On Some Low Distortion Metric Ramsey Problems

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Abstract

In this note, we consider the metric Ramsey problem for the normed spaces ℓ_p. Namely, given some $1 \leq p \leq \infty$ and $\alpha \geq 1$, and an integer n, we ask for the largest m such that every n-point metric space contains an m-point subspace which embeds into ℓ_p with distortion $\leq \alpha$. In [1] it is shown that in the case of ℓ_2, the dependence of m on α undergoes a phase transition at $\alpha = 2$. Here we consider this problem for other ℓ_p, and specifically the occurrence of a phase transition for $p \neq 2$. It is shown that a phase transition occurs at 2 for every $p \in [1, 2]$. For $p > 2$ we are unable to determine the answer, but estimates are provided for the possible location of such a phase transition. We also study the analogous problem for isometric embedding and show that for every $1 < p < \infty$ there are arbitrarily large metric spaces, no four points of which embed isometrically in ℓ_p .

1 Introduction

A Ramsey-type theorem states that large systems necessarily contain large, highly structured sub-systems. Here we consider Ramsey-type problem for finite metric spaces, interpreting "highly structured" as having low distortion embedding in ℓ_p.

A mapping between two metric spaces $f : M \to X$, is called an embedding of M in X. The *distortion* of the embedding is defined as

$$\operatorname{dist}(f) = \sup_{\substack{x,y \in M \\ x \neq y}} \frac{d_X(f(x), f(y))}{d_M(x,y)} \cdot \sup_{\substack{x,y \in M \\ x \neq y}} \frac{d_M(x,y)}{d_X(f(x), f(y))}.$$

The least distortion required to embed M in X is denoted by $c_X(M)$. When $c_X(M) \leq \alpha$ we say that M α-embeds in X. In this note we study the following notion.

Definition 1 (Metric Ramsey function). We denote by $R_X(\alpha, n)$ the largest integer m such that every n-point metric space has a subspace of size m that α-embeds into X.

When $X = \ell_p$ we use the notations c_p and R_p. Note that for $p \in [1, \infty]$, it is always true that $R_p(\alpha, n) \geq R_2(\alpha, n)$. When $\alpha = 1$ we drop it from the notation, i.e., $R_X(n) = R_X(1, n)$.

Bourgain, Figiel, and Milman [4] study this function for $X = \ell_2$, as a non-linear analog of Dvoretzky's theorem [6]. They prove

Theorem 1 ([4]). *For any $\alpha > 1$ there exists $C(\alpha) > 0$ such that $R_2(\alpha, n) \geq C(\alpha) \log n$. Furthermore, there exists $\alpha_0 > 1$ such that $R_2(\alpha_0, n) = O(\log n)$.*

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In [1] the metric Ramsey problem is studied comprehensively. In particular, the following phase transition is established in the case of $X = \ell_2$.

Theorem 2 ([1]). *Let $n \in \mathbb{N}$. Then:*

1. *For every $1 < \alpha < 2$: $c(\alpha)\log n \le R_2(\alpha, n) \le 2\log n + C(\alpha)$, where $c(\alpha), C(\alpha)$ may depend only on α.*
2. *For every $\alpha > 2$: $n^{c'(\alpha)} \le R_2(\alpha, n) \le n^{C'(\alpha)}$, where $c'(\alpha), C'(\alpha)$ depend only on α and $0 < c'(\alpha) \le C'(\alpha) < 1$. Moreover, $c'(\alpha)$ tends to 1 as α tends to ∞.*

By Dvoretzky's theorem, the lower bound in part 2 of Theorem 2 implies in particular that if $\alpha > 2$, and X is any infinite dimensional normed space, then $R_X(\alpha, n) \ge n^{c'(\alpha)}$. Therefore, in our search for a possible phase transition for $R_p(\cdot, n)$, $p \ne 2$, it is natural to extend the upper bound in part 1 of Theorem 2 to this range. The main result proved in this note is the following:

Theorem 3. *There is an absolute constant $c > 0$ such that for every $0 < \delta < 1$,*

1. *For $1 \le p < 2$, $R_p(2 - \delta, n) \le e^{\frac{c}{\delta^2}} \log n$.*
2. *For $2 < p < \infty$, $R_p(2^{2/p} - \delta, n) \le e^{\frac{c}{p^2\delta^2}} \log n$.*

Thus we extend the result of [1] to show that a phase transition occurs in the metric Ramsey problem for ℓ_p, $p \in [1, 2)$, at $\alpha = 2$. The asymptotic behavior of $R_p(\alpha, n)$ for $p > 2$, and $\alpha \in [2^{2/p}, 2]$, is left as an open problem. In particular, we do not know whether or not this function undergoes a similar phase transition. We find this problem potentially significant: if there is a phase transition at 2 also in the range $2 < p < \infty$, then this result will certainly be of great interest. On the other hand, if it is possible to improve the lower bound in part 2 of Theorem 2 for $p > 2$ and certain distortions strictly less than 2, then this would involve an embedding technique that is different from the method used in [1], which doesn't distinguish between the various ℓ_p spaces.

The proof of the upper bound on $R_2(\alpha, n)$ for $\alpha < 2$ stated in Theorem 2 uses the Johnson-Lindenstrauss dimension reduction lemma for ℓ_2 [9]. For ℓ_p, $p \ne 2$, no such dimension reduction is known to hold. Our proof is based on a non-trivial modification of the random construction in [4], in the spirit of Erdös' upper bound on the Ramsey numbers [8, 3]. In the process we prove tight bounds on the embeddability of the metrics of complete bipartite graphs in ℓ_p. Specifically we show that

$$c_p(K_{n,n}) = \begin{cases} 2 - \Theta(n^{-1}) & p \in [1, 2] \\ 2^{2/p} - \Theta((pn)^{-1}) & p > 2. \end{cases}$$

The second part of this note addresses the isometric Ramsey problem for $p \in (1, \infty)$. It turns out that this problem is naturally tackled within the class of uniformly convex normed spaces (see Section 3 for the definition).

