Euler’s Formula

Where does Euler’s formula
e^{i\theta} = \cos \theta + i \sin \theta
come from? How do we even define, for example, \( e^{i\theta} \)? We can’t multiple \( e \) by itself the square root of minus one times.

The answer is to use the Taylor series for the exponential function. For any complex number \( z \) we define \( e^z \) by

\[
e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.
\]

Since \( |z^n| = |z|^n \), this series converges absolutely: \( \sum_{n=0}^{\infty} \frac{|z|^n}{n!} \) is a real series that we already know converges.

If we multiply the series for \( e^z \) term-by-term with the series for \( e^w \), collect terms of the same total degree, and use a certain famous theorem of algebra, we find that the law of exponents

\[
e^{z+w} = e^z \cdot e^w
\]
continues to hold for complex numbers.

Now for Euler’s formula:

\[
e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}
\]

\[
= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \cdots
\]

\[
= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)
\]

\[
= \cos \theta + i \sin \theta.
\]

The special case \( \theta = 2\pi \) gives

\[
e^{2\pi i} = 1.
\]

This celebrated formula links together three numbers of totally different origins: \( e \) comes from analysis, \( \pi \) from geometry, and \( i \) from algebra.

Here is just one application of Euler’s formula. The addition formulas for \( \cos(\alpha + \beta) \) and \( \sin(\alpha + \beta) \) are somewhat hard to remember, and their geometric proofs usually leave something to be desired. But it is impossible to forget that
\[ e^{i(\alpha + \beta)} = e^{i\alpha} \cdot e^{i\beta}. \]

Now use Euler’s formula thrice:

\[
\cos(\alpha + \beta) + i \sin(\alpha + \beta) = [\cos \alpha + i \sin \alpha] \cdot [\cos \beta + i \sin \beta]
= (\cos \alpha \cos \beta - \sin \alpha \sin \beta) + i(\cos \alpha \sin \beta + \sin \alpha \sin \beta).
\]

Equate the real and imaginary parts and presto! we have

\[
\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta
\sin(\alpha + \beta) = \cos \alpha \sin \beta + \sin \alpha \cos \beta.
\]