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SOME APPLICATIONS OF BALL'S EXTENSION THEOREM

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Abstract. We present two applications Ball's extension theorem. First we observe that Ball's extension theorem, together with the recent solution of Ball's Markov type 2 problem due to Naor, Peres, Schramm and Sheffilield, imply a generalization, and an alternative proof of, the Johnson-Lindenstrauss extension theorem. Secondly, we prove that the distortion required to embed the integer lattice $\{0,1,\ldots,m\}^n$, equipped with the ℓ_p^n metric, in any 2-uniformly convex Banach space is of order $\min\left\{n^{\frac12-\frac1p},m^{1-\frac2p}\right\}$.

1. Introduction

Let (Y,d_Y), (Z,d_Z) be metric spaces, and for every $X\subseteq Y$, denote by $e(X,Y,Z)$ the infimum over all constants K such that every Lipschitz function $f:X\to Z$ can be extended to a function $\widetilde{f}:Y\to Z$ satisfying $\|\widetilde{f}\|_{\mathrm{Lip}}\le K\|f\|_{\mathrm{Lip}}$. (If no such K exists, we set $e(X,Y,Z)=\infty$). We also define $e(Y,Z)=\sup\{e(X,Y,Z):\ X\subseteq Y\}$ and for every integer n, $e_n(Y,Z)=\sup\{e(X,Y,Z):\ X\subseteq Y,\ |X|\le n\}$. We refer to [11, 13, 22] and the references therein for a discussion of the rich history and many results dealing with these notions.

The investigation of these parameters has often involved the introduction of influential probabilistic tools. The first result on the parameter $e_n(X,Y)$ is due to Marcus and Pisier [16], who used the theory of stable processes to show that for every $1<p\le 2$, $e_n(L_p,L_2)=O\left((\log n)^{\frac1p-\frac12}\right)$. Johnson and Lindenstrauss [9] have shown that for every metric space X, $e_n(X,L_2)=O(\sqrt{\log n})$, and that $e_n(L_1,L_2)=\Omega\left(\sqrt{\frac{\log n}{\log\log n}}\right)$. Their paper used the method of *random projections*, and contained the celebrated Johnson-Lindenstrauss dimension reduction lemma, which has since found numerous applications in asymptotic geometry and theoretical computer science (see the monograph [27]). The best known general upper bound is due to Lee and Naor [11, 13], who used random partitions of metric spaces to show that for every metric space X and every Banach space Z, $e_n(X,Z)=O\left(\frac{\log n}{\log\log n}\right)$.

In an important paper, K. Ball [1] discovered a striking connection between the behavior of Markov chains in metric spaces and the Lipschitz extension problem. In

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particular, he introduced the notion of Markov type, and used it to prove a fundamental extension criterion. This criterion, in combination with the recent solution of Ball's Markov type 2 problem due to Naor, Peres, Schramm and Sheffield [22], yields the following extension theorem:

Theorem 1.1. *For every $2 \le p < \infty$ and every 2-uniformly convex Banach space Y,*

$$e(L_p, Y) \le 24K_2(Y)\sqrt{p-1}.$$

(Recall that a Banach space Y is said to be 2-uniformly convex with constant K if for every $x, y \in Y$,

$$2\|x\|^2 + \frac{2}{K^2}\|y\|^2 \le \|x+y\|^2 + \|x-y\|^2.$$

The least such K is called the 2-uniform convexity constant of Y, and is denoted $K_2(Y)$. See [14, 2] for a discussion of this notion.)

Several applications of Theorem 1.1 to the theory of Lipschitz extensions and the theory of bi-Lipschitz embeddings were presented in [22]. In this note we present additional applications of Theorem 1.1 to both of these theories. Our first observation is that the fact that L_p, $p > 2$, has Markov type 2 with constant $O(\sqrt{p})$ (proved in [22]) answers positively a question posed by K. Ball in [1], showing that indeed the Markov type approach to Lipschitz extension introduced in [1] gives an alternative proof of the Johnson-Lindenstrauss extension theorem [9]. Moreover, this approach yields a significant generalization of this theorem to the case when the target space is an arbitrary 2-uniformly convex Banach space. Although this is a rather straightforward corollary of the results of [1, 22], we believe that it is worthwhile to point out an alternative approach to the Johnson-Lindenstrauss extension theorem which does not depend on dimension reduction (especially since it is known that Johnson-Lindenstrauss type dimension reduction is not always possible [7, 12, 10]).

