Pisier's inequality revisited

Tuomas Hytönen
Department of Mathematics and Statistics
University of Helsinki
P.O.B. 68, FI-00014 Helsinki, Finland
E-mail: tuomas.hytonen@helsinki.fi

Assaf Naor
Courant Institute of Mathematical Sciences
New York University
251 Mercer Street, New York NY, 10012, USA
E-mail: naor@cims.nyu.edu

Abstract

Given a Banach space X, for $n \in \mathbb{N}$ and $p \in (1,\infty)$ we investigate the smallest constant $\mathfrak{P} \in (0,\infty)$ for which every n-tuple of functions $f_1,\ldots,f_n : \{-1,1\}^n \to X$ satisfies

$$\int_{\{-1,1\}^n} \left\| \sum_{j=1}^n \partial_j f_j(\varepsilon) \right\|^p d\mu(\varepsilon) \leqslant \mathfrak{P}^p \int_{\{-1,1\}^n} \int_{\{-1,1\}^n} \left\| \sum_{j=1}^n \delta_j \Delta f_j(\varepsilon) \right\|^p d\mu(\varepsilon) d\mu(\delta),$$

where μ is the uniform probability measure on the discrete hypercube $\{-1,1\}^n$ and $\{\partial_j\}_{j=1}^n$ and $\Delta = \sum_{j=1}^n \partial_j$ are the hypercube partial derivatives and the hypercube Laplacian, respectively. Denoting this constant by $\mathfrak{P}_p^n(X)$, we show that

$$\mathfrak{P}_p^n(X) \leqslant \sum_{k=1}^n \frac{1}{k}$$

for every Banach space $(X, \|\cdot\|)$. This extends the classical Pisier inequality, which corresponds to the special case $f_j = \Delta^{-1}\partial_j f$ for some $f : \{-1,1\}^n \to X$. We show that $\sup_{n\in\mathbb{N}} \mathfrak{P}_p^n(X) < \infty$ if either the dual X^* is a UMD$^+$ Banach space, or for some $\theta \in (0,1)$ we have $X = [H,Y]_\theta$, where H is a Hilbert space and Y is an arbitrary Banach space. It follows that $\sup_{n\in\mathbb{N}} \mathfrak{P}_p^n(X) < \infty$ if X is a Banach lattice of nontrivial type.

1 Introduction

Fix a Banach space $(X, \|\cdot\|)$ and $n \in \mathbb{N}$. For every $f : \{-1,1\}^n \to X$ and $j \in \{1,\ldots,n\}$ the hypercube jth partial derivative of f, which is denoted $\partial_j f : \{-1,1\}^n \to X$, is defined as

$$\partial_j f(\varepsilon) \stackrel{\text{def}}{=} \frac{f(\varepsilon) - f(\varepsilon_1,\ldots,\varepsilon_{j-1},-\varepsilon_j,\varepsilon_{j+1},\ldots,\varepsilon_n)}{2}. \tag{1.1}$$

The hypercube Laplacian of f, denoted $\Delta f : \{-1,1\}^n \to X$, is

$$\Delta f(\varepsilon) \stackrel{\text{def}}{=} \sum_{j=1}^n \partial_j f(\varepsilon). \tag{1.2}$$

It is immediate to check that Δ is invertible on the space of all mean zero functions $f : \{-1,1\}^n \to X$. Below Δ^{-1} is understood to be defined for every $f : \{-1,1\}^n \to X$ by setting $\Delta^{-1} f = \Delta^{-1}\overline{f}$. Here $\overline{f} = f - \int_{\{-1,1\}^n} f(\delta) d\mu(\delta)$, where μ denotes the uniform probability measure on $\{-1,1\}^n$.

The following inequality is due to Pisier [28]. Throughout this paper the asymptotic notation $\lesssim, \gtrsim$ indicates the corresponding inequalities up to universal constant factors. We will also denote equivalence up to universal constant factors by $\asymp$, i.e., $A \asymp B$ is the same as $(A \lesssim B) \wedge (A \gtrsim B)$.

Theorem 1.1 (Pisier's inequality)**.** For every Banach space $(X, \|\cdot\|)$, every $n \in \mathbb{N}$, every $p \in [1,\infty]$ and every $f : \{-1,1\}^n \to X$, we have

$$\left(\int_{\{-1,1\}^n} \left\| f(\varepsilon) - \int_{\{-1,1\}^n} f(\delta) d\mu(\delta) \right\|^p d\mu(\varepsilon)\right)^{1/p} \lesssim \log n \left(\int_{\{-1,1\}^n}\int_{\{-1,1\}^n} \left\| \sum_{j=1}^n \delta_j \partial_j f(\varepsilon) \right\|^p d\mu(\varepsilon) d\mu(\delta)\right)^{1/p}. \tag{1.3}$$

Due to the application of Pisier's inequality to the theory of nonlinear type (see [28, 25, 12, 23]), it is of great interest to understand when (1.3) holds true with the $\log n$ term replaced by a constant that may depend on the geometry of X but is independent of n. Talagrand proved [30] that the $\log n$ term in (1.3) is asymptotically optimal for general Banach spaces X, Wagner proved [31] that the $\log n$ term in (1.3) can be replaced by a universal constant if $p = \infty$ and X is a general Banach space, and in [25] it is shown that the $\log n$ term in (1.3) can be replaced by a constant that is independent of n if X is a UMD Banach space. It remains an intriguing open question whether every Banach space of nontrivial type satisfies (1.3) with

the $\log n$ term replaced by a constant that is independent of n. If true, this would resolve a 1976 question of Enflo [9] by establishing that Rademacher type p and Enflo type p coincide (see [25, 23] and Section 6 below).

Here we obtain a new class of Banach spaces that satisfies a dimension-independent Pisier inequality. Our starting point is the following extension of Pisier's inequality.

