A Note on Simultaneous Polar and Cartesian Decomposition

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Abstract

We study measures on $\mathbb{R}^n$ which are product measures for the usual Cartesian product structure of $\mathbb{R}^n$ as well as for the polar decomposition of $\mathbb{R}^n$ induced by a convex body. For finite atomic measures and for absolutely continuous measures with density $d\mu/dx = e^{-V(x)}$, where V is locally integrable, a complete characterization is presented.

1 Introduction

A subset $K \subset \mathbb{R}^n$ is called a star-shaped body if it is star-shaped with respect to the origin, compact, has non-empty interior, and for every $x \neq 0$ there is a unique $r > 0$ such that $x/r \in \partial K$. We denote this r by $\|x\|_K$ ($\|\cdot\|_K$ is the Minkowski functional of K). Note that $\|x\|_K$ is automatically continuous (If x_n tends to $x \neq 0$, then for every subsequence x_{n_k} such that $\|x_{n_k}\|_K$ converges to r, the compactness ensures that $x/r \in \partial K$, so that $r = \|x\|_K$ by the uniqueness assumption). Any star-shaped body $K \subset \mathbb{R}^n$ induces a polar product structure on $\mathbb{R}^n \setminus \{0\}$ through the identification

$$x \mapsto \left(\|x\|_K, \frac{x}{\|x\|_K} \right).$$

In this note we study the measures on $\mathbb{R}^n$, $n \geq 2$ which are product measures with respect to the Cartesian coordinates, and the above polar decomposition.

In measure theoretic formulation, we will be interested in the measures μ on $\mathbb{R}^n$ which are product measures with respect to the product structures $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R} = \mathbb{R}^+ \cdot \partial K$. Here $\times$ is the usual Cartesian product and for $R \subset \mathbb{R}^+$, $\Omega \in \partial K$, the polar product is by definition $R \cdot \Omega = \{r\omega;\ r \in R \text{ and } \omega \in \Omega\}$. We

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adopt similar notation for product measures: $\otimes$ will be used for Cartesian-product measures and $\odot$ for polar-product measures. With this notation, we say that μ has a simultaneous product decomposition with respect to K if there are measures $\mu_1, \dots \mu_n$ on $\mathbb{R}$ such that $\mu = \mu_1 \otimes \cdots \otimes \mu_n$, and there is a measure τ on $\mathbb{R}^+$ and a measure ν on ∂K such that $\mu = \tau \odot \nu$ (in what follows, all measures are Borel). Notation like A^k or $\prod_i A_i$ always refer to the Cartesian product.

For probability measures one can formulate the notion of simultaneous product decomposition as follows. A measure μ on $\mathbb{R}^n$ has a simultaneous product decomposition with respect to K if and only if there are independent real valued random variables $X_1, \dots, X_n$ such that if we denote $X = (X_1, \dots, X_n)$ then $\mu(A) = P(X \in A)$ and $X/\|X\|_K$ is independent of $\|X\|_K$.

The standard Gaussian measure on $\mathbb{R}^n$ is obviously a Cartesian product. A consequence of its rotation invariance is that it is also a polar-product measure for the usual polar structure induced by the Euclidean ball. Many characterizations of the Gaussian distribution have been obtained so far. The motivations for such characterizations arise from several directions. Maxwell proved that the Gaussian measure is the only rotation invariant product probability measure on $\mathbb{R}^3$, and deduced that this is the distribution of the velocities of gas particles. The classical Cramer and Bernstein characterizations of the Gaussian measure, as well as the numerous related results that appeared in the literature arose from various probabilistic and statistical motivations. We refer to the book [2] and the reference therein for a detailed account. The more modern characterization due to Carlen [3] arose from the need to characterize the equality case in a certain functional inequality.

To explain the motivation for the present paper, we begin by noting that the Gaussian density is in fact one member of a wider family of measures with simultaneous product decomposition, involving bodies other than the Euclidean ball. They will be easily introduced after setting notation. The cone measure on the boundary of K, denoted by μ_K is defined as:

$$\mu_K(A) = \mathrm{vol}([0,1] \cdot A).$$

This measure is natural when studying the polar decomposition of the Lebesgue measure with respect to K, i.e. for every integrable $f : \mathbb{R}^n \to \mathbb{R}$, one has

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^{+\infty} n r^{n-1} \int_{\partial K} f(r\omega)\, d\mu_K(\omega) dr.$$

For the particular case $K = B_p^n = \{x \in \mathbb{R}^n;\ \|x\|_p \leq 1\}$, where $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$, a fundamental result of Schechtman and Zinn [7] (see also Rachev and Rüschendorff [6]), gives a concrete representation of μ_K:

Theorem 1 *Let g be a random variable with density $e^{-|t|^p}/(2\Gamma(1+1/p))$, $t \in \mathbb{R}$. If $g_1, ..., g_n$ are i.i.d. copies of g, set:*

$$S = \left(\sum_{i=1}^n |g_i|^p\right)^{1/p},$$

and consider the random vector:

$$Z = \left(\frac{g_1}{S}, ..., \frac{g_n}{S}\right) \in \mathbb{R}^n.$$

Then the random vector Z is independent of S. Moreover, for every measurable $A \subset \partial B_p^n$ we have:

$$\frac{\mu_{B_p^n}(A)}{\text{vol}(B_p^n)} = P\left(Z \in A\right).$$

The independence of Z and S, that is the simultaneous product decomposition, turns out to be very useful for probabilistic as well as geometric purposes ([8],[5],[4],[1]). One might hope that such a statement holds true for other norms and other densities. The aim of this note is to show that the ℓ_p^n norm is in fact characterized by this property, although we will show that such an independence result holds for other measures. Section 2 is devoted to absolutely continuous measures. Section 3 presents a classifications for finite atomic measures, when K is convex. As the reader will see there are more examples. Some of them, however, are not interesting and we will discard them by suitable assumptions. For example: a constant random variable is independent of any other. This observation allows to produce several measures with simultaneous product decomposition. Any random variable X with values in the half-line $\{x;\ x_1 > 0\}$ works. Its law is clearly a Cartesian product measure, and $X/\|X\|_K$ is constant regardless of K, so it is independent of $\|X\|_K$. Similarly, if X has independent components and takes values in only one sphere $r\partial K$ it has a simultaneous product decomposition. If K is not assumed to be convex, many different examples may be produced: take two sets of positive numbers $\{x_1, \ldots, x_N\}$ and $\{y_1, \ldots, y_N\}$ and consider $\mu_1 = \sum \delta_{x_i}$ and $\mu_2 = \sum \delta_{y_i}$. If one assumes that the numbers y_i/x_j are all different then the measure $\mu_1 \otimes \mu_2$ is supported on points (x_j, y_i) all having different directions. So there are several origin-star-shaped bodies K such that $\mu_1 \otimes \mu_2$ is supported on the boundary of K. For such K's, $\mu_1 \otimes \mu_2$ admits a polar decomposition.

Finally, if μ has simultaneous product decomposition, and $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$ then the restriction of μ to $\{x;\ x_i\epsilon_i > 0\}$ still has this property. This remark allows us to restrict the study to the positive "octant" $(0, +\infty)^n$ (and one has to glue pieces together at the end).