Our second application of Theorem 1.1 is to the problem of embedding the integer lattice into 2-uniformly convex normed spaces, improving a result of Bourgain [5]. Our main result is that if Y is an infinite dimensional 2-uniformly convex Banach space, $p > 2$, and m, n are integers, then the distortion required to embed the integer lattice $\{0, \ldots, m\}^n$, equipped with the ℓ_p^n metric, is $\Theta\left(\min\left\{n^{\frac{1}{2}-\frac{1}{p}}, m^{1-\frac{2}{p}}\right\}\right)$. This shows that up to constants which may depend only on p, the optimal embedding of the integer lattice is the better of two natural embeddings: the identity map and a natural snowflake-type embedding à la Schoenberg (the phase transition occurs at $m = \sqrt{n}$. See Section 3 for more details).

2. A generalization of the Johnson-Lindenstrauss theorem

The main purpose of this section is to present a very simple proof of the following strengthning of the Johnson-Lindenstrauss extension theorem:

Theorem 2.1. *Let Y be a 2 uniformly convex Banach space. Then for every metric space X and every $n \in \mathbb{N}$*

$$e_n(X, Y) \le 60K_2(Y)\sqrt{\log n}.$$

Proof. Fix $A = \{a_1, \ldots, a_n\} \subseteq X$ and $f : A \to Y$. Consider the Fréchet map $\mathcal{F} : X \to \ell_\infty^n$ given by

$$\mathcal{F}(x) = (d(x, a_1), d(x, a_2), \ldots, d(x, a_n)).$$

Then $\mathcal{F}|_A$ is easily seen to be an isometry (see [4, Lemma 1.1]), and $\|\mathcal{F}\|_{\mathrm{Lip}} \le 1$. Fix $p > 2$. By Hölder's inequality the identity mappings $\mathrm{Id}_{\infty\to p} : \ell_\infty^n \to \ell_p^n$ and $\mathrm{Id}_{p\to\infty} : \ell_p^n \to \ell_\infty^n$ satisfy $\|\mathrm{Id}_{\infty\to p}\|_{\mathrm{Lip}} \le n^{1/p}$, $\|\mathrm{Id}_{p\to\infty}\|_{\mathrm{Lip}} \le 1$. The mapping $g : \mathrm{Id}_{\infty\to p} \circ \mathcal{F}(A) \to Y$ given by $g = f \circ (\mathcal{F}|_A)^{-1} \circ \mathrm{Id}_{p\to\infty}$ can be extended using Theorem 1.1 to a mapping $\widetilde{g} : \ell_p^n \to Y$ with $\|\widetilde{g}\|_{\mathrm{Lip}} \le 24K_2(Y)\sqrt{p} \cdot \|g\|_{\mathrm{Lip}} \le 24K_2(Y)\sqrt{p} \cdot \|f\|_{\mathrm{Lip}}$. Define $\widetilde{f} : X \to Y$ by $\widetilde{f} = \widetilde{g} \circ \mathrm{Id}_{\infty\to p} \circ \mathcal{F}$. Then $\widetilde{f}$ extends f and $\|\widetilde{f}\|_{\mathrm{Lip}} \le 24K_2(X)\sqrt{p} \cdot n^{1/p} \cdot \|f\|_{\mathrm{Lip}}$. Choosing $p = 2\log n$ yields the required result. □