Definition 1.2 (Pisier constant of X)**.** The n-dimensional Pisier constant of X (with exponent p), denoted $\mathfrak{P}_p^n(X)$, is the infimum over those $\mathfrak{P} \in (0,\infty)$ such that every $f_1,\ldots,f_n : \{-1,1\}^n \to X$ satisfy

$$\left(\int_{\{-1,1\}^n} \left\| \sum_{j=1}^n \partial_j f_j(\varepsilon) \right\|^p d\mu(\varepsilon)\right)^{1/p} \leqslant \mathfrak{P}\left(\int_{\{-1,1\}^n}\int_{\{-1,1\}^n} \left\| \sum_{j=1}^n \delta_j \Delta f_j(\varepsilon) \right\|^p d\mu(\varepsilon) d\mu(\delta)\right)^{1/p}. \tag{1.4}$$

We also set

$$\mathfrak{P}_p(X) \stackrel{\text{def}}{=} \sup_{n\in\mathbb{N}} \mathfrak{P}_p^n(X).$$

Inequality (1.4) reduces to Pisier's inequality if we choose $f_j = \Delta^{-1}\partial_j f$ for some $f : \{-1,1\}^n \to X$. The generalized inequality (1.4) has the advantage of being well-behaved under duality, as explained in Section 2. The following theorem yields a logarithmic bound on $\mathfrak{P}_n^p(X)$, thus extending Pisier's inequality.

Theorem 1.3. For every Banach space X, every $p \in [1,\infty]$ and every $n \in \mathbb{N}$,

$$\mathfrak{P}_p^n(X) \leqslant \sum_{k=1}^n \frac{1}{k}.$$

Our approach yields a quantitative improvement over Pisier's inequality only in lower order terms: an optimization of Pisier's argument (as carried out in [23]) shows that the $O(\log n)$ term in (1.3) can be taken to be at most $\log n + O(\log\log n)$, while Theorem 1.3 shows that this term can be taken to be $\log n + O(1)$.

In [25] it was shown that the logarithmic term in (1.3) can be replaced by a constant that is independent of n if X is a UMD Banach space. Recall that X is a UMD Banach space if for every $p \in (1,\infty)$ there exists a constant $\beta \in (0,\infty)$ such that if $\{M_j\}_{j=0}^n$ is a p-integrable X-valued martingale

defined on some probability space $(\Omega, \mathbb{P})$, then for every $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ we have

$$\int_\Omega \left\| M_0 + \sum_{j=1}^n \varepsilon_j (M_j - M_{j-1}) \right\|^p d\mathbb{P} \leqslant \beta^p \int_\Omega \|M_n\|^p d\mathbb{P}. \tag{1.5}$$

The infimum over those $\beta \in (0, \infty)$ for which (1.5) holds true is denoted $\beta_p(X)$. It can be shown (see [7]) that $\beta_p(X) \lesssim \frac{p^2}{p-1}\beta_2(X)$, so in order to define the UMD property it suffices to require the validity of (1.5) for $p = 2$. UMD Banach spaces are known to be superreflexive [20, 1], and one also has $\beta_q(X^*) = \beta_p(X)$, where $q = p/(p-1)$ (see e.g. [7]).

In [10] Garling investigated the natural weakening of (1.5) in which the desired inequality is required to hold true in expectation over $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ rather than for every $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$. Specifically, say that X is a UMD$^+$ Banach space if for every $p \in (1, \infty)$ there exists a constant $\beta \in (0, \infty)$ such that if $\{M_j\}_{j=0}^n$ is a p-integrable X-valued martingale defined on some probability space $(\Omega, \mathbb{P})$ then

$$\int_{\{-1,1\}^n} \int_\Omega \left\| M_0 + \sum_{j=1}^n \varepsilon_j (M_j - M_{j-1}) \right\|^p d\mathbb{P} d\mu(\varepsilon) \leqslant \beta^p \int_\Omega \|M_n\|^p d\mathbb{P}. \tag{1.6}$$

The infimum over those β for which (1.6) holds true is denoted $\beta_p^+(X)$.

Theorem 1.4. If X is a Banach space such that X^* is UMD$^+$ then the following inequality holds true. Fix $p \in (1, \infty)$ and $n \in \mathbb{N}$. For every function $F : \{-1, 1\}^n \times \{-1, 1\}^n \to X$ and $j \in \{1, \ldots, n\}$ denote

$$F_j(\varepsilon) \stackrel{\text{def}}{=} \int_{\{-1,1\}^n} \delta_j F(\varepsilon, \delta) d\mu(\delta).$$

Then

$$\left(\int_{\{-1,1\}^n} \left\| \sum_{j=1}^n \Delta^{-1} \partial_j F_j(\varepsilon) \right\|^p d\mu(\varepsilon) \right)^{1/p} \leqslant \beta_q^+(X^*) \left(\int_{\{-1,1\}^n} \int_{\{-1,1\}^n} \|F(\varepsilon, \delta)\|^p d\mu(\varepsilon) d\mu(\delta) \right)^{1/p}, \tag{1.7}$$

where $q = p/(p-1)$.

For every $f_1, \ldots, f_n : \{-1, 1\}^n \to X$, an application of Theorem 1.4 to the function $F(\varepsilon, \delta) = \sum_{j=1}^n \delta_j f_j(\varepsilon)$ yields the following estimate on the Pisier constant of a UMD$^+$ Banach space.

Corollary 1.5. $\mathfrak{P}_p(X) \leqslant \beta_q^+(X^*)$.

It is unknown if a UMD$^+$ Banach space must also be a UMD Banach space, though it seems reasonable to conjecture that there are UMD$^+$ spaces that are not UMD. Regardless of this, Theorem 1.4 and Corollary 1.5 are conceptually different from the result of [25], which relies on the full force of the UMD condition, i.e. it requires the validity of (1.5) for every choice of signs $\varepsilon_1, \ldots, \varepsilon_n$, while our argument needs such estimates to hold true only for an average choice of signs. We also have a quantitative improvement: in [25] it was shown that Pisier's inequality holds true with the $O(\log n)$ term in (1.3) replaced by $\beta_p(X) = \beta_q(X^*)$, while we obtain the same estimate with the $O(\log n)$ term in (1.3) replaced by $\beta_q^+(X^*) \leqslant \beta_q(X^*)$. Geiss proved [11] that for every $\eta \in (0,1)$ there is $C_\eta \in (0,\infty)$ such that for every $M > 1$ there is a Banach space X that satisfies

$$\infty > \beta_q(X^*) \geqslant C_\eta \beta_q^+(X^*)^{2-\eta} \geqslant M.$$

Remark 1.1. *Inequality* (1.7) *is an extension of the generalized Pisier inequality* (1.4)*, but for general Banach spaces it behaves very differently: unlike the logarithmic behavior of Theorem 1.3, the best constant appearing in the right hand side of* (1.7) *for a general Banach space X must be at least a constant multiple of $\sqrt{n}$, as exhibited by the case $X = L_1((\{-1,1\}^n, \mu), \mathbb{R})$ and $F : \{-1,1\}^n \times \{-1,1\}^n \to X$ given by $F(\varepsilon,\delta)(\eta) = \prod_{i=1}^n (1 + \varepsilon_i \delta_i \eta_i)$.*

