1. Evaluate $\int_2^3 xe^{x^2}\,dx$.

Call the integral I, and let $t = x^2$. Then $dt = 2x\,dx$.

When $x = 3$, $t = 9$ and when $x = 2$, $t = 4$. Therefore

$$I = \frac{1}{2}\int_4^9 e^t\,dt = \frac{1}{2}e^t\Big|_4^9 = \frac{e^9}{2} - \frac{e^4}{2}.$$

2. Evaluate $\int \frac{9x+9}{(x-1)(x^2+4x+13)}\,dx$.

The second factor is irreducible. Set up partial fractions:

$$\frac{9x+9}{(x-1)(x^2+4x+13)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4x+13},$$

so

$$9x+9 = A(x^2+4x+13) + (Bx+C)(x-1).$$

Choosing $x=1$ we find $18 = A(18)$, so $A=1$. Hence

$$9x+9 = x^2+4x+13+(Bx+C)(x-1) = (1+B)x^2+(4-B+C)x+(13-C).$$

Equating coefficients, we find

$$\begin{aligned} 0 &= 1+B \\ 9 &= 4-B+C \\ 9 &= 13-C \end{aligned}$$

Thus $B=-1$ and $C=4$. Call the integral I. Then

$$I = \int \frac{1}{x-1}\,dx + \int \frac{-x+4}{x^2+4x+13}\,dx.$$

The first integral is $\ln|x-1|$ plus a constant. Call the second integral J.

To find J, complete the square: $x^2+4x=13=(x+2)^2+9$. Now let $t=x+2$, so $dx=dt$ and $x=t-2$. Hence

$$J = \int \frac{-(t-2)+4}{t^2+9}\,dt = -\int \frac{t}{t^2+9}\,dt + \int \frac{6}{t^2+9}\,dt = -\frac{1}{2}\ln(t^2+9) + \frac{6}{3}\tan^{-1}\frac{t}{3} + C.$$

The final answer is

$$\ln|x-1| - \frac{1}{2}\ln\left((x+2)^2+9\right) + 2\tan^{-1}\frac{x+2}{3} + C.$$

3. Evaluate $\int e^x \sin x \, dx$.

Call the integral I and integrate by parts. Choose $u = e^x$. Then

$$\begin{array}{ll} u = e^x & dv = \sin x \, dx \\ du = e^x \, dx & v = -\cos x \end{array}$$

Therefore

$$I = -e^x \cos x + \int e^x \cos x \, dx.$$

Integrate by parts again. Choose $U = e^x$. Then

$$\begin{array}{ll} U = e^x & dV = \cos x \, dx \\ dU = e^x \, dx & V = \sin x \end{array}$$

Then

$$I = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - I.$$

Solving for I by algebra, we find

$$I = -\frac{1}{2} e^x \cos x + \frac{1}{2} e^x \sin x + C.$$

4. Evaluate $\displaystyle\int \frac{\cos x\,dx}{(\sin^2 x + 4)^{5/2}}$.

Call the integral I and let $t = \sin x$, so $dt = \cos x\,dx$. Then

$$I = \int \frac{dt}{(t^2+4)^{\frac{5}{2}}}.$$

Now use trigonometric substitution: let $t = 2\tan\theta$, so $dt = 2\sec^2\theta$ and $(t^2+4)^{\frac{5}{2}} = 32\sec^5\theta$. Hence

$$\begin{aligned} I &= \frac{1}{16}\int \frac{\sec^2\theta}{\sec^5\theta}\,d\theta = \frac{1}{16}\int \cos^3\theta\,d\theta = \frac{1}{16}\int (1-\sin^2\theta)\cos\theta\,d\theta \\ &= \frac{1}{16}\int \cos\theta\,d\theta - \frac{1}{16}\int \sin^2\theta\cos\theta\,d\theta = \frac{\sin\theta}{16} - \frac{\sin^3\theta}{48} + C. \end{aligned}$$

Draw a right triangle with angle θ, label the opposite side t and the adjacent side 2, so $\tan\theta = \frac{t}{2}$. Then the hypotenuse is $\sqrt{t^2+4}$, so $\sin\theta = \frac{t}{\sqrt{t^2+4}}$, and

$$I = \frac{1}{16}\frac{t}{\sqrt{t^2+4}} - \frac{1}{48}\left(\frac{t}{\sqrt{t^2+4}}\right)^3 + C = \frac{1}{16}\frac{\sin x}{\sqrt{\sin^2 x + 4}} - \frac{1}{48}\left(\frac{\sin x}{\sqrt{\sin^2 x+4}}\right)^3 + C.$$

5. Does the following integral converge or diverge? Give your reasons fully and clearly. If the integral converges, find its value.

$$\int_{\pi/4}^{\pi/2} \frac{\sec^2\theta}{\tan^2\theta - 1}\, d\theta$$

Call the integral I and let $t = \tan\theta$. Then $dt = \sec^2\theta\, d\theta$. When $\theta \to \pi/2$, $t \to \infty$ and when $\theta = \pi/4$, $t = 1$. Therefore

$$I = \int_1^\infty \frac{dt}{t^2 - 1}\, dt.$$

There are two bad points: 1 and ∞, so write $I = I_1 + I_2$ where

$$I_1 = \int_1^2 \frac{dt}{t^2 - 1}$$

$$I_2 = \int_2^\infty \frac{dt}{t^2 - 1}$$

Now $t^2 - 1 = (t+1)(t-1)$ so $\dfrac{1}{t^2-1} \sim \dfrac{1}{2(t-1)}$ as $t \to 1$ (since $t + 1 \to 2$ as $t \to 1$). Let $y = t - 1$, so $dt = dy$. When $t = 2$, $y = 1$ and when $t = 1$, $y = 0$. Thus

$$I_1 = \frac{1}{2}\int_0^1 \frac{dy}{y},$$

which diverges by the p-test at 0 ($p = 1 \geq 1$). Therefore I_1 diverges, and so I diverges (even though I_2 converges).

6. Does the following series converge or diverge? Give your reasons fully and clearly. If the series converges, find its value.

$$\sum_{n=1}^{\infty} e^{-2\pi n}$$

This is a geometric series with ratio $e^{-2\pi}$, and $|e^{-2\pi}| < 1$, so the series converges. It begins with $n = 1$ rather than $n = 0$, so its value is

$$\frac{e^{-2\pi}}{1 - e^{-2\pi}}.$$

7. Does the following integral converge or diverge? Give your reasons fully and clearly.

$$\int_1^\infty \frac{dx}{e^{-x} + \sqrt{x - 1}}$$

Call the integral I. The only bad point is ∞ and

$$\frac{1}{e^{-x} + \sqrt{x - 1}} \sim \frac{1}{\sqrt{x}}$$

as $x \to \infty$. But

$$\int_1^\infty \frac{dx}{\sqrt{x}}$$

diverges by the p-test at ∞ ($p = 1/2 \leq 1$), so I diverges by limit comparison.

8. Does the following series converge or diverge? Give your reasons fully and clearly.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Use the integral test. Let $I = \int_2^{\infty} \frac{dx}{x \ln x}$. The integrand is continuous and decreasing for $2 \leq x < \infty$, so the series and the integral both converge or both diverge.

Let $t = \ln x$, so $dt = \frac{dx}{x}$. When $x \to \infty$, $t \to \infty$ and when $x = 2$, $t = \ln 2$. Hence

$$I = \int_{\ln 2}^{\infty} \frac{dx}{x},$$

which diverges by the p-test at ∞ ($p = 1 \leq 1$). Therefore the series diverges by the integral test.