Name *Gotfried W. Leibniz*

1. Evaluate $\displaystyle\int \frac{5\,dx}{x^3+2x^2+5x}$.

Factor: $x^3+2x^2+5x = x(x^2+2x+5)$. The second factor is irreducible.

Set up partial fractions:

$$\frac{5}{x^3+2x^2+5x} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+5},$$

$$5 = A(x^2+2x+5) + Bx^2 + Cx$$

$$= (A+B)x^2 + (2A+C)x + 5A, \text{ so}$$

$$\begin{cases} 0 = A+B \\ 0 = 2A+C \\ 5 = 5A \end{cases}$$

$A=1$, $B=-1$, $C=-2$ and

$$\int \frac{5\,dx}{x^3+2x^2+5x} = \int \frac{dx}{x} - \int \frac{x+2}{x^2+2x+5}\,dx$$

$$= \int \frac{dx}{x} - \int \frac{x+2}{(x+1)^2+4}\,dx$$

Let $y=x+1$, so $dy=dx$ and $x=y-1$, so $\displaystyle\int \frac{x+2}{(x+1)^2+4}\,dx = \int \frac{y+1}{y^2+4}\,dy$ $\displaystyle= \frac{1}{2}\int \frac{2y}{y^2+4}\,dy + \int \frac{dy}{y^2+4} = \frac{1}{2}\ln(y^2+4) + \frac{1}{2}\tan^{-1}\frac{y}{2} + C$. The answer is $\displaystyle \ln|x| - \frac{1}{2}\ln\left((x+1)^2+4\right) \;\; \frac{1}{2}\tan^{-1}\frac{x+1}{2} + C.$

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2. For each of the following integrals, state whether it converges or diverges, and give your reasons carefully and clearly.

a. $\int_{-\infty}^{\infty} \cos 2t\, dt.$ There are two bad points, ∞ and $-\infty$, and for convergence both $\int_0^\infty \cos 2t\, dt$ and $\int_{-\infty}^0 \cos 2t\, dt$ would have to converge. But

$$\int_0^\infty \cos 2t\, dt = \lim_{b\to\infty} \int_0^b \cos 2t\, dt = \lim_{b\to\infty} \tfrac{1}{2}\sin 2t\Big|_0^b$$

$= \lim_{b\to\infty} \frac{1}{2}\sin 2b$ does not exist. The integral diverges.

b. $\int_1^\infty \frac{x^3\, dx}{1+x^4}.$ As $x \to \infty$, $\frac{x^3}{1+x^4} \sim \frac{x^3}{x^4} = \frac{1}{x}$. By the p-test, $\int_1^\infty \frac{dx}{x}$ diverges, so $\int_1^\infty \frac{x^3\, dx}{1+x^4}$ diverges by limit comparison.

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3. For each of the following series, state whether it converges or diverges, and give your reasons carefully and clearly.

a. $\sum_{n=1}^{\infty} e^{-n \ln n}$. Apply the root test:

$$\left(e^{-n \ln n}\right)^{\frac{1}{n}} = e^{-\ln n} = \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Since $\rho = 0 < 1$, the series converges.

b. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{1+\frac{1}{n}}$. The series diverges by the n'th term test, since the sequence $(-1)^n \frac{1}{1+\frac{1}{n}}$ does not tend to 0 as $n \to \infty$.

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4. Find the Taylor series, centered at -1, of $f(x) = \dfrac{1}{x}$.

$$\frac{1}{x} = \frac{1}{(x+1)-1} = -\frac{1}{1-(x+1)}.$$

But this is minus the sum of a geometric series with ratio $x+1$, so

$$\frac{1}{x} = -\sum_{n=0}^{\infty} (x+1)^n$$

which converges for $-2 < x < 0$.

Alternatively, make a table:

n	$f^{(n)}(x)$	$f^{(n)}(a)$	$f^{(n)}(a)/n!$
0	x^{-1}	-1	-1
1	$-x^{-2}$	-1	-1
2	$2x^{-3}$	-2	-1
3	$-3!x^{-4}$	$-3!$	-1
4	$4!x^{-5}$	$-4!$	-1
5	$-5!x^{-6}$	$-5!$	-1

so again the Taylor series is $-\sum_{n=0}^{\infty} (x+1)^n$.

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5. Estimate $\int_0^{1/2} e^{-x^3}\,dx$ with an error no bigger than $1/100$. Give your reasons.

Since $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots$ for all u,

we have $e^{-x^3} = 1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \cdots$ for all x.