Theorem 4 (Isometric Ramsey Problem). *Let X be a uniformly convex normed space with $\dim(X) \ge 2$. Then $R_X(1, n) = 3$ for $n \ge 3$.*

Since for $p \in (1, \infty)$, ℓ_p is uniformly convex, the conclusion of Theorem 4 holds in these cases. Note that the theorem does not apply for ℓ_1 and ℓ_∞ which are not uniformly convex. Specifically, it is known that ℓ_∞ is universal in that it contains an isometric copy of every finite metric space, whence $R_\infty(n) = n$. It is known [5] that any 4-point metric space is isometrically embeddable in ℓ_1, and therefore $R_1(n) \ge 4$ for $n \ge 4$. The determination of $R_1(n)$ is left as an open problem.

2 An Upper Bound For $\alpha < 2$

In this section we prove that for any $\alpha < \min\{2, 2^{2/p}\}$, $R_p(\alpha, n) = O(\log n)$. Our technique both improves and simplifies the technique of [4], which is itself in the spirit of Erdös' original upper bound for the Ramsey coloring numbers. The basic idea is to exploit a universality property of random graphs $G \in G(n, 1/2)$. Namely, that any fixed graph of constant size appears as induced subgraph of every induced subgraph of G of size $\Omega(\log n)$. More precisely, we define the following notion of universality.

Definition 2. Let H be a graph. A graph G is called (H, s)-universal if every set of s vertices in G contains an induced subgraph isomorphic to H.

Proposition 1. *For every k-vertex graph H there exists a constant $C > 0$ and an integer n_0 such that for any $n > n_0$ there exists a $(H, C\log n + 1)$-universal graph. Furthermore,*

$$C \leq O\left(k^2 2^{\binom{k}{2}}\right) \quad \text{and} \quad n_0 \leq O\left(k^3 2^{\binom{k}{2}}\right).$$

Such facts are well-known in random graph theory, and similar arguments can be found for example in [11]. We sketch the standard details for the sake of completeness.

Recall that a family of sets $\mathcal{F}$ is called *almost disjoint* if $|A \cap B| \leq 1$ for every $A, B \in \mathcal{F}$.

Lemma 2. *For every integer k and a finite set S with $k < \sqrt{|S|/2}$, there exists an almost disjoint family $K \subset \binom{S}{k}$, such that $|K| \geq \left\lfloor \frac{s}{2k} \right\rfloor^2$.*

Proof. Let p be a prime satisfying $\frac{s}{2k} \leq p \leq \frac{|S|}{k}$, and assume that

$$L = \{(i,j); i, j \in \mathbb{Z}_p,\ i \in \{0, \ldots, k-1\}\} \subseteq S.$$

For each $a, b \in \mathbb{Z}_p$, define

$$A_{a,b} = \{(i,j);\ j = ai + b \pmod{p},\ i \in \{0, \ldots, k-1\}\},$$

and take $K = \{A_{a,b} | a, b \in \mathbb{Z}_p\}$. The set K is easily checked to satisfy the requirements. □

Lemma 3. *Let H be a k-vertex graph and let $s > 2k^2$. The probability that a random graph $G \in G(s, 1/2)$ does not contain an induced subgraph isomorphic to H, is at most $(1 - 2^{-\binom{k}{2}})^{\left\lfloor \frac{s}{2k} \right\rfloor^2}$.*

Proof. Construct, as in Lemma 2, an almost disjoint family $\mathcal{F}$ of $\left\lfloor \frac{s}{2k} \right\rfloor^2$ subsets of $\{1, \ldots, s\}$, the vertex set of G. If $F_1 \neq F_2 \in \mathcal{F}$, then the event that the restriction of G to F_1 (resp. F_2) is isomorphic to H are independent. Hence, the probability that none of the sets $F \in \mathcal{F}$ spans a subgraph isomorphic to H is at most $(1 - 2^{-\binom{k}{2}})^{\left\lfloor \frac{s}{2k} \right\rfloor^2}$. □

Proof of Proposition 1. Let G be a random graph in $G(n, 1/2)$. By the previous lemma, the expected number of sets of s vertices which contain no induced isomorphic copy of H is at most $\binom{n}{s}\left(1 - 2^{-\binom{k}{2}}\right)^{\left\lfloor \frac{s}{2k} \right\rfloor^2}$. If this number is < 1, then there is an (H, s)-universal graph, as claimed. It is an easy matter to check that this holds with the parameters as stated. □

A class $\mathcal{C}$ of finite metric spaces is called a *metric class* if it is closed under isometries. $\mathcal{C}$ is said to be *hereditary*, if $M \in \mathcal{C}$ and $N \subset M$ imply $N \in \mathcal{C}$. We call a metric space (X, d) a $\{0, 1, 2\}$ metric space if for all $x, y \in X$, $d(x, y) \in \{0, 1, 2\}$. There is a simple 1:1 correspondence between graphs and $\{0, 1, 2\}$ metrics. Namely, associated with a $\{0, 1, 2\}$ metric space $M = (X, d)$ is the graph $G = (X, E)$ where $[x, y] \in E$ iff $d_M(x, y) = 1$.

