations."** Papers are due in class at 1:30 this Friday.

1. Let F be a field and let V_n be the set of all polynomials in $F[x]$ of degree at most n (together with 0). For c in F, define T_c on V_n by $(T_c p)(x) = p(x - c)$, and let D be differentiation. Find the determinants and traces of T_c and xD.

2. Notice that for f in $\mathbb{R}[x]$, it makes sense to evaluate f at a complex number, such as $i = \sqrt{-1}$. Let $M = \{f \in \mathbb{R}[x] : f(i) = 0\}$. Show that M is an ideal. Find its monic generator.

3. Let F be a field, and let M be the smallest ideal containing all f in $F[x]$ such that $f(0)f(1) = 0$. Find its monic generator.

4. Let F be a field and let A be a matrix in $F[x]^{n\times n}$, where $n \geq 5$, such that

$$A_{11} = A_{21} = A_{32} = A_{42} = A_{52} = x$$

(read the subscripts carefully) and all other entries of A are elements of F. For c in F, let $A(c)$ be the matrix in $F^{n\times n}$ obtained by replacing each x by c. Prove that either $A(c)$ is never invertible or it is invertible for all but at most two values of c in F.

5. Let F be a field and let A be a diagonal matrix in $F^{100\times 100}$ with distinct diagonal elements, so that $A_{ij} = c_i\delta_{ij}$ with $c_i \neq c_j$ for $i \neq j$. Find the dimension of the space of all matrices C that are of the form $C = AB - BA$ for some B in $F^{100\times 100}$.

6. Theorem 13 of the text, on page 23, is concerned with matrices with entries in a field F. Give an example to show that the theorem can be false if F is replaced by K, where K is a commutative ring with identity. Prove that nevertheless the corollary, on page 24, is true in this context.