2 Absolutely Continuous Measures

As in the classical characterizations of the Gaussian measure, the assumption that the measure is absolutely continuous reduces the characterization problem to a solution of a functional equation which holds almost everywhere (with respect to the Lebesgue measure). Unless we add some smoothness assumptions on the densities, the next step is to apply a smoothing procedure. Of course, after "guessing" the family of solutions of the equations, we must come up with a smoothing procedure which sends each member of this family to another member of the family. The classical Cramer and Bernstein characterizations use Fourier transform techniques (see [2]), while Carlen [3] applies the heat semigroup. The particular form of the equation we will derive will force us to use yet another smoothing procedure.

When μ is absolutely continuous the following (easy) characterization holds:

Lemma 2 *Assume that μ is an absolutely continuous measure on $\mathbb{R}^n$, then it has a simultaneous product decomposition with respect to K if and only if there are locally integrable non-negative functions $f_1, \dots, f_n$ defined on $\mathbb{R}$, g defined on ∂K (locally integrable with respect to μ_K) and h on $(0, \infty)$ such that*

$$\frac{d\mu}{dx}(x) = \prod_{i=1}^{n} f_i(x_i) = g\left(\frac{x}{\|x\|_K}\right) \cdot h(\|x\|_K),$$

Lebesgue almost everywhere.

Proof: Assume that μ has a simultaneous product decomposition with respect to K. In the above notation, write $\mu = \mu_1 \otimes \cdots \otimes \mu_n = \tau \odot \nu$. For every measurable $B \subset \partial K$:

$$\nu(B) = \mu(\mathbb{R}^+ \cdot B) = \int_{R^+ \cdot B} \frac{d\mu}{dx}(x)dx =$$

$$= \int_B \left(\int_0^\infty n \cdot r^{n-1} \frac{d\mu}{dx}(r\omega)dr\right) d\mu_K(\omega).$$

Similarly, for every measurable $A \subset \mathbb{R}^+$

$$\tau(A) = \mu(A \cdot \partial K) = \int_A n \cdot \left(\int_{\partial K} \frac{d\mu}{dx}(r\omega)d\mu_K(\omega)\right) dr.$$

This shows that both τ and ν are absolutely continuous. Similarly $\mu_1, \dots, \mu_n$ are absolutely continuous.

Now, for every measurable $A \subset \mathbb{R}^+$, $B \subset \partial K$, $C_1, \dots C_n \subset \mathbb{R}$:

$$\mu(A \cdot B) = \tau(A)\nu(B) = \int_0^\infty \int_{\partial K} \frac{d\tau}{dr}(r) \cdot \frac{d\nu}{d\mu_K}(\omega)drd\mu_K(\omega) =$$

$$= \int_{A \cdot B} \frac{1}{n \cdot \|x\|_K^{n-1}} \cdot \frac{d\tau}{dr}(\|x\|_K) \cdot \frac{d\nu}{d\mu_K}\left(\frac{x}{\|x\|_K}\right) dx,$$

and

$$\mu(C_1 \times \ldots \times C_n) = \int_{C_1 \times \ldots \times C_n} \prod_{i=1}^{n} \frac{d\mu_i}{dx_i}(x_i)dx.$$

Since the product Borel σ-algebras on $\mathbb{R}^+ \cdot \partial K$ and $\mathbb{R} \times \ldots \times \mathbb{R}$ (n times) coincide, this shows that:

$$\frac{d\mu}{dx}(x) = \prod_{i=1}^{n} \frac{d\mu_i}{dx_i}(x_i) = \frac{1}{n \cdot \|x\|_K^{n-1}} \cdot \frac{d\tau}{dr}(\|x\|_K) \cdot \frac{d\nu}{d\mu_K}\left(\frac{x}{\|x\|_K}\right),$$

Lebesgue almost everywhere. The reverse implication is even simpler. □

Fix some $p > 0$, $b_1, \dots b_n > -1$ and $a_1, \dots a_n > 0$. Let $X_1, \dots, X_n$ be independent random variables, such that the density of X_i is:

$$\frac{p a_i^{(b_i+1)/p}}{2\Gamma\left(\frac{b_i+1}{p}\right)} \cdot |t|^{b_i} e^{-a_i |t|^p}.$$

Note that:

$$\prod_{i=1}^n |x_i|^{b_i} e^{-a_i|x_i|^p} =$$

$$= \left(\sum_{i=1}^n a_i|x_i|^p\right)^{\frac{1}{p}\sum_{i=1}^n b_i} \cdot e^{-\sum_{i=1}^n a_i|x_i|^p} \prod_{i=1}^n \left[\frac{|x_i|}{\left(\sum_{j=1}^n a_j|x_j|^p\right)^{1/p}}\right]^{b_i}.$$

Hence, if we denote $X = (X_1, \ldots, X_n)$ then by Lemma 2, $X/\left(\sum_{i=1}^n a_i|X_i|^p\right)^{1/p}$ and $\left(\sum_{i=1}^n a_i|X_i|^p\right)^{1/p}$ are independent. Moreover, if $b_1 = \ldots = b_n = 0$ and $a_1 = \ldots = a_n = 1$ then it follows from the proof of Lemma 2 that $X/\|X\|_p$ generates the cone measure on the sphere of ℓ_p^n. We have therefore obtained a generalization of Theorem 1.

The main goal of this section is to prove that the above densities are the only way to obtain a measure with a simultaneous product decomposition with respect to a star shaped body $K \subset \mathbb{R}^n$ (and that K must then be a weighted ℓ_p^n ball). In solving the functional equation of Lemma 2 we will require a smoothing procedure. Clearly, we require a way to smooth a function such that a function of the form $c|t|^b e^{-a|t|^p}$ is transformed to a function of the same form. Let $\psi : \mathbb{R} \to \mathbb{R}$ be locally integrable. For any infinitely differentiable $\rho : (0,\infty) \to [0,\infty)$ which is compactly supported in $(0,\infty)$ define:

$$(\rho \star \psi)(x) = \int_0^\infty \rho(t)\psi\left(\frac{x}{t}\right)dt = \int_0^\infty x\rho(sx)\psi\left(\frac{1}{s}\right)ds.$$

It is easy to verify that $\rho \star \psi$ is infinitely differentiable on $(-\infty,0)\cup(0,\infty)$. Fix some $\epsilon < 1/2$ and let $\rho_\epsilon : (0,\infty) \to [0, 1/(2\epsilon)]$ be any infinitely differentiable function such that $\rho_\epsilon(t) = 1/(2\epsilon)$ when $|t-1| \le \epsilon$ and $\rho_\epsilon(t) = 0$ when $|t-1| > \epsilon + \epsilon^2$. Now, for $x > 0$ (and similarly when $x < 0$):

$$\left|(\rho_\epsilon \star \psi)(x) - \frac{1}{2x\epsilon}\int_{x/(1+\epsilon+\epsilon^2)}^{x/(1-\epsilon-\epsilon^2)} \psi(u)du\right| =$$

$$= \frac{1}{x}\int_{x/(1+\epsilon+\epsilon^2)}^{x/(1-\epsilon-\epsilon^2)} \left|\frac{1}{2\epsilon} - \left(\frac{x}{u}\right)^2 \rho_\epsilon\left(\frac{x}{u}\right)\right| |\psi(u)|du \le$$

$$\le \frac{1}{x}\int_{x/(1+\epsilon)}^{x/(1-\epsilon)} |\psi(u)|du + \frac{1}{2\epsilon x}\left[\int_{x/(1+\epsilon+\epsilon^2)}^{x/(1+\epsilon)} |\psi(u)|du + \int_{x/(1-\epsilon)}^{x/(1-\epsilon-\epsilon^2)} |\psi(u)|du\right],$$

which, by the Lebesgue density theorem, implies that $\lim_{\epsilon\to 0} \rho \star \psi = \psi$ almost everywhere. Since, $\lim_{\epsilon\to 0}\int_0^\infty \rho_\epsilon(t)dt = 1$, the same holds for $\beta_\epsilon = \rho_\epsilon / \int_0^\infty \rho_\epsilon$.