Remark 2.2. A similar argument shows that for every $2 \le p < \infty$,

$$e_n(L_p, L_1) \le 10\sqrt{p \log n}.$$

Indeed, fix $A = \{a_1, \ldots, a_n\} \subseteq L_p$ and $f : A \to L_1$. Denote $Z = \mathrm{span}(f(A))$. Then $\dim(Z) \le n$, so that by a theorem of Talagrand [25] there is an invertible linear mapping $T : Z \to \ell_1^m$ such that $\|T\|_{\mathrm{Lip}} = 1$ and $\|T^{-1}\|_{\mathrm{Lip}} \le 2$, where $m = O(n \log n)$. For $1 < q \le 2$ the identity mappings $\mathrm{Id}_{1\to q} : \ell_1^m \to \ell_q^m$ and $\mathrm{Id}_{q\to 1} : \ell_q^m \to \ell_1^m$ satisfy $\|Id_{1\to q}\|_{\mathrm{Lip}} \le 1$ and $\|Id_{q\to 1}\|_{\mathrm{Lip}} \le m^{1-1/q}$. The Banach space $W := \mathrm{Id}_{1\to q} \circ T(Z)$ is a subspace of ℓ_q^m, so that $K_2(W) \le 1/\sqrt{q-1}$ (see [2]). The mapping $h : A \to W$ given by $h = \mathrm{Id}_{1\to q} \circ T \circ f$ can be extended using Theorem 1.1 to a mapping $\widetilde{h} : L_p \to W$ such that $\|\widetilde{h}\|_{\mathrm{Lip}} \le 24\sqrt{p-1} \cdot K_2(W) \cdot \|h\|_{\mathrm{Lip}} \le 24\sqrt{\frac{p-1}{q-1}} \cdot \|f\|_{\mathrm{Lip}}$. The mapping $\widetilde{f} := T^{-1} \circ \mathrm{Id}_{q\to 1} \circ \widetilde{h} : L_p \to Z \subseteq L_1$ extends f and satisfies

$$\|\widetilde{f}\|_{\mathrm{Lip}} \le 48 \cdot m^{1-1/q} \sqrt{\frac{p-1}{q-1}} \cdot \|f\|_{\mathrm{Lip}} = O\left(\frac{(n \log n)^{1-1/q}\sqrt{p}}{\sqrt{q-1}}\right) \cdot \|f\|_{\mathrm{Lip}}.$$

Choosing $q = 1 + \frac{1}{\log n}$ yields the required result.

Remark 2.3. Using the arguments in [21] it is possible to prove variants of Theorem 2.1 which deal with extensions of Hölder functions. For example, for every metric space (X, d) and $\alpha \in (0, 1]$ let X^α denote the metric space (X, d^α). A combination of the above proof and [21] shows that for $q \ge 2$, and every metric space X, $e_n(X^{2/q}, L_q) = O((\log n)^{1/q})$.

3. Embedding the integer lattice into uniformly convex spaces

Given two metric spaces X, Y and an injection $f : X \hookrightarrow Y$ we define the distortion of f to be $\mathrm{dist}(f) = \|f\|_{\mathrm{Lip}} \cdot \|f^{-1}\|_{\mathrm{Lip}}$. The least distortion with which X may be embedded into Y is denoted $c_Y(X) = \inf\{\mathrm{dist(f)} : \ f : X \hookrightarrow Y\}$. When $Y = L_p$ we write $c_Y(X) = c_p(X)$. The parameter $c_2(X)$ is called the Euclidean distortion of X.

We recall the notions of type and cotype of a Banach space X: we say that X has type p if for every $x_1,\ldots,x_n\in X$,

$$\left(\int_{\{-1,1\}^n}\left\|\sum_{j=1}^n\varepsilon_jx_j\right\|^2d\varepsilon\right)^{1/2}\le T\left(\sum_{j=1}^n\|x_j\|^p\right)^{1/p}.$$

The least such constant T is called the type p constant of X, and is denoted $T_p(X)$. X is said to have cotype q if for every $x_1,\ldots,x_n\in X$,

$$\left(\int_{\{-1,1\}^n}\left\|\sum_{j=1}^n\varepsilon_jx_j\right\|^2d\varepsilon\right)^{1/2}\ge \frac{1}{C}\left(\sum_{j=1}^n\|x_j\|^p\right)^{1/p}.$$

The least such constant C is called the cotype q constant of X, and is denoted $C_q(X)$. It is well known that $C_2(X)\le K_2(X)$ (see for example [14]).

