Lipschitz sums of convex functions

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Abstract

In this paper we give a geometric characterization of the convex subsets of a Banach space with the property that for any two convex continuous functions on this set, if their sum is Lipschitz, then the functions must be Lipschitz. We apply this result to the theory of Δ-convex functions.

1 Introduction

This paper deals with the following problem: let K be a convex subset of a Banach space X. Let $h_1, h_2 \colon K \to \mathbb{R}$ be convex continuous functions such that $h_1 + h_2$ is Lipschitz. Does this necessarily imply that h_1 and h_2 are themselves Lipschitz? More precisely, we are interested in the geometric properties of K which imply such a statement. Under some mild assumptions, we give here a necessary and sufficient condition for K to have such a property.

Throughout this paper, all Banach spaces are real. If X is a Banach space, $x \in X$ and $r > 0$ we use the notation $B(x,r) = \{y \in X;\ \|x-y\| \le r\}$ and $S(x,r) = \{y \in X;\ \|x-y\| = r\}$. We also denote $B_X = B(0,1)$ and $S_X = S(0,1)$. X^* denotes the dual space of X. For any two sets A, B we denote by $d(A,B)$ the distance between A and B.

The method of proof forces us to impose some assumptions on K. Apart from the natural assumptions $0 \in K$ and $\overline{\text{span}}(K) = X$, we essentially assume that either $K - K$ or its complement is nowhere dense. This will

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be formulated more precisely later, but we prefer to begin by stating the theorem in a simpler, somewhat less general form. We do not presently know if the result is true for a general convex set which is not contained in any hyperplane.

Theorem 1 *Let X be a Banach space and suppose that K is a convex subset of X for which*

- $0 \in K$;
- $\overline{\text{span}}(K) = X$;

and either

- *K is locally weakly compact*

or

- *K has nonempty interior.*

Then the following are equivalent:

(i) If $h_1, h_2 : K \to \mathbb{R}$ are continuous convex functions such that $h_1 + h_2$ is Lipschitz, then both h_1 and h_2 must be Lipschitz.

(ii) There exists a constant $0 < c \leq 2$ such that for every $x \in K$, $0 < r < \text{diam K}$ and $y \in S_X$ there are $u, v \in K \cap B(x, r)$ such that $u - v = cry$.

(iii) There is a $c > 0$ such that for every $x^*, y^* \in S_{X^*}$ and for every $x, y \in K$ with $y^*(x) < y^*(y)$ there are $u, v \in K$ such that $y^*(x) \leq y^*(u) \leq y^*(v) \leq y^*(y)$ and $|x^*(u) - x^*(v)| > c\big(y^*(y) - y^*(x)\big)$.

Remark 2 We remark that the implication (ii) $\Longrightarrow$ (i) requires only the assumptions $0 \in K$ and $\overline{\text{span}}(K) = X$. Thus in our later applications, we will be able to state results for arbitrary convex subsets of Banach spaces satisfying these two assumptions.

If K is bounded and has nonempty interior, then (ii) must be satisfied. Indeed, by convexity, if $B(x_0, r_0) \subset K$ for some x_0 and r_0, then $K \cap B(x, r)$ contains a segment of length $r_0 r/\text{diam K}$ in each direction. It is also easy to see that every convex set K which contains an infinite open cone satisfies condition (ii).

If $\dim X < \infty$, then every convex set K with $\overline{\text{span}}(K) = X$ and $0 \in K$ has nonempty interior. It is also easy to see that if $\dim X < \infty$, and if a convex set $K \subset X$ has nonempty interior then conditions (i)–(iii) of Theorem 1 imply that K is either bounded or contains an infinite open cone:

Corollary 3 *Conditions (i)–(iii) of Theorem 1 hold for any convex set K with nonempty interior, which is either bounded or contains an infinite open cone.*

Suppose $\dim X < \infty$ *and K is a convex set for which $0 \in K$ and* $\overline{\text{span}}(K) = X$. *Then (i)–(iii) of Theorem 1 are satisfied by K if and only if either K is bounded or K contains an infinite cone C for which* $\overline{\text{aff}}(C) = X$.

In contrast, in Remark 13 we give an example of an *open* unbounded convex set K in an infinite dimensional Banach space, which fulfills conditions (i)–(iii) but does not contain a half-line. We also give an example of an open convex set K which contains an infinite cone C such that $\overline{\text{span}}(C) = X$, but for which conditions (i)–(iii) fail.

For any convex set K, $x \in K$ and $r > 0$ set

$$K_{x,r} = K \cap B(x, r).$$

It is easy to see that if K has nonempty interior or if $K_{x,r}$ is weakly compact for every $x \in K$, $r > 0$, then $K_{x,r} - K_{x,r}$ has nonempty interior or is weakly compact, respectively. Every weakly compact set is either nowhere dense or has nonempty interior.

In fact, our proof uses only the following fact:

($*$) *For every $x \in K$, $r > 0$ and $y \notin K_{x,r} - K_{x,r}$ there exists a z arbitrarily close to y and there exists $z^* \in S_{X^*}$ which separates z and $K_{x,r} - K_{x,r}$.*

By the Hahn-Banach Theorem, ($*$) is satisfied if and only if $K_{x,r} - K_{x,r}$ is either nowhere dense or has nonempty interior. We prove, in fact, the following strengthening of Theorem 1:

Theorem 4 *Let X be a Banach space and suppose that K is a convex subset of X for which*

- $0 \in K$;
- $\overline{\text{span}}(K) = X$;

and either

- $K_{x,r} - K_{x,r}$ *is nowhere dense for every* $x \in K$ *and* $r > 0$

or

- $K_{x,r} - K_{x,r}$ *has nonempty interior for every* $x \in K$ *and* $r > 0$.