Since for every function of the form $f(t) = c|t|^b e^{-a|t|^p}$, the function $\exp(\rho_\epsilon \star (\log f))$ has the same form, the above smoothing procedure allows us to prove our main result. In what follows $\varepsilon(x) \in \{-1, 1\}$ denotes the sign of x (any convention for the sign of zero will do).

Theorem 3 *Let $K \subset \mathbb{R}^n$ be a star shaped body. Assume that μ is an absolutely continuous probability measure on $\mathbb{R}^n$ which has a simultaneous product decomposition with respect to K. Assume in addition that* $\log\left(\frac{d\mu}{dx}\right)$ *is locally integrable. Then there is some $p > 0$ and there are $b_1, \ldots, b_n > -1$ and*

$r, a_1(1), a_1(-1), c_1(1), c_1(-1) \ldots, a_n(1), a_n(-1), c_n(1), c_n(-1) > 0$ such that:

$$K = \left\{ x \in \mathbb{R}^n;\ \sum_{i=1}^{n} a_i(\varepsilon(x_i))|x_i|^p \leq r \right\},$$

and

$$d\mu(x) = \prod_{i=1}^{n} c_i(\varepsilon(x_i))|x_i|^{b_i} e^{-a_i(\varepsilon(x_i))|x_i|^p}\, dx_i.$$

Conversely, for K and μ as above, μ has simultaneous product decomposition with respect to K.

We will require the following elementary lemma:

Lemma 4 *Fix $\alpha, \alpha' > 0$. Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function such that for every $x > 0$,*

$$f(\alpha x) = 2f(x) \qquad \text{and} \qquad f(\alpha' x) = \frac{3}{2} f(x),$$

then for every $x > 0$, $f(x) = f(1)\, x^{\frac{\log 2}{\log \alpha}}$ (If $\alpha = 1$, $f(x) = 0$ for all x).

Proof: We may assume that f is not identically zero. Then $\alpha \neq \alpha'$ and $\alpha, \alpha' \neq 1$. Consider the set

$$P = \Big\{ \beta > 0; \text{ there is } c_\beta > 0 \text{ s.t. } f(\beta x) = c_\beta f(x), \text{ for all } x > 0 \Big\}.$$

It is a multiplicative subgroup of $(0, \infty)$. By classical results P is either dense in $(0, \infty)$ or discrete. Assume first that it is dense. Fix some x_0 such that $f(x_0) \neq 0$. For any $\beta \in P$, $c_\beta = f(\beta x_0)/f(x_0)$. By continuity of f and by the density of P it follows that for every $x, \beta > 0$ one has

$$f(\beta x) = \frac{f(\beta x_0)}{f(x_0)} f(x).$$

So, if for some β, $f(\beta x_0) = 0$ then f is identically zero. Therefore, f does not vanish. We can chose $x_0 = 1$ and setting $g = f/f(1)$, we have that for every $x, y > 0$, $g(xy) = g(x)g(y)$. It is well known that the continuity of g ensures that it is a power function.

To finish, let us note that P cannot be discrete. Indeed, if P is discrete, since it contains α and $\alpha' \neq \alpha$, it is of the form $\{T^h;\ h \in \mathbb{Z}\}$ for some positive $T \neq 1$. So there are $k, k' \in \mathbb{Z} \setminus \{0\}$ such that $\alpha = T^k$ and $\alpha' = T^{k'}$. Our hypothesis is that for all $x > 0$

$$2f(x) = f(\alpha x) = f(T^k x) = c_T^k f(x),$$
$$\frac{3}{2} f(x) = f(\alpha' x) = f(T^{k'} x) = c_T^{k'} f(x).$$

For an x such that $f(x) \neq 0$ we get $2 = c_T^k$ and $\frac{3}{2} = c_T^{k'}$. It follows that $3^k = 2^{k+k'}$. This is impossible because $k \neq 0$. □

Proof of Theorem 3: Using the notation and the result of Lemma 2,

$$\frac{d\mu}{dx}(x) = \prod_{i=1}^{n} f_i(x_i) = g\left(\frac{x}{\|x\|_K}\right) \cdot h(\|x\|_K).$$

For $i = 1, \dots, n$ denote $F_i = \log f_i$. Denote also $G = \log g$ and $H = \log h$. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a compactly supported continuous function. For every $t > 0$:

$$\sum_{i=1}^{n} \int_{\mathbb{R}^n} \varphi(tx) F_i(x_i) dx = \int_{\mathbb{R}^n} \varphi(tx) G\left(\frac{x}{\|x\|_K}\right) dx + \int_{\mathbb{R}^n} \varphi(tx) H\left(\|x\|_K\right) dx.$$

Changing variables this translates to

$$\sum_{i=1}^{n} \int_{\mathbb{R}^n} \varphi(y) F_i\left(\frac{y_i}{t}\right) dy = \int_{\mathbb{R}^n} \varphi(y) G\left(\frac{y}{\|y\|_K}\right) dy + \int_{\mathbb{R}^n} \varphi(y) H\left(\frac{\|y\|_K}{t}\right) dy.$$

Fix some $\epsilon > 0$. Multiplying by β_ϵ and integrating, we get,

$$\sum_{i=1}^{n} \int_{\mathbb{R}^n} \varphi(y)(\beta_\epsilon \star F_i)(y_i) dy = \int_{\mathbb{R}^n} \varphi(y) G\left(\frac{y}{\|y\|_K}\right) dy + \int_{\mathbb{R}^n} \varphi(y)(\beta_\epsilon \star H)(\|y\|_K) dy.$$

Denote $\phi_i = \beta_\epsilon \star F_i$ and $\eta = \beta_\epsilon \star H$. By the above identity for almost every $y \in \mathbb{R}^n$:

$$\sum_{i=1}^{n} \phi_i(y_i) = G\left(\frac{y}{\|y\|_K}\right) + \eta(\|y\|_K).$$

Since ϕ_i and η are continuous on $\mathbb{R} \setminus \{0\}$, we can change G on a set of measure zero such that the latter identity holds for every $y \in \mathbb{R}^n$ with non-zero coordinates.

Fix some $y \in \mathbb{R}^n$ with non-zero coordinates. For every $\lambda > 0$, one has

$$\sum_{i=1}^{n} \phi_i(\lambda y_i) = G\left(\frac{y}{\|y\|_K}\right) + \eta(\lambda \|y\|_K).$$

Since both sides of the equation are differentiable in λ, taking derivatives at $\lambda = 1$ yields:

$$\sum_{i=1}^{n} \chi_i(y_i) = \zeta(\|y\|_K), \tag{1}$$

where for simplicity, we write $\chi_i(t) = t\phi_i'(t)$ and $\zeta(t) = t\eta'(t)$. From this, we shall deduce that ζ, χ_i are power functions. This can be proved by differentiation along the boundary of K, under smoothness assumptions. Since we want to deal with general star shaped bodies, we present now another reasoning.