Fix $p\ge 1$, and two integers $m,n\in\mathbb{N}$. We denote by $[m]_p^n$ the set $\{0,1,\ldots,m\}^n$ equipped with the metric induced by ℓ_p^n. We first deal with the case $2\le p\le\infty$ (the case $p\in[1,2]$ is discussed below). Observe that for every $x,y\in[m]_p^n$,

$$\|x-y\|_2^{2/p}=\left(\sum_{j=1}^n(x_j-y_j)^2\right)^{1/p}\le\left(\sum_{j=1}^n|x_j-y_j|^p\right)^{1/p}\le m^{1-\frac{2}{p}}\cdot\|x-y\|_2^{2/p}.$$

By a theorem of Schoenberg (see for example [28]), $\mathbb{R}^n$ equipped with the metric $\|x-y\|_2^{2/p}$ is isometric to a subset of L_2. Thus we have shown that $c_2\left([m]_p^n\right)\le m^{1-\frac{2}{p}}$. Additionally, by Hölder's inequality, the identity mapping between ℓ_p^n and ℓ_2^n is $n^{\frac12-\frac1p}$ bi-Lipschitz. This shows that

$$c_2\left([m]_p^n\right)\le\min\left\{n^{\frac12-\frac1p},m^{1-\frac2p}\right\}.$$

This bound on the Euclidean distortion of the integer lattice, in combination with Dvoretzky's theorem (see [20]), implies the following fact:

Fact 3.1. Let Y be an infinite dimensional Banach space. Then for every $p\ge 2$

$$c_Y\left([m]_p^n\right)\le\min\left\{n^{\frac12-\frac1p},m^{1-\frac2p}\right\}.$$

The main result of this section is a matching lower bound for 2-uniformly convex spaces. The proof is a modification of an argument of Bourgain [5], combined with Theorem 1.1.

Theorem 3.2. *Let Y be a 2-uniformly convex Banach space. Then for every $2\le p<\infty$,*

$$c_Y\left([m]_p^n\right)\ge\frac{1}{400K_2(Y)^2\sqrt{p}}\cdot\min\left\{n^{\frac12-\frac1p},m^{1-\frac2p}\right\}.$$

Proof. Let $f:[m]_p^n\to Y$ be a one to one mapping with $\|f\|_{\mathrm{Lip}}=L$ and $\|f^{-1}\|_{\mathrm{Lip}}=1$. Assume first of all that $m\ge 4\sqrt{n}$ (which implies that $m^{1-\frac2p}\ge n^{\frac12-\frac1p}$). We may assume in this case that $L\le\frac{m}{200K_2(Y)n^{1/p}\sqrt{p}}$, since otherwise the required result holds true.

By Theorem 1.1 f can be extended to a function $\widetilde{f}:\ell_p^n\to Y$ with $\|\widetilde{f}\|_{\mathrm{Lip}}\le 24K_2(Y)\sqrt{p}\cdot L$. By Rademacher's theorem (see e.g. [4]) $\widetilde{f}$ is differentiable almost everywhere. Let $x\in\ell_p^n$ be a point of differentiability of $\widetilde{f}$ and $\varepsilon\in\{-1,1\}^n$. Then

$$\begin{aligned}\left\|\left.\frac{d}{dt}\right|_{t=0}\widetilde{f}(x+t\varepsilon)\right\|_Y &= \lim_{t\to 0}\frac{\left\|\widetilde{f}(x+t\varepsilon)-\widetilde{f}(x)\right\|_Y}{|t|}\\ &\le \|\varepsilon\|_p\cdot\|\widetilde{f}\|_{\mathrm{Lip}}\\ &\le 24n^{1/p}K_2(Y)L\sqrt{p}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\{-1,1\}^n}\left\|\left.\frac{d}{dt}\right|_{t=0}\widetilde{f}(x+t\varepsilon)\right\|_Y^2 d\varepsilon &= \int_{\{-1,1\}^n}\left\|\sum_{j=1}^n\varepsilon_j\frac{\partial}{\partial x_j}\widetilde{f}(x)\right\|_Y^2 d\varepsilon\\ &\ge \frac{1}{C_2(Y)^2}\sum_{j=1}^n\left\|\frac{\partial}{\partial x_j}\widetilde{f}(x)\right\|_Y^2.\end{aligned}$$

