The Jordan Canonical Form

The Jordan canonical form describes the structure of an arbitrary linear transformation on a finite-dimensional vector space over an algebraically closed field. Here we develop it using only the most basic concepts of linear algebra, with no reference to determinants or ideals of polynomials.

THEOREM 1. *Let $\beta_1, \ldots, \beta_n$ be linearly independent vectors in a vector space. If they are in the span of $\alpha_1, \ldots, \alpha_k$ then $k \geq n$.*

Proof. We prove the following claim:

Let $\beta_1, \ldots, \beta_n$ be linearly independent vectors in a vector space. For all j with $0 \leq j \leq n$ and all vectors $\alpha_1, \ldots, \alpha_k$, if $\beta_1, \ldots, \beta_n$ are in the span of $\beta_1, \ldots, \beta_j, \alpha_1, \ldots, \alpha_k$, then $j + k \geq n$.

The proof of the claim is by induction on k. For $k = 0$, the claim is obvious since $\beta_1, \ldots, \beta_n$ are linearly independent. Suppose the claim is true for $k-1$, and suppose that $\beta_1, \ldots, \beta_n$ are in the span of the vectors $\beta_1, \ldots, \beta_j, \alpha_1, \ldots, \alpha_k$. Then in particular we have

$$\beta_{j+1} = b_1\beta_1 + \cdots + b_j\beta_j + a_1\alpha_1 + \cdots + a_k\alpha_k. \tag{1}$$

For some i we must have $a_i \neq 0$ since $\beta_1, \ldots, \beta_n$ are linearly independent, so we can solve (1) for α_i as a linear combination of

$$\beta_1, \ldots, \beta_j, \beta_{j+1}, \alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_k. \tag{2}$$

Hence the vectors (2) span $\beta_1, \ldots, \beta_n$. By the induction hypothesis, $(j + 1) + (k - 1) \geq n$, so $j + k \geq n$. This proves the claim. The case $j = 0$ of the claim gives the theorem. □

By this theorem, any two bases of a finite-dimensional vector space have the same number of elements, the *dimension* of the vector space.

Let T be a linear transformation on the finite-dimensional vector space V over the field F. An *annihilating polynomial* for T is a non-zero polynomial p such that $p(T) = 0$.

THEOREM 2. *Let T be a linear transformation on the finite-dimensional vector space V. Then there exists an annihilating polynomial for T.*

Proof. Let $\alpha_1, \ldots, \alpha_n$ be a basis for V. For each i with $1 \le i \le n$, by Theorem 1 there exist scalars $a_0, \ldots, a_n$, not all 0, such that

$$a_0\alpha_i + a_1T\alpha_i + \cdots + a_nT^n\alpha_i = 0.$$

That is, $p_i(T)\alpha_i = 0$ where $p_i = a_0 + a_1x + \cdots + a_nx^n$. Let p be the product of all the p_i. Then p is an annihilating polynomial for T since $p(T)\alpha_i = 0$ for each basis vector α_i. □

We denote the null space of the linear transformation T by $\mathcal{N}T$ and its range by $\mathcal{R}T$.

THEOREM 3. *Let T be a linear transformation on the finite dimensional vector space V. Then $\mathcal{N}T$ and $\mathcal{R}T$ are linear subspaces of V invariant under T, with*

$$\dim\mathcal{N}T + \dim\mathcal{R}T = \dim V. \tag{3}$$

If $\mathcal{N}T \cap \mathcal{R}T = \{0\}$ then

$$V = \mathcal{N}T \oplus \mathcal{R}T \tag{4}$$

is a decomposition of V as a direct sum of subspaces invariant under T.

Proof. It is clear that $\mathcal{N}T$ and $\mathcal{R}T$ are linear subspaces of V invariant under T. Let $\alpha_1, \ldots, \alpha_k$ be a basis for $\mathcal{N}T$ and extend it by the vectors $\alpha_{k+1}, \ldots, \alpha_n$ to be a basis for V. Then

$$T\alpha_{k+1}, \ldots, T\alpha_n$$

are a basis for $\mathcal{R}T$: they span $\mathcal{R}T$, and if $a_{k+1}T\alpha_{k+1} + \cdots + a_nT\alpha_n = 0$ then $a_{k+1}\alpha_{k+1} + \cdots + a_n\alpha_n \in \mathcal{N}T$, so all of the coefficients are 0. This proves (3), from which (4) follows if $\mathcal{N}T \cap \mathcal{R}T = \{0\}$. □

