Quiz 4 MAT104 Spring2003 (60 minutes)

1. *(12 points)* For each of the series below, write **AC** if the series converges absolutely, **CC** if the series converges conditionally and **D** if the series diverges. *Please circle your answer* and briefly justify it — that is, tell which tests you are using to back up your conclusions. (You do not need to work out all the details explicitly.)

   (a) $\sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi)}{n}$ Ans: **D**

   Since $\cos(n\pi) = (-1)^n$ the numerator becomes $(-1)^{2n} = 1$ so we have the harmonic series, which diverges by the p-test with $p = 1$.

   (b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(\ln 2)^n}$ Ans: **D**

   This series is geometric with $r = -1/\ln 2$. Since $2 < e$ we know that $\ln 2 < 1$. So $|r| > 1$ and the series diverges.

   (c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n)!}{5^n \cdot n! \cdot n!}$ Ans: **AC** by the absolute ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2n+2)!}{5^{n+1}(n+1)!(n+1)!} \cdot \frac{5^n n! n!}{(2n)!} \sim \frac{4n^2}{5n^2} = \frac{4}{5} < 1 \text{ as } n \to \infty$$

   (d) $\sum_{n=1}^{\infty} (-1)^{n+1}(\sqrt[n]{e} - 1)$ Ans: **CC** by the Alternating Series Test.

   Since $e > 1$ we know that $\sqrt[n]{e} > \sqrt[n]{1} = 1$. So we see that this is an alternating series. Next question we would ask if whether it passes the nth term test. Here we use the Taylor expansion. Since

$$\sqrt[n]{e} = e^{1/n} = 1 + \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3!n^3} + \cdots$$

   then as n goes to infinity all the higher degree terms will die out faster than $1/n$. So $\sqrt[n]{e} - 1$ is asymptotic to $1/n$ as n goes to infinity.

   At this point, by the limit comparison test we can conclude that the series will not be absolutely convergent. The nth term in absolute value is asymptotic to $1/n$ which gives a divergent series.

   What about conditional convergence? We need to check the conditions of the alternating series test. We already know that the given series alternates and that the nth term goes to 0 in absolute value. We still must check that

$$\sqrt[n+1]{e} - 1 < \sqrt[n]{e} - 1 \text{ or in other words, that } e^{1/n+1} < e^{1/n}$$

   but this is clear since $1/(n+1) < 1/n$ and the exponential function is always increasing so it preserves inequalities.

2. *(10 points)* Find the second-order Taylor polynomial of the function $f(x) = \sqrt[3]{x}$ near $x = 8$. Compute the approximate value of $\sqrt[3]{10}$ given by this polynomial. (Leave your answer in fraction form, but simplify it.)

Start by computing derivatives. $f(x) = x^{1/3}$, $f'(x) = (1/3)x^{-2/3}$, $f''(x) = (-2/9)x^{-5/3}$. Evaluating at $x = 8$ we get $f(8) = 2$, $f'(8) = (1/3)(1/2^2) = 1/12$ and $f''(8) = (-2/9)(1/2^5) = -1/144$. So

$$P_2(x, 8) = 2 + \frac{x-8}{12} - \frac{(x-8)^2}{2 \cdot 144} = 2 + \frac{x-8}{12} - \frac{(x-8)^2}{288}$$

Our approximation will be

$$P_2(10, 8) = 2 + \frac{2}{12} - \frac{4}{288} = 2 + \frac{1}{6} - \frac{1}{72} = 2\frac{11}{72}$$

3. *(8 points)* For which values of x does the power series $\sum_{n=2}^{\infty} \frac{(x-4)^n}{n \ln^2 n}$ converge? Justify your answer.

Here we use the absolute ratio test of course.

$$\frac{|x-4|^{n+1}}{(n+1)\ln^2(n+1)} \cdot \frac{n \ln^2 n}{|x-4|^n} \to |x-4| \text{ as } n \to \infty$$

(Use L'Hôpital's Rule to compute the limits.) So the power series is absolutely convergent on the interval $(3, 5)$ and it is divergent outside this interval, except possibly at the endpoints. These must be checked separately:

For $x = 5$: The series becomes $\sum_{2}^{\infty} \frac{1}{n \ln^2 n}$. This will converge by the integral test. For $x = 3$ we simply get the alternating version of this $\sum_{2}^{\infty} \frac{(-1)^n}{n \ln^2 n}$ and our work at $x = 5$ shows that this series is absolutely convergent.

So the power series is absolutely convergent on $[3, 5]$ and divergent everywhere else.

4. *(10 points)*

(a) State (or compute) the Taylor series centered at 0 of $f(x) = \ln(1 + x)$.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots + \frac{(-1)^{n-1}x^n}{n} + \cdots$$

(b) State (or compute) the Taylor series centered at 0 of $g(x) = \frac{1}{1+x^2}$.

Viewing this as the sum of a geometric series with $r = -x^2$, we have

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots + (-1)^n x^{2n} + \cdots$$

(c) Compute the first four nonzero terms of the Taylor series centered at 0 of $h(x) = \dfrac{\ln(1+x)}{1+x^2}$.

$$(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots)(1 - x^2 + x^4 - x^6 + \cdots)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} \cdots \quad \text{(Multiplying the ln series by 1)}$$

$$-x^3 + \frac{x^4}{2} - \frac{x^5}{3} + \frac{x^6}{4} - \frac{x^7}{5} + \frac{x^8}{6} - \cdots \quad \text{(Multiplying the ln series by } -x^2\text{)}$$

$$+x^5 - \frac{x^6}{2} + \frac{x^7}{3} - \frac{x^8}{4} + \frac{x^9}{5} - \frac{x^{10}}{6} + \cdots \quad \text{(Multiplying the ln series by } x^4\text{)}$$

$\cdots$

$$= x - \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} + \cdots \quad \text{(gathering like terms)}$$

5. *(10 points)* Compute the following limits:

(a) $\displaystyle\lim_{x\to 0} \frac{\sin 3x(1-\cos 2x)}{e^{x^3}-1} = 6$

$$\begin{aligned} \sin 3x &= (3x) - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \cdots \\ \cos 2x &= 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots \\ e^{x^3} &= 1 + x^3 + \frac{x^6}{2!} + \cdots \end{aligned}$$

As x goes to zero, the higher powers of x die out much more quickly. So the lowest power of x dominates. Thus $\sin 3x \sim 3x$, $1 - \cos 2x \sim 4x^2/2 = 2x^2$ and $e^{x^3} - 1 \sim x^3$ as x goes to zero. Thus

$$\frac{\sin 3x(1-\cos 2x)}{e^{x^3}-1} \sim \frac{(3x)(2x^2)}{x^3} = 6 \quad \text{as } x \to 0$$

(b) $\displaystyle\lim_{x\to\infty} x^2\left(e^{-1/x^2} - 1\right) = -1.$

The Taylor series for e^u converges to e^u for every real number u. So in particular if $u = -1/x^2$ then

$$e^{-1/x^2} = 1 - \frac{1}{x^2} + \frac{1}{2x^4} - \frac{1}{3!x^6} + \cdots$$

Thus

$$x^2(e^{-1/x^2} - 1) = x^2(-\frac{1}{x^2} + \frac{1}{2x^4} - \frac{1}{6x^6} + \cdots) = -1 + \frac{1}{2x^2} - \frac{1}{6x^4} + \cdots$$

As x goes to ∞ all the terms except the first die out, so the limit is -1.