1. *(10 points)* Find $\int \frac{e^{\sin x}}{\tan x \csc x}\, dx$.

Since $\tan x \csc x = \sec x$ we have

$$\int e^{\sin x} \cos x\, dx = \int e^u\, du = e^u + C = e^{\sin x} + C.$$

2. *(12 points)* Find $\int \frac{dx}{\sqrt[3]{x+1} - 1}$.

Here make the substitution $u = \sqrt[3]{x+1}$. Then

$$du = (1/3)(x+1)^{-2/3}\, dx \longrightarrow 3(x+1)^{2/3}\, du = dx \longrightarrow 3u^2\, du = dx.$$

So the integral becomes

$$3\int \frac{u^2}{u-1}\, du$$

Make the substitution $t = u - 1$, which gives $dt = du$ and $u^2 = (t+1)^2$ so the integral becomes

$$\begin{aligned} 3\int \frac{t^2+2t+1}{t}\, dt &= 3\int (t + 2 + 1/t) dt \\ &= 3t^2/2 + 6t + 3\ln|t| + C \\ &= \dots \\ &= \frac{3(\sqrt[3]{x+1}-1)^2}{2} + 6(\sqrt[3]{x+1} - 1) + 3\ln|\sqrt[3]{x+1} - 1| + C \end{aligned}$$

3. *(12 points)* Find $\int_1^e x(\ln x)^2\, dx$.

First compute the definite integral, using integration by parts with $u = (\ln x)^2$ and $dv = x\, dx$. We get

$$\int x(\ln x)^2\, dx = \frac{x^2(\ln x)^2}{2} - \int x \ln x\, dx.$$

Again using integration by parts, with $u = \ln x$ and $dv = x\, dx$ we find

$$\int x(\ln x)^2\, dx = \frac{x^2(\ln x)^2}{2} - \left(\frac{x^2 \ln x}{2} - \int \frac{x}{2}\, dx\right) = \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} + C.$$

Evalutating at the limits gives

$$\int_1^e x(\ln x)^2\, dx = \frac{e^2(\ln e)^2}{2} - \frac{e^2 \ln e}{2} + \frac{e^2}{4} - \frac{(\ln 1)^2}{2} + \frac{\ln 1}{2} - 1/4 = \frac{e^2}{4} - \frac{1}{4}.$$

4. *(12 points)* Find $\int x^3(x^2-4)^{3/2}\, dx$. (Assume that $x > 0$.)

Make the substitution $x = 2\sec\theta$. Then $x^2 - 4 = 4\tan^2\theta$ and $dx = 2\sec\theta\tan\theta\, d\theta$.

So our integral becomes

$$\begin{aligned} \int (8\sec^3\theta)(4\tan^2\theta)^{3/2}(2\sec\theta\tan\theta)\, d\theta &= \int 8\cdot 8\cdot 2\sec^4\theta\tan^4\theta\, d\theta \\ &= 128\int \sec^2\theta\tan^4\theta\sec^2\theta\, d\theta \\ &= 128\int (\tan^4\theta + \tan^6\theta)\sec^2\theta\, d\theta. \end{aligned}$$

Making the substitution $u = \tan\theta$ gives

$$128\int(\tan^4\theta + \tan^6\theta)\sec^2\theta\, d\theta = 128\left(\frac{u^5}{5} + \frac{u^7}{7}\right) + C = 128\left(\frac{\tan^5\theta}{5} + \frac{\tan^7\theta}{7}\right) + C.$$

Since $x = 2\sec\theta$, we will have $\tan\theta = \dfrac{\sqrt{x^2-4}}{2}$ and this gives

$$\int x^3(x^2-4)^{3/2}\, dx = \frac{4(\sqrt{x^2-4})^5}{5} + \frac{(\sqrt{x^2-4})^7}{7} + C.$$

5. *(10 points)* Find $\displaystyle\int \frac{dx}{4x^3+4x^2+x}$.

