Overview of Sequences and Series

MAT 104 – Frank Swenton, Summer 2000

Sequences are ordered lists of real numbers, such as "$a_1, a_2, a_3, \ldots$", sometimes written $\{a_n\}$

- Just as with limits of functions on the real line, we can talk about limits of sequences; if the numbers in the sequence approach some real number L, then the sequence has a limit. Otherwise, it may approach ∞ or $-\infty$, or it may not approach anything at all in particular.
- Note that *if* $\lim_{x\to\infty} f(x)$ either exists or is infinite, then the limit $\lim_{n\to\infty} f(n)$ is the same (the graph of the sequence lies on the graph of the function, so if the function approaches a limit, the sequence is stuck to the same limit). This fact can be useful in computing limits of some sequences, since limits of functions can often be more easily evaluated by l'Hôpital's rule. For example:

  Using l'Hôpital's rule, $\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$, so $\lim_{n\to\infty} \frac{n^2}{e^n} = 0$ as well.
- Some useful limits to know:

  If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$

  $\lim_{n\to\infty} \sqrt[n]{n} = 1$ (and the same for $\sqrt[n]{n^2}$, $\sqrt[n]{n^3}$, etc.)

  $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$

  Some limits that equal e: $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$, $\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$

  Some limits that equal $\frac{1}{e}$: $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n$, $\lim_{n\to\infty} \left(\frac{n-1}{n}\right)^n$, $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$
- Remember the hierarchy of functions (on the "limit comparison test" page);

  $$\text{If } f(n) \ll g(n), \text{ then } \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0, \text{ and } \lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty.$$

  For example, $\lim_{n\to\infty} \frac{100^n}{n!} = 0$, $\lim_{n\to\infty} \frac{n^{100}}{2^n} = 0$, etc.
- Remember the rules of exponentiation: $a^{m+n} = a^m \cdot a^n$, $a^{-n} = \frac{1}{a^n}$, and $a^{mn} = (a^m)^n$

Series (sums of infinitely many terms)

- Given a sequence $a_1, a_2, a_3, \ldots$, we may want to add up all of the values, i.e. $a_1 + a_2 + a_3 + \cdots$. This is called a **series**, and it is denoted $\sum_{n=1}^{\infty} a_n$. The individual numbers being added together are called the **terms** of the series. To add up an infinite number of terms, we first define the **partial sums** of the series, $S_n = a_1 + a_2 + \cdots + a_n$, i.e. S_n is the sum of the first n terms of the series. We then define the meaning of the infinite sum by $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$. If this limit exists as a real number, we call the series **convergent**; if the limit doesn't exist, we call the series **divergent**.
- The term **positive series** refers to a series all of whose *terms* are positive.
- Properties of series:
  - If $\sum_{n=1}^{\infty} a_n = A$, then for any constant c, we have $\sum_{n=1}^{\infty} (c \cdot a_n) = c \cdot A$
  - If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then we have $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$

Geometric Series

- A series of nonzero terms is called a **geometric series** if the ratio between successive terms in the series is constant. Thus, given a series $\sum_{n=1}^{\infty} a_n$, compute $\frac{a_{n+1}}{a_n}$; if this number is independent of n, then the series is a geometric series and this value is its **ratio** r. The **initial term** a of the series is the first term of the series (substitute the bottom value in for n). The initial term a and ratio r of a geometric series determine its properties.
  - if $|r| \geq 1$, then the series diverges
  - if $|r| < 1$, then the series converges, and its sum is given by $\frac{a}{1-r}$

Determining convergence or divergence of complicated series

n$^{\text{th}}$ term test for divergence: If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ *diverges*

- Note that this is *only* a test for *divergence*–if the limit is zero you can't conclude anything, so you may *never* use this test to show that a series converges. It is, nonetheless, still a useful test; if you see a series and aren't sure that the individual terms of the series go to zero as $n \to \infty$, then try this test and you may be able to conclude very quickly that the series diverges.

