Overview of Improper Integrals
MAT 104 – Frank Swenton, Summer 2000

Definitions

- A **proper integral** is a definite integral where the interval is *finite* and the integrand is *defined* and *continuous* at all points in the interval.
  - Proper integrals always converge, that is, always give a finite area
- The **trouble spots** of a definite integral are the points in the interval of integration that make it an improper integral, i.e., keep it from being proper. They are of two types:
  a. Points where the integrand is undefined or discontinuous
  b. ∞ and $-\infty$ are always trouble spots when they appear as limits of integration
- A **simple improper integral** is an improper integral with only one trouble spot, that trouble spot being at an endpoint of the interval. Simple improper integrals are defined to be the appropriate limits of proper integrals, e.g.:
$$\int_0^1 \frac{1}{x}\,dx = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{1}{x}\,dx$$
  - If the limit exists as a real number, then the simple improper integral is called **convergent**.
  - If the limit doesn't exist as a real number, the simple improper integral is called **divergent**.

Dealing with improper integrals

- First step: always locate *all* trouble spots and split the integral into *simple* improper integrals, then deal with the pieces individually.
  - If each and every one of the pieces converges, the original integral converges to their sum
  - If even just one of the pieces diverges, the original integral diverges
- For certain simple improper integrals, it's worthwhile to know offhand whether they converge or diverge (though they aren't difficult to compute directly):
$$\int_1^\infty \frac{1}{x^a}\,dx \text{ \textit{converges} if } a > 1\text{; it \textit{diverges} if } a \le 1$$
$$\int_0^1 \frac{1}{x^a}\,dx \text{ \textit{converges} if } a < 1\text{; it \textit{diverges} if } a \ge 1$$
$$\textstyle\int_0^\infty e^{-x}\,dx \text{ converges}$$

Determining convergence or divergence: If a simple improper integral can be reasonably integrated directly, then this is a fine way to determine whether it's convergent or divergent. If not, there are four primary tools at your disposal for determining whether a simple improper integral converges or diverges (below, all integrals are presumed to be simple improper integrals, but for simplicity the limits have been left off)

- Make sure that you've applied tests properly; *explain* what you've done and why your answer is what it is. Usually, you will *work* backward, starting from the given integral and applying tests. Your final answer, however, should be given starting with the last integral and working up to the one that was initially given. For example, you might say:
$$\int_1^\infty \frac{1}{x^2}\,dx \text{ is convergent, so since } 0 \le \frac{|\sin x|}{x^2} \le \frac{1}{x^2},$$
$$\text{the comparison test for convergence shows that } \int_1^\infty \frac{|\sin x|}{x^2}\,dx \text{ converges,}$$
$$\text{thus by the absolute convergence test, we know that } \int_1^\infty \frac{\sin x}{x^2}\,dx \text{ converges.}$$

Absolute convergence test: If $\int |f(x)|\,dx$ converges, then $\int f(x)\,dx$ converges as well.

Note that this test is only useful for showing *convergence*; it's often used to make the integrand nonnegative so that the comparison test for convergence can be used.

Comparison test for convergence: If $0 \le f \le g$ and $\int g(x)\,dx$ converges, then $\int f(x)\,dx$ converges.

Remember the picture: To apply this test, you need a *larger* function whose integral *converges.*

Comparison test for divergence: If $0 \le f \le g$ and $\int f(x)\,dx$ diverges, then $\int g(x)\,dx$ diverges.

Remember the picture: To apply this test, you need a *smaller* function whose integral *diverges.*

Asymptotic functions (Recall that $f(x) \sim g(x)$ near a in case $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = 1$)

To determine the convergence or divergence of a simple improper integral, the integrand may be replaced by *any other function asymptotic to it near the trouble spot*, and the convergence/divergence will remain the same. (Note that new trouble spots should never be introduced when using asymptotics.)

- Asymptotics **near 0**
  - Anything having a *nonzero* limit as $x \to 0$ is asymptotic to that limit;
    e.g., $\cos x \sim 1$, $e^{-x} \sim 1$
  - If $\displaystyle\lim_{x\to 0} f(x) = 0$, then $\sin(f(x)) \sim f(x)$;
    e.g., $\sin x \sim x$, $\sin\sqrt{x} \sim \sqrt{x}$, etc.
  - A sum of powers of x is asymptotic to the *lowest-powered* term (constants are power zero);
    e.g., $2 + 3\sqrt{x} + x^2 \sim 2$, $\quad 10 + \dfrac{5}{x} + \dfrac{2}{\sqrt{x}} + 3x^3 \sim \dfrac{5}{x}$.
  - Asymptotics behave nicely with respect to products, quotients, and *fixed* powers;
    e.g., $(x+2)\left(\dfrac{3}{\sqrt{x}} + \dfrac{4}{x}\right) \sim 2 \cdot \dfrac{4}{x}$, $\quad \dfrac{\sqrt{x}+x}{1+\sqrt{x}} \sim \dfrac{\sqrt{x}}{1}$, $\quad \sqrt{x+1+\dfrac{1}{x}} \sim \sqrt{\dfrac{1}{x}}$
  - Oscillating terms (e.g., $\sin\frac{1}{x}$ or $\cos\frac{1}{x^2}$) *do not* have any good asymptotic.
    For such terms, use comparison instead.
- Asymptotics **near ∞**
  - Anything having a *nonzero* limit as $x \to \infty$ is asymptotic to that limit;
    e.g., $\tan^{-1} x \sim \frac{\pi}{2}$
  - If $\displaystyle\lim_{x\to\infty} f(x) = 0$, then $\sin(f(x)) \sim f(x)$;
    e.g., $\sin\frac{1}{x} \sim \frac{1}{x}$, $\sin\frac{1}{\sqrt{x}} \sim \frac{1}{\sqrt{x}}$, etc.
  - A sum of powers of x is asymptotic to the *highest-powered* term (constants are power zero);
    e.g., $2 + 3\sqrt{x} + x^2 \sim x^2$, $\quad \dfrac{5}{x} + \dfrac{2}{\sqrt{x}} + \dfrac{3}{x^2} \sim \dfrac{2}{\sqrt{x}}$.
  - Asymptotics behave nicely with respect to products, quotients, and *fixed* powers;
    e.g., $(x+2)\left(\dfrac{3}{\sqrt{x}} + \dfrac{4}{x}\right) \sim x \cdot \dfrac{3}{\sqrt{x}}$, $\quad \dfrac{\sqrt{x}+x}{1+\sqrt{x}} \sim \dfrac{x}{\sqrt{x}}$, $\quad \sqrt{x+1+\dfrac{1}{x}} \sim \sqrt{x}$
  - Oscillating terms (e.g., $\sin x$ or $\cos(x^2)$) *do not* have any good asymptotic.
    For such terms, use comparison instead.