Then the following are equivalent:

(i) *If* $h_1, h_2 : K \to \mathbb{R}$ *are continuous convex functions such that* $h_1 + h_2$ *is Lipschitz, then both* h_1 *and* h_2 *must be Lipschitz.*

(ii) *There exists a constant* $0 < c \leq 2$ *such that for every* $x \in K$, $r < \operatorname{diam K}$ *and* $y \in S_X$ *there are* $u, v \in K \cap B(x, r)$ *such that* $u - v = cry$.

(iii) *There is a* $c > 0$ *such that for every* $x^*, y^* \in S_{X^*}$ *and for every* $x, y \in K$ *with* $y^*(x) < y^*(y)$ *there are* $u, v \in K$ *such that* $y^*(x) \leq y^*(u) \leq y^*(v) \leq y^*(y)$ *and* $|x^*(u) - x^*(v)| > c\big(y^*(y) - y^*(x)\big)$.

A rather unexpected consequence of our result is the following fact:

Corollary 5 *Suppose that* K *is an unbounded open set which satisfies the hypotheses of Theorem 4. If* $H \subset X$ *is convex and* $K \subset H$*, then* H *satisfies (i).*

Proof: By Remark 2, all we have to do is to check that condition (ii) for K implies condition (ii) for H. Fix some $x \in H$, $r > 0$, $y \in S_X$ and $z \in K$.

If $r \geq d(x, z)$ then, using (ii) for K, there are

$$u, v \in B(z, r) \cap K \subset B(x, 2r) \cap H$$

with $u - v = cry$. Then of course $u' = (u + x)/2$ and $v' = (v + x)/2$ are in $B(x, r) \cap H$ and $u' - v' = (c/2)ry$.

On the other hand, if $r < d(x, z)$ then, using (ii) for K, there are

$$u, v \in B\big(z, d(x, z)\big) \cap K \subset B\big(x, 2d(x, z)\big) \cap H$$

such that $u - v = cd(x, z)y$. Now, the points

$$u' = x + \frac{r}{2d(x, z)}(u - x) \quad \text{and} \quad v' = x + \frac{r}{2d(x, z)}(v - x)$$

are in $B(x,r) \cap H$ and

$$u' - v' = \frac{crd(x,z)}{2d(x,z)} y = (c/2)ry.$$

Therefore H satisfies (ii) for $c/2$. ■

Before passing to the proof of Theorem 4, we discuss an application of this result to the theory of Δ-convex mappings.

Let X and Y be Banach spaces, and $K \subset X$ convex. A mapping $F\colon K \to Y$ is called Δ-convex if there is a continuous convex function $f\colon K \to \mathbb{R}$ such that for every $y^* \in S_{Y^*}$, $y^* \circ F + f$ is continuous and convex. Such an f is called a control function for F.

Such functions are known to have many interesting properties. For instance, it is known that the Δ-convex mappings from $\mathbb{R}^n$ to $\mathbb{R}$ form an algebra, and that the composition of two Δ-convex mappings is itself Δ-convex. Δ-convex mappings have applications in optimization theory, approximation theory, and the theory of Gâteaux differentiability of convex functions. For more information we refer to the papers [VZ], [DVZ] and the references therein. The geometric importance of Δ-convex mappings may be appreciated from a result of M. Cepedello Boiso [C1, C2] (see also the book [BL]), which states that a Banach space X is superreflexive if and only if every real-valued Lipschitz function on X can be uniformly approximated by Δ-convex functions.

The properties of a control function f for a Δ-convex function F impose restrictions on the behavior of F. For instance, it is proved in [VZ] that F is differentiable (Fréchet or Gâteaux) at some point whenever f is. One can ask: is F Lipschitz whenever f is? Theorem 1 and a simple Banach-Steinhaus argument give the following improvement of Theorem 18 in [DVZ]:

Theorem 6 *Let X and Y be Banach spaces and $K \subset X$ a convex set satisfying (ii). If $F\colon K \to Y$ is Δ-convex with Lipschitz control function f, then F is Lipschitz.*

Proof: For each $y^* \in S_{Y^*}$, the functions $g_1 = y^* \circ F + f$ and $g_2 = -y^* \circ F + f$ are continuous and convex on K, and $g_1 + g_2$ is Lipschitz. By Remark 2, K satisfies also (i), so that g_1 and g_2 are Lipschitz. Consequently $y^* \circ F$ is Lipschitz for each $y^* \in Y^*$. The Banach-Steinhaus theorem now implies that F is Lipschitz. ■

The example in Remark 13 shows that this is a strict improvement of the result of [DVZ].

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2 Proof of Theorem 4

Let $K \subset X$ be a convex set which satisfies the assumptions of Theorem 4. We break the proof of Theorem 4 down into several lemmas. We prove the theorem by showing that (ii) $\implies$ (i), (iii) $\implies$ (ii) and $\neg$(iii) $\implies$ $\neg$(i).

For every pair of vectors $x, y \in X \setminus \{0\}$ define

$$\alpha(0,x,y) = \min\left(d\left(\frac{x}{\|x\|}, \mathbb{R}y\right), d\left(\frac{y}{\|y\|}, \mathbb{R}x\right)\right).$$

For any $p \in X$ put $\alpha(p,x,y) = \alpha(0, x-p, y-p)$.

Similarly, if $x, y \in X$ and $p \in X \setminus \{x, y\}$ then we define:

$$\beta(p,x,y) = \frac{\|x-y\|}{\max(\|p-x\|, \|p-y\|)}.$$

Obviously $\beta(p,x,y) \geq \alpha(p,x,y)$.

We begin by showing that condition (ii) of Theorem 4 implies an apparently stronger geometric condition.

Lemma 7 *Assume that condition (ii) of Theorem 4 holds for some constant $0 < c \leq 2$. Then for every $x \in K$, $r < \operatorname{diam} K$ and $y \in S_X$ there are $u, v \in K \cap B(x,r)$ such that:*

(1) $u - v = \frac{c^2 r}{8+c} y$;

(2) $d(x + \mathbb{R}y, u + \mathbb{R}y) \geq \frac{cr}{10}$;

(3) $\beta(x,u,v) \geq \alpha(x,u,v) \geq \frac{c^3}{120}$.

