Mathematics 217 – Spring 2002

Problems in lieu of final exam

You may consult any written or online source, but no other person. References other than the text should be explicitly acknowledged. Please write and sign the following statement on your problem set: **"This paper represents my own work in accordance with University regulations."**

Always give your reasons for your answer.

Papers are due, outside or under the door of Fine 1208, by the end of Tuesday, May 14.

You can email me at `nelson@math.princeton.edu` if any problem seems to require clarification. Please check the Web page at

`http://math.princeton.edu/~nelson/217.html`

to see if any clarifications have been made.

1. Suppose that an $n \times n$ matrix satisfies

$$\sum_{j=1}^{n} A_{ij} = 1$$

for all i. Prove that there is a solution to

$$\sum_{i=1}^{n} x_i A_{ij} = x_j$$

other than the 0 solution. [That is, prove that there exist scalars $x_1, \ldots, x_n$, not all 0, satisfying this equation for all j.]

2. Find

$$\begin{pmatrix} 14 & -6 \\ 30 & -13 \end{pmatrix}^{1000}.$$

Your answer may contain powers of scalars.

3. Consider a system of differential equations

$$\frac{d\alpha}{dt} = A\alpha\,; \quad \alpha(0) = \alpha_0$$

where A is an $n \times n$ complex matrix. Prove that the solution satisfies

$$\|\alpha(t)\| \leq \|\alpha_0\|$$

for all $t \geq 0$ and all α_0 if and only if

$$\mathrm{Re}(A\beta|\beta) \leq 0$$

for all β. [Here $\|\ \|$ is the norm with respect to the standard inner product on $\mathbb{C}^n$.]

4. (a) Classify up to similarity all 2×2 complex matrices of trace 0 and determinant 0.

(b) Classify up to similarity all 3×3 complex matrices of trace 0 and determinant 0.

(c) Classify up to similarity all 4×4 complex matrices of trace 0 and determinant 0.

5. Determine all possible Jordan canonical forms for a matrix with the characteristic polynomial $(x-2)^3(x-3)^2$.

6. (a) Show that a 3×3 complex matrix satisfying $A^3 = A$ can be diagonalized.

(b) Is this necessarily true for a complex 4×4 matrix? [Some people have misread this question. It means, "Is it necessarily true that a 4×4 complex matrix satisfying $A^3 = A$ can be diagonalized?"]

(c) Is this necessarily true for a real 3×3 matrix?

7. Give an example of an inner product space V and a linear functional f on V such that there is no vector β in V such that $f(\alpha) = (\alpha|\beta)$ for all α in V.

8. Classify nilpotent linear transformations on a two-dimensional complex inner product space up to unitary equivalence. (In matrix language, find a set of 2×2 complex nilpotent matrices A such that for every 2×2 complex nilpotent matrix B, there is a unique matrix A in the set and a unitary matrix U such that $U^{-1}BU = A$.)

9. Let V be the vector space of all real polynomials with the inner product

$$(p|q) = \int_0^\infty p(x)q(x)e^{-x}\,dx.$$

Construct orthonormal polynomials of degree 0, 1, and 2.