Integral test: If $f(x)$ is *continuous*, *positive*, and *decreasing* for $x \geq 1$,

a. If $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} f(n)$ converges, as well.

b. If $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} f(n)$ diverges, as well.

- Note that the number "1" appearing throughout the test could be any other number and the test would still apply.
- The two most common uses for the integral test are:
  - For **p-series**: $\sum_{n=1}^{\infty} \frac{1}{n^p}$: this series *converges* if $p > 1$, but it *diverges* if $p \leq 1$ (worth remembering)
  - For series that look like functions that are easily integrable; often "$\ln n$" appears.

Comparison tests (only for *positive* series!)

These tests are used compare a series with complicated terms to one whose terms are simpler; they're usually used to deal with complicating **factors**–often *oscillating terms*, such as $\sin n$ or $\cos n^2$.

- **Comp. test for convergence**: If $0 \leq p_n \leq c_n$ and $\sum c_n$ is convergent, then $\sum p_n$ is convergent.
- **Comp. test for divergence**: If $0 \leq d_n \leq p_n$ and $\sum d_n$ is divergent, then $\sum p_n$ is divergent.
- Remember that for these tests to apply, you need either a *larger* series that *converges*, or else a *smaller* series that *diverges*.
- Examples:

$$\sum_{n=3}^{\infty} \frac{1}{n^2 \ln n} \text{ has } 0 \leq \frac{1}{n^2 \ln n} \leq \frac{1}{n^2}, \text{ so since } \sum_{n=3}^{\infty} \frac{1}{n^2} \text{ converges, it does too.}$$

$$\sum_{n=3}^{\infty} \frac{\ln n}{n} \text{ has } 0 \leq \frac{1}{n} \leq \frac{\ln n}{n}, \text{ so since } \sum_{n=3}^{\infty} \frac{1}{n} \text{ diverges, it does too.}$$

Limit comparison tests (only for *positive* series!)

The limit comparison tests are used compare a series with complicated terms to one whose terms are simpler; they're usually used to deal with complicating **sums** appearing in the terms.

- Essentially, when you see a sum within the terms of a series, you want to try replacing it by the one piece that dominates the sum (a sum will act like its biggiest piece). Thus, an important skill to develop is that of recognizing which part of a sum will dominate–the *heirarchy of functions* listed below should help to guide you in this. We'll use the notation "$\sim$" to denote the idea of "acting like"; you may want to *think* of things in this way, but be sure to carefully apply the limit-comparison test once you've chosen the series you'd like to limit-compare with.
- **Limit comparison test for convergence**: If $\sum p_n$ is a positive series, and if $\sum c_n$ is a *convergent* positive series such that $\lim_{n\to\infty} \frac{p_n}{c_n}$ *exists*, then $\sum p_n$ is convergent, as well.
- **Limit comparison test for divergence**: If $\sum p_n$ is a positive series, and if $\sum d_n$ is a *divergent* positive series such that $\lim_{n\to\infty} \frac{p_n}{d_n}$ either (a) exists and is *not zero* or (b) is infinite, then $\sum p_n$ is divergent, as well.
- Note: *If* you find a limit-comparison series where the above limit is **nonzero and finite**, then the test works *both ways*, and you basically can't lose–this is the ideal case that you're shooting for. But do be aware that "0" is still ok for convergence and "∞" is still ok for divergence, should they occur.
- Note also that using this "dominant term" idea and the limit comparison test, you can compare *any* series whose terms only involve fixed powers of n to a *p-series*, and convergence or divergence is then simple to determine.
- Example 1: Since $\frac{n^2+2^n}{n^4+3^n} \sim \frac{2^n}{3^n}$ and $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$ is a convergent geometric series, we can use the limit comparison test for convergence to show that the series $\sum_{n=1}^{\infty} \frac{n^2+2^n}{n^4+3^n}$ is a convergent series (check the hypotheses!!!).
- Example 2: Since $\frac{\frac{3}{n}+5+\sqrt{n}}{n+2^{-n}} \sim \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series, we can use the limit comparison test for divergence to show that the series $\sum_{n=1}^{\infty} \frac{\frac{3}{n}+5+\sqrt{n}}{n+2^{-n}}$ is a divergent series (again, check the hypotheses!!!).