Proof: Define $r' = r/(1+c/8)$. Note that there is some $z \in K \cap B(x,r')$ such that $d(z, x+\mathbb{R}y) \geq cr'/4$. Indeed, if y^* is a norm-one functional with $y^*(y) = 0$ then for every $w \in S_X$ with $y^*(w) > 1/2$, by our assumption, there are $a, b \in K \cap B(x,r')$ for which $a - b = cr'w$. Then

$$d(a, x+\mathbb{R}y) + d(b, x+\mathbb{R}y) \geq |y^*(a-x)| + |y^*(b-x)| \geq y^*(cr'w) > cr'/2,$$

thus $\max\{d(a, x+\mathbb{R}y), d(b, x+\mathbb{R}y)\} \geq cr'/4$.

Fix any such z. Using the assumption again, there are

$$u, v \in K \cap B(z, cr'/8) \subset K \cap B(x,r)$$

for which

$$u - v = c(cr'/8)y = \frac{c^2 r}{8+c} y.$$

Now

$$d(x+\mathbb{R}y, u+\mathbb{R}y) \geq \frac{cr'}{4} - \frac{cr'}{8} = \frac{cr}{8+c} \geq \frac{cr}{10}.$$

For any $q \in x + \mathbb{R}(u-x)$ write $q = \lambda u + (1-\lambda)x$ and define

$$q' = x + \frac{v-q}{1-\lambda} = u + \frac{v-u}{1-\lambda}.$$

Clearly q' is on the line $u + \mathbb{R}y$, thus $\|q'-x\| \geq cr/10$. Therefore

$$\|v-q\| = \|q'-x\||1-\lambda| = \|q'-x\| \cdot \frac{\|u-q\|}{\|u-x\|} \geq \frac{cr}{10} \cdot \frac{\|u-v\| - \|v-q\|}{\|u-x\|}.$$

Hence

$$\|v-q\| \geq \frac{\frac{cr}{10} \cdot \|u-v\|}{\|u-x\| + \frac{cr}{10}} \geq \frac{\frac{cr}{10} \cdot \frac{c^2 r}{10}}{r + \frac{cr}{10}} \geq \frac{c^3 r}{120}.$$

Finally,

$$\frac{\|v-q\|}{\|v-x\|} \geq \frac{c^3}{120},$$

which by symmetry proves (3). ■

Lemma 8 *Assume that $f, g \colon K \to \mathbb{R}$ are such that $h_1 = g + f$ and $h_2 = g - f$ are continuous convex functions, and g is Lipschitz with constant L. Then for every line ℓ which intersects K in at least two points and for every $x, y, z, w \in \ell \cap K$, $x \neq y$, $z \neq w$*

$$\frac{|f(x) - f(y)|}{\|x-y\|} - \frac{|f(z) - f(w)|}{\|z-w\|} \leq 4L.$$

Proof: Without loss of generality we may assume that $\ell = \mathbb{R}$. Put $u = \min(x, y, z, w)$ and $v = \max(x, y, z, w)$. Then for any $a, b \in [u, v]$, $a \neq b$ and $i = 1, 2$

$$D^+ h_i(u) \leq \frac{h_i(a) - h_i(b)}{a - b} \leq D^- h_i(v),$$

where D^+ and D^- denote the right and left derivatives, respectively. In particular, $0 \leq D^- h_i(v) - D^+ h_i(u)$, and so

$$0 \leq \big(D^- h_2(v) - D^+ h_2(u)\big) + \big(D^- h_1(v) - D^+ h_1(u)\big) = 2D^- g(v) - 2D^+ g(u) \leq 4L.$$

It follows that either $0 \leq D^- h_1(v) - D^+ h_1(u) \leq 2L$, or $0 \leq D^- h_2(v) - D^+ h_2(u) \leq 2L$. If $0 \leq D^- h_1(v) - D^+ h_1(u) \leq 2L$, then

$$\begin{aligned} 4L \geq & \quad \big(D^- h_1(v) + L\big) - \big(D^+ h_1(u) - L\big) \\ \geq & \quad \frac{\big(h_1(x) - h_1(y)\big) - \big(g(x) - g(y)\big)}{x - y} - \frac{\big(h_1(z) - h_1(w)\big) - \big(g(z) - g(w)\big)}{z - w} \\ = & \quad \frac{f(x) - f(y)}{x - y} - \frac{f(z) - f(w)}{z - w}. \end{aligned}$$

If $0 \leq D^- h_2(v) - D^+ h_2(u) \leq 2L$, then a similar calculation gives the same bound. ■

Proof that (ii) $\Longrightarrow$ (i): Suppose that h_1, h_2 are continuous convex functions on K, for which $h_1 + h_2$ is Lipschitz. We write $h_1 = g + f$ and $h_2 = g - f$. Then g is Lipschitz with constant L, say. Our goal is to prove that f is also Lipschitz.

Fix two points $p_1, p_2 \in K$ and some $\varepsilon > 0$. Since f is continuous at p_1 and p_2, there is a $0 < \delta \leq \|p_1 - p_2\|/4$ such that $|f(p_1) - f(q)| < \varepsilon$ for every $q \in B(p_1, \delta)$ and $|f(p_2) - f(q)| < \varepsilon$ for every $q \in B(p_2, \delta)$. Let ℓ be a line which intersects K in at least two points. Take any p which is in the relative interior of the interval $K \cap \ell$. Clearly $\max\{\|p - p_1\|, \|p - p_2\|\} \geq 2\delta$, so that we may assume that $\|p - p_1\| = d \geq 2\delta$.