Using the fact that $C_2(Y)\le K_2(Y)$, and integrating with respect to $x\in[0,m]^n$ we deduce that

$$\sum_{j=1}^n\frac{1}{m^n}\int_{[0,m]^n}\left\|\frac{\partial}{\partial x_j}\widetilde{f}(x)\right\|_Y^2 dx\le 576pn^{2/p}L^2K_2(Y)^4. \tag{3.1}$$

Fix an integer $j\in\{1,\ldots,n\}$. For every $x=(x_1,\ldots,x_n)\in[0,m]^n$ there are $a,b\in[m]_p^n$ such that

$$\|a-(x_1,\ldots,x_{j-1},m,x_{j+1},\ldots,x_n)\|_p\le n^{1/p},$$

and

$$\|b-(x_1,\ldots,x_{j-1},0,x_{j+1},\ldots,x_n)\|_p\le n^{1/p}.$$

In particular it follows that $\|a-b\|_p\ge m-2n^{1/p}$. Now, since $\widetilde{f}$ extends f we have that

$$\begin{aligned}\|\widetilde{f}(x_1,\ldots,x_{j-1},m,x_{j+1},\ldots,x_n) &- \widetilde{f}(x_1,\ldots,x_{j-1},0,x_{j+1},\ldots,x_n)\|_Y\\ &\ge \|\widetilde{f}(a)-\widetilde{f}(b)\|_Y-\\ &\quad\ \|\widetilde{f}(a)-\widetilde{f}(x_1,\ldots,x_{j-1},m,x_{j+1},\ldots,x_n)\|_Y-\\ &\quad\ \|\widetilde{f}(b)-\widetilde{f}(x_1,\ldots,x_{j-1},0,x_{j+1},\ldots,x_n)\|_Y\\ &\ge m-2n^{1/p}-2\|\widetilde{f}\|_{\mathrm{Lip}}\cdot n^{1/p}\\ &\ge \frac{m}{4}, \qquad (3.2)\end{aligned}$$

where we have used our assumption that $L \le \frac{m}{200K_2(Y)n^{1/p}\sqrt{p}}$. On the other hand,

$$\begin{aligned}
&\|\widetilde{f}(x_1,\ldots,x_{j-1},m,x_{j+1},\ldots,x_n)-\widetilde{f}(x_1,\ldots,x_{j-1},0,x_{j+1},\ldots,x_n)\|_Y\\
&= \left\|\int_0^m \frac{\partial}{\partial x_j}\widetilde{f}(x_1,\ldots,x_{j-1},s,x_{j+1},\ldots,x_n)\,ds\right\|_Y\\
&\le \int_0^m \left\|\frac{\partial}{\partial x_j}\widetilde{f}(x_1,\ldots,x_{j-1},s,x_{j+1},\ldots,x_n)\right\|_Y ds\\
&\le \sqrt{m\cdot\int_0^m \left\|\frac{\partial}{\partial x_j}\widetilde{f}(x_1,\ldots,x_{j-1},s,x_{j+1},\ldots,x_n)\right\|_Y^2 ds}.
\end{aligned}$$

Squaring this inequality, integrating with respect to $x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n$, and using (3.2), we get that

$$\frac{1}{m^n}\int_{[0,m]^n}\left\|\frac{\partial}{\partial x_j}\widetilde{f}(x)\right\|_Y^2 dx \ge \frac{1}{16}.$$

Plugging this estimate into (3.1) we see that

$$\frac{n}{16}\le 576pn^{2/p}L^2K_2(Y)^4,$$

or

$$L\ge \frac{n^{\frac12-\frac1p}}{96K_2(Y)^2\sqrt{p}},$$

implying the required result.

It remains to deal with the case $m\le 4\sqrt{n}$, but in this case $[m]_p^n$ contains an isometric copy of $[m]_p^{\lfloor m^2/16\rfloor}$ for so that the required result follows from the previous argument. □

We end this section with some remarks and open problems.

Remarks.