A **hierarchy of functions** (in terms of "size" at infinity) is shown below, and should guide you in how to indentify the dominant parts of a series' terms:

$$[0] \ll \left[\frac{1}{n^n}\right] \ll \left[\frac{1}{n!}\right] \ll \begin{bmatrix}\text{small exponentials}\\ \ldots 10^{-n} \ldots e^{-n} \ldots\end{bmatrix} \ll \begin{bmatrix}\text{small } \textit{fixed} \text{ powers}\\ \ldots \frac{1}{n^2} \ldots \frac{1}{\sqrt{n}} \ldots\end{bmatrix} \ll \left[\frac{1}{\ln n}\right] \ll \quad \text{(go to 0)}$$

$$\begin{bmatrix}\text{positive, bounded}\\ \ldots 0.1 \ldots 1 \ldots \tan^{-1} n \ldots 2 \ldots\end{bmatrix} \quad \text{(go to a positive real number)}$$

$$\ll [\ln n] \ll \begin{bmatrix}\text{big } \textit{fixed} \text{ powers}\\ \ldots \sqrt{n} \ldots n^2 \ldots\end{bmatrix} \ll \begin{bmatrix}\text{big exponentials}\\ \ldots e^n \ldots 10^n \ldots\end{bmatrix} \ll [n!] \ll [n^n] \ll [\infty] \qquad \text{(go to } \infty)$$

This hierarchy has a useful property:

$$\text{If } f(n) \ll g(n), \text{ then } \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0, \text{ and } \lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty.$$

For example, $\lim_{n\to\infty} \frac{n!}{n^n} = 0$, $\lim_{n\to\infty} \frac{n^{100}}{2^n} = 0$, etc.

Absolute Ratio Test (for a series $\sum a_n$)

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, the series *converges absolutely.*
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ (or is ∞), the series *diverges.*
- Important note: If the limit is *equal* to 1 or *does not exist*, then no conclusion can be drawn! In this case, try another test.
- Example 1: $\sum_{n=1}^{\infty} \frac{10^n}{n!}$: applying the ratio test, we find

$$\lim_{n\to\infty} \left(\frac{10^{n+1}}{(n+1)!}\right) \Big/ \left(\frac{10^n}{n!}\right) = \lim_{n\to\infty} \frac{10}{n+1} = 0$$

so since $0 < 1$, the series converges.

- Example 2: $\sum_{n=1}^{\infty} n^2 2^{-n}$: applying the ratio test, we find

$$\lim_{n\to\infty} \frac{(n+1)^2 2^{-(n+1)}}{n^2 2^{-n}} = \cdots = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 2^{-1} = (1+0)^2 2^{-1} = \frac{1}{2}$$

so since $\frac{1}{2} < 1$, the series converges.

Absolute Root Test (for a series $\sum a_n$)–very similar to the absolute ratio test

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, the series *converges absolutely.*
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ (or is ∞), the series *diverges.*
- Important note: If the limit is *equal* to 1 or *does not exist*, then no conclusion can be drawn! In this case, try another test.
- Usually, the ratio test is easier to apply than the root test; the notable exception is when the terms of the series are such that it's reasonably easy to take their n-th roots, as in the example below.
- Example: $\sum_{n=1}^{\infty} \left(\frac{4n+2}{3n+\sqrt{n}}\right)^n$: applying the root test, we find

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{4n+2}{3n+\sqrt{n}}\right)^n} = \lim_{n\to\infty} \frac{4n+2}{3n+\sqrt{n}} = \lim_{n\to\infty} \frac{4+\frac{2}{n}}{3+\frac{1}{\sqrt{n}}} = \frac{4+0}{3+0} = \frac{4}{3}$$

so since $\frac{4}{3} > 1$, the series diverges.