Applying Lemma 7 for $x = p$, $r = d - \delta$ and y a unit vector in the direction of ℓ, we find $u, v \in K \cap B(p, d - \delta)$ which satisfy conclusions (1), (2) and (3) of Lemma 7. For every $a \in B(p, d - \delta) \cap K$ we have $\|p_1 - a\| \geq \delta$. By convexity, this implies that $a' = p_1 + \frac{\delta}{\|a - p_1\|}(a - p_1)$ is in K. Lemma 8 applied to the line connecting a and p_1 shows that for any two points b, c on this line

$$\frac{|f(b) - f(c)|}{\|b - c\|} \leq \frac{|f(a') - f(p_1)|}{\|a' - p_1\|} + 4L \leq \frac{\varepsilon}{\delta} + 4L$$

holds. By Lemma 7 the distance between p and u and the distance between p and v is at least $c(d-\delta)/10 \geq cd/20$. On the other hand, the distance between p_1 and any of the points p, u, v is at most $2d$. Hence

$$\begin{aligned} |f(u)-f(p)| &\leq |f(u)-f(p_1)|+|f(p)-f(p_1)| \\ &\leq \left(\frac{\varepsilon}{\delta}+4L\right)\cdot 4d \\ &\leq \left(\frac{\varepsilon}{\delta}+4L\right)\cdot \frac{80\|u-p\|}{c}. \end{aligned}$$

That is,

$$\frac{|f(u)-f(p)|}{\|u-p\|} \leq \frac{80}{c}\left(\frac{\varepsilon}{\delta}+4L\right),$$

and, similarly,

$$\frac{|f(v)-f(p)|}{\|v-p\|} \leq \frac{80}{c}\left(\frac{\varepsilon}{\delta}+4L\right).$$

In addition to the above, we know that $\alpha(p,u,v) \geq c^3/120$.

Now we use the continuity of f to find some $p' = p + \lambda y \in K \cap \ell$ where $\lambda > 0$ is small enough to ensure that

$$\max\left(\frac{|f(u)-f(p')|}{\|u-p'\|}, \frac{|f(v)-f(p')|}{\|v-p'\|}\right) \leq \frac{100}{c}\left(\frac{\varepsilon}{\delta}+4L\right).$$

Let s be the intersection point of the line segments between p and u, and p' and v (note that $p'-p$ and $u-v$ have the same direction, therefore this intersection point exists). By Lemma 8

$$\frac{|f(p)-f(s)|}{\|p-s\|} \leq \frac{|f(p)-f(u)|}{\|p-u\|} + 4L \leq \frac{100}{c}\left(\frac{\varepsilon}{\delta}+5L\right),$$

and similarly, with p' replacing p and v replacing u. Hence if λ is small enough then

$$\frac{|f(p')-f(s)|}{\|p'-s\|} \leq \frac{100}{c}\left(\frac{\varepsilon}{\delta}+6L\right).$$

By choosing λ to be small enough, we can also ensure that $\alpha(s,u,v) =$

$\alpha(s,p,p') \geq c^3/200$. Hence

$$\begin{aligned}
|f(p)-f(p')| &\leq |f(p)-f(s)|+|f(p')-f(s)| \\
&\leq 2\max\left(\|p-s\|,\|p'-s\|\right)\cdot\frac{100}{c}\cdot\left(\frac{\varepsilon}{\delta}+6L\right) \\
&= \frac{\|p-p'\|}{\beta(s,p,p')}\cdot\frac{200}{c}\cdot\left(\frac{\varepsilon}{\delta}+6L\right) \\
&\leq \frac{\|p-p'\|}{\alpha(s,p,p')}\cdot\frac{200}{c}\cdot\left(\frac{\varepsilon}{\delta}+6L\right) \\
&\leq \frac{40000\|p-p'\|}{c^4}\left(\frac{\varepsilon}{\delta}+6L\right).
\end{aligned}$$

This proves that f is Lipschitz on each line ℓ with Lipschitz constant

$$\frac{40000}{c^4}\left(\frac{\varepsilon}{\delta}+6L\right). \qquad \blacksquare$$

Proof that (iii) $\Longrightarrow$ (ii): It is convenient to distinguish three cases:

Case 1: K is bounded.
Case 2: K is unbounded, and there is an $x^* \in S_{X^*}$ for which

$$\sup_{x\in K}|x^*(x)| < \infty.$$

Case 3: For every $x^* \in S_{X^*}$ we have

$$\sup_{x\in K}|x^*(x)| = \infty.$$

Proof for Case 1: In this case condition (ii) of Theorem 4 is equivalent to the fact that for some $\varepsilon > 0$, K contains a line segment of length ε in each direction. Indeed, (ii) clearly implies the existence of such line segments. On the other hand, if there are segments of length ε in each direction, then by convexity for every $x \in K$ and $r < \operatorname{diam} \mathrm{K}$ the ball $B(x,r)$ contains a segment of length $\varepsilon r/\operatorname{diam}\mathrm{K}$ in every direction.

Fix any $y^* \in S_{X^*}$ and $x, y \in K$ such that $y^*(x) < y^*(y)$. Assume that (ii) does not hold. Then there are unit vectors $\{y_n\}_{n=1}^{\infty} \subset S_X$ and a sequence of positive numbers $\{\varepsilon_n\}_{n=1}^{\infty}$ tending to zero such that $K \cap (K+\varepsilon_n y_n) = \emptyset$ for every n. In other words, $\varepsilon_n y_n \notin K - K$. Our assumption $(*)$ on K ensures that there is a z_n of norm arbitrarily close to ε_n which can be separated from $K-K$. Therefore there is an $x_n^* \in S_{X^*}$ such that $x_n^*(u) - x_n^*(v) \le 3\varepsilon_n$ for every $u,v \in K$. But condition (iii) implies the existence of $u,v \in K$ for which

$$c\big(y^*(y) - y^*(x)\big) < x^*(u) - x^*(v) \le 3\varepsilon_n,$$

which is a contradiction when n is large enough. ■

Proof for Case 2: In this case it is easy to see that (iii) cannot hold. Indeed, since K is unbounded, we can find $y^* \in S_{X^*}$ for which $\sup_{x\in K}|y^*(x)| = \infty$. Then for every $c > 0$ there are $x, y \in K$ such that

$$c\big(y^*(y) - y^*(x)\big) > 2\sup_{x\in K}|x^*(x)| \ge \sup_{u,v\in K}|x^*(u) - x^*(v)|. \quad ■$$