Note that $\lim_{t\to 0} \chi_i(t)$ exists. Indeed since η is smooth on $(0, \infty)$ and $\|\cdot\|_K$ is continuous, the above equation for $y^t = \sum_{j\neq i} e_j + te_i$ gives

$$\lim_{t\to 0} \chi_i(t) = \zeta\left(\left\| \sum_{j\neq i} e_j \right\|_K\right) - \sum_{j\neq i} \chi_i(1).$$

Similarly, ζ may be extended by continuity at 0. Hence, (1) holds on $\mathbb{R}^n$. Applying (1) to λe_i, gives

$$\chi_i(\lambda) = \zeta(|\lambda| \cdot \|\varepsilon(\lambda) e_i\|_K) + \gamma_i,$$

for some constant γ_i. Plugging this into (1) for $y_i \geq 0$ we obtain an equation in ζ only:

$$\sum_{i=1}^{n} \zeta(y_i \|e_i\|_K) + \gamma_i = \zeta(\|y\|_K).$$

Choosing $y = \lambda y_1 e_1/\|e_1\|_K + \lambda y_2 e_2/\|e_2\|_K$, with $\lambda, y_1, y_2 > 0$, we get

$$\zeta(\lambda y_1) + \zeta(\lambda y_2) + \sum_{i=1}^{n} \gamma_i + (n-2)\zeta(0) = \zeta\Big(\lambda \Big\| y_1 \frac{e_1}{\|e_1\|_K} + y_2 \frac{e_2}{\|e_2\|_K} \Big\|_K\Big).$$

Differentiating in λ at $\lambda = 1$ and setting $f(t) = t\zeta'(t), \quad t > 0$,

$$f(y_1) + f(y_2) = f\Big(\Big\| y_1 \frac{e_1}{\|e_1\|_K} + y_2 \frac{e_2}{\|e_2\|_K} \Big\|_K\Big). \tag{2}$$

For $y_1 = y_2 = t > 0$, we obtain $2f(t) = f(\alpha t)$, with $\alpha = \Big\| \frac{e_1}{\|e_1\|_K} + \frac{e_2}{\|e_2\|_K} \Big\|_K$. Combining this relation with (2) gives

$$\frac{1}{2} f(y_1 \alpha) + f(y_2) = f\Big(\Big\| y_1 \frac{e_1}{\|e_1\|_K} + y_2 \frac{e_2}{\|e_2\|_K} \Big\|_K\Big).$$

For $y_1 = t/\alpha$ and $y_2 = t > 0$ we get $\frac{3}{2} f(t) = f(t\alpha')$, with $\alpha' = \Big\| \frac{1}{\alpha} \frac{e_1}{\|e_1\|_K} + \frac{e_2}{\|e_2\|_K} \Big\|_K$. Lemma 4 ensures that $f(t) = f(1)t^p$ with $p = \log 2/\log \alpha \neq 0$, and with the convention that $p = 0$ if $\alpha = 1$ (in this case $f(1) = 0$). It follows that

$$\zeta(t) = \zeta'(1)t^p/p + \zeta(0)$$

if $p \neq 0$ and $\zeta(t) = \zeta(0)$ otherwise. Integrating again, we get an expression for $\eta = \beta_\epsilon * \log h$. Letting ϵ tend to zero shows that there are constants $a, b, c \in \mathbb{R}$ such that for a.e. $t > 0$, $h(t) = ct^b e^{-at^p}$ (this is valid even if $p = 0$).

Next we find an expression of the functions f_i. We start with the relation

$$t\phi'(t) = \chi_i(t) = \zeta(|t| \cdot \|\varepsilon(t)e_i\|_K) + \gamma_i = a_i(\varepsilon(t))|t|^p + b_i,$$

for some constants $a_i(1), a_i(-1), b_i$. Thus for $t \neq 0$, $\phi'(t) = a_i(\varepsilon(t))|t|^{p-1}\varepsilon(t) + b_i/t$. Integrating (with different constants on $(-\infty, 0)$ and on $(0, \infty)$) and taking the limit $\epsilon \to 0$ as before we arrive at $f_i(t) = c_i(\varepsilon(t))|t|^{b_i} e^{-a_i|t|^p}$, for almost every t and for some constants $c_i(1), c_i(-1)$. For f_i to have finite integral, p has to be non-zero. Our initial equation reads as: for a.e. x,

$$\prod_{i=1}^{n} c_i(\varepsilon(x_i))|x_i|^{b_i} e^{-a_i|x_i|^p} = c\|x\|_K^b e^{-a\|x\|_K^p} g\left(\frac{x}{\|x\|_K}\right).$$

By continuity this holds on $\mathbb{R}^n \setminus \{x;\ \prod_{i=1}^n x_i = 0\}$. For such an x and $\lambda > 0$, the equation becomes

$$\prod_{i=1}^{n} c_i(\varepsilon(x_i))\lambda^{\sum_{i=1}^n b_i} e^{-\lambda^p \sum_{i=1}^n a_i(\varepsilon(x_i))|x_i|^p} \prod_{i=1}^{n} c_i|x_i|^{b_i}$$

$$= c\lambda^b \|x\|_K^b e^{-a\lambda^p \|x\|_K^p} g\left(\frac{x}{\|x\|_K}\right).$$

This clearly implies that $a\|x\|_K^p = \sum_{i=1}^n a_i(\varepsilon(x_i))|x_i|^p$. Since μ is a probability measure, necessarily $a, a_i(1), a_i(-1) > 0$ and $b_i > -1$. Thus K is determined. The boundedness of K forces $p > 0$. The proof is complete. □

We now pass to the case of μ being an infinite measure. In this case, every star-shaped body gives rise to a measure with a simultaneous product decomposition. Indeed, for every $b_1, \dots, b_n > -1$ in $\mathbb{R}$, and every star-shaped body K, the measure $d\mu(x) = \prod_{i=1}^n |x_i|^{b_i}\, dx$ admits such a decomposition, due to the identity:

$$\prod_{i=1}^n |x_i|^{b_i} = \prod_{i=1}^n \left(\frac{|x_i|}{\|x\|_K}\right)^{b_i} \cdot \|x\|_K^{\sum_{i=1}^n b_i}.$$

We can however prove that the above example is the only additional case. For simplicity we work with measures on $(0,\infty)^n$.

Theorem 5 *Let $K \subset \mathbb{R}^n$ be a star shaped body. Assume that μ is an absolutely continuous measure on $(0,\infty)^n$ which has a simultaneous product decomposition with respect to K. Assume in addition that $\log\left(\frac{d\mu}{dx}\right)$ is locally integrable. Then one of the following assertions holds:*
1) There are $p, r > 0$, $b_1, \dots, b_n \in \mathbb{R}$ and $c \geq 0$ and $a_1, \dots, a_n \neq 0$ all having the same sign such that:

$$K \cap (0,\infty)^n = \left\{ x \in (0,\infty)^n;\ \sum_{i=1}^n |a_i| \cdot |x_i|^p \leq r \right\},$$

and

$$d\mu = c \prod_{i=1}^n \left(x_i^{b_i} e^{-a_i x_i^p} \mathbf{1}_{\{x_i>0\}} dx_i \right).$$

2) K is arbitrary and there are $b_1, \dots, b_n \in \mathbb{R}$ and $c > 0$ such that

$$d\mu = c \prod_{i=1}^n \left(x_i^{b_i} \mathbf{1}_{\{x_i>0\}} dx_i \right).$$

Conversely if K and μ satisfy 1) or 2) then μ has a simultaneous product decomposition with respect to K.