THEOREM 4. *Let T be a linear transformation on a non-zero finite-dimensional vector V over an algebraically closed field F. Then T has an eigenvector.*

Proof. By Theorem 2 there exists an annihilating polynomial p for T. Since F is algebraically closed, p is a non-zero scalar multiple of

$$(x - c_k)\cdots(x - c_1)$$

for some scalars $c_k, \ldots, c_1$. Let α be a non-zero vector and let i be the least number such that

$$(T - c_iI)\cdots(T - c_1I)\alpha = 0.$$

If $i = 1$, then α is an eigenvector with the eigenvalue c_1; otherwise,

$$\beta = (T - c_{i-1}I)\cdots(T - c_1I)\alpha$$

is an eigenvector with the eigenvalue c_i. □

THEOREM 5. *Let T be a linear transformation on the finite-dimensional vector space V with an eigenvalue c. Let*

$$V_c = \{\, \alpha \in V : \text{ for some } j,\ (T - cI)^j \alpha = 0 \,\}. \tag{5}$$

Then there exists r such that

$$V_c = \mathcal{N}(T - cI)^r \tag{6}$$

and

$$V = \mathcal{N}(T - cI)^r \oplus \mathcal{R}(T - cI)^r \tag{7}$$

is a decomposition of V as a direct sum of subspaces invariant under T.

Proof. If $\alpha \in V_c$ and $c \in F$, then $c\alpha \in V_c$; if $\alpha_1 \in V_c$ (so that $(T - cI)^{j_1}\alpha_1 = 0$ for some j_1) and $\alpha_2 \in V_c$ (so that $(T - cI)^{j_2}\alpha_2 = 0$ for some j_2), then $\alpha_1 + \alpha_2 \in V_c$ (since $(T - cI)^j(\alpha_1 + \alpha_2) = 0$ whenever $j \geq j_1, j_2$). Thus V_c is a linear subspace of V. It has a finite basis, since V is finite dimensional, so there is an r such that $(T - cI)^r\alpha = 0$ for each basis element α and consequently for all α in V_c. This proves (6).

I claim that $V_c = \mathcal{N}(T - cI)^r$ and $\mathcal{R}(T - cI)^r$ have intersection $\{0\}$. Suppose that α is in both spaces. Then $\alpha = (T - cI)^r\beta$ for some β since α is in $\mathcal{R}(T - cI)^r$. Since it is in $\mathcal{N}(T - cI)^r$,

$$(T - cI)^r\alpha = (T - cI)^{2r}\beta = 0,$$

so $\beta \in V_c$ by the definition (5) of V_c. Hence $(T - cI)^r\beta = 0$ by (6), so $\alpha = 0$. This proves the claim.

By Theorem 3 we have (7). Each of the two spaces is invariant under $T - cI$, and under cI, so also under $T = (T - cI) + cI$. □

THEOREM 6. *Let T be a linear transformation on the finite-dimensional vector space V over the algebraically closed field F, and let the scalars $c_1, \ldots, c_k$ be the distinct eigenvalues of T. Then there exist numbers r_i, for $1 \leq i \leq k$, such that*

$$V = \mathcal{N}(T - c_1I)^{r_1} \oplus \cdots \oplus \mathcal{N}(T - c_kI)^{r_k} \tag{8}$$

is a direct sum decomposition of V into subspaces invariant under T.

Proof. From Theorem 5 by induction on the number of distinct eigenvalues. □

A linear transformation N is *nilpotent of degree r* in case $N^r = 0$ but $N^{r-1} \neq 0$; it is *nilpotent* in case it is nilpotent of degree r for some r. Notice that on each of the subspaces of the direct sum decomposition (8), the operator T is a scalar multiple of I plus a nilpotent operator. Thus our remaining task is to find the structure of a nilpotent operator.

THEOREM 7. *Let N be nilpotent of degree r on the vector space V. Then we have strict inclusions*

$$\mathcal{N}N \subset \mathcal{N}N^2 \subset \cdots \subset \mathcal{N}N^{r-1} \subset \mathcal{N}N^r = V. \tag{9}$$

Proof. The inclusions are obvious. They are strict inclusions because by definition there is a vector α in V such that $N^r\alpha = 0$ but $N^{r-1}\alpha \neq 0$. Then $N^{r-i}\alpha$ is in $\mathcal{N}N^i$ but not $\mathcal{N}N^{i-1}$. □

We say that the vectors $\beta_1, \ldots, \beta_k$ are *linearly independent of* the linear subspace W in case $b_1\beta_1 + \cdots + b_k\beta_k$ is in W only if $b_1 = \cdots = b_k = 0$.