Proof for Case 3: Assume first that (ii) fails for some constant $0 < c < 1/3$. In other words, there is $0 < c < 1/3$, $x \in K$, $y \in S_X$ and $r > 0$ for which

$$cry \notin K_{x,r} - K_{x,r}.$$

Without loss of generality, we may assume that $x = 0$. Hence, for every $\varepsilon > 0$ we can find $z^* \in S_{X^*}$ such that $z^*(u) - z^*(v) \le (c+\varepsilon)r$ for all $u, v \in K \cap B(0,r)$, that is, by choosing $\varepsilon < 1/3 - c$ and replacing c by $c+\varepsilon$ we have a $z^* \in S_{X^*}$ such that $z^*(u) - z^*(v) \le cr$ for all $u,v \in K \cap B(0,r)$. In particular, for every $u \in K \cap B(0,r)$ we have $|z^*(u)| \le cr$. Define

$$\begin{aligned} A^+ &= \{x \in X;\ z^*(x) > cr \text{ and } z^*(x) > c\|x\|\}, \\ A^- &= \{x \in X;\ z^*(x) < -cr \text{ and } z^*(x) < -c\|x\|\}. \end{aligned}$$

We claim that both A^+ and A^- are disjoint from K. Indeed, if $z^*(x) > cr$ and $z^*(x) > c\|x\|$ for some $x \in K$, then there is a $\lambda > 1$ such that $z^*(x) > \lambda cr$ and $z^*(x) > \lambda c\|x\|$. Then $\lambda crx/|z^*(x)| \in K \cap B(0,r)$, so $|z^*(\lambda crx/|z^*(x)|)| \le cr$, which is a contradiction. Similarly for A^-.

By the Hahn-Banach Theorem, since A^+, A^- are convex with nonempty interior, we can find $x^*, v^* \in S_{X^*}$ and $\alpha \ge 0 \ge \beta$ for which

$$\inf_{u\in A^+} x^*(u) \ge \alpha \ge \sup_{u\in K} x^*(u), \quad \inf_{u\in K} v^*(u) \ge \beta \ge \sup_{u\in A^-} v^*(u).$$

Since K is not contained in any hyperplane, we have $x^* \ne v^*$. Let $\delta > 0$, and take $x, z \in B(0, 1+\delta)$ for which $x^*(x) = z^*(z) = 1$. Note that $z^*(u) \le c\|u\|$ for any $u \in \ker(x^*)$, since otherwise for λ large enough and

$u' = \lambda u \in \ker(x^*)$ we would have $z^*(u') \in \operatorname{int}(A^+)$, a contradiction. Since $z = x^*(z)x + \big(z - x^*(z)x\big)$ and $z - x^*(z)x \in \ker(x^*)$, we have for δ sufficiently small,

$$\begin{aligned} 1 = z^*(z) &= x^*(z)z^*(x) + z^*\big(z - x^*(z)x\big) \\ &\leq x^*(z)z^*(x) + c\|z - x^*(z)x\| \\ &\leq x^*(z)z^*(x) + 3c. \end{aligned}$$

It is also easy to see that $\lambda z \in A^+$ if λ is large enough and δ is small enough, therefore $x^*(z) \geq 0$. Then, since $c < 1/3$ and $1 \leq x^*(z)z^*(x) + 3c$, it is immediate to see that $x^*(z) > 0$ and $z^*(x) > 0$. Therefore

$$1 \leq x^*(z)z^*(x) + 3c \leq (1+\delta)z^*(x) + 3c,$$

that is, $z^*(x) \geq (1-3c)/(1+\delta) > 1 - 4c$ if δ is small enough. Hence, for any $u \in S_X$

$$\begin{aligned} c(2+\delta) &\geq c\|u - x^*(u)x\| \geq \big|z^*\big(u - x^*(u)x\big)\big| \\ &= |z^*(u) - x^*(u)z^*(x)| \\ &= \big|z^*(u) - x^*(u) + x^*(u)\big(1 - z^*(x)\big)\big| \\ &\geq |z^*(u) - x^*(u)| - 4c. \end{aligned}$$

Thus $\|z^* - x^*\| \leq c(6+\delta)$ for every δ small enough, therefore $\|z^* - x^*\| \leq 6c$. Similarly, we find that $\|z^* - v^*\| \leq 6c$. Let

$$c_0 = \|x^* - v^*\| \leq 12c,$$

take an arbitrary point $x_1 \in K$, and define

$$y^* = \frac{x^* - v^*}{c_0} \in S_{X^*}.$$

By assumption, no functional is bounded on K, and yet

$$y^*(x) \leq \frac{\alpha - \beta}{c_0} \quad \forall x \in K.$$

Therefore for every $t < y^*(x_1)$ we can find an $x_2 \in K$ for which $y^*(x_2) = t$. Fix x_2 with

$$y^*(x_2) = 2y^*(x_1) - \frac{\alpha - \beta}{c_0}.$$

Since $v^*(x) \geq \beta$, we have $x^*(x) \geq \beta + x^*(x_2) - v^*(x_2)$ for any $x \in K$ for which $y^*(x_2) \leq y^*(x)$. Thus, for any $y_1, y_2 \in K$ for which $y^*(x_2) \leq y^*(y_i) \leq y^*(x_1)$, $\alpha \geq x^*(y_i) \geq \beta + x^*(x_2) - v^*(x_2)$ is satisfied. That is,

$$
\begin{aligned}
|x^*(y_1) - x^*(y_2)| &\leq \alpha - \beta - x^*(x_2) + v^*(x_2) \\
&= \alpha - \beta - y^*(x_2)c_0 \\
&= \big(2y^*(x_1) - y^*(x_2)\big)c_0 - y^*(x_2)c_0 \\
&= 2c_0\big(y^*(x_1) - y^*(x_2)\big) \\
&\leq 24c\big(y^*(x_1) - y^*(x_2)\big).
\end{aligned}
$$

Summarizing, we have proved that if condition (iii) is satisfied with a constant $c < 1/72$, then condition (ii) must be satisfied with constant $c/24$. But if (iii) is satisfied for some c then it is also satisfied for every $c' < c$, which completes our proof. ■

This finishes the proof of the implication (iii) $\Longrightarrow$ (ii). ■

For later purposes, we record here that from the proof of Case 2 it follows that if K is unbounded and (iii) fails for some constant c, then there are $x^*, y^* \in S_{X^*}$ and $x, y \in K$ such that (iii) fails for c, x, y, x^*, y^* and y^* is not bounded on K. By symmetry, we can assume that $\sup_{x \in K} y^*(x) = +\infty$.