Suppose that $\theta \in (0,1)$ and $X = [H, Y]_\theta$, where H is a Hilbert space and Y is an arbitrary Banach space. Here $[\cdot,\cdot]_\theta$ denotes complex interpolation (see [3]). Theorem 1.6 below shows that in this case $\mathfrak{P}_p(X) < \infty$, and therefore Pisier's inequality holds true with the $\log n$ term in (1.3) replaced by a constant that is independent of n. Pisier proved [27] that every Banach lattice of nontrivial type (see [19]) is of the form $[H, Y]_\theta$ for some $\theta \in (0,1)$, so we thus obtain the desired dimension independence in Pisier's inequality for Banach lattices of nontrivial type. This result does not follow from previously known cases in which a dimension-independent Pisier inequality has been proved, since, as shown by Bourgain [4, 5], there exist Banach lattices of nontrivial type which are not UMD. Note, however, that we are still far from proving the conjectured dimension-independent Pisier inequality for Banach spaces with nontrivial type: any space of the form $[H, Y]_\theta$ admits an equivalent norm whose modulus of smoothness has power type $2/(1+\theta)$ (see [27, 8]), while there exist Banach spaces with nontrivial type that do not admit such an equivalent norm (see [14, 16, 15, 29]).

Theorem 1.6. Let X, Y be Banach spaces and let H be a Hilbert space. Suppose that for some $\theta \in (0,1)$ we have $X = [H, Y]_\theta$. Then for every $p \in (1,\infty)$,

$$\mathfrak{P}_p(X) \leqslant \frac{2\max\{p, p/(p-1)\}}{1-\theta}.$$

Remark 1.2. *If $r \in (2,\infty)$ then the $O(\log n)$ term in Pisier's inequality (1.3), when $p = 2$ and $X = \ell_r$, can be replaced by $O(r)$, due to the fact that $\beta_2^+(\ell_r) \asymp r$ (this follows from Hitczenko's work [13], as explained to us by Mark Veraar). This bound also follows from Theorem 1.6. At the same time, an inspection of Talagrand's example in [30] shows that this term must be at least a constant multiple of $\log r$. Determining the correct order of magnitude as $r \to \infty$ of the constant in Pisier's inequality when $X = \ell_r$ remains an interesting open problem.*

2 Duality

The dimension $n \in \mathbb{N}$ will be fixed from now on. For $p \in [1,\infty]$ and a Banach space X, let $L_p(X)$ denote the vector-valued Lebesgue space $L_p((\{-1,1\}^n, \mu), X)$. Thus $L_p(L_p(X))$ can be naturally identified with the space $L_p((\{-1,1\}^n \times \{-1,1\}^n, \mu\times\mu), X)$.

For $f \in L_p(X)$ we denote its Fourier expansion by

$$f = \sum_{A\subseteq\{1,\ldots,n\}} \widehat{f}(A) W_A,$$

where the Walsh function $W_A : \{-1,1\}^n \to \{-1,1\}$ corresponding to $A \subseteq \{1,\ldots,n\}$ is given by $W_A(\varepsilon_1,\ldots,\varepsilon_n) = \prod_{i\in A}\varepsilon_i$, and the Fourier coefficient $\widehat{f}(A) \in X$ is given by $\widehat{f}(A) = \int_{\{-1,1\}^n} f(x) W_A(x) d\mu(x)$. Using this (standard) notation, we have

$$\forall\, i \in \{1,\ldots,n\},\ \forall\, f \in L_p(X), \quad \partial_i f = \sum_{\substack{A\subseteq\{1,\ldots,n\}\\ i\in A}} \widehat{f}(A) W_A,$$

$$\forall\, f \in L_p(X), \quad \Delta f = \sum_{A\subseteq\{1,\ldots,n\}} |A| \widehat{f}(A) W_A,$$

and

$$\forall\, f \in L_p(X), \quad \Delta^{-1} f \stackrel{\text{def}}{=} \sum_{\substack{A\subseteq\{1,\ldots,n\}\\ A\neq\emptyset}} \frac{1}{|A|} \widehat{f}(A) W_A.$$

The Rademacher projection of $f \in L_p(X)$ is defined as usual by

$$\mathbf{Rad}(f) \stackrel{\text{def}}{=} \sum_{i=1}^n \widehat{f}(\{i\}) W_{\{i\}}.$$

We denote below $\mathbf{Rad}_X \stackrel{\text{def}}{=} \mathbf{Rad}(L_p(X))$ and $\mathbf{Rad}_X^{\perp} \stackrel{\text{def}}{=} (I - \mathbf{Rad})(L_p(X))$. The dual of $(\mathbf{Rad}_X, \|\cdot\|_{L_p(X)})$ is naturally identified with the quotient $L_q(X^*)/\mathbf{Rad}_{X^*}^{\perp}$, where $q = p/(p-1)$.

Define an operator $S : L_p(L_p(X)) \to L_p(X)$ by

$$\forall\, F \in L_p(L_p(X)), \quad S(F) \stackrel{\text{def}}{=} \sum_{j=1}^{n} \Delta^{-1}\partial_j \widehat{F}(\{j\}). \tag{2.1}$$

Using this notation, Theorem 1.4 is nothing more than the following operator norm bound.

$$\|S\|_{L_p(L_p(X))\to L_p(X)} \leqslant \beta_q^+(X^*).$$

The adjoint operator $S^* : L_q(X^*) \to L_q(L_q(X^*))$ is given by

$$\forall\, g \in L_q(X^*),\ \forall\, \delta \in \{-1,1\}^n, \quad S^*(g)(\delta) = \sum_{j=1}^{n} \delta_j \Delta^{-1}\partial_j g.$$

Therefore Theorem 1.4 has the following equivalent dual formulation.

Theorem 2.1 (Dual formulation of Theorem 1.4)**.** Let Z be a UMD$^+$ Banach space. Then for every $q \in (1,\infty)$ and every $g \in L_q(Z)$ we have

$$\left(\int_{\{-1,1\}^n} \left\| \sum_{j=1}^{n} \delta_j \Delta^{-1}\partial_j g \right\|_{L_q(Z)}^q d\mu(\delta)\right)^{1/q} \leqslant \beta_q^+(Z)\|g\|_{L_q(Z)}.$$

Theorem 2.1, and consequently also Theorem 1.4, will be proven in Section 3.

