Mathematics 217 – Spring 2002

Problems in lieu of midterm exam

Here are brief indications of solutions.

1. Let F be a field and let V_n be the set of all polynomials in $F[x]$ of degree at most n (together with 0). For c in F, define T_c on V_n by $(T_c p)(x) = p(x - c)$, and let D be differentiation. Find the determinants and traces of T_c and xD.

With respect to the basis $x^n, \ldots\ x, 1$, both of these operators have triangular matrices, so by one of the homework problems their determinants are just the products of the diagonal entries, namely $1, \ldots, 1$ for T_c and $n, n-1, \ldots, 1, 0$ for xD. Hence det T_c=1 and trace $T_c = n + 1$, and det $xD = 0$ and trace $xD = \frac{1}{2}n(n + 1)$.

2. Notice that for f in $\mathbb{R}[x]$, it makes sense to evaluate f at a complex number, such as $i = \sqrt{-1}$. Let $M = \{f \in \mathbb{R}[x] : f(i) = 0\}$. Show that M is an ideal. Find its monic generator.

$x^2 + 1$.

3. Let F be a field, and let M be the smallest ideal containing all f in $F[x]$ such that $f(0)f(1) = 0$. Find its monic generator.

Remember that an ideal is also closed under addition. The answer is 1, since $1 = (1 - x) + x$ is in M.

4. Let F be a field and let A be a matrix in $F[x]^{n\times n}$, where $n \geq 5$, such that

$$A_{11} = A_{21} = A_{32} = A_{42} = A_{52} = x$$

(read the subscripts carefully) and all other entries of A are elements of F. For c in F, let $A(c)$ be the matrix in $F^{n\times n}$ obtained by replacing each x by c. Prove that $A(c)$ is invertible for all but at most two values of c in F.

All of the x entries in A lie in the first and second columns. Hence for any permutation σ, at most two of $A_{1\,\sigma 1} \ldots A_{n\,\sigma n}$ are x; that is, det A is a polynomial in x of degree at most two, and hence is either identically 0 (in which case $A(c)$ is never invertible) or it has at most two roots c_1 and c_2 in F. For all other values of c, det $A(c) \neq 0$ and $A(c)$ is invertible.

5. Let F be a field and let A be a diagonal matrix in $F^{100\times 100}$ with distinct diagonal elements, so that $A_{ij} = c_i\delta_{ij}$ with $c_i \neq c_j$ for $i \neq j$. Find the dimension of the space of all matrices C that are of the form $C = AB - BA$ for some B in $F^{100\times 100}$.

The map $B \mapsto AB - BA$ is linear on the 10000-dimensional vector space $F^{100\times 100}$. Its null space—the space of B commuting with A—is the 100-dimensional space of all diagonal matrices, by a simple matrix computation. By the rank plus nullity theorem (Theorem 2 of Chapter 3, on page 71), the dimension of the range (which is what we are asked to find) is $10000 - 100 = 9900$.

6. Theorem 13 of the text, on page 23, is concerned with matrices with entries in a field F. Give an example to show that the theorem can be false if F is replaced by K, where K is a commutative ring with identity. Prove that nevertheless the corollary, on page 24, is true in this context.

Let $K = \mathbb{Z}$, let $n = 1$, and let $A = (2)$. Then the only solution to $AX = 0$ is $X = 0$, but A is not invertible.

Suppose that A has a left inverse B, so that $BA = I$. Then $\det B \cdot \det A = \det I = 1$, so $\det A$ is invertible in K. Then A is invertible by Theorem 4 of Chapter 5, on page 160. (Similarly when A has a right inverse.)