It remains to prove that if (iii) fails then there are continuous functions $f, g \colon K \to \mathbb{R}$ such that f is not Lipschitz, g is Lipschitz and $h_1 = g + f$, $h_2 = g - f$ are convex.

We begin by defining some auxiliary functions:

Lemma 9 *Fix $0 < \varepsilon < 1/2$, $\beta > 0$ and $L > 1$. Let $\alpha = \varepsilon^2\beta$ and $\gamma = e^L\beta$. Then there are continuous functions $f, g \colon \mathbb{R}^2 \to \mathbb{R}$ for which*

(A) $g + f$, $g - f$ are both convex on $\big([-\alpha, \alpha] \times \mathbb{R}\big) \cup \big(\mathbb{R} \times (\mathbb{R} \setminus [0, 2\gamma])\big)$;

(B) $|f_y| < \varepsilon$, $|g_x|, |g_y| < 40\varepsilon$ on $\big([-\alpha, \alpha] \times \mathbb{R}\big) \cup \big(\mathbb{R} \times (\mathbb{R} \setminus [0, 2\gamma])\big)$;

(C) $|f(x, y)| \leq \alpha L$ on $[-\alpha, \alpha] \times \mathbb{R}$ and $f(x, y) = 0$ on $\mathbb{R} \times (\mathbb{R} \setminus [\beta, 2\gamma - \beta])$;

(D) $f_x(x, \gamma) = L$ and $|f_x(x, y)| \leq L$ for every $x, y \in \mathbb{R}$.

(Here f_x, f_y, g_x, g_y denote the partial derivatives of f and g.)

Proof: Denote the sets $\{y \le \beta\}$, $\{\beta \le y \le \gamma\}$, $\{\gamma \le y \le 2\gamma - \beta\}$ and $\{2\gamma - \beta \le y\}$ by S_1, S_2, S_3 and S_4, respectively. Let

$$f(x,y) = \begin{cases} 0 & y \in S_1 \\ x\log(y/\beta) & y \in S_2, \end{cases}$$

and extend f to $\mathbb{R} \times S_3$ and $\mathbb{R} \times S_4$ by $f(x,y) = f(x, 2\gamma - y)$. Clearly (C) and (D) hold. Let $g = p + q + r$, where

$$\begin{aligned} p(x,y) &= \begin{cases} 0 & y + |x| \le \beta \\ \frac{\varepsilon(y+|x|-\beta)^2}{\alpha} & y \le \beta \le y + |x| \\ \frac{\varepsilon x^2}{\alpha} & y \in S_2 \end{cases} \\ q(x,y) &= \begin{cases} 0 & y \in S_1 \\ -\frac{\alpha \log(y/\beta)}{\varepsilon} & y \in S_2 \end{cases} \\ r(x,y) &= \begin{cases} 0 & y \in S_1 \\ 12\varepsilon(y-\beta) & y \in S_2 \\ 12\varepsilon(2y - \beta - \gamma) & y \in S_3 \\ 12\varepsilon(3y - 3\gamma) & y \in S_4, \end{cases} \end{aligned}$$

and extend p, q to the whole plane by $p(x,y) = p(x, 2\gamma - y)$, $q(x,y) = q(x, 2\gamma - y)$. These are well-defined continuous functions.

It is easy to check that the restriction of g to $\mathbb{R} \times S_1$ is convex, and thus, as the restriction of f to $\mathbb{R} \times S_1$ is 0, it follows that $g+f$ and $g-f$ are convex on $\mathbb{R} \times S_1$. It is also straightforward to verify that the restrictions of $g+f$ and $g-f$ to $\{-\alpha \le x \le \alpha,\ y \in S_2\}$ are convex: Indeed, on this set $g+f$ and $g-f$ may be written as the sum of linear functions together with the functions h^+ and h^-, respectively, where $h^\pm(x,y) = \pm x \log y + \varepsilon x^2/\alpha - (\alpha \log y)/\varepsilon$. On calculating, the second partial derivatives of $h^\pm$ we find $h^\pm_{xx} = 2\varepsilon/\alpha > 0$ and

$$h^\pm_{xx} h^\pm_{yy} - (h^\pm_{xy})^2 = \frac{2\frac{\varepsilon}{\alpha}(\frac{\alpha}{\varepsilon} \mp x) - 1}{y^2} = \frac{2 \mp 2\varepsilon x/\alpha - 1}{y^2} \ge \frac{1 - 2\varepsilon}{y^2} > 0,$$

that is, the functions $h^\pm$ are convex on $\mathbb{R} \times S_2$. This also shows that $g+f$ and $g-f$ are convex on the sets $\mathbb{R} \times S_4$ and $\{-\alpha \le x \le \alpha,\ y \in S_3\}$. Therefore, in order to show that $g \pm f$ are convex on $\big([-\alpha, \alpha] \times \mathbb{R}\big) \cup \big(\mathbb{R} \times (\mathbb{R} \setminus [0, 2\gamma])\big)$, it only remains to check their convexity around the points of the segments $\{y = \beta\}$, $\{y = \gamma\}$ and $\{y = 2\gamma - \beta\}$.