Lemma 4. *Let $\mathcal{C}$ be a hereditary metric class of finite metric spaces, and suppose that there exists some finite $\{0,1,2\}$ metric space M_0 which is not in P. Then there exist metric spaces $M = M_n$ of arbitrarily large size n such that every subspace $S \subset M_n$ with at least $C \log n$ points is not in $\mathcal{C}$. The constant C depends only on the cardinality of M_0.*

Proof. Let H_0 be the graph corresponding to the metric space M_0. We apply Proposition 1, to construct arbitrarily large graphs $G_n = (V_n, E_n)$ with $|V_n| = n$, in which every set of $\geq C \log n$ vertices contains an induced subgraph isomorphic to H_0. Let M_n be the n-point metric space corresponding to G_n. It follows that every subspace of M_n of size $\geq C \log n$ contains a metric subspace that is isometric to M_0. Since $\mathcal{C}$ is hereditary, $S \notin \mathcal{C}$. □

Note that $\{M; M$ is a metric space, $c_p(M) \leq \alpha\}$ is a hereditary metric class. Therefore, in order to show that for $\alpha < 2$, $R_p(\alpha, n) = O(\log n)$, it is enough to find a $\{0,1,2\}$ metric space whose ℓ_p distortion is greater than α. We use the complete bipartite graphs $K_{n,n}$. The ℓ_p-distortion of $K_{n,n}$, $1 \leq p < \infty$, is estimated in the following proposition.

Proposition 5. *For every $1 \leq p \leq 2$,*

$$2\left(\frac{n-1}{n}\right)^{1/p} \leq c_p(K_{n,n}) \leq 2\sqrt{\frac{n-1}{n}}$$

For every $2 \leq p < \infty$,

$$2^{2/p}\left(\frac{n-1}{n}\right)^{1/p} \leq c_p(K_{n,n}) \leq 2^{2/p}\left(1 - \frac{1}{2n}\right)^{1/p}.$$

Before proving Proposition 5, we will deduce the main result of this section:

Theorem 5. *There is an absolute constant $c > 0$ such that for every $0 < \delta < 1$, if $1 \leq p \leq 2$ then:*

$$R_p(2-\delta, n) \leq e^{\frac{c}{\delta^2}} \log n,$$

and if $2 < p < \infty$ then:

$$R_p(2^{2/p}-\delta, n) \leq e^{\frac{c}{p^2\delta^2}} \log n.$$

Proof. Proposition 1 implies that there is an absolute constant C such that for every $n \geq 2^{Ck^3}$ there exists a $\{0,1,2\}$ metric space M_n such that any subset $S \subset M_n$ of cardinality at least $2^{Ck^2} \log n$ contains an isometric copy of $K_{k,k}$.

We start with $1 \leq p \leq 2$. Let $k = \left\lfloor \frac{2}{\delta} \right\rfloor + 1$. By Proposition 5,

$$c_p(K_{k,k}) \geq 2\left(1 - \frac{1}{k}\right)^{1/p} > 2\left(1 - \frac{\delta}{2}\right) = 2 - \delta,$$

so that for n large enough ($\geq e^{\frac{C'}{\delta^3}}$), and hence for all n (by proper choice of constants),

$$R_p(2-\delta, n) \leq e^{\frac{C'}{\delta^2}} \log n.$$

When $p > 2$ take $k = 2\left\lfloor \frac{4}{p\delta} \right\rfloor$. In this case one easily verifies that:

$$c_p(K_{k,k}) \geq 2^{2/p}\left(1 - \frac{1}{k}\right)^{1/p} \geq 2^{2/p} - \delta,$$

from which the required result follows as above. □

Preliminary to the proof of Proposition 5, we require some preparation.

Lemma 6. *Let $A = (a_{ij})$ be an $n \times n$ matrix and $2 \le p < \infty$. Then:*

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left|\sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{jk}\right|^p + \left|\sum_{k=1}^{n} a_{ki} - \sum_{k=1}^{n} a_{kj}\right|^p\right) \le \frac{(2n)^p}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^p.$$

Proof. We identify $\ell_p^{n^2}$ with space of all $n \times n$ matrices $A = (a_{ij})$, equipped with the ℓ_p norm:

$$\|A\|_p = \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^p\right)^{1/p}.$$

Define a linear operator $T : \ell_p^{n^2} \to \ell_p^{n^2} \oplus \ell_p^{n^2}$ by:

$$T(a_{ij}) = \left(\sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{jk}\right)_{ij} \oplus \left(\sum_{k=1}^{n} a_{ki} - \sum_{k=1}^{n} a_{kj}\right)_{ij}.$$

Our goal is to show that $\|T\|_{p\to p} \le 2^{1-1/p}n$. By standard results from the complex interpolation theory (see [2]), it is enough to prove this estimate for $p = 2$ and $p = \infty$. The case $p = \infty$ is simple:

$$\|T(A)\|_\infty = \max_{1\le i,j\le n} \max\left\{\left|\sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{jk}\right|, \left|\sum_{k=1}^{n} a_{ki} - \sum_{k=1}^{n} a_{kj}\right|\right\} \le 2n\|A\|_\infty.$$

For $p = 2$ we have to show that:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left|\sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{jk}\right|^2 + \left|\sum_{k=1}^{n} a_{ki} - \sum_{k=1}^{n} a_{kj}\right|^2\right) \le 2n^2\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2.$$

This inequality follows from the following elementary identity:

$$\begin{aligned} 2n^2 &\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(\sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{jk}\right)^2 + \left(\sum_{k=1}^{n} a_{ki} - \sum_{k=1}^{n} a_{kj}\right)^2\right] + \\ &\quad + 2\sum_{i=1}^{n}\sum_{j=1}^{n}\left(na_{ij} - \sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{kj}\right)^2. \end{aligned}$$

□

Corollary 7. *Let $1 \le p < \infty$ and $x_1, \dots, x_n, y_1 \dots y_n \in \ell_p$. Then if $2 \le p < \infty$,*

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\|x_i - x_j\|_p^p + \|y_i - y_j\|_p^p\right) \le 2^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\|x_i - y_j\|_p^p.$$

If $1 \le p \le 2$ then:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\|x_i - x_j\|_p^p + \|y_i - y_j\|_p^p\right) \le 2\sum_{i=1}^{n}\sum_{j=1}^{n}\|x_i - y_j\|_p^p.$$

Proof. By summation it is clearly enough to prove these inequalities for $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$. If $2 \le p < \infty$ then the required result follows from an application of Lemma 6 to the matrix $a_{ij} = x_i - y_j$. If $1 \le p \le 2$ then consider ℓ_p equipped with the metric $d(x, y) = \|x - y\|_p^{p/2}$. It is well known (see [12]) that (ℓ_p, d) embeds isometrically in ℓ_2, so that the case $1 \le p \le 2$ follows from the case $p = 2$.

