Partial Fractions

Irreducible quadratics in the denominator

Suppose that in working a problem by partial fractions you encounter a fraction with irreducible quadratic denominator. How do you integrate it?

For example, let $I = \displaystyle\int \frac{x+3}{x^2+4x+7}\,dx$.

First, complete the square: $x^2+4x+7 = (x+2)^2+3$. Thus $I = \displaystyle\int \frac{x+3}{(x+2)^2+3}\,dx$.

WARNING: Do not split this up as $I = \displaystyle\int \frac{x}{(x+2)^2+3}\,dx + \int \frac{3}{(x+2)^2+3}\,dx$. In the first of these two integrals, the numerator x is not a constant multiple of the derivative $2(x+2)$ of the denominator, so substitution does not work.

Substitute for the quantity in the completion of the square: let $y = x+2$, so $x = y-2$ and $dx = dy$. Hence $I = \displaystyle\int \frac{y+1}{y^2+3}\,dy$. Now split it up. I is the sum of $\displaystyle\int \frac{y}{y^2+3}\,dy = \frac{1}{2}\ln|y^2+3| + C$ and $\displaystyle\int \frac{1}{y^2+3}\,dy = \frac{1}{\sqrt{3}}\arctan\left(\frac{y}{\sqrt{3}}\right) + C$.

The final answer, expressed in terms of x is

$$\frac{1}{2}\ln|x^2+4x+7| + \frac{1}{\sqrt{3}}\arctan\left(\frac{x+2}{\sqrt{3}}\right) + C.$$

(The absolute value sign after ln is unnecessary since x^2+4x+7 is always positive.)

Repeated factors

This section will not help you work problems; it is intended to answer a question that many people wonder about.

Again let us take an example. Consider $\displaystyle\int \frac{dx}{x(x+1)^2}$. We are told to set up partial fractions as follows:

$$\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}. \tag{1}$$

Why $\dfrac{C}{(x+1)^2}$ and not, say, $\dfrac{C+Dx}{(x+1)^2}$?

First of all, the number of unknown constants $A, B, \ldots$ must equal the total degree of the denominator in the problem, so that we have as many equations as unknowns.

Suppose we set up partial fractions as

$$\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{E+Fx}{(x+1)^2}, \tag{2}$$

so there are three unknown constants. Actually, this will work, but the second term is hard to integrate once we have found E and F. It can be rewritten as

$$\frac{E+Fx}{(x+1)^2} = \frac{(E-F)+F(x+1)}{(x+1)^2},$$

which makes it easy to integrate. But this is just the original form (1), with $B = F$ and $C = E - F$. So we set it up this way from the start to save work!

Almost certainly, you will never encounter higher powers of irreducible quadratics in the denominator, either in this course or in the big wide world. If you do, complete the square and use trigonometric substitution.