At the points $(x,y) \in \big([-\alpha,\alpha]\times\mathbb{R}\big) \cup \big(\mathbb{R}\times(\mathbb{R}\setminus[0,2\gamma])\big)$ we have

$$f_x(x,y) = \begin{cases} 0 & y\in S_1 \\ \log(y/\beta) & y \in S_2 \end{cases}$$

$$|f_y(x,y)| \le \frac{\alpha}{\beta} < \varepsilon$$

$$|p_x| \le 2\varepsilon,\ |p_y| \le 2\varepsilon,\ q_x = 0,\ |q_y| \le \frac{\alpha}{\varepsilon\beta} = \varepsilon,\ r_x = 0$$

$$r_y = \begin{cases} 0 & y \in S_1 \\ 12\varepsilon & y\in S_2 \\ 24\varepsilon & y \in S_3 \\ 36\varepsilon & y\in S_4. \end{cases}$$

Since $|f_y|, |p_y|, |q_y|$ are at most 2ε, and $r_y|_{S_{i+1}} - r_y|_{S_i} = 12\varepsilon$, it follows that

$$(g_y \pm f_y)|_{S_{i+1}} - (g_y \pm f_y)|_{S_i} \ge 0.$$

From this it easily follows that $g \pm f$ are also convex around the points of the required segments. This also shows that (B) is satisfied. ■

Proof that (i) $\Longrightarrow$ (iii): Suppose that (iii) does not hold. For every n fix $c_n > 0$, $x_n, y_n \in K$ and $x_n^*, y_n^* \in S_{X^*}$ such that $c_n \to 0$, $y_n^*(x_n) < y_n^*(y_n)$ and for every n and for every $u,v \in K$ with $y_n^*(x_n) \le y_n^*(u) \le y_n^*(v) \le y_n^*(y_n)$ we have

$$|x_n^*(u) - x_n^*(v)| \le c_n(y_n^*(y_n) - y_n^*(x_n)).$$

For simplicity we use the notation

$$\begin{aligned} u_n(x) &= x_n^*(x) - x_n^*(x_n) \\ v_n(x) &= y_n^*(x) - y_n^*(x_n) \\ \varphi_n(x) &= \big(u_n(x), v_n(x)\big) \in \mathbb{R}^2 \\ \alpha_n &= c_n\big(y_n^*(y_n) - y_n^*(x_n)\big) \\ \eta_n &= \frac{y_n^*(y_n) - y_n^*(x_n)}{2}. \end{aligned}$$

Using this notation, we know that $0 < \eta_n$ and

$$u_n(x_n) = v_n(x_n) = 0,\ v_n(y_n) = 2\eta_n,$$

$$\alpha_n = 2c_n\eta_n,$$

and for all $x \in K$,

$$0 \le v_n(x) \le 2\eta_n \implies -\alpha_n \le u_n(x) \le \alpha_n. \qquad (\star)$$

If K is bounded, then $\alpha_n = 2c_n\eta_n \le c_n \mathrm{diam\,K}$, and so $\alpha_n \to 0$. Since u_n and v_n are both affine, $\varphi_n(K)$ must be convex, and as $\varphi_n(x_n) = (0,0) \in \varphi_n(K)$, it is clear that $(\star)$ remains true if we replace α_n, η_n by $N\alpha_n, N\eta_n$ for any $N \ge 1$. If K is unbounded then, as we noticed after the proof that (iii) $\implies$ (ii), for every $N > 1$ there is an $x \in K$ for which $v_n(x) = N\eta_n$. That is, replacing α_n and η_n by $N\alpha_n$ and $N\eta_n$ if it is necessary, we obtain the following lemma:

Lemma 10 *Suppose that condition (iii) of Theorem 4 fails. Then for every constant $c > 0$ there is a sequence of positive numbers c_n with $c_n \to 0$ as $n \to \infty$ and there are sequences of positive numbers α_n, η_n, such that:*

(a) $0 \le v_n(x) \le 2\eta_n \implies -\alpha_n \le u_n(x) \le \alpha_n \quad \forall x \in K$;

(b) $\alpha_n = 2c_n\eta_n$;

(c) there is an $x \in K$ for which $v_n(x) = \eta_n$;

(d) At least one of the following is satisfied:

$$\begin{array}{rll} \text{either} & (d1) & \alpha_n \to 0 \\ \text{or} & (d2) & c\alpha_n - \|x_n\| \to \infty. \end{array}$$

In order to complete the proof that (i) $\implies$ (iii), it is suffices to prove the following lemma:

Lemma 11 *Assume that for every $c > 0$ there are sequences of positive numbers α_n, η_n, c_n with $c_n \to 0$, for which conditions (a)-(d) of Lemma 10 are satisfied. Then for every $0 < \varepsilon < 1$ and $\delta, L > 0$ there are continuous functions $F, G\colon K \to \mathbb{R}$ such that:*

- *$G + F$ and $G - F$ are convex;*
- $\mathrm{Lip}(G) \le \varepsilon$;
- $2L \ge \mathrm{Lip}(F) \ge L$; *and*

- *either (d1) holds and* $\sup |F| \le \delta$*, or (d2) holds and* $F(x) = 0$ *for every* $\|x\| \le \delta$*.*