□

Remark. In [7] P. Enflo defined the notion on generalized roundness of a metric space. A metric space (M, d) is said to have generalized roundness $q \ge 0$ if for every $x_1, \ldots, x_n, y_1, \ldots, y_n \in M$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(d(x_i, x_j)^q + d(y_i, y_j)^q) \le 2\sum_{i=1}^{n}\sum_{j=1}^{n} d(x_i, y_j)^q.$$

Enflo proved that Hilbert space has generalized roundness 2 and in [10] the concept of generalized roundness was investigated and was shown to be equivalent to the notion of negative type (see [5, 12] for the definition). Particularly, it was proved in [10] that for $1 \le p < 2$, ℓ_p has generalized roundness p, which is precisely the second statement in Corollary 7. In [10] it was also shown that for $2 \le p < \infty$, ℓ_p doesn't have generalized roundness q for any $q > 0$. Finally, we would like to remark that the case $p = 1$ of Corollary 7 has a particularly simple proof using density functions. As before it is enough to prove that for $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(|x_i - x_j| + |y_i - y_j|) \le 2\sum_{i=1}^{n}\sum_{j=1}^{n} |x_i - y_j|. \tag{1}$$

Define for $x \in \mathbb{R}$:

$$I(x) = |\{i \in \{1, \ldots, n\}; x_i \le x\}|,$$
$$J(x) = |\{i \in \{1, \ldots, n\}; y_i \le x\}|.$$

Let:

$$f_L(x) = I(x)(n - I(x)) + J(x)(n - J(x)),$$
$$f_R(x) = I(x)(n - J(x)) + J(x)(n - I(x)).$$

Then:

$$\begin{aligned} 2\sum_{i=1}^{n}\sum_{j=1}^{n} |x_i - y_j| &- \sum_{i=1}^{n}\sum_{j=1}^{n}(|x_i - x_j| + |y_i - y_j|) \\ &= 2\int_{-\infty}^{\infty} f_R(x)dx - 2\int_{-\infty}^{\infty} f_L(x)dx \\ &= 2\int_{-\infty}^{\infty} [I(x)(n - J(x)) + J(x)(n - I(x)) - I(x)(n - I(x)) - J(x)(n - J(x))]dx \\ &= 2\int_{-\infty}^{\infty} [I(x) - J(x)]^2 dx \ge 0. \end{aligned}$$

Proof of Proposition 5. We identify $K_{n,n}$ with the metric on $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ where $d(u_i, u_j) = d(v_i, v_j) = 2$ for all $i \ne j$, and $d(u_i, v_j) = 1$ for every $1 \le i, j \le n$. Fix some

$1 \leq p < \infty$ and let $f : \{u_1, \ldots, u_n, v_1, \ldots, v_n\} \to \ell_p$ be an embedding such that for every $x, y \in K_{n,n}$, $d(x,y) \leq \|f(x) - f(y)\|_p \leq L d(x,y)$. Then,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (\|f(u_i) - f(u_j)\|_p^p + \|f(v_i) - f(v_j)\|_p^p) \geq 2n(n-1)2^p,$$

and

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \|f(u_i) - f(v_j)\|_p^p \leq n^2 L^p.$$

For $1 \leq p \leq 2$ Corollary 7 gives:

$$2n(n-1)^p 2^p \leq 2n^2 L^p \Longrightarrow L \geq 2 \left(\frac{n-1}{n}\right)^{1/p}.$$

For $2 \leq p < \infty$ we get that:

$$2n(n-1)2^p \leq 2^{p-1} n^2 L^p \Longrightarrow L \geq 2^{2/p} \left(\frac{n-1}{n}\right)^{1/p}.$$

This proves the required lower bounds on $c_p(K_{n,n})$.

To prove the upper bound assume first that $p = 2$ and denote by $\{e_i\}_{i=1}^{\infty}$ the standard unit vectors in ℓ_2. Define $f : K_{n,n} \to \ell_2^{2n}$ by:

$$f(u_i) = \sqrt{2}\Big(e_i - \frac{1}{n}\sum_{j=1}^{n} e_j\Big),$$

$$f(v_i) = \sqrt{2}\Big(e_{n+i} - \frac{1}{n}\sum_{j=1}^{n} e_{n+j}\Big).$$

Then for $i \neq j$, $\|f(u_i) - f(u_j)\|_2 = \|f(v_i) - f(v_j)\|_2 = d(u_i, u_j) = d(v_i, v_j)$. On the other hand:

$$\begin{aligned}\|f(u_i) - f(v_j)\|_2 &= \sqrt{\|f(u_i)\|_2^2 + \|f(v_j)\|_2^2} \\ &= \sqrt{4\left(1 - \frac{1}{n}\right)^2 + 4(n-1)\cdot\frac{1}{n^2}} = 2\sqrt{\frac{n-1}{n}}.\end{aligned}$$

This finishes the calculation of $c_2(K_{n,n})$. For $1 \leq p < 2$, since for every $\epsilon > 0$ and for every k, ℓ_p contains a $(1+\epsilon)$ distorted copy of ℓ_2^k, we get the estimate $c_p(K_{n,n}) \leq 2\sqrt{\frac{n-1}{n}}$.

