An application of metric cotype to quasisymmetric embeddings

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Abstract

We apply the notion of metric cotype to show that L_p admits a quasisymmetric embedding into L_q if and only if $p \le q$ or $q \le p \le 2$.

This note is a companion to [4]. After the final version of [4] was sent to the journal for publication I learned from Juha Heinonen and Leonid Kovalev of a long-standing open problem in the theory of quasisymmetric embeddings, and it turns out that this problem can be resolved using the methods of [4]. The argument is explained below. I thank Juha Heinonen and Leonid Kovalev for bringing this problem to my attention.

Let (X, d_X) and (Y, d_Y) be metric spaces. An embedding $f : X \to Y$ is said to be a quasisymmetric embedding with modulus $\eta : (0,\infty) \to (0,\infty)$ if η is increasing, $\lim_{t\to 0} \eta(t) = 0$, and for every distinct $x, y, z \in X$ we have

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \le \eta\left(\frac{d_X(x,y)}{d_X(x,z)}\right).$$

We refer to [1] and the references therein for a discussion of this notion.

It was not known whether every two separable Banach spaces are quasisymetrically equivalent. This is asked explicitly in [6] (see problem 8.3.1 there). We will show here that the answer to this question is negative. Moreover, it turns out that under mild assumptions the cotype of a Banach space is preserved under quasisymmetric embeddings. Thus, in particular, our results imply that L_p does not embed quasisymetrically into L_q if $p > 2$ and $q < p$. The question of determining when L_p is quasisymetrically equivalent to L_q was explicitly asked in [6] (see problem 8.3.3 there). We also deduce, for example, that the separable space c_0 does not embed quasisymetrically into any Banach space which has an equivalent uniformly convex norm.

We recall some definitions. A Banach space X is said to have (Rademacher) type $p > 0$ if there exists a constant $T < \infty$ such that for every n and every $x_1, \dots, x_n \in X$,

$$\mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|_X^p \le T^p \sum_{j=1}^n \|x_j\|_X^p.$$

where the expectation $\mathbb{E}_\varepsilon$ is with respect to a uniform choice of signs $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1,1\}^n$. X is said to have (Rademacher) cotype $q > 0$ if there exists a constant $C < \infty$ such that for every n and every $x_1, \dots, x_n \in X$,

$$\mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|_X^q \ge \frac{1}{C^q} \sum_{j=1}^n \|x_j\|_X^q.$$

We also write

$$p_X = \sup\{p \ge 1 : \ X \text{ has type } p\} \quad \text{and} \quad q_X = \inf\{q \ge 2 : \ X \text{ has cotype } q\}.$$

X is said to have non-trivial type if $p_X > 1$, and X is said to have non-trivial cotype if $q_X < \infty$. For example, L_p has type $\min\{p, 2\}$ and cotype $\max\{p, 2\}$ (see for example [5].

Theorem 1. *Let X be a Banach space with non-trivial type. Assume that Y is a Banach space which embeds quasisymmetrically into X. Then $q_Y \le q_X$.*

Proof. Let $f : Y \to X$ be a quasisymmetric embedding with modulus η. Assume for the sake of contradiction that X has cotype q and that $p := q_Y > q$. By the Maurey-Pisier theorem [2] for every $n \in \mathbb{N}$ there is a linear operator $T : \ell_p^n \to Y$ such that for all $x \in \ell_p^n$ we have $\|x\|_p \le \|T(x)\|_Y \le 2\|x\|_p$. For every integer $m \in \mathbb{N}$ consider the mapping $g : \mathbb{Z}_m^n \to X$ given by

$$g(x_1, \dots, x_n) = f \circ T\left(e^{\frac{2\pi i x_1}{m}}, \dots, e^{\frac{2\pi i x_n}{m}}\right).$$

By Theorem 4.1 in [4] there exist constants $A, B > 0$ which depend only on the type and cotype constants of X such that for every integer $m \ge An^{1/q}$ which is divisible by 4 and every $h : \mathbb{Z}_m^n \to X$ we have

$$\sum_{j=1}^{n} \mathbb{E}_x \left[\left\| h\left(x + \frac{m}{2} e_j\right) - h(x) \right\|_X^q \right] \le B^q m^q \, \mathbb{E}_{\varepsilon,x} \left[\|h(x+\varepsilon) - h(x)\|_X^q \right], \tag{1}$$

where the expectations above are taken with respect to uniformly chosen $x \in \mathbb{Z}_m^n$ and $\varepsilon \in \{-1, 0, 1\}^n$ (here, and in what follows we denote by $\{e_j\}_{j=1}^n$ the standard basis of $\mathbb{R}^n$).

From now on we fix m to be be the smallest integer which is divisible by 4 and $m \ge An^{1/q}$. Thus $m \le 8An^{1/q}$. For every $x \in \mathbb{Z}_m^n$, $j \in \{1, \dots, n\}$ and $\varepsilon \in \{-1, 0, 1\}^n$ we have

$$\frac{\|g(x+\varepsilon) - g(x)\|_X}{\left\| g\left(x + \frac{m}{2} e_j\right) - g(x) \right\|_X} \le \eta \left(\frac{\left\| T\left(\sum_{k=1}^n \left(e^{\frac{\pi i \varepsilon_k}{m}} - 1 \right) e_j \right) \right\|_Y}{\|T(2e_j)\|_Y} \right) \le \eta\left(\frac{\pi n^{1/p}}{m} \right) \le \eta\left(\frac{\pi}{A} n^{\frac{1}{p} - \frac{1}{q}} \right).$$

Thus, using (1) for $g = h$ we see that

$$\begin{aligned} n \, \mathbb{E}_{\varepsilon,x} \|g(x+\varepsilon) - g(x)\|_X^q &\le \eta\left(\frac{\pi}{A} n^{\frac{1}{p} - \frac{1}{q}} \right)^q \sum_{j=1}^n \mathbb{E}_x \left\| g\left(x + \frac{m}{2} e_j\right) - g(x) \right\|_X^q \\ &\le \eta\left(\frac{\pi}{A} n^{\frac{1}{p} - \frac{1}{q}} \right)^q (8AB)^q n \, \mathbb{E}_{\varepsilon,x} \|g(x+\varepsilon) - g(x)\|_X^q. \end{aligned}$$

Canceling the term $\mathbb{E}_{\varepsilon,x} \|g(x+\varepsilon) - g(x)\|_X^q$ we deduce that

$$\eta\left(\frac{\pi}{A} n^{\frac{1}{p} - \frac{1}{q}} \right) \ge \frac{1}{8AB}.$$

Since $p > q$ this contradicts the fact that $\lim_{t \to 0} \eta(t) = 0$. □

Using the same argument as in [4] (and noting that the snowflake embedding from [3] is a quasisymmetric embedding), we obtain the following complete answer to the question when L_p embeds quasisymmetrically into L_q.

Corollary 2. *For $p, q > 0$, L_p embeds quasisymmetrically into L_q if and only if $p \le q$ or $q \le p \le 2$.*

References

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