Let T be the restriction of S to $\mathbf{Rad}_{L_p(X)}$. Thus

$$\mathfrak{P}_p^n(X) = \|T\|_{\mathbf{Rad}_{L_p(X)}\to L_p(X)} = \|T^*\|_{L_q(X^*)\to L_q(L_q(X^*))/\mathbf{Rad}_{L_q(X^*)}^{\perp}}.$$

The adjoint $T^* : L_q(X^*) \to L_q(L_q(X^*))/\mathbf{Rad}_{L_q(X^*)}^{\perp}$ is given by

$$\forall\, g \in L_q(X^*),\ \forall\, \delta \in \{-1,1\}^n, \quad T^*(g) = \sum_{j=1}^{n} \delta_j \Delta^{-1}\partial_j g + \mathbf{Rad}_{L_q(X^*)}^{\perp}.$$

Therefore Theorem 1.3 has the following equivalent dual formulation.

Theorem 2.2 (Dual formulation of Theorem 1.3)**.** Let Z be a Banach space and $q \in [1,\infty]$. Then for every $g \in L_q(Z)$ we have

$$\inf_{\Phi\in\mathbf{Rad}_{L_q(Z)}^{\perp}} \left(\int_{\{-1,1\}^n} \left\| \Phi(\delta) + \sum_{j=1}^{n} \delta_j \Delta^{-1}\partial_j g \right\|_{L_q(Z)}^q d\mu(\delta)\right)^{1/q} \leqslant \left(\sum_{k=1}^{n} \frac{1}{k}\right) \|g\|_{L_q(Z)}.$$

Theorem 2.2, and consequently also Theorem 1.3, will be proven in Section 3. Since $[H,Y]^*_\theta = [H,Y^*]_\theta$ (see [3]), we also have the following equivalent dual formulation of Theorem 1.6.

Theorem 2.3 (Dual formulation of Theorem 1.6)**.** Let H be a Hilbert space, W a Banach space, and $\theta \in (0,1)$. Set $Z = [H,W]_\theta$. Then for every $q \in (1,\infty)$ and $g \in L_q(Z)$,

$$\inf_{\Psi \in \mathbf{Rad}^+_{L_q(Z)}} \left(\int_{\{-1,1\}^n} \left\| \Psi(\delta) + \sum_{j=1}^n \delta_j \Delta^{-1} \partial_j g \right\|^q_{L_q(Z)} d\mu(\delta) \right)^{1/q} \leqslant \frac{2\max\{q, q/(q-1)\}}{1-\theta} \|g\|_{L_q(Z)}.$$

Theorem 2.3, and consequently also Theorem 1.6, will be proven in Section 5.

3 Proof of Theorem 2.1

Fix $q \in (1,\infty)$ and $g \in L_q(Z)$. Let S_n denote the symmetric group on $\{1,\ldots,n\}$. For $\sigma \in S_n$ and $k \in \{0,\ldots,n\}$ define $g^\sigma_k \in L_q(Z)$ by

$$g^\sigma_k(\varepsilon) \stackrel{\text{def}}{=} \sum_{A \subseteq \{\sigma^{-1}(1),\ldots,\sigma^{-1}(k)\}} \widehat{g}(A) W_A(\varepsilon) \tag{3.1}$$

$$= \frac{1}{2^{n-k}} \sum_{\delta_{\sigma^{-1}(k+1)},\ldots,\delta_{\sigma^{-1}(n)} \in \{-1,1\}} g\left(\sum_{i=1}^k \varepsilon_{\sigma^{-1}(i)} e_{\sigma^{-1}(i)} + \sum_{i=k+1}^n \delta_{\sigma^{-1}(i)} e_{\sigma^{-1}(i)} \right),$$

where here, and in what follows, $e_1,\ldots,e_n$ denotes the standard basis of $\mathbb{R}^n$. Then $\{g^\sigma_k\}_{k=0}^n$ is a Z-valued martingale with $g^\sigma_n = g$ and $g^\sigma_0 = \widehat{g}(\emptyset)$, implying that

$$\left(\int_{\{-1,1\}^n} \left\| \sum_{k=1}^n \delta_k \left(g^\sigma_k - g^\sigma_{k-1} \right) \right\|^q_{L_q(Z)} d\mu(\delta) \right)^{1/q} \leqslant \beta^+_q(Z) \|g\|_{L_q(Z)}. \tag{3.2}$$

In (3.2) we may replace $\{\delta_k\}_{k=1}^n$ by $\{\delta_{\sigma^{-1}(k)}\}_{k=1}^n$, since these two sequences of signs have the same joint distribution. Then we make the change of variable $j = \sigma^{-1}(k)$, so that $k = \sigma(j)$. Averaging the resulting inequality over $\sigma \in S_n$, and using the convexity of the norm, we see that

$$\left(\int_{\{-1,1\}^n} \left\| \frac{1}{n!} \sum_{\sigma \in S_n} \sum_{j=1}^n \delta_j \left(g^\sigma_{\sigma(j)} - g^\sigma_{\sigma(j)-1} \right) \right\|^q_{L_q(Z)} d\mu(\delta) \right)^{1/q} \leqslant \beta^+_q(Z) \|g\|_{L_q(Z)}. \tag{3.3}$$

It remains to note that for each $\delta \in \{-1,1\}^n$ we have

$$\begin{aligned}
&\frac{1}{n!}\sum_{\sigma\in S_n}\sum_{j=1}^{n}\delta_j\left(g^\sigma_{\sigma(j)}-g^\sigma_{\sigma(j)-1}\right)\\
&=\frac{1}{n!}\sum_{\sigma\in S_n}\sum_{j=1}^{n}\delta_j\sum_{\substack{\emptyset\subsetneq A\subseteq\{1,\ldots,n\}\\ \max\sigma(A)=\sigma(j)}}\widehat{g}(A)W_A\\
&=\sum_{\substack{A\subseteq\{1,\ldots,n\}\\ A\neq\emptyset}}\sum_{j\in A}\delta_j\frac{|\{\sigma\in S_n:\max\sigma(A)=\sigma(j)\}|}{n!}\widehat{g}(A)W_A\\
&=\sum_{\substack{A\subseteq\{1,\ldots,n\}\\ A\neq\emptyset}}\frac{\sum_{j\in A}\delta_j}{|A|}\widehat{g}(A)W_A\\
&=\sum_{j=1}^{n}\delta_j\Delta^{-1}\partial_j g. \qquad (3.4)
\end{aligned}$$

Due to (3.3) and (3.4) the proof of Theorem 2.1 is complete. □

4 Proof of Theorem 2.2

The following lemma introduces an auxiliary function which is a variant of a similar function that was used by Pisier in [28].