The case $2 < p < \infty$ requires a different embedding. We begin by describing an embedding with distortion $2^{2/p}$ and then explain how to modify it so as to reduce the distortion by a factor of $\left(1 - \frac{1}{2n}\right)^{1/p}$. Let $z_1, \ldots, z_n$ be a collection of n mutually orthogonal ± 1 vectors of dimension $m = O(n)$. (For example the first n rows in an $m \times m$ Hadamard matrix). In our first embedding we define $f(u_i)$ as the $(2m)$-dimensional vector $(z_i, 0)$, namely, z_i concatenated with m zeros. Likewise, $f(v_i) = (0, z_i)$ for all i. Now $\|f(u_i) - f(u_j)\|_p = 2\left(\frac{m}{2}\right)^{1/p}$ and $\|f(u_i) - f(v_j)\|_p = (2m)^{1/p}$, and so f has distortion $2^{2/p}$. To get the $\left(1 - \frac{1}{2n}\right)^{1/p}$ improvement, note that for some $m \leq 4n$ it is possible to select the z_i so that the m-th coordinate in all of them is $+1$. Modify the previous construction to an embedding into $2m - 1$ dimensions as follows: Now $g(u_i)$ is z_i concatenated with $m - 1$ zeros, whereas $g(v_i)$ has zeros in the first $m - 1$ coordinates, 1 in the m-th and this is followed by the first $m - 1$ coordinates of the vector z_i. The easy details are omitted. □

Remark: The upper bounds in Proposition 5 were not used in the proof of Theorem 5. Apart from their intrinsic interest, these upper estimates show that the above technique cannot prove an upper bound of $O(\log n)$ on $R_2(2-\epsilon, n)$ which is independent of ϵ. In fact, this can never be achieved using $\{0,1,2\}$ metric spaces due to the following proposition.

Proposition 8. *Let X be an n-point $\{0,1,2\}$ metric space. Then $c_2(X) \le 2\sqrt{\frac{n-1}{n}}$.*

Proof. We think of X as a metric on $\{1,\dots,n\}$ and denote $d(i,j) = d_{ij}$. Define an $n \times n$ matrix A as follows:

$$A_{ij} = \begin{cases} 2 & \text{if } i = j \\ 0 & \text{if } d_{ij} = 2 \\ \frac{2}{n} & \text{if } d_{ij} = 1 \end{cases}.$$

We claim that A is positive semi-definite. Indeed, for any $z \in \mathbb{R}^n$

$$\begin{aligned} \langle Az, z\rangle &= \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} z_i z_j \\ &\ge \sum_{i=1}^{n} 2z_i^2 - \sum_{i\neq j} \frac{2}{n}|z_i|\cdot|z_j| \\ &\ge \sum_{i=1}^{n} 2z_i^2 - \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{2}{n}|z_i|\cdot|z_j| \\ &= 2\|z\|_2^2 - \frac{2}{n}\|z\|_1^2 \ge 2\|z\|_2^2 - \frac{2}{n} n\|z\|_2^2 = 0. \end{aligned}$$

Let $e_1,\dots,e_n$ be the standard unit vectors in $\mathbb{R}^n$. Define $f : X \to \mathbb{R}^n$ by $f(i) = A^{1/2}e_i$. Now,

$$\|f(i) - f(j)\|_2^2 = \langle Ae_i, e_i\rangle + \langle Ae_j, e_j\rangle - 2\langle Ae_i, e_j\rangle = A_{ii} + A_{jj} - 2A_{ij},$$

so that if $d_{ij} = 1$ then $\|f(i) - f(j)\|_2 = \sqrt{4 - \frac{4}{n}}$ and if $d_{ij} = 2$ then $\|f(i) - f(j)\|_2 = 2$. It follows that

$$\mathrm{dist}(f) = 2\sqrt{\frac{n-1}{n}}.$$

□

3 The Isometric Ramsey Problem

In this section we prove that for $n \ge 3$, $1 < p < \infty$, $R_p(n) = R_p(1, n) = 3$. In fact, we show that this is true for any uniformly convex normed space. We begin by sketching an argument that is specific to ℓ_2:

Proposition 9. $R_2(n) = 3$ *for* $n \ge 3$.

Proof. That $R_2(n) \ge 3$ follows since any metric space on 3 points embeds isometrically in ℓ_2^2. To show that $R_2(n) \le 3$, we construct a metric space on $n > 3$ points, no 4 point subspace of which embeds isometrically in ℓ_2. Fix an integer $n > 3$ and let $\{a_i\}_{i=0}^n$ be an increasing sequence such that $a_0 = 0$, $a_1 = 1$ and for $1 \le i < n$, $a_{i+1} \ge 2(n+1)a_i$. Fix some

$0 < \epsilon < 1/(2a_n)$. It is easily verified that $d(i,j) = |i-j| - \epsilon a_{|i-j|}$ is a metric on $\{1,2,\ldots,n\}$. We show that for ϵ small enough no four points in $(\{1,\ldots,\},d)$ embed isometrically in ℓ_2. Fix four integers $1 \le i_1 < i_2 < i_3 < i_4 \le n$ and set $j = i_2 - i_1$, $k = i_3 - i_2$, $l = i_4 - i_3$. Suppose that for every $\epsilon > 0$ there exists an isometric embedding $f : (\{i_1,i_2,i_3,i_4\},d) \to \ell_2^3$. Without loss of generality we may assume that $f(i_1) = (\alpha,\beta,\gamma)$, $f(i_2) = (0,0,0)$, $f(i_3) = (k-\epsilon a_k, 0, 0)$ and $f(i_4) = (p,q,0)$. Then:

$$\begin{aligned} 2\alpha(k-\epsilon a_k) &= 2\langle f(i_1), f(i_3)\rangle \\ &= \|f(i_1) - f(i_2)\|_2^2 + \|f(i_3) - f(i_2)\|_2^2 - \|f(i_3) - f(i_1)\|_2^2 \\ &= (j-\epsilon a_j)^2 + (k-\epsilon a_k)^2 - (j+k-\epsilon a_{j+k})^2. \end{aligned}$$