Proof: This result follows from the proof of Theorem 3. The writing is simpler since we work on $(0,\infty)^n$. We present the modifications. If in the argument $p = 0$, then $f_i(t) = c_i t^{b_i}$ and we are done. If $p \neq 0$ then the argument provides $a, a_1, \dots, a_n$ such that whenever $x_i > 0$

$$a\|x\|_K^p = \sum_{i=1}^n a_i x_i^p.$$

If $a = 0$ then then $a_i = 0$ for all i's, $f_i(t) = c_i t^{b_i}$ and there is no constraint on K. If $a \neq 0$ then the previous relation gives $a_i = a\|e_i\|_K^p$, so the a_i's are not zero and have same sign. Since $\|x\|_K = \left(\sum_{i=1}^n \frac{a_i}{a}|x_i|^p\right)^{1/p}$, the set $K \cap (0,\infty)^n$ is a weighted ℓ_p^n-ball. By boundedness $p > 0$. As before $f_i(t) = c_i t^{b_i} e^{-a_i t^p}$. This ends the proof. □

3 Atomic Measures

In this section we focus on finite atomic measures $\sum_{P\in S}\alpha_P\delta_P$, where δ_P is the Dirac measure at P and $S\subset\mathbb{R}^n$ is countable. For convenience, we write $\mu(P)$ for $\mu(\{P\})$. We also restrict to convex sets K. The following result deals with measures which are not supported on a sphere. Measures which concentrate on a sphere, when K is convex are much easier to classify and we leave this to the reader (one of the μ_i's has to be a Dirac mass).

Theorem 6 *Assume that $K\subset\mathbb{R}^n$ is convex, symmetric and contains the origin in its interior, and that μ is a finite (and non-zero) atomic measure on $(0,\infty)^n$, which admits a simultaneous polar and Cartesian decomposition with respect to K: $\mu=\mu_1\otimes\cdots\otimes\mu_n=\tau\odot\nu$. Assume in addition that τ is not a Dirac measure. Then the following assertions hold:*
a) There are $\lambda_1,\dots,\lambda_n>0$ such that $K\cap[0,\infty)^n=\prod_{i=1}^n[0,\lambda_i]$.
b) There are $c,r,\alpha_1,\dots\alpha_n>0$, $0<q<1$ and $D=\prod_{i=1}^n\{r\lambda_i q^k;\ k\in\mathbb{N}\}$ such that :

$$\mu(x)=\begin{cases} c\prod_{i=1}^n x_i^{\alpha_i} & \text{if } x\in D\\ 0 & \text{if } x\notin D\end{cases}$$

Conversely, if K and μ satisfy a) and b) then μ has a simultaneous product decomposition with respect to K.

As a matter of illustration, we check that conditions $a)$ and $b)$ ensure simultaneous Cartesian and polar product decompositions. Let $x\in(0,\infty)^n$. Since the set D is a Cartesian product, it is clear that μ defined in $b)$ is a Cartesian-product measure. Next, write $x=\rho\omega$ with $\rho>0$ and $\omega\in\partial K$ then

$$\mu(x)=\mu(\rho\omega)=c\,\rho^{\sum_{i=1}^n\alpha_i}\prod_{i=1}^n\omega_i^{\alpha_i}\delta_D(\rho\omega),$$

so we just need to check that D is a polar product set in order to show that the latter is the product of a quantity depending only on ρ times a quantity depending on ω. But this is easy: if $x\in D$ then for each i, one has $x_i=r\lambda_i q^{k_i}$. By $a)$, $\|x\|_K=r\,q^{\min_i k_i}\in T=\{rq^k;\ k\in\mathbb{N}\}$ and

$$\frac{x}{\|x\|_K}=\left(\lambda_i q^{k_i-\min_j k_j}\right)_{i=1}^n\in\Omega=\Big\{(\lambda_i q^{h_i})_{i=1}^n;\ h_i\geq 0\quad\text{and}\prod_i h_i=0\Big\}.$$

This shows that $D\subset T\cdot\Omega$. The converse inclusion is easily checked. Hence μ has simultaneous product decomposition.

The rest of this section is devoted to the proof of the necessary condition in Theorem 6. From now on we assume that μ and K satisfy the assumption of the theorem. We begin with some notation. If λ is a measure on a measurable space (Ω,Σ), let $\mathcal{M}_\lambda=\{x\in\Omega;\lambda(x)=\sup_{\omega\in\Omega}\lambda(\omega)\}$. Clearly, if λ is a finite measure then $|\mathcal{M}_\lambda|<\infty$. We also put

$$\mathcal{M}_\lambda^2=\left\{x\in\Omega;\lambda(x)=\sup_{\omega\in\Omega\setminus\mathcal{M}_\lambda}\lambda(\omega)\right\}.$$

When λ is countably supported we define $\mathrm{supp}(\lambda)=\{\omega\in\Omega;\lambda(\omega)>0\}$.

Returning to the setting of Theorem 6, we clearly have that $\mathrm{supp}(\mu) = \mathrm{supp}(\mu_1) \times \cdots \times \mathrm{supp}(\mu_n) = \mathrm{supp}(\tau) \cdot \mathrm{supp}(\nu)$ and $\mathcal{M}_\mu = \mathcal{M}_{\mu_1} \times \cdots \times \mathcal{M}_{\mu_n} = \mathcal{M}_\tau \cdot \mathcal{M}_\nu$.

The next lemma will be used for a measure other than μ of the main theorem. This is why the fact that τ in the product decomposition is not Dirac is specifically stated as an hypothesis there.

Lemma 7 *Define the slope function $s_1(x) = \frac{x_1}{\|(x_2,\ldots,x_n)\|_2}$. If τ is not a Dirac measure, then s_1 does not attain its minimum on* $\mathrm{supp}(\mu)$*.*

Proof: Assume that $\mu(x) > 0$. Then $\mu_1(x_1), \ldots \mu_n(x_n) > 0$, $\nu(x/\|x\|_K) > 0$ and $\tau(\|x\|_K) > 0$. Since τ is not a Dirac measure, there is $r > 0$, $r \neq \|x\|_K$ such that $\tau(r) > 0$. Let $y = \frac{r}{\|x\|_K}x$. Then $\mu(y) = \tau(r)\nu(x/\|x\|_K) > 0$, so that for every $i = 1, \ldots, n$, $\mu_i\left(\frac{r}{\|x\|_K}x_i\right) > 0$. Therefore for

$$u = \left(\frac{r}{\|x\|_K}x_1, x_2, \ldots, x_n\right),$$

$$v = \left(x_1, \frac{r}{\|x\|_K}x_2, \frac{r}{\|x\|_K}x_3, \ldots, \frac{r}{\|x\|_K}x_n\right),$$

$\mu(u) > 0$ and $\mu(v) > 0$. But $s_1(u) = \frac{r}{\|x\|_K}s_1(x)$, $s_1(v) = \frac{\|x\|_K}{r}s_1(x)$ and either $\frac{r}{\|x\|_K} < 1$ or $\frac{\|x\|_K}{r} < 1$. □

Corollary 8 *Under the assumptions of Theorem 6, $|\mathrm{supp}(\mu)| = \infty$ and $|\mathcal{M}_\tau| = 1$.*

Proof: The first assertion is obvious, and the second assertion follows since $\mu|_{\mathcal{M}_\mu}$ has a simultaneous product decomposition. Indeed $\mu|_{\mathcal{M}_\mu}$ has also a finite support, therefore s_1 attains its minimum on it. It follows that the radial measure $\tau|_{\mathcal{M}_\mu}$ has to be a Dirac measure. □

Put $\mathcal{M}_\tau = \{r\}$.