Lemma 4.1. Let Z be a Banach space. Fix $n\in\mathbb{N}$, $q\in[1,\infty]$ and $t\in(0,1)$. For $g\in L_q(Z)$ define $G_t\in L_q(L_q(Z))$ by

$$G_t(\delta)\stackrel{\text{def}}{=}\frac{1}{1-t}\sum_{A\subseteq\{1,\ldots,n\}}\widehat{g}(A)W_A\prod_{i\in A}(t+(1-t)\delta_i)-\frac{t^n}{1-t}g. \qquad (4.1)$$

Then

$$\mathbf{Rad}(G_t)(\delta)=\sum_{\substack{A\subseteq\{1,\ldots,n\}\\ A\neq\emptyset}}t^{|A|-1}\sum_{j\in A}\delta_j\widehat{g}(A)W_A, \qquad (4.2)$$

and

$$\|G_t\|_{L_q(L_q(Z))}\leqslant\frac{1-t^n}{1-t}\|g\|_{L_q(Z)}. \qquad (4.3)$$

Proof. Identity (4.2) follows from (4.1) since for every $A\subseteq\{1,\ldots,n\}$,

$$\mathbf{Rad}\left(\prod_{i\in A}(t+(1-t)\delta_i)\right)=t^{|A|-1}(1-t)\sum_{j\in A}\delta_j.$$

To prove (4.3) observe that for every $\varepsilon, \delta \in \{-1,1\}^n$,

$$\begin{aligned}
&(1-t)G_t(\delta)(\varepsilon)\\
&= \sum_{A\subseteq\{1,\ldots,n\}} \widehat{g}(A) W_A(\varepsilon) \prod_{i=1}^n \left(t + (1-t)\delta_i^{\mathbf{1}_A(i)}\right) - t^n g(\varepsilon) \qquad (4.4)\\
&= \sum_{A\subseteq\{1,\ldots,n\}} \widehat{g}(A) W_A(\varepsilon) \sum_{B\subseteq\{1,\ldots,n\}} t^{|B|}(1-t)^{n-|B|} W_{A\smallsetminus B}(\delta) - t^n g(\varepsilon)\\
&= \sum_{B\subsetneq\{1,\ldots,n\}} t^{|B|}(1-t)^{n-|B|} \sum_{A\subseteq\{1,\ldots,n\}} \widehat{g}(A) W_{A\cap B}(\varepsilon) W_{A\smallsetminus B}(\varepsilon\delta)\\
&= \sum_{B\subsetneq\{1,\ldots,n\}} t^{|B|}(1-t)^{n-|B|} g_B(\varepsilon,\delta), \qquad (4.5)
\end{aligned}$$

where in (4.4) we use (4.1) and in (4.5) for every $B \subseteq \{1,\ldots,n\}$ we set

$$g_B(\varepsilon,\delta) \stackrel{\text{def}}{=} g\left(\sum_{j\in B} \varepsilon_j e_j + \sum_{j\in\{1,\ldots,n\}\smallsetminus B} \varepsilon_j\delta_j e_j\right).$$

Since g_B is equidistributed with g, it follows from (4.5) that

$$\frac{\|G_t\|_{L_q(L_q(Z))}}{\|g\|_{L_q(Z)}} \leqslant \frac{1}{1-t}\sum_{B\subsetneq\{1,\ldots,n\}} t^{|B|}(1-t)^{n-|B|} = \frac{1-t^n}{1-t}. \qquad \square$$

Proof of Theorem 2.2. Observe that for every $\delta \in \{-1,1\}^n$ we have

$$\begin{aligned}
\sum_{j=1}^n \delta_j \Delta^{-1}\partial_j g &= \sum_{\substack{A\subseteq\{1,\ldots,n\}\\ A\neq\emptyset}} \frac{1}{|A|}\sum_{j\in A} \delta_j \widehat{g}(A) W_A\\
&= \sum_{\substack{A\subseteq\{1,\ldots,n\}\\ A\neq\emptyset}} \left(\int_0^1 t^{|A|-1}dt\right)\sum_{j\in A}\delta_j\widehat{g}(A)W_A\\
&\stackrel{(4.2)}{=} \mathbf{Rad}\left(\int_0^1 G_t(\delta)dt\right). \qquad (4.6)
\end{aligned}$$

It follows that if we set

$$\Phi \stackrel{\text{def}}{=} \int_0^1 G_t dt - \mathbf{Rad}\left(\int_0^1 G_t dt\right). \qquad (4.7)$$

then $\Phi \in \mathbf{Rad}_{L_q(Z)}^{\perp}$ and

$$\begin{aligned}
&\left(\int_{\{-1,1\}^n} \left\|\Phi(\delta) + \sum_{j=1}^n \delta_j\Delta^{-1}\partial_j g\right\|_{L_q(Z)}^q d\mu(\delta)\right)^{1/q}\\
&\qquad \stackrel{(4.6)}{=} \left\|\int_0^1 G_t dt\right\|_{L_q(L_q(Z))} \stackrel{(4.3)}{\leqslant} \left(\int_0^1 \frac{1-t^n}{1-t}dt\right)\|g\|_{L_q(Z)}.
\end{aligned}$$

It remains to note that $\int_0^1 \frac{1-t^n}{1-t}dt = \sum_{k=0}^{n-1}\int_0^1 t^k dt = \sum_{k=1}^n \frac{1}{k}$. $\square$

5 Proof of Theorem 2.3

For $t \in (0,1)$ define a linear operator $V_t : L_q(Z) \to L_q(L_q(Z))$ by

$$V_t(g)(\delta) \stackrel{\text{def}}{=} G_t(\delta) - \widehat{G_t}(\emptyset) \stackrel{(4.1)}{=} \frac{1}{1-t} \sum_{A \subseteq \{1,\ldots,n\}} \widehat{g}(A) W_A \left(\prod_{i \in A} (t + (1-t)\delta_i) - t^{|A|} \right). \tag{5.1}$$