First factor the denominator:

$$4x^3 + 4x^2 + x = x(2x+1)^2.$$

Then use the method of partial fractions. One can show that

$$\frac{1}{4x^3+4x^2+x} = \frac{1}{x} - \frac{2}{2x+1} - \frac{2}{(2x+1)^2}.$$

Integrating this gives

$$\int \frac{dx}{4x^3+4x^2+x} = \ln|x| - \ln|2x+1| + \frac{1}{2x+1} + C.$$

6. *(12 points)* Find $\displaystyle\int \frac{dx}{x+4\sqrt{x}+13}$.

First make the substitution $u = \sqrt{x}$. This gives $2u\, du = dx$ and

$$\int \frac{dx}{x+4\sqrt{x}+13} = \int \frac{2u}{u^2+4u+13}\, du = \int \frac{2u}{(u+2)^2+9}\, du.$$

Making the substitution $t = u+2$ in this integral gives

$$\int \frac{2t-4}{t^2+9}\, dt = \int \frac{2t}{t^2+9}\, dt - 4\int \frac{dt}{t^2+9} = \ln(t^2+9) - \frac{4}{3}\arctan\left(\frac{t}{3}\right) + C.$$

Going back to the original variable gives

$$\int \frac{dx}{x+4\sqrt{x}+13} = \ln(x+4\sqrt{x}+13) - \frac{4}{3}\arctan\left(\frac{\sqrt{x}+2}{3}\right) + C.$$

7. *(10 points)* The region enclosed by the curves $y = x^2$ and $y = x^3$ is revolved around the line $x = 5$. Find the volume of the resulting solid.

Rotates to give a shell of radius $5 - x$, height $x^2 - x^3$. So

$$V = 2\pi\int_0^1 (5-x)(x^2-x^3)\, dx = 2\pi\int_0^1 (5x^2 - 6x^3 + x^4)\, dx = 2\pi\left(\frac{5x^3}{3} - \frac{6x^4}{4} + \frac{x^5}{5}\right)\Bigg|_0^1 = \ldots = \frac{11\pi}{15}$$

8. *(10 points)* The base of a solid is the region below the curve $y = \sqrt{\arctan x}$ and above the x-axis, for $0 \le x \le 1$. (See diagram.) The cross-section through each plane perpendicular to the x-axis is a square lying above the base. Find the volume.

$$V = \int_0^1 (\sqrt{\arctan x})^2\, dx = \int_0^1 \arctan x\, dx$$

Use integration by parts with $u = \arctan x$ and $dv = dx$ to obtain

$$V = x \arctan x \Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx = \arctan 1 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\, dx = \arctan 1 - \frac{1}{2}\ln(1+x^2)\Big|_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

9. *(12 points)* Sketch the curve given in polar coordinates by

$$r = 3\theta, \qquad \text{for } 0 \le \theta \le \pi,$$

and find the length of this curve.

This is very similar to example 3 on page 521 of the text.

To compute the length of a polar curve we use the formula

$$\text{Length} = \int_\alpha^\beta \sqrt{r^2 + (dr/d\theta)^2}\, d\theta.$$

In this case, since $r = 3\theta$ we have $dr/d\theta = 3$ and the integral runs from $\theta = 0$ to $\theta = \pi$, giving

$$\text{Length} = \int_{\theta=0}^{\theta=\pi} \sqrt{9\theta^2 + 9}\, d\theta = 3\int_0^\pi \sqrt{\theta^2+1}\, d\theta.$$

Make the substitution $\theta = \tan x$ to get

$$3\int_{\theta=0}^{\theta=\pi} \sec^3 x\, dx$$

Using integration by parts we can compute that

$$\int \sec^3 x\, dx = \frac{3}{2}\sec x \tan x + \frac{3}{2}\ln|\sec x + \tan x| + C.$$

Since $\theta = \tan x$, we have $\sqrt{\theta^2+1} = \sec x$ and

$$\frac{3}{2}\sec x \tan x + \frac{3}{2}\ln|\sec x + \tan x| = \frac{3}{2}\sqrt{\theta^2+1}\cdot\theta + \frac{3}{2}\ln|\sqrt{\theta^2+1} + \theta| + C.$$

Evaluating at the limits gives

$$\frac{3}{2}\sqrt{\pi^2+1}\cdot\pi + \frac{3}{2}\ln|\sqrt{\pi^2+1} + \pi| + C.$$