Mathematics 217 – Spring 2002

Hints for the problems in lieu of final exam

1. Suppose that an $n \times n$ matrix satisfies

$$\sum_{j=1}^{n} A_{ij} = 1$$

for all i. Prove that there is a solution to

$$\sum_{i=1}^{n} x_i A_{ij} = x_j$$

other than the 0 solution. [That is, prove that there exist scalars $x_1, \ldots, x_n$, not all 0, satisfying this equation for all j.]

The vector α all of whose components are 1 is an eigenvector for A with eigenvalue 1. Since the transpose of a matrix has the same eigenvalues, there is an eigenvector for the transpose with eigenvalue 1. This is the sought-for solution.

2. Find

$$\begin{pmatrix} 14 & -6 \\ 30 & -13 \end{pmatrix}^{1000}.$$

Your answer may contain powers of scalars.

The characteristic equation has roots 2 and -1, so the matrix is similar to

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is easy to find the 1000-th power of this matrix. Find the eigenvectors, and therefore the change of basis, to express the result explicitly.

3. Consider a system of differential equations

$$\frac{d\alpha}{dt} = A\alpha\,; \quad \alpha(0) = \alpha_0$$

where A is an $n \times n$ complex matrix. Prove that the solution satisfies

$$\|\alpha(t)\| \leq \|\alpha_0\|$$

for all $t \geq 0$ and all α_0 if and only if

$$\mathrm{Re}(A\beta|\beta) \leq 0$$

for all β. [Here $\|\ \|$ is the norm with respect to the standard inner product on $\mathbb{C}^n$.]

The solution satisfies $\|\alpha(t)\| \le \|\alpha_0\|$ for all $t \ge 0$ and all α_0 if and only if $\|\alpha(t)\|$ is decreasing; i.e., if and only if

$$\frac{d}{dt}\big(\alpha(t)\big|\alpha(t)\big) \le 0.$$

But

$$\frac{d}{dt}\left(\alpha(t)\middle|\alpha(t)\right) = \left(\frac{d}{dt}\alpha(t)\middle|\alpha(t)\right) + \left(\alpha(t)\middle|\frac{d}{dt}\alpha(t)\right) = 2\mathrm{Re}\left(A\alpha(t)\middle|\alpha(t)\right).$$

4. (a) Classify up to similarity all 2×2 complex matrices of trace 0 and determinant 0.

(b) Classify up to similarity all 3×3 complex matrices of trace 0 and determinant 0.

(c) Classify up to similarity all 4×4 complex matrices of trace 0 and determinant 0.

Use the Jordan canonical form.

5. Determine all possible Jordan canonical forms for a matrix with the characteristic polynomial $(x-2)^3(x-3)^2$.

Use the Jordan canonical form.

6. (a) Show that a 3×3 complex matrix satisfying $A^3 = A$ can be diagonalized.

(b) Is this necessarily true for a complex 4×4 matrix? [Some people have misread this question. It means, "Is it necessarily true that a 4×4 complex matrix satisfying $A^3 = A$ can be diagonalized?"]

(c) Is this necessarily true for a real 3×3 matrix?

$x^3 - x$ is an annihilating polynomial for A. This factors into distinct linear factors, $x^3 - x = x(x+1)(x-1)$, so in all three cases the matrix is diagonalizable.

7. Give an example of an inner product space V and a linear functional f on V such that there is no vector β in V such that $f(\alpha) = (\alpha|\beta)$ for all α in V.

The example must be infinite dimensional. Let V be all polynomials, with inner product

$$(p|q) = \int_0^1 f(t)\overline{q(t)}\,dt$$

and let $f(p) = p(0)$.

8. Classify nilpotent linear transformations on a two-dimensional complex inner product space up to unitary equivalence. (In matrix language, find a set of 2×2 complex nilpotent matrices A such that for every 2×2 complex nilpotent matrix B, there is a unique matrix A in the set and a unitary matrix U such that $U^{-1}BU = A$.)

If A is not the 0 matrix, then the null space is one dimensional. Let α_1 be a unit vector in the null space and let α_2 be a unit vector orthogonal to it. With respect to this orthonormal basis, the matrix is

$$\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$$

where $c \in \mathbb{C}$. Use the fact that α_2 (or α_1) can be multiplied by any complex number of absolute value 1 to show that we can take $c \geq 0$, and then argue that two such matrices with different positive values of c cannot be unitarily equivalent.

9. Let V be the vector space of all real polynomials with the inner product

$$(p|q) = \int_0^\infty p(x)q(x)e^{-x}\,dx.$$

Construct orthonormal polynomials of degree 0, 1, and 2.

Use the Gram-Schmidt process.