Lemma 5.1. Let H be a Hilbert space. Then for every $t \in (0,1)$,

$$\|V_t\|_{L_2(H) \to L_2(L_2(H))} \leqslant \frac{1}{\sqrt{1-t^2}} \leqslant \frac{1}{\sqrt{1-t}}. \tag{5.2}$$

Proof. Observe that for every $A \subseteq \{1,\ldots,n\}$ we have

$$\int_{\{-1,1\}^n} \left(\prod_{i \in A} (t + (1-t)\delta_i) - t^{|A|} \right)^2 d\mu(\delta) = \sum_{B \subsetneq A} t^{2|B|} (1-t)^{2(|A|-|B|)} = \left(t^2 + (1-t)^2\right)^{|A|} - t^{2|A|}. \tag{5.3}$$

It follows from (5.1), (5.3), and the orthogonality of $\{W_A\}_{A \subseteq \{1,\ldots,n\}}$, that

$$\|V_t\|_{L_2(H) \to L_2(L_2(H))} = \max_{a \in \{1,\ldots,n\}} \frac{\sqrt{(t^2 + (1-t)^2)^a - t^{2a}}}{1-t}. \tag{5.4}$$

Now, for every $a \in \{1,\ldots,n\}$ and $t \in (0,1)$ we have

$$\begin{aligned} \left(t^2 + (1-t)^2\right)^a - t^{2a} &= (1-t)^2 \sum_{k=0}^{a-1} \left(t^2 + (1-t)^2\right)^{a-1-k} t^{2k} \\ &\leqslant (1-t)^2 \sum_{k=0}^{a-1} t^{2k} = (1-t)^2 \frac{1-t^{2a}}{1-t^2} \leqslant \frac{1-t}{1+t}, \end{aligned} \tag{5.5}$$

where in the first inequality of (5.5) we used the estimate $t^2 + (1-t)^2 \leqslant 1$, which holds for every $t \in [0,1]$. The desired estimate (5.2) now follows from a substitution of (5.5) into (5.4). □

Lemma 5.2. Let H be a Hilbert space and let W be a Banach space. Fix $\theta \in (0,1)$ and $q \in (1,\infty)$. Set $Z = [H,W]_\theta$. Then for every $t \in (0,1)$ we have

$$\|V_t\|_{L_q(Z) \to L_q(L_q(Z))} \leqslant \frac{2}{(1-t)^{1-(1-\theta)\min\{1/q,1-1/q\}}}. \tag{5.6}$$

Proof. For every $r \in [1,\infty]$ we have

$$\|V_t(g)\|_{L_r(L_r(W))} \overset{(5.1)}{=} \left\|G_t - \widehat{G_t}(\emptyset)\right\|_{L_r(L_r(W))} \leqslant 2\|G_t\|_{L_r(L_r(W))} \overset{(4.3)}{\leqslant} \frac{2\|g\|_{L_r(W)}}{1-t}.$$

Consequently,

$$\forall\, r \in [1,\infty], \quad \|V_t\|_{L_r(W)\to L_r(L_r(W))} \leqslant \frac{2}{1-t}. \tag{5.7}$$

If $q \in [2,\infty)$ then we interpolate (see [3]) between (5.2) and (5.7) with $W = H$ and $r = \infty$. If $q \in (1,2]$ then we interpolate between (5.2) and (5.7) with $W = H$ and $r = 1$. The norm bound thus obtained implies the estimate

$$\forall\, q \in (1,\infty), \quad \|V_t\|_{L_q(H)\to L_q(L_q(H))} \leqslant \frac{2}{(1-t)^{\max\{1/q,1-1/q\}}}. \tag{5.8}$$

Finally, interpolation between (5.8) and (5.7) with $r = q$ gives the desired norm bound (5.6). □

Proof of Theorem 2.3. By (5.1) we have $\mathbf{Rad}(V_t(g)) = \mathbf{Rad}(G_t)$. Therefore, analogously to (4.7), if we set

$$\Psi \overset{\text{def}}{=} \int_0^1 V_t(g)dt - \mathbf{Rad}\left(\int_0^1 G_t dt\right) = \int_0^1 V_t(g)dt - \mathbf{Rad}\left(\int_0^1 V_t(g)dt\right),$$

then $\Psi \in \mathbf{Rad}^{\perp}_{L_q(Z)}$ and by (4.6) for every $\delta \in \{-1,1\}^n$ we have

$$\Psi(\delta) + \sum_{j=1}^n \delta_j \Delta^{-1}\partial_j g = \int_0^1 V_t(g)(\delta)dt. \tag{5.9}$$

Hence,

$$\left(\int_{\{-1,1\}^n} \left\|\Psi(\delta) + \sum_{j=1}^n \delta_j \Delta^{-1}\partial_j g\right\|^q_{L_q(Z)} d\mu(\delta)\right)^{1/q} \overset{(5.9)\wedge(5.6)}{\leqslant} \int_0^1 \frac{2\|g\|_{L_q(Z)}}{(1-t)^{1-(1-\theta)\min\{1/q,1-1/q\}}}dt = \frac{2\|g\|_{L_q(Z)}}{(1-\theta)\min\{1/q,1-1/q\}}.$$

This is precisely the assertion of Theorem 2.3. □

6 Enflo type in uniformly smooth Banach spaces

A Banach space X has Rademacher type $p \in [1,2]$ (see e.g. [21]) if there exists $T_R \in (0,\infty)$ such that for all $n \in \mathbb{N}$ and all $x_1,\ldots,x_n \in X$,

$$\int_{\{-1,1\}^n} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^p d\mu(\varepsilon) \leqslant T_R^p \sum_{j=1}^n \|x_j\|^p. \tag{6.1}$$

X has Enflo type p (see [9, 6, 28, 25]) if there exists $T_E \in (0,\infty)$ such that for all $n \in \mathbb{N}$ and all $f : \{-1,1\}^n \to X$,

$$\int_{\{-1,1\}^n} \frac{\|f(\varepsilon) - f(-\varepsilon)\|^p}{2^p} d\mu(\varepsilon) \leqslant T_E^p \sum_{j=1}^n \|\partial_j f\|_{L_p(X)}^p. \tag{6.2}$$

By considering the function $f(\varepsilon) = \sum_{j=1}^n \varepsilon_j x_j$ one sees that (6.1) is a special case of (6.2). It is a long-standing open problem [9] whether, conversely, (6.1) implies (6.2). A crucial feature of (6.2) is that it is a purely metric condition (thus one can define when a *metric space* has Enflo type p), while (6.1) is a linear condition. See [22] for a purely metric condition (which is more complicated than, but inspired by, Enflo type) that is known to be equivalent to Rademacher type.