(1) For $p\in[1,2]$ we have that $c_q([m]_p^n)=1$ for every $q\in[1,p]$, since ℓ_p embeds isometrically into L_q in this case (see for example [28]). More generally, by the Maurey-Pisier theorem [18, 20]), for an infinite dimensional Banach space Y satisfying $\sup\{q:\ Y \text{ has type } q\}\le p$ we have that $c_Y([m]_p^n)=1$. Observe that for $m\ge 1$, $[m]_p^n$ contains an isometric copy of $\{0,1\}^n$. Thus, by a theorem of Enflo [8], if $q\in[p,2]$ then $c_q([m]_p^n)=n^{\frac1p-\frac1q}$, where the upper bound $c_q([m]_p^n)\le n^{\frac1p-\frac1q}$ follows from Hölder's inequality. More generally, if Y has type $q\in[p,2]$ then $c_q([m]_p^n)\ge\frac{cn^{\frac1p-\frac1q}}{\log n}$, where c is a universal constant. This follows from a result of Pisier [24] (see also an earlier slightly weaker result of Bourgain, Milman and Wolfson [6]). If we assume that Y has type $q\in[p,2]$ and that Y has the UMD property (see [23] for the definition), then the results of Naor and Schechtman [23] imply that $c_Y([m]_p^n)\ge C(Y)n^{\frac1p-\frac1q}$, where the constant $C(Y)$ depends only on the UMD constant of Y. In particular, for $2\le q<\infty$ we get that $c_q([m]_p^n)\ge C(q)n^{\frac1p-\frac12}$, and this is optimal (up to constants depending only on q) since the Banach-Mazur distance between ℓ_p^n and ℓ_q^n is of order $n^{\frac1p-\frac12}$ (see [26]).

(2) For the ℓ_∞ integer lattice we can take $p = \log(\min\{m,n\})$ in Theorem 3.2 to get that

$$c_Y([m]_\infty^n) \geq \frac{1}{400K_2(Y)^2} \cdot \min\left\{\sqrt{\frac{n}{\log n}}, \frac{m}{\sqrt{\log m}}\right\}.$$

We conjecture that this lower bound is not optimal, namely that whenever Y is a 2-uniformly convex Banach space, $c_Y([m]_p^n) = \Omega(\min\{\sqrt{n}, m\})$. In [5] Bourgain shows that if Y has cotype q then $c_Y([m]_\infty^m) = \Omega(n^{1/q})$, provided that $m \geq n^{1+1/q}$. Thus, the above conjecture is true for $m \geq n^{3/2}$. The results of [15, 17] imply that there exist arbitrarily large n-point metric spaces X_n (namely expander graphs) with the property that for every $x, y \in X_n$, $d(x,y) \in [1, 10\log n]$, and for $p \geq 1$, $c_p(X_n) = \Omega\left(\frac{\log n}{p}\right)$. Using the Fréchet embedding, we can embed X_n isometrically into ℓ_∞^n. Moreover, since the distances in X_n are in the range $[1, 10\log n]$, we can embed X_n with distortion 2, say, into $[\lceil 20\log n\rceil]_\infty^n$. Thus, it follows that $c_p([m]_\infty^n) = \Omega(m/p)$, at least for $m \leq 20\log n$.

(3) A natural approach to the above conjecture would be to show that for a 2-uniformly convex space Y, any Lipschitz mapping from $[m]_\infty^n$ into Y can be extended to ℓ_∞^n with only a universally bounded multiplicative loss in the Lipschitz constant. The above argument shows that this would imply that $c_Y([m]_\infty^n) = \Omega(\min\{\sqrt{n}, m\})$, and it would improve the results of [5, 3].

(4) The lower bounds on distortion which were presented in this paper do not belong to the usual type of non-embeddability results, since they are not based on Poincaré type inequalities. In our forthcoming paper [19] we introduce a family of Poincaré type inequalities which can be used to prove non-embeddability results for the integer lattice, and which are a non-linear analogue of the notion of cotype. We defer the discussion on this (more complicated) argument to [19], but we wish to state that it shows in particular that at least for Hilbert space, $c_2([m]_\infty^n) = \Theta(\min\{\sqrt{n}, m\})$.

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