Hence,

$$\alpha \le -j + \frac{\epsilon}{k}[(k+j)a_{k+j} - ja_j - ka_k - ja_k] + O(\epsilon^2).$$

Similarly:

$$p \ge (k+l) + \frac{\epsilon}{k}[(k+l)a_k - (k+l)a_{k+l} - ka_k + la_l] + O(\epsilon^2).$$

Now:

$$\begin{aligned} & j+k+l-\epsilon a_{j+k+l} = \\ &\quad = \|f(i_4) - f(i_1)\|_2 \\ &\quad \ge p - \alpha \\ &\quad \ge j+k+l + \frac{\epsilon}{k}[(k+l)a_k - (k+l)a_{k+l} + la_l - (k+j)a_{k+j} + ja_j + ja_k] + O(\epsilon^2). \end{aligned}$$

Letting ϵ tend to zero we deduce that:

$$\begin{aligned} & a_{j+k+l} \le \\ &\quad \le \left(1+\frac{j}{k}\right)a_{k+j} + \left(1+\frac{l}{k}\right)a_{k+l} - \frac{l}{k}a_l - \frac{j}{k}a_j - \frac{j+k+l}{k}a_k < 2(n+1)a_{j+k+l-1}, \end{aligned}$$

which is a contradiction. □

The argument above is quite specific to ℓ_2, and so we now consider any uniformly convex normed space. The modulus of uniform convexity of a normed space X is defined by:

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|a+b\|}{2};\ \|a\|,\|b\| \le 1 \quad \text{and} \quad \|a-b\| \ge \epsilon\right\}.$$

X is said to be uniformly convex if $\delta_X(\epsilon) > 0$ for every $0 < \epsilon \le 2$. The L_p spaces $1 < p < \infty$, are known to be uniformly convex. For a uniformly convex space X, δ_X is known to be continuous and strictly increasing on $(0,2]$.

Assume that X is a uniformly convex normed space and $a,b \in X \setminus \{0\}$. Then:

$$\begin{aligned} & \left\|\frac{a}{\|a\|} + \frac{b}{\|b\|}\right\| = \\ &\qquad = \left\|\left(\frac{1}{\|a\|} + \frac{1}{\|b\|}\right)(a+b) - \frac{a}{\|b\|} - \frac{b}{\|a\|}\right\| \\ &\qquad \ge \left(\frac{1}{\|a\|} + \frac{1}{\|b\|}\right)\|a+b\| - \frac{\|a\|}{\|b\|} - \frac{\|b\|}{\|a\|} \\ &\qquad = 2 - \left(\frac{1}{\|a\|} + \frac{1}{\|b\|}\right)(\|a\| + \|b\| - \|a+b\|). \end{aligned}$$

Now,

$$\delta_X\left(\left\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\right\|\right)\le$$

$$\le 1-\frac{1}{2}\cdot\left\|\frac{a}{\|a\|}+\frac{b}{\|b\|}\right\|\le\frac{1}{2}\cdot\left(\frac{1}{\|a\|}+\frac{1}{\|b\|}\right)(\|a\|+\|b\|-\|a+b\|).$$

Hence

$$\left\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\right\|\le\delta_X^{-1}\left(\frac{1}{2}\cdot\left(\frac{1}{\|a\|}+\frac{1}{\|b\|}\right)(\|a\|+\|b\|-\|a+b\|)\right).$$

Take $x,y,z\in X$ and apply this inequality for $a=x-y$, $b=y-z$. It follows that:

$$\left\|y-\left(\frac{\|y-z\|}{\|x-y\|+\|y-z\|}\cdot x+\frac{\|x-y\|}{\|x-y\|+\|y-z\|}\cdot z\right)\right\|\le$$

$$\le\frac{\|x-y\|\cdot\|y-z\|}{\|x-y\|+\|y-z\|}\cdot\delta_X^{-1}\left(\frac{\|x-y\|+\|y-z\|-\|x-z\|}{\min\{\|x-y\|,\|y-z\|\}}\right).$$

This inequality is the way uniform convexity is going to be applied in the sequel. Indeed, we have the following "metric" consequence of it:

Lemma 10. *Let X be a uniformly convex normed space and $x_1,x_2,x_3,x_4\in X$ be distinct. Then:*

$$\frac{\|x_1-x_2\|+\|x_2-x_3\|-\|x_1-x_3\|}{2\|x_2-x_3\|}\le$$

$$\le\delta_X^{-1}\left(\frac{\|x_1-x_3\|+\|x_3-x_4\|-\|x_1-x_4\|}{\min\{\|x_1-x_3\|,\|x_3-x_4\|\}}\right)+$$

$$+\delta_X^{-1}\left(\frac{\|x_2-x_3\|+\|x_3-x_4\|-\|x_2-x_4\|}{\min\{\|x_2-x_3\|,\|x_3-x_4\|\}}\right).$$

Proof. Define:

$$\lambda=\frac{\|x_3-x_4\|}{\|x_1-x_3\|+\|x_3-x_4\|}\quad\text{and}\quad\mu=\frac{\|x_3-x_4\|}{\|x_2-x_3\|+\|x_3-x_4\|}.$$

An application of the above inequality twice gives:

$$\|x_3-(\lambda x_1+(1-\lambda)x_4)\|\le\frac{\|x_1-\|x_3\|\cdot\|x_3-x_4\|}{\|x_1-x_3\|+\|x_3-x_4\|}\cdot\delta_X^{-1}\left(\frac{\|x_1-x_3\|+\|x_3-x_4\|-\|x_1-x_4\|}{\min\{\|x_1-x_3\|,\|x_3-x_4\|\}}\right),$$

and

$$\|x_3-(\mu x_2+(1-\mu)x_4)\|\le\frac{\|x-2-x_3\|\cdot\|x_3-x_4\|}{\|x_2-x_3\|+\|x_3-x_4\|}\cdot\delta_X^{-1}\left(\frac{\|x_2-x_3\|+\|x_3-x_4\|-\|x_2-x_4\|}{\min\{\|x_2-x_3\|,\|x_3-x_4\|\}}\right)$$