Lemma 9 $\mathrm{supp}(\mu) \subset \{x; \|x\|_K \leq r\}$.

Proof: If $\mathrm{supp}(\mu) \not\subset \{x; \|x\|_K \leq r\}$ then since τ is a finite measure there is $R > r$ such that $\tau(R) > 0$ is maximal on (r, ∞). For every $i = 1, \ldots, n$ let $M_i = \max \mathcal{M}_{\mu_i} > 0$. Now, $x = (M_1, M_2, \ldots, M_n) \in \mathcal{M}_\mu$, so that $\|x\|_K = r$. Put $y = \frac{R}{r}x$. Clearly $\mu(y) = \tau(R)\nu(x/\|x\|_K)$ is maximal on the set $\{x; \|x\|_K > r\}$. For every $i = 1, \ldots, n$ define

$$x^i = \left(M_1, \ldots, M_{i-1}, \frac{R}{r}M_i, M_{i+1}, \ldots, M_n\right).$$

Note that for every $j = 1, \ldots, n$, $M_j = \max \mathcal{M}_{\mu_j} < \frac{R}{r}M_j = y_j$ so that $y_j \notin \mathcal{M}_{\mu_j}$. It follows that for every $i = 1, \ldots, n$, $\mu(x^i) > \mu(y)$, so that $\|x^i\|_K \leq r$. Now, using the convexity of K we have

$$r = \|x\|_K = \left\| \frac{1}{n-1+\frac{R}{r}} \sum_{i=1}^{n} x^i \right\|_K \leq$$

$$\leq \frac{1}{n-1+\frac{R}{r}} \sum_{i=1}^{n} \|x^i\|_K \leq \frac{nr}{n-1+\frac{R}{r}} < r,$$

which is a contradiction. □

Since τ isn't Dirac, there is some $r' \in \mathcal{M}_\tau^2$. By Lemma 9, $r' < r$.

Lemma 10 *For every* $i = 1, \ldots, n$, $\inf \text{supp}(\mu_i) = 0$.

Proof: As in Lemma 7, we will study the function:

$$s_i(x) = \frac{x_i}{\|\sum_{j \neq i} x_j e_j\|_2}.$$

Since Lemma 9 implies in particular that $\text{supp}(\mu)$ is bounded, our claim will follow once we show that $\inf_{x \in \text{supp}(\mu)} s_i(x) = 0$. Let $\sigma_i = \inf_{x \in \text{supp}(\mu)} s_i(x)$ and assume that $\sigma_i > 0$. For every $\epsilon > 0$ there is $x \in \text{supp}(\mu)$ such that $s_i(x) \leq (1+\epsilon)\sigma_i$. From the proof of Lemma 7 it follows that for every $\rho \in \text{supp}(\tau)$ there are $u, v \in \text{supp}(\tau)$ such that $s_i(u) = \frac{\rho}{\|x\|_K} s_i(x)$ and $s_i(v) = \frac{\|x\|_K}{\rho} s_i(x)$. Hence, $\min\left\{\frac{\rho}{\|x\|_K}, \frac{\|x\|_K}{\rho}\right\} \geq \frac{1}{1+\epsilon}$. If $\|x\|_K = r$ take $\rho = r'$. Otherwise, $\|x\|_K \leq r'$, in which case take $\rho = r$. In both cases we get that $r \leq (1+\epsilon)r'$, which is a contradiction when ϵ is small enough. □

In what follows we will continue to use the notation $M_i = \max \mathcal{M}_{\mu_i}$, and we will also put $m_i = \min \mathcal{M}_{\mu_i}$. Let $x = (M_1, \ldots, M_n)$, $x' = (m_1, \ldots, m_n)$.

Corollary 11 *For every* $J \subset \{1, \ldots, n\}$*:*

$$\left\| \sum_{i \in J} M_i e_i \right\|_K \leq r.$$

Proof: By Lemma 10 for every $\epsilon > 0$ and $i = 1, \ldots, n$ there is $z_i \in \text{supp}(\mu_i)$ with $z_i < \epsilon$. Now:

$$\sum_{i \in J} M_i e_i + \sum_{i \notin J} z_i e_i \in \text{supp}(\mu),$$

so that by Lemma 9 we get:

$$\left\| \sum_{i \in J} M_i e_i + \sum_{i \notin J} z_i e_i \right\|_K \leq r.$$

The result follows by taking $\epsilon \to 0$. □

Corollary 12 $\prod_{i=1}^{n} [0, M_i] \subset rK$.

Lemma 13 *Let* J *be a non-empty subset of* $\{1, \ldots, n\}$*. Then:*

$$\left\| \sum_{i \in J} M_i e_i + \sum_{i \notin J} \frac{r'}{r} m_i e_i \right\|_K = r.$$

Proof: Denote $y = \frac{r'}{r}x'$. Since $x' \in M_\mu$, $\|x'\|_K = r$. Now, $\mu(y) = \tau(r')\nu(x'/\|x'\|_K)$, and because $r' \in M_\tau^2$ and $x'/\|x'\|_K \in M_\nu$ we deduce that $\mu(y)$ is maximal on the set $\{x; \|x\|_K \neq r\}$. But since $r' < r$, for any $j = 1, \dots, n$ one has $y_j = \frac{r'}{r}m_j < m_j = \min M_{\mu_j}$, so that $y_j \notin M_{\mu_j}$. It follows that since $J \neq \emptyset$,

$$\mu\left(\sum_{i\in J} M_i e_i + \sum_{i\notin J} \frac{r'}{r} m_i e_i\right) = \left(\prod_{i\in J} \mu_i(M_i)\right)\left(\prod_{i\notin J} \mu_i(y_i)\right) > \prod_{i=1}^{n} \mu_i(y_i) = \mu(y),$$

so that $\left\|\sum_{i\in J} M_i e_i + \sum_{i\notin J} \frac{r'}{r} m_i e_i\right\|_K = r$. □

We can now prove the first part of Theorem 6:

Proposition 14 $K \cap [0,\infty)^n = \prod_{i=1}^{n} \left[0, \frac{M_i}{r}\right]$.

Proof: We set $Q = \prod_{i=1}^{n}[0, M_i]$. Let $1 \leq i \leq n$. Since $0 < m_i r'/r < M_i$ the point

$$P_i = M_i e_i + \sum_{j\neq i} \frac{r'}{r} m_j e_j$$

lies in the interior of the facet $Q \cap \{x;\ x_i = M_i\}$ of Q. It is also a boundary point of rK by Lemma 13. As guaranteed by Corollary 12, $Q \subset rK$, so that any supporting hyperplane of rK at P_i is a supporting hyperplane of Q at this point. Therefore at P_i the convex set rK admits $\{x;\ x_i = M_i\}$ as a (unique) supporting hyperplane. It follows that $rK \subset \{x;\ x_i \leq M_i\}$. This is true for every $1 \leq i \leq n$ and the proof is complete. □

We now pass to the proof of the final assertion of Theorem 6. We have proved that there are real numbers $t_i = r/M_i > 0,\ i = 1 \dots n$ such that for every $x \in [0,\infty)^n$, $\|x\|_K = \max_{1\leq i\leq n} t_i x_i$. Moreover for every $x \in \mathrm{supp}(\mu)$, $\|x\|_K \leq r$ and $\mu(r/t_1, \dots, r/t_n) > 0$.