For $0 \le x \le \frac{1}{2}$, this is an alternating decreasing series, so the error $R(x)$ is less than the first omitted term. Therefore

$$e^{-x^3} = 1 - x^3 + R(x) \quad \text{with } |R(x)| \le \frac{x^6}{2!}.$$

Hence $\int_0^{\frac{1}{2}} e^{-x^3}\,dx = \int_0^{\frac{1}{2}} (1-x^3)\,dx + \int_0^{\frac{1}{2}} R(x)\,dx$

$= \left(x - \frac{x^4}{4}\right)\Big|_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} R(x)\,dx$

$= \frac{1}{2} - \frac{1}{64} + \int_0^{\frac{1}{2}} R(x)\,dx$

$= \frac{31}{64} + \int_0^{\frac{1}{2}} R(x)\,dx$. The error $\int_0^{\frac{1}{2}} R(x)\,dx \le \int_0^{\frac{1}{2}} \frac{x^6}{2}\,dx$

$= \frac{x^7}{14}\Big|_0^{\frac{1}{2}} = \frac{1}{2^7 \cdot 14}$, which is way less than $\frac{1}{100}$.

Thus $\int_0^{\frac{1}{2}} e^{-x^3}\,dx \approx \frac{31}{64}$ with an error less than $\frac{1}{100}$.

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6. Find $\lim_{x\to 0} \frac{(x-\sin x)^2}{x^6}$.

$\sin x = x - \frac{x^3}{3!} + \cdots$,

so $x - \sin x = \frac{x^3}{3!} - \cdots$ and

$x - \sin x \sim \frac{x^3}{3!}$ as $x \to 0$. Hence

$$\lim_{x\to 0} \frac{(x-\sin x)^2}{x^6} = \lim_{x\to 0} \frac{\left(\frac{x^3}{3!}\right)^2}{x^6} = \frac{1}{(3!)^2} = \frac{1}{36}.$$

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7. Find the area between the origin and the curve given in polar coordinates by $r = \theta e^{\theta}$ for $0 \leq \theta \leq \pi$.

$$A = \int_0^{\pi} \frac{1}{2} r^2 d\theta = \int_0^{\pi} \frac{1}{2} (\theta e^{\theta})^2 d\theta = \frac{1}{2} \int_0^{\pi} \theta^2 e^{2\theta} d\theta.$$

Let $I = \int \theta^2 e^{2\theta} d\theta$,

$u = \theta^2$, $dv = e^{2\theta} d\theta$

$du = 2\theta d\theta$, $v = \frac{1}{2} e^{2\theta}$, so

$I = \frac{1}{2}\theta^2 e^{2\theta} - \int \theta e^{2\theta} d\theta.$

Let $J = \int \theta e^{2\theta} d\theta$,

$U = \theta$, $dV = e^{2\theta} d\theta$

$dU = d\theta$, $V = \frac{1}{2} e^{2\theta}$, so

$J = \frac{1}{2}\theta e^{2\theta} - \frac{1}{2}\int e^{2\theta} d\theta = \frac{1}{2}\theta e^{2\theta} - \frac{1}{4} e^{2\theta} + C.$

Therefore

$$A = \frac{1}{2}\left(\frac{1}{2}\theta^2 e^{2\theta} - \left[\frac{1}{2}\theta e^{2\theta} - \frac{1}{4} e^{2\theta}\right]\right)\Big|_0^{\pi}$$

$$= \left(\frac{1}{4}\theta^2 e^{2\theta} - \frac{1}{4}\theta e^{2\theta} + \frac{1}{8} e^{2\theta}\right)\Big|_0^{\pi}$$

$$= \frac{1}{4}\pi^2 e^{2\pi} - \frac{1}{4}\pi e^{2\pi} + \frac{1}{8} e^{2\pi} - \frac{1}{8}.$$

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8. Find all roots of $x^6 - 3x^3 + 9 = 0$ in polar form: $x = re^{i\theta}$.

This is a quadratic equation for x^3, so first find x^3:

$$x^3 = \frac{3 \pm \sqrt{9-36}}{2} = 3\left(\frac{1}{2} \pm \frac{\sqrt{1-4}}{2}\right)$$

$$= 3\left(\frac{1}{2} \pm \frac{\sqrt{3}\,i}{2}\right)$$

$$= 3e^{i\frac{\pi}{3}},\ 3e^{-i\frac{\pi}{3}}.$$

Now take the cube roots of these two numbers. The obvious cube roots are $3^{\frac{1}{3}}e^{i\frac{\pi}{9}}$, $3^{\frac{1}{3}}e^{-i\frac{\pi}{9}}$. Multiply each of these by the three cube roots of 1, namely 1, $e^{i\frac{2\pi}{3}}$, $e^{i\frac{4\pi}{3}}$ (or 1, $e^{i\frac{6\pi}{9}}$, $e^{i\frac{12\pi}{9}}$):

$$3^{\frac{1}{3}}e^{i\frac{\pi}{9}},\quad 3^{\frac{1}{3}}e^{-i\frac{\pi}{9}}$$

$$3^{\frac{1}{3}}e^{i\frac{7\pi}{9}},\quad 3^{\frac{1}{3}}e^{i\frac{5\pi}{9}}$$

$$3^{\frac{1}{3}}e^{i\frac{13\pi}{9}},\quad 3^{\frac{1}{3}}e^{i\frac{11\pi}{9}}.$$