By symmetry, we may assume without loss of generality that $\lambda \leq \mu$. Now,

$$\begin{aligned}
&\left\| x_2 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)} x_1 - \frac{\mu-\lambda}{\mu(1-\lambda)} x_3 \right\| \leq \\
&\qquad = \frac{1}{\mu} \left\| \mu x_2 + (1-\mu) x_4 - x_3 + \frac{1-\mu}{1-\lambda}(x_3 - \lambda x_1 - (1-\lambda) x_4) \right\| \\
&\qquad \leq \frac{1}{\mu} \|x_3 - \mu x_2 - (1-\mu) x_4\| + \frac{1-\mu}{\mu(1-\lambda)} \cdot \|x_3 - \lambda x_1 - (1-\lambda) x_4\| \\
&\qquad \leq \frac{\|x_2 - x_3\| + \|x_3 - x_4\|}{\|x_3 - x_4\|} \cdot \frac{\|x_2 - x_3\| \cdot \|x_3 - x_4\|}{\|x_2 - x_3\| + \|x_3 - x_4\|} \cdot \\
&\qquad \quad \cdot \delta_X^{-1} \left(\frac{\|x_2 - x_3\| + \|x_3 - x_4\| - \|x_2 - x_4\|}{\min\{\|x_2 - x_3\|, \|x_3 - x_4\|\}} \right) + \\
&\qquad \quad + \frac{\|x_2 - x_3\|}{\|x_3 - x_4\|} \frac{\|x_1 - x_3\| + \|x_3 - x_4\|}{\|x_1 - x_3\|} \frac{\|x_1 - \|x_3\| \cdot \|x_3 - x_4\|}{\|x_1 - x_3\| + \|x_3 - x_4\|} \cdot \\
&\qquad \quad \cdot \delta_X^{-1} \left(\frac{\|x_1 - x_3\| + \|x_3 - x_4\| - \|x_1 - x_4\|}{\min\{\|x_1 - x_3\|, \|x_3 - x_4\|\}} \right) \\
&\qquad = \|x_2 - x_3\| \delta_X^{-1} \left(\frac{\|x_1 - x_3\| + \|x_3 - x_4\| - \|x_1 - x_4\|}{\min\{\|x_1 - x_3\|, \|x_3 - x_4\|\}} \right) + \\
&\qquad \quad + \|x_2 - x_3\| \delta_X^{-1} \left(\frac{\|x_2 - x_3\| + \|x_3 - x_4\| - \|x_2 - x_4\|}{\min\{\|x_2 - x_3\|, \|x_3 - x_4\|\}} \right).
\end{aligned}$$

Additionally,

$$\begin{aligned}
&\|x_2 - x_1\| \leq \\
&\qquad \leq \left\| x_2 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)} x_1 - \frac{\mu-\lambda}{\mu(1-\lambda)} x_3 \right\| + \left\| x_1 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)} x_1 - \frac{\mu-\lambda}{\mu(1-\lambda)} x_3 \right\| \\
&\qquad = \left\| x_2 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)} x_1 - \frac{\mu-\lambda}{\mu(1-\lambda)} x_3 \right\| + \frac{\mu-\lambda}{\mu(1-\lambda)} \|x_1 - x_3\|,
\end{aligned}$$

and

$$\begin{aligned}
&\|x_2 - x_3\| \leq \\
&\qquad \leq \left\| x_2 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)} x_1 - \frac{\mu-\lambda}{\mu(1-\lambda)} x_3 \right\| + \left\| x_3 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)} x_1 - \frac{\mu-\lambda}{\mu(1-\lambda)} x_3 \right\| \\
&\qquad = \left\| x_2 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)} x_1 - \frac{\mu-\lambda}{\mu(1-\lambda)} x_3 \right\| + \frac{\lambda(1-\mu)}{\mu(1-\lambda)} \|x_1 - x_3\|.
\end{aligned}$$

Summing up these estimates gives the required result. □

We can now prove the main result of this section:

Theorem 6. *Let X be a uniformly convex normed space with $\dim(X) \geq 2$. Then for every $n \geq 3$, $R_X(n) = 3$. Moreover, for every $\delta : (0,2] \to (0,\infty)$ which is continuous, increasing and $\delta \leq \delta_{\ell_2}$, let UC_δ be the class of all normed spaces X with $\delta_X \geq \delta$. Then for each $n \geq 3$ there is a constant $\epsilon_n(\delta) > 0$ such that $R_{UC_\delta}(1 + \epsilon_n(\delta), n) = 3$.*

Proof. That $R_X(n) \geq 3$ follows since any 3 point metric embeds isometrically into any 2 dimensional normed space, by a standard continuity argument.

Fix some $\delta : (0,2] \to (0,\infty)$ which is continuous, increasing and $\delta \le \delta_{\ell_2}$. We shall construct inductively a sequence $\{M_n\}_{n=3}^{\infty}$ of metric spaces and numbers $\{\eta_n\}_{n=3}^{\infty}$ such that:

a) For every $n \ge 3$, $\eta_n > 0$. Each M_n is a metric on $\{1,\ldots,n\}$, and we denote $d^n_{ij} = d_{M_n}(i,j)$.

b) For every $1 \le i < j < k \le n$,

$$d^n_{i,j} + d^n_{jk} - d^n_{i,k} - \eta_n \ge 2d^n_{j,k}\left[\delta^{-1}\left(\frac{d^n_{i,k} + d^n_{k,n} - d^n_{i,n}}{\min\{d^n_{i,k}, d^n_{k,n}\}}\right) + \delta^{-1}\left(\frac{d^n_{j,k} + d^n_{k,n} - d^n_{j,n}}{\min\{d^n_{j,k}, d^n_{k,n}\}}\right)\right].$$

Lemma 10 immediately implies that there is a constant $\epsilon_n(\delta) > 0$ such that for every $1 \le i < j < k < l \le n$ and for every normed space X with $\delta_X \ge \delta$:

$$c_X(\{i,j,k,l\}, d_{M_n}) \ge 1 + \epsilon_n(\delta),$$

as required.