Lemma 15 *If $p, q \in \mathrm{supp}(\tau)$ and $q > p$ then $\frac{pr}{q}, \frac{pq}{r} \in \mathrm{supp}(\tau)$.*

Proof: Since $\mu\left(\frac{r}{t_1}, \dots, \frac{r}{t_n}\right) > 0$, $\nu\left(\frac{1}{t_1}, \dots, \frac{1}{t_n}\right) > 0$. Now,

$$0 < \tau(p)\nu\left(\frac{1}{t_1}, \dots, \frac{1}{t_n}\right) = \mu\left(\frac{p}{t_1}, \dots, \frac{p}{t_n}\right) = \prod_{i=1}^{n} \mu_i\left(\frac{p}{t_i}\right),$$

so that for every $i = 1, \dots, n$, $\frac{p}{t_i} \in \mathrm{supp}(\mu_i)$. Similarly, $\frac{q}{t_i} \in \mathrm{supp}(\mu_i)$. Hence, since $p < q$:

$$0 < \mu\left(\frac{p}{t_1}, \frac{q}{t_2}, \dots, \frac{q}{t_n}\right) = \tau(q)\nu\left(\frac{p}{qt_1}, \frac{1}{t_2}, \dots, \frac{1}{t_n}\right),$$

so that $\nu\left(\frac{p}{qt_1}, \frac{1}{t_2}, \dots, \frac{1}{t_n}\right) > 0$. Now, using again the fact that $p < q$ we have:

$$\mu_1\left(\frac{pr}{qt_1}\right) = \frac{\mu\left(\frac{pr}{qt_1}, \frac{r}{t_2}, \dots, \frac{r}{t_n}\right)}{\prod_{i=2}^{n} \mu_i\left(\frac{r}{t_i}\right)} =$$

$$= \frac{\tau(r)\nu\left(\frac{p}{qt_1}, \frac{1}{t_2}, \ldots, \frac{1}{t_n}\right)}{\prod_{i=2}^n \mu_i\left(\frac{r}{t_i}\right)} > 0.$$

This shows that $\frac{pr}{qt_1} \in \mathrm{supp}(\mu_1)$. Similarly, for every i, $\frac{pr}{qt_i} \in \mathrm{supp}(\mu_i)$. Hence,

$$0 < \prod_{i=1}^n \mu_i\left(\frac{pr}{qt_i}\right) = \mu\left(\frac{pr}{qt_1}, \ldots, \frac{pr}{qt_n}\right) = \tau\left(\frac{pr}{q}\right)\nu\left(\frac{1}{t_1}, \ldots, \frac{1}{t_n}\right).$$

This shows that $\frac{pr}{q} \in \mathrm{supp}(\tau)$. Now, since the remark preceding Lemma 15 implies that $p \le r$,

$$\mu_1\left(\frac{pq}{rt_1}\right)\prod_{i=2}^n \mu_i\left(\frac{q}{t_i}\right) = \mu\left(\frac{pq}{rt_1}, \frac{q}{t_2}, \ldots, \frac{q}{t_n}\right) = \tau(q)\nu\left(\frac{p}{rt_1}, \frac{1}{t_2}, \ldots, \frac{1}{t_n}\right) =$$

$$= \frac{\tau(q)\mu\left(\frac{p}{t_1}, \frac{r}{t_2}, \ldots, \frac{r}{t_n}\right)}{\tau(r)} = \frac{\tau(q)}{\tau(r)}\mu_1\left(\frac{p}{t_1}\right)\prod_{i=2}^n \mu_i\left(\frac{r}{t_i}\right) > 0.$$

Hence, $\frac{pq}{rt_1} \in \mathrm{supp}(\mu_1)$. Similarly, for every i, $\frac{pq}{rt_i} \in \mathrm{supp}(\mu_i)$, so that:

$$0 < \mu\left(\frac{pq}{rt_1}, \ldots, \frac{pq}{rt_n}\right) = \tau\left(\frac{pq}{r}\right)\nu\left(\frac{1}{t_1}, \ldots, \frac{1}{t_n}\right),$$

which shows that $\frac{pq}{r} \in \mathrm{supp}(\tau)$. □

Lemma 16 *For every $i = 1, \ldots, n$, $t_i\mathrm{supp}(\mu_i) = \mathrm{supp}(\tau)$.*

Proof: In the proof of Lemma 15 we have seen that $\mathrm{supp}(\tau) \subset t_i\mathrm{supp}(\mu_i)$. We show the other inclusion. First, note that $\inf \mathrm{supp}(\tau) = 0$. Indeed for every $\epsilon > 0$, Lemma 10 ensures the existence of $x_i \in \mathrm{supp}(\mu_i)$ such that $x_i < \epsilon/t_i$. Now,

$$0 < \prod_{i=1}^n \mu_i(x_i) = \tau\left(\max_{1\le i\le n} t_i x_i\right)\nu\left(\frac{x}{\|x\|_K}\right).$$

So $\inf \mathrm{supp}(\tau) \le \epsilon$.

Take any $p \in \mathrm{supp}(\mu_i)$. There is some $q \in \mathrm{supp}(\tau)$ such that $q < t_i p$. By the proof of Lemma 15, for every j, $q/t_j \in \mathrm{supp}(\mu_j)$, so that:

$$0 < \mu_i(p)\prod_{j\ne i}\mu_j\left(\frac{q}{t_j}\right) = \mu\left(pe_i + \sum_{j\ne i}\frac{q}{t_j}e_j\right) = \tau(t_i p)\nu\left(\frac{1}{t_i}e_i + \sum_{j\ne i}\frac{q}{t_i t_j p}e_j\right),$$

so that $t_i p \in \mathrm{supp}(\tau)$. □

Lemma 17 *For every $i = 1, \ldots, n$ and for every $p, q \in \mathrm{supp}(\mu_i)$,*

$$\mu_i\left(\frac{t_i pq}{r}\right) = \frac{\mu_i(p)\mu_i(q)}{\mu_i\left(\frac{r}{t_i}\right)}.$$

Proof: Note that since $t_i p \in \text{supp}(\tau)$, $t_i p \leq r$. Hence, using the fact that $\frac{t_i q}{t_j} \in \text{supp}(\mu_j)$ we have:

$$0 < \mu_i\left(\frac{t_i pq}{r}\right)\prod_{j\neq i}\mu_j\left(\frac{t_i q}{t_j}\right) = \mu\left(\frac{t_i pq}{r}e_i + \sum_{j\neq i}\frac{t_i q}{t_j}e_j\right) =$$

$$= \tau\left(\max\left\{\frac{t_i^2 pq}{r}, t_i q\right\}\right)\nu\left(\frac{1}{\max\left\{\frac{t_i^2 pq}{r}, t_i q\right\}}\left(\frac{t_i pq}{r}e_i + \sum_{j\neq i}\frac{t_i q}{t_j}e_j\right)\right) =$$

$$= \tau\left(t_i q\right)\nu\left(\frac{p}{r}e_i + \sum_{j\neq i}\frac{1}{t_j}e_j\right) = \tau(t_i q)\frac{\mu\left(pe_i + \sum_{j\neq i}\frac{r}{t_j}e_j\right)}{\tau(r)} =$$

$$= \frac{\tau(t_i q)}{\tau(r)}\mu_i(p)\prod_{j\neq i}\mu_j\left(\frac{r}{t_j}\right) = \mu_i(p)\frac{\mu\left(qe_i + \sum_{j\neq i}\frac{t_i q}{t_j}e_j\right)}{\nu\left(\sum_{j=1}^n\frac{1}{t_j}e_j\right)}\cdot\frac{\mu\left(\sum_{j=1}^n\frac{r}{t_j}e_j\right)}{\mu_i\left(\frac{r}{t_i}\right)\tau(r)} =$$

$$= \frac{\mu_i(p)\mu_i(q)}{\mu_i\left(\frac{r}{t_i}\right)}\prod_{j\neq i}\mu_j\left(\frac{t_i q}{t_j}\right).$$