Observe that if (6.1) holds then it follows from (1.4) that for every $f_1,\ldots,f_n : \{-1,1\}^n \to X$,

$$\left\| \sum_{j=1}^n \Delta^{-1}\partial_j f_j \right\|_{L_p(X)} \leqslant T_R \mathfrak{P}_p^n(X) \left(\sum_{j=1}^n \|f_j\|_{L_p(X)}^p \right)^{1/p}. \tag{6.3}$$

The special case $f_j = \partial_j f$ shows that (6.3) implies (6.2) with

$$T_E \leqslant T_R \mathfrak{P}_p^n(X).$$

For this reason it is worthwhile to investigate (6.3) on its own right.

Let $\mathfrak{Q}_p^n(X)$ be the infimum over those $\mathfrak{Q} \in (0,\infty)$ such that every $f_1,\ldots,f_n : \{-1,1\}^n \to X$ satisfy

$$\left\| \sum_{j=1}^n \Delta^{-1}\partial_j f_j \right\|_{L_p(X)} \leqslant \mathfrak{Q} \left(\sum_{j=1}^n \|f_j\|_{L_p(X)}^p \right)^{1/p}. \tag{6.4}$$

We also set

$$\mathfrak{Q}_p(X) \stackrel{\text{def}}{=} \sup_{n\in\mathbb{N}} \mathfrak{Q}_p^n(X).$$

By duality, $\mathfrak{Q}_p^n(X)$ equals the infimum over those $\mathfrak{Q} \in (0,\infty)$ for which every $g \in L_q(X^*)$ satisfies

$$\left(\sum_{j=1}^n \left\|\Delta^{-1}\partial_j g\right\|_{L_q(X^*)}^q\right)^{1/q} \leqslant \mathfrak{Q}\|g\|_{L_q(X^*)}. \tag{6.5}$$

Letting $S_X = \{x \in X : \ \|x\| = 1\}$ denote the unit sphere of X, recall that the modulus of uniform convexity of X is defined for $\varepsilon \in [0,2]$ as

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \ x,y \in S_X, \ \|x-y\| = \varepsilon\right\}.$$

The modulus of uniform smoothness of X is defined for $\tau \in (0,\infty)$ as

$$\rho_X(\tau) \stackrel{\text{def}}{=} \sup\left\{\frac{\|x+\tau y\|+\|x-\tau y\|}{2} - 1 : \ x,y \in S_X\right\}.$$

These moduli relate to each other via the following classical duality formula of Lindenstrauss [18].

$$\delta_{X^*}(\varepsilon) = \sup\left\{\frac{\tau\varepsilon}{2} - \rho_X(\tau) : \ \tau \in [0,1]\right\}. \tag{6.6}$$

Theorem 6.1. For every $K, p \in (1,\infty)$ there exists $C(K,p) \in (0,\infty)$ such that if X is a Banach space that satisfies $\rho_X(\tau) \leqslant K\tau^p$ for all $\tau \in (0,\infty)$, then $\mathfrak{Q}_p(X) \leqslant C(K,p)$.

Proof. We shall use here the notation introduced in the proof of Theorem 2.1 (Section 3). It follows from (6.6) that $\delta_{X^*}(\varepsilon) \gtrsim_{K,p} \varepsilon^q$ for every $\varepsilon \in [0,2]$ (here, and it what follows, the notation $\lesssim_{K,p}$ suppresses constant factors that may depend only on K and p). Hence, for $g \in L_q(X^*)$ and $\sigma \in S_n$, since $\{g_k^\sigma\}_{k=0}^n$, as defined in (3.1), is an X^*-valued martingale, it follows from Pisier's martingale inequality [26] that

$$\left(\sum_{k=1}^n \|g_k^\sigma - g_{k-1}^\sigma\|_{L_q(X^*)}^q\right)^{1/q} \lesssim_{K,p} \|g\|_{L_q(X^*)}. \tag{6.7}$$

By reindexing (6.7) with $k = \sigma(j)$, averaging over $\sigma \in S_n$, and using the convexity of the norm, we obtain the estimate

$$\left(\sum_{j=1}^n \left\|\frac{1}{n!}\sum_{\sigma\in S_n}\left(g_{\sigma(j)}^\sigma - g_{\sigma(j)-1}^\sigma\right)\right\|_{L_q(X^*)}^q\right)^{1/q} \lesssim_{K,p} \|g\|_{L_q(X^*)}. \tag{6.8}$$

Arguing as in (3.4), for every $j \in \{1, \ldots, n\}$ we have the identity

$$(6.9)\quad \frac{1}{n!}\sum_{\sigma\in S_n}\left(g^\sigma_{\sigma(j)} - g^\sigma_{\sigma(j)-1}\right) = \frac{1}{n!}\sum_{\sigma\in S_n}\sum_{\substack{\emptyset\subsetneq A\subseteq\{1,\ldots,n\}\\ \max\sigma(A)=\sigma(j)}}\widehat{g}(A)W_A$$
$$= \sum_{\substack{A\subseteq\{1,\ldots,n\}\\ j\in A}}\frac{|\{\sigma\in S_n:\ \max\sigma(A)=\sigma(j)\}|}{n!}\widehat{g}(A)W_A = \Delta^{-1}\partial_j g.$$

Consequently, (6.8) combined with (6.9) imply that (6.5) holds true with $\mathfrak{Q} \lesssim_{K,p} 1$. This concludes the proof of Theorem 6.1. □

Remark 6.1. *It follows from [17, Sec. 6] that a Banach space X satisfying the assumption of Theorem 6.1 has Enflo type p. Theorem 6.1 can be viewed as a generalization of this fact to yield the inequality* (6.4)*. In [24] it was shown that any Banach space satisfying the assumption of Theorem 6.1 actually has K. Ball's Markov type p property [2], a property which is a useful strengthening of Enflo type p.*

Acknowledgements

T. H. is supported in part by the European Union through the ERC Starting Grant "Analytic probabilistic methods for borderline singular integrals", and by the Academy of Finland through projects 130166 and 133264. A. N. is supported in part by NSF grant CCF-0832795, BSF grant 2010021, the Packard Foundation and the Simons Foundation. Part of this work was completed while A. N. was visiting Université Pierre et Marie Curie, Paris, France.