M_3 is the equilateral metric on $\{1,2,3\}$, in which case $\eta_3 = 1$. We construct $M_{n+1} = (\{1,\ldots,n+1\}, d^{n+1})$ as an extension of M_n, by setting

$$d^{n+1}_{n,n+1} = 1 - s/2 \quad \text{and} \quad \forall 1 \le i < n,\ d^{n+1}_{i,n+1} = d^n_{in} + 1 - s.$$

This is indeed a definition of a metric as long as $0 < s \le \min\{1, 2\min_{1\le i<n} d^n_{i,n}\}$ (this fact follows from a simple case analysis).

We are left to check condition **b**). Fix $1 \le i < j < k \le n$. If $k \ne n$ then:

$$\begin{aligned}
d^{n+1}_{i,j} + d^{n+1}_{j,k} - d^{n+1}_{i,k} - \eta_n &= d^n_{i,j} + d^n_{j,k} - d^n_{i,k} - \eta_n \\
&\ge 2d^n_{j,k}\left[\delta^{-1}\left(\frac{d^n_{i,k} + d^n_{k,n} - d^n_{i,n}}{\min\{d^n_{i,k}, d^n_{k,n}\}}\right) + \delta^{-1}\left(\frac{d^n_{j,k} + d^n_{k,n} - d^n_{j,n}}{\min\{d^n_{j,k}, d^n_{k,n}\}}\right)\right] \\
&\ge 2d^n_{j,k}\left[\delta^{-1}\left(\frac{d^n_{i,k} + (d^n_{k,n} + 1 - s) - (d^n_{i,n} + 1 - s)}{\min\{d^n_{i,k}, d^n_{k,n} + 1 - s\}}\right)\right. \\
&\qquad \left. + \delta^{-1}\left(\frac{d^n_{j,k} + (d^n_{k,n} + 1 - s) - (d^n_{j,n} + 1 - s)}{\min\{d^n_{j,k}, d^n_{k,n} + 1 - s\}}\right)\right] \\
&= 2d^{n+1}_{j,k}\left[\delta^{-1}\left(\frac{d^{n+1}_{i,k} + d^{n+1}_{k,n+1} - d^{n+1}_{i,n+1}}{\min\{d^{n+1}_{i,k}, d^{n+1}_{k,n+1}\}}\right) + \delta^{-1}\left(\frac{d^{n+1}_{j,k} + d^{n+1}_{k,n+1} - d^{n+1}_{j,n+1}}{\min\{d^{n+1}_{j,k}, d^{n+1}_{k,n+1}\}}\right)\right].
\end{aligned}$$

It remains to check **b**) for the quadruple $\{i,j,n,n+1\}$. Condition **b**) for M_n implies that:

$$d^{n+1}_{ij} + d^{n+1}_{jn} - d^{n+1}_{in} \ge \eta_n.$$

On the other hand,

$$\begin{aligned}
&2d^{n+1}_{j,n}\left[\delta^{-1}\left(\frac{d^{n+1}_{i,n} + d^{n+1}_{n,n+1} - d^{n+1}_{i,n+1}}{\min\{d^{n+1}_{i,n}, d^{n+1}_{n,n+1}\}}\right) + \delta^{-1}\left(\frac{d^{n+1}_{j,n} + d^{n+1}_{n,n+1} - d^{n+1}_{j,n+1}}{\min\{d^{n+1}_{j,n}, d^{n+1}_{n,n+1}\}}\right)\right] = \\
&\qquad = 2d^n_{j,n}\left[\delta^{-1}\left(\frac{s/2}{\min\{d^n_{i,n}, 1 - s/2\}}\right) + \delta^{-1}\left(\frac{s/2}{\min\{d^n_{j,n}, 1 - s/2\}}\right)\right],
\end{aligned}$$

so that condition **b**) will hold when s is small enough such that the quantity above is at most $\eta_n/2$ and with $\eta_{n+1} = \eta_n/2$. □

Corollary 11. *For all $1 < p < \infty$, $R_p(n) = 3$ for $n \geq 3$.*

We end this section with a simple lower bound for the isometric Ramsey problem for graphs. We do not know what is the asymptotically correct bound in this setting.

Proposition 12. *Let G be an un-weighted graph of order n. Then there is a set of $\Omega\left(\sqrt{\frac{\log n}{\log\log n}}\right)$ vertices in G whose metric embeds isometrically into ℓ_2.*

Proof. Let Δ be the diameter of G. The shortest path between two diameterically far vertices is isometrically embeddable in ℓ_2. On the other hand, the Bourgain,Figiel, Milman theorem [4] yields that for every $0 < \epsilon < 1$ a subset $N \subset V$ which is $(1+\epsilon)$ embeddable in Hilbert space and $|N| = \Omega\left(\frac{\epsilon}{\log(2/\epsilon)}\log n\right)$. When $\epsilon = \frac{1}{2\Delta}$, such an embedding is an isometry. Hence we can always extract a subset of V which is isometrically embeddable in ℓ_2 with cardinality

$$\Omega\left(\max\left\{\Delta, \frac{\log n}{\Delta\log\Delta}\right\}\right) = \Omega\left(\sqrt{\frac{\log n}{\log\log n}}\right),$$

as claimed. □

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