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9. Consider the region under the curve $y = e^{-x}$ and above the x-axis for $0 \leq x < \infty$.

a. Revolve it around the x-axis and find the volume.

$$V = \int_0^{\infty} \underbrace{\pi (e^{-x})^2}_{\text{cross-section area}} dx = \pi \int_0^{\infty} e^{-2x} dx$$

$$= \pi \lim_{b \to \infty} \int_0^b e^{-2x} dx = \pi \lim_{b \to \infty} \left(-\tfrac{1}{2} e^{-2x} \right) \Big|_0^b$$

$$= \pi \lim_{b \to \infty} \left(-\tfrac{1}{2} e^{-2b} + \tfrac{1}{2} \right) = \frac{\pi}{2}.$$

b. Revolve it around the y-axis and find the volume.

$$V = \int_0^{\infty} \underbrace{2\pi x}_{\text{circumference}} \underbrace{e^{-x}}_{\text{height}} dx = 2\pi \lim_{b \to \infty} \int_0^b x e^{-x} dx.$$

To find $\int x e^{-x} dx$, let

$u = x$, $dv = e^{-x} dx$

$du = dx$, $v = -e^{-x}$, so

$\int x e^{-x} dx = -x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x} + C.$ Hence

$$V = 2\pi \lim_{b \to \infty} \left(-x e^{-x} - e^{-x} \right) \Big|_0^b$$

$$= 2\pi \lim_{b \to \infty} \left(-b e^{-b} - e^{-b} + 1 \right) = 2\pi.$$

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10. Find the arc length of the curve given by $y = x^2$ for $0 \le x \le \sqrt{2}$. (You may find the formula

$$\int \sec^3\theta \, d\theta = \frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\int \sec\theta \, d\theta$$

useful.)

$$L = \int_0^{\sqrt{2}} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{\sqrt{2}} \sqrt{1+4x^2}\, dx.$$

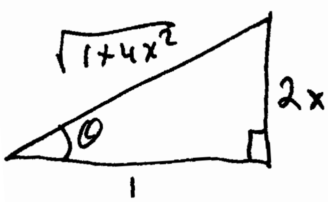
Let $2x = \tan\theta$, so $1+4x^2 = 1+\tan^2\theta = \sec^2\theta$, and

$dx = \frac{1}{2}\sec^2\theta \, d\theta$. Then

$$\int \sqrt{1+4x^2}\,dx = \frac{1}{2}\int \sec\theta \cdot \sec^2\theta \, d\theta$$

$$= \frac{1}{2}\left(\frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\int \sec\theta \, d\theta\right)$$

$$= \frac{1}{4}\sec\theta\tan\theta + \frac{1}{4}\ln|\sec\theta + \tan\theta| + C.$$



$$= \frac{1}{4}\sqrt{1+4x^2}\cdot 2x + \frac{1}{4}\ln\left|\sqrt{1+4x^2} + 2x\right| + C.$$ Hence

$$L = \left(\frac{1}{4}\sqrt{1+4x^2}\cdot 2x + \frac{1}{4}\ln\left|\sqrt{1+4x^2} + 2x\right|\right)\Bigg|_0^{\sqrt{2}}$$

$$= \frac{1}{4}\sqrt{9}\cdot 2\sqrt{2} + \frac{1}{4}\ln\left|\sqrt{9} + 2\sqrt{2}\right|$$

$$= \frac{3}{\sqrt{2}} + \frac{1}{4}\ln(3 + 2\sqrt{2}).$$

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11. The mass m of a crystal in a solution grows at a rate *proportional* to $m^{2/3}$. The original mass is 1 gram and the mass after 24 hours is 8 grams. Find the exact value of the mass as a function of time.

We are given

① $\frac{dm}{dt} = k m^{2/3}$, k a constant

② $m(0)=1$

③ $m(24)=8$.

Separate variables in ① and integrate:

$m^{-\frac{2}{3}} dm = k\,dt$

$3m^{\frac{1}{3}} = kt + C$

$m^{\frac{1}{3}} = \frac{1}{3}(kt+C)$

$m = \frac{1}{27}(kt+C)^3$.

Use ②:

$1 = \frac{1}{27}C^3$, $C = 3$, so

$m = \frac{1}{27}(kt+3)^3$.

Use ③:

$8 = \frac{1}{27}(k24+3)^3$, $2 = \frac{1}{3}(k24+3)$, $k = \frac{2\cdot 3 - 3}{24} = \frac{1}{8}$.

$m = \frac{1}{27}\left(\frac{1}{8}t+3\right)^3$ or $m = \left(\frac{1}{24}t+1\right)^3$.