References

[1] D. J. Aldous. Unconditional bases and martingales in $L_p(F)$. *Math. Proc. Cambridge Philos. Soc.*, 85(1):117–123, 1979.

[2] K. Ball. Markov chains, Riesz transforms and Lipschitz maps. *Geom. Funct. Anal.*, 2(2):137–172, 1992.

[3] J. Bergh and J. Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[4] J. Bourgain. Some remarks on Banach spaces in which martingale difference sequences are unconditional. *Ark. Mat.*, 21(2):163–168, 1983.

[5] J. Bourgain. On martingales transforms in finite-dimensional lattices with an appendix on the K-convexity constant. *Math. Nachr.*, 119:41–53, 1984.

[6] J. Bourgain, V. Milman, and H. Wolfson. On type of metric spaces. *Trans. Amer. Math. Soc.*, 294(1):295–317, 1986.

[7] D. L. Burkholder. Martingales and Fourier analysis in Banach spaces. In *Probability and analysis (Varenna, 1985)*, volume 1206 of *Lecture Notes in Math.*, pages 61–108. Springer, Berlin, 1986.

[8] M. Cwikel and S. Reisner. Interpolation of uniformly convex Banach spaces. *Proc. Amer. Math. Soc.*, 84(4):555–559, 1982.

[9] P. Enflo. Uniform homeomorphisms between Banach spaces. In *Séminaire Maurey-Schwartz (1975–1976), Espaces, L^p, applications radonifiantes et géométrie des espaces de Banach, Exp. No. 18*, page 7. Centre Math., École Polytech., Palaiseau, 1976.

[10] D. J. H. Garling. Random martingale transform inequalities. In *Probability in Banach spaces 6 (Sandbjerg, 1986)*, volume 20 of *Progr. Probab.*, pages 101–119. Birkhäuser Boston, Boston, MA, 1990.

[11] S. Geiss. A counterexample concerning the relation between decoupling constants and UMD-constants. *Trans. Amer. Math. Soc.*, 351(4):1355–1375, 1999.

[12] O. Giladi and A. Naor. Improved bounds in the scaled Enflo type inequality for Banach spaces. *Extracta Math.*, 25(2):151–164, 2010.

[13] P. Hitczenko. Domination inequality for martingale transforms of a Rademacher sequence. *Israel J. Math.*, 84(1-2):161–178, 1993.

[14] R. C. James. A nonreflexive Banach space that is uniformly nonoctahedral. *Israel J. Math.*, 18:145–155, 1974.

[15] R. C. James. Nonreflexive spaces of type 2. *Israel J. Math.*, 30(1-2):1–13, 1978.

[16] R. C. James and J. Lindenstrauss. The octahedral problem for Banach spaces. In *Proceedings of the Seminar on Random Series, Convex Sets and Geometry of Banach Spaces (Mat. Inst., Aarhus Univ., Aarhus, 1974; dedicated to the memory of E. Asplund)*, pages 100–120. Various Publ. Ser., No. 24, Aarhus Univ., Aarhus, 1975. Mat. Inst.

[17] S. Khot and A. Naor. Nonembeddability theorems via Fourier analysis. *Math. Ann.*, 334(4):821–852, 2006.

[18] J. Lindenstrauss. On the modulus of smoothness and divergent series in Banach spaces. *Michigan Math. J.*, 10:241–252, 1963.

[19] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces. II*, volume 97 of *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]*. Springer-Verlag, Berlin, 1979. Function spaces.

[20] B. Maurey. Système de Haar. In *Séminaire Maurey-Schwartz 1974–1975: Espaces L_p, applications radonifiantes et géométrie des espaces de Banach, Exp. Nos. I et II*, pages 26 pp. (erratum, p. 1). Centre Math., École Polytech., Paris, 1975.

[21] B. Maurey. Type, cotype and K-convexity. In *Handbook of the geometry of Banach spaces, Vol. 2*, pages 1299–1332. North-Holland, Amsterdam, 2003.

[22] M. Mendel and A. Naor. Scaled Enflo type is equivalent to Rademacher type. *Bull. Lond. Math. Soc.*, 39(3):493–498, 2007.

[23] A. Naor. An introduction to the Ribe program. *Jpn. J. Math.*, 7(2):167–233, 2012.

[24] A. Naor, Y. Peres, O. Schramm, and S. Sheffield. Markov chains in smooth Banach spaces and Gromov-hyperbolic metric spaces. *Duke Math. J.*, 134(1):165–197, 2006.

[25] A. Naor and G. Schechtman. Remarks on non linear type and Pisier's inequality. *J. Reine Angew. Math.*, 552:213–236, 2002.

[26] G. Pisier. Un exemple concernant la super-réflexivité. In *Séminaire Maurey-Schwartz 1974–1975: Espaces L^p applications radonifiantes et géométrie des espaces de Banach, Annexe No. 2*, page 12. Centre Math. École Polytech., Paris, 1975.

[27] G. Pisier. La méthode d'interpolation complexe: applications aux treillis de Banach. In *Séminaire d'Analyse Fonctionnelle (1978–1979)*, pages Exp. No. 17, 18. École Polytech., Palaiseau, 1979.

[28] G. Pisier. Probabilistic methods in the geometry of Banach spaces. In *Probability and analysis (Varenna, 1985)*, volume 1206 of *Lecture Notes in Math.*, pages 167–241. Springer, Berlin, 1986.

[29] G. Pisier and Q. H. Xu. Random series in the real interpolation spaces between the spaces v_p. In *Geometrical aspects of functional analysis (1985/86)*, volume 1267 of *Lecture Notes in Math.*, pages 185–209. Springer, Berlin, 1987.

[30] M. Talagrand. Isoperimetry, logarithmic Sobolev inequalities on the discrete cube, and Margulis' graph connectivity theorem. *Geom. Funct. Anal.*, 3(3):295–314, 1993.

[31] R. Wagner. Notes on an inequality by Pisier for functions on the discrete cube. In *Geometric aspects of functional analysis*, volume 1745 of *Lecture Notes in Math.*, pages 263–268. Springer, Berlin, 2000.