THEOREM 8. *Let N be a nilpotent linear transformation of degree r on the finite-dimensional vector space V. Then there exist a number m and vectors $\alpha_1, \ldots, \alpha_m$ such that the non-zero vectors of the form $N^j\alpha_l$, for $j \geq 0$ and $1 \leq l \leq m$, are a basis for V. Any vectors linearly independent of $\mathcal{N}N^{r-1}$ can be included among the $\alpha_1, \ldots, \alpha_m$.*

For $1 \leq l \leq m$, let V_l be the subspace with basis $\alpha_l, \ldots, N^{s_l-1}\alpha_l$, where s_l is the least number such that $N^{s_l}\alpha_l = 0$. Then

$$V = V_1 \oplus \cdots \oplus V_m \tag{10}$$

is a direct sum decomposition of V into s_l-dimensional subspaces invariant under N, and N is nilpotent of degree s_l on V_l. For $1 \leq i \leq r$, let $\varphi(i)$ be the number of subspaces in the decomposition (10) *of dimension at least i. Then*

$$\dim \mathcal{N}N^i - \dim \mathcal{N}N^{i-1} = \varphi(i),$$

so the number of subspaces in (10) *of any given dimension is determined uniquely by N.*

Proof. We prove the statements of the first paragraph by induction on r. For $r = 1$, we have $N = 0$ and the result is trivial. Suppose that the result holds for $r - 1$, and consider a nilpotent linear transformation of degree r.

Given vectors linearly independent of $\mathcal{N}N^{r-1}$, extend them to a maximal such set $\beta_1, \ldots, \beta_k$ (so that they together with any basis for $\mathcal{N}N^{r-1}$ are a basis for V). Then the vectors $N\beta_1, \ldots, N\beta_k$ are in $\mathcal{N}N^{r-1}$ and are linearly independent of $\mathcal{N}N^{r-2}$, for if $b_1N\beta_1 + \cdots + b_kN\beta_k \in \mathcal{N}N^{r-2}$ then $b_1\beta_1 + \cdots + b_k\beta_k \in \mathcal{N}N^{r-1}$ so $b_1 = \cdots = b_k = 0$. Now N restricted to $\mathcal{N}N^{r-1}$ is nilpotent of degree $r - 1$, so by the induction hypothesis there are vectors $\alpha_1, \ldots, \alpha_m$, including $N\beta_1, \ldots, N\beta_k$ among them, such that the non-zero vectors of the form $N^j\alpha_l$ are a basis for $\mathcal{N}N^{r-1}$. Adjoin the vectors $\beta_1, \ldots, \beta_k$ to them; then this is a basis for V of the desired form.

Now the statements of the second paragraph follow directly. (See the following example, in which N is nilpotent of degree 5 on a 24 dimensional space. The bottom i rows are $\mathcal{N}N^i$.) □

V_1	V_2	V_3	V_4	V_5	V_6
α_1	α_2				
$N\alpha_1$	$N\alpha_2$	α_3	α_4	α_5	
$N^2\alpha_1$	$N^2\alpha_2$	$N\alpha_3$	$N\alpha_4$	$N\alpha_5$	
$N^3\alpha_1$	$N^3\alpha_2$	$N^2\alpha_3$	$N^2\alpha_4$	$N^2\alpha_5$	α_6
$N^4\alpha_1$	$N^4\alpha_2$	$N^3\alpha_3$	$N^3\alpha_4$	$N^3\alpha_5$	$N\alpha_6$

We have done all the work necessary to establish the Jordan canonical form; it remains only to put the pieces together. It is convenient to express the result in matrix language.

Let $B(r;c)$ be the $r \times r$ lower triangular matrix with c along the diagonal, 1 everywhere immediately below the diagonal, and 0 everywhere else. Such a matrix is called a *Jordan block*. Notice that in the decomposition (10), the matrix of N on V_l, with respect to the basis described in Theorem 8, is the Jordan block $B(s_l;0)$. (With the basis in reverse order, the entries 1 are immediately above the diagonal. Either convention is acceptable.) A matrix that is a direct sum of Jordan blocks is in *Jordan form*.

THEOREM 9. *Let T be a linear transformation on the finite-dimensional vector space V over the algebraically closed field F. Then there exists a basis of V such that the matrix of T is in Jordan form. This matrix is unique except for the order of the Jordan blocks.*

Proof. By Theorems 6 and 8. □

The proof shows that the same result holds for a field that is not algebraically closed provided that T has some annihilating polynomial that factors into first degree factors.