# SERIES

We say that the *series* $\sum_{n=0}^{\infty} a_n$ **converges to** $L$, or $\sum_{n=0}^{\infty} a_n = L$, in case the limit of the *sequence* of **partial sums** $\sum_{n=0}^{N} a_n$ is $L$; i.e., $\lim_{N\to\infty} \sum_{n=0}^{N} = L$. The series $\sum_{n=0}^{\infty} a_n$ **converges** in case to converges to some number $L$; otherwise, it **diverges**.

From the definition, we find that the **geometric series** $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$. It follows that $\sum_{n=k}^{\infty} r^n = \frac{r^k}{1-r}$ if $|r| < 1$ by factoring out the common factor $r^k$.

## Tests for series with positive terms

**comparison** If $0 \leq a_n \leq b_n$ and the big series $\sum_{n=0}^{\infty} b_n$ converges, then the small series $\sum_{n=0}^{\infty} a_n$ converges. (But if the big series diverges, this gives no information about the small series.)

**limit comparison** (very useful) If $0 < a_n$, $0 < b_n$, and $a_n \sim b_n$ as $n \to \infty$ (that is, $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$), then the two series both converge or both diverge. Read $\sim$ as "is asymptotic to", or "behaves like". A polynomial in $n$ behaves like the leading term as $n \to \infty$.

**integral test** If $f$ is continuous, positive, and decreasing on $[1, \infty)$, then the integral $\int_1^{\infty} f(x)\, dx$ and the series $\sum_{n=1}^{\infty} f(n)$ both converge or both diverge. This is seldom used except for the following special case:

**p-test** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

**root and ratio tests** If $a_n > 0$ and $\lim_{n\to\infty} a_n^{1/n} = \rho$, then the series converges if $\rho < 1$ and diverges if $\rho > 1$. If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$, then the series converges if $\rho < 1$ and diverges if $\rho > 1$. In both cases, there is no information if $\rho = 1$.

**Tests for series that may have both negative and positive terms**

**n'th term test for divergence** If the *sequence* $a_n$ does not tend to 0 as $n \to \infty$ (i.e., if $\lim_{n\to\infty} = l$ and $l \neq 0$, or if the limit does not exist), then the *series* $\sum_{n=0}^{\infty} a_n$ diverges. (But if $\lim_{n\to\infty} a_n = 0$, this gives no information.)

**absolute convergence** If $\sum_{n=0}^{\infty} |a_n|$ converges, then $\sum_{n=0}^{\infty} a_n$ converges. (But if $\sum_{n=0}^{\infty} |a_n|$ diverges, this gives no information.)

**alternating series test** If the sequence $a_n$ decreases to 0 (that is, if $a_0 > a_1 > a_2 > \cdots$ and $\lim_{n\to\infty} a_n = 0$), then the series $\sum_{n=0}^{\infty} (-1)^n a_n$ converges.

**Tips**

First make an informed guess as to whether the series converges or diverges. Is $a_{100}$ very small? Try the n'th term test for divergence.

Use limit comparison to find a simpler series you are familiar with where the terms behave like the terms of the given series.

If you see a factorial (!), use the ratio test.

If you see $c^n$ (where c is a constant), try the root or ratio test. If you just see $n^c$, try the p-test.