□

Lemma 18 *Assume that $A \subset (0,1]$, $A \neq \{1\}$, $A \neq \emptyset$, has the property that xy and x/y are in A whenever $x, y \in A$ and $x \leq y$. Let $f : A \to (0,\infty)$ be a function such that if $x, y \in A$ then $f(xy) = f(x)f(y)$ and:*

$$\sum_{a\in A} f(a) < \infty.$$

Then there are $\alpha > 0$ and $0 < q < 1$ such that $f(a) = a^\alpha$ and $A = \{q^n\}_{n=0}^\infty$.

Proof: For any $a \in A\setminus\{1\}$ and $n \in \mathbb{N}$, $a^n \in A$ and $f(a^n) = f(a)^n$. Since $\sum_{n=1}^\infty f(a^n) < \infty$, $f(a) < 1$. Now, if $a, b \in A\setminus\{1\}$ and $\frac{a^n}{b^m} < 1$, $\frac{a^n}{b^m} \in A$, so that $1 > f\left(\frac{a^n}{b^m}\right) = \frac{f(a)^n}{f(b)^m}$. We have shown that for every $n, m \in \mathbb{N}$ and $a, b \in A\setminus\{1\}$:

$$\frac{n}{m} > \frac{\log b}{\log a} \Longrightarrow \frac{n}{m} > \frac{\log f(b)}{\log f(a)}.$$

Hence, $\frac{\log b}{\log a} \geq \frac{\log f(b)}{\log f(a)}$ for every $a, b \in A\setminus\{1\}$. By symmetry, there is $\alpha \in \mathbb{R}$ such that for every $a \in A$, $\frac{\log f(a)}{\log a} = \alpha$. This proves the first assertion ($\alpha > 0$ since $f(a) < 1$).

Put $B = \{-\log a; a \in A\}$. Clearly:

$$a, b \in B \Longrightarrow a + b, |a - b| \in B.$$

Since $f(a) = a^\alpha$ and $\sum_{a\in A} f(a) < \infty$, for every $x > 0$ there are only finitely many $a \in A$ with $a \geq x$. In other words, for every $x > 0$ there are only finitely many $b \in B$ with $b \leq x$. In particular, if we let $p = \inf B\setminus\{0\}$ then $p > 0$ and $p \in B$. Now, for every $n = 0, 1, 2, \ldots$, $np \in B$. We claim that $B = \{0, p, 2p, 3p, \ldots\}$. Indeed, if $x \in B\setminus\{0, p, 2p, 3p, \ldots\}$ then there is an integer n such that $0 < |x - np| < p$. But, $|x - np| \in B$, and this contradicts the definition of p. Finally, for $q = e^{-p}$, $A = \{1, q, q^2, \ldots\}$. □

Remark: All the assumptions in Lemma 18 are necessary. Apart from the trivial examples such as $A = (0, 1]$ and $A = (0, 1] \cap \mathbb{Q}$ we would like to point out the more interesting example $A = \{2^n 3^m; m, n \in \mathbb{Z} \text{ and } 2^n 3^m \leq 1\}$, $f(2^n 3^m) = 2^{\alpha m} 3^{\beta m}$ where α and β are distinct real numbers (of course in this case the condition $\sum_{a \in A} f(a) < \infty$ is not satisfied).

Proof of Theorem 6: Assertion $a)$ is given by Proposition 14. To prove $b)$ fix some $1 \leq i \leq n$ and define: $A = \frac{t_i}{r}\mathrm{supp}(\mu_i)$. By Lemma 16 and Lemma 9, $A = \frac{1}{r}\mathrm{supp}(\tau) \subset \frac{1}{r}(0, r] = (0, 1]$. Additionally, if $x, y \in A$, $x \leq y$, then there are $p, q \in \mathrm{supp}(\tau)$, $p \leq q$, such that $x = \frac{p}{r}$, $y = \frac{q}{r}$. Lemma 15 implies that $\frac{x}{y} = \frac{1}{r} \cdot \frac{pr}{q} \in \frac{1}{r}\mathrm{supp}(\tau) = A$. Similarly, $xy \in A$. Define $f_i : A \to \mathbb{R}$ by

$$f_i(x) = \frac{\mu_i\left(\frac{r}{t_i}x\right)}{\mu_i\left(\frac{r}{t_i}\right)}.$$

Clearly for every $x \in A$, $f_i(x) > 0$ and since μ_i is a finite measure, $\sum_{a \in A} f_i(a) < \infty$. An application of Lemma 17 gives that for every $x, y \in A$, $f_i(xy) = f_i(x)f_i(y)$. Now, Lemma 18 implies that there are $\alpha_i > 0$ and $0 < q < 1$ such that $f_i(a) = a^{\alpha_i}$ and $A = \{q^n\}_{n=0}^{\infty}$. So, $\mathrm{supp}(\tau) = \{rq^k;\ k \in \mathbb{N}\}$ and by Lemma 16, one gets $\mathrm{supp}(\mu_i) = \{rq^k/t_i;\ k \in \mathbb{N}\}$. Moreover, $\mu_i(rq^k/t_i) = \mu_i(r/t_i)q^{k\alpha_i}$. This concludes the proof of the theorem. □

4 Concluding Remarks

In this sections we list some remarks and open problems that arise from the results of the previous two sections.

1) There are examples when K is allowed to be unbounded (of course in this case it is no longer a body). Indeed the "unit ball" of ℓ_p^n for non-positive p gives such a decomposition with $f_i(t) = |t|^{b_i} \exp(-|t|^p)$.

2) Theorem 3 does not cover the case of the uniform measure on $B_\infty^n = [-1, 1]^n$, which clearly has simultaneously the Cartesian and the polar decomposition with respect to $K = B_\infty^n$. It is the natural limit case of the examples with the densities $\exp(-|t|^p)$. Under strong conditions on the density and its support, results can be obtained which encompass measures supported on the cube. It would be very nice to get rid of the conditions. It seems that one of the necessary steps would be to understand the structure of sets in $\mathbb{R}^n$ which are products with respect to the Cartesian structure and for the polar structure generated by a convex set K. This is a problem of independent interest.

3) The classification of simultaneous product measures, without additional hypothesis, is a very challenging problem. Note that our results may be used. Indeed if μ has simultaneous product decomposition, then its absolutely continuous part has it too. Similarly, if a singular measure has the property, then its atomic part has it too, so Theorem 6 applies. The main obstacle seems to be dealing with singular continuous measures.

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