Indeed, by Lemma 11, for every sequence $\varepsilon_n \to 0$, $L_n \to \infty$ and $\delta_n > 0$ we can find $F_n, G_n : K \to \mathbb{R}$ and points $a_n, b_n \in K$ such that $G_n \pm F_n$ are convex, $\mathrm{Lip}(G_n) \le \varepsilon_n$, $\mathrm{Lip}(F_n) \le 2L_n$, $|F_n(b_n) - F_n(a_n)| > L_n\|b_n - a_n\|$, and either (*d1*) holds and $\sup |F_n| \le \delta_n$ or (*d2*) holds and $F_n(x) = 0$ for every $\|x\| \le \delta_n$. We can also assume that for a fixed point $x^0 \in K$, $G_n(x^0) = 0$ for every n. Then, by choosing $\sum_n \varepsilon_n < \infty$, $g = \sum_n G_n$ is Lipschitz. By requiring $\sum_n \delta_n < \infty$ in case (*d1*) and $\delta_n \to \infty$ in case (*d2*), we can also ensure that $f = \sum_n F_n$ exists and is continuous. Since the sum of convex functions is convex, $g + f$ and $g - f$ are convex. If we also require that $\sum_{i<n} 2L_i < L_n/3$ for every n, and $\sum_{i>n} 2\delta_i < (L_n/3)\|b_n - a_n\|$ for every n if (*d1*) holds, or , $\delta_i > \max(\|a_n\|, \|b_n\|)$ for every $i > n$ if (*d2*) holds, then, in the first case,

$$|f(b_n) - f(a_n)| > L_n\|b_n - a_n\| - \sum_{i<n} 2L_i\|b_n - a_n\| - 2\sum_{i>n} \delta_i \ge (L_n/3)\|b_n - a_n\|,$$

and, in the second case

$$\begin{aligned} |f(b_n) - f(a_n)| &= \Big|\sum_{i=1}^{n} F_i(b_n) - F_i(a_n)\Big| > L_n\|b_n - a_n\| - \sum_{i<n} 2L_i\|b_n - a_n\| \\ &> (2L_n/3)\|b_n - a_n\|. \end{aligned}$$

In either case, we deduce that f is not Lipschitz and (i) $\implies$ (iii), as required. ■

Before passing to the proof of Lemma 11, we prove the following simple result. This is the only place in our proof at which the assumption $0 \in K$ is crucial, since the statement is clearly false for the set $\{1\} \times \mathbb{R} \subset \mathbb{R}^2$.

Lemma 12 *For every* $\eta, \varrho > 0$*,* $x \in K$ *and* $w \in S_X$ *there are* $y, z \in B(x, \varrho) \cap K$ *such that*

$$\left\| \frac{y - z}{\|y - z\|} - w \right\| < \eta.$$

Proof: Since for every u, v in $B(0, \varrho) \cap K$ and $x \in K$ the vectors

$$u' = \frac{\varrho}{\|x\| + \varrho} u + \frac{\|x\|}{\|x\| + \varrho} x$$

and

$$v' = \frac{\varrho}{\|x\| + \varrho} v + \frac{\|x\|}{\|x\| + \varrho} x$$

are in $B(x, \varrho) \cap K$ and $(u' - v')/\|u' - v'\| = (u - v)/\|u - v\|$, it is enough to prove the lemma for $x = 0$.

Since $\overline{\text{span}}(K) = X$, there are *positive* numbers $a_1, ..., a_n$, $b_1, ..., b_m$ and there are $u_1, ..., u_n, v_1, ..., v_m \in K$ such that

$$\left\| \frac{\sum_{i=1}^n a_i u_i - \sum_{j=1}^m b_j v_j}{\| \sum_{i=1}^n a_i u_i - \sum_{j=1}^m b_j v_j \|} - w \right\| < \eta.$$

Then for $M = \max \left(\sum_{i=1}^n a_i, \sum_{j=1}^m b_j \right)$, the vectors $u = \sum_{i=1}^n a_i u_i / M$ and $\sum_{j=1}^m b_j v_j / M$ are in K, and satisfy the claim of the lemma. ∎

Proof of Lemma 11: Set

$$c = \frac{1}{(\varepsilon/80)^2},$$

and choose n to be so large that

$$c_n < \frac{(\varepsilon/80)^2}{2e^{L+2\varepsilon}}.$$

In addition, in case (*d1*), for n large enough we know that

$$\alpha_n < \frac{\delta}{L + 2\varepsilon},$$

and in case (*d2*), for n large enough,

$$\frac{\alpha_n}{(\varepsilon/80)^2} - \|x_n\| > \delta.$$

Fix an n for which these inequalities are satisfied, and set $\eta = \eta_n$, $\alpha = \alpha_n$,

$$\beta = \frac{\alpha_n}{(\varepsilon/80)^2},$$

$$\gamma = e^{L+2\varepsilon} \beta < \eta,$$

and $u(x) = u_n(x)$, $v(x) = v_n(x)$, $\varphi(x) = \big(u(x), v(x)\big)$. Let f and g be the functions given by Lemma 9 when the parameters are $\varepsilon/80, L+2\varepsilon, \alpha, \beta$ and γ. Define

$$F(x) = f\big(\varphi(x)\big),\ \ G(x) = g\big(\varphi(x)\big).$$

Since $\gamma < \eta$, from (a) of Lemma 10 we can see

$$\varphi(K) \subset \big([-\alpha, \alpha] \times \mathbb{R}\big) \cup \big(\mathbb{R} \times (\mathbb{R} \setminus [0, 2\gamma])\big).$$

Thus $G + F$ and $G - F$ are continuous convex functions. We also know from (C) of Lemma 9 that if (*d1*) holds, then $\sup|F| \leq \alpha(L + 2\varepsilon) < \delta$, and if (*d2*) holds, then $F(x) = 0$ for every x with $v(x) \leq \beta$. Indeed, if $v(x) > \beta$ then, $\|x\| \geq \beta - \|x_n\| > \delta$, that is, $F(x) = 0$ for every x with $\|x\| \leq \delta$. We also see from (B), (D) of Lemma 9 that $\mathrm{Lip}(g) < \varepsilon$ and that $\mathrm{Lip}(F) \leq \mathrm{Lip}(f) \leq \sqrt{L^2 + \varepsilon^2} \leq 2L$.

By continuity and the fact that $\gamma < \eta$, by applying (c) of Lemma 10 we can find an $x = x^0$ for which $v(x^0) = \gamma$. By Lemma 12, for every $0 < \varrho$, $\eta < 1$ we can find $y, z \in B(x^0, \varrho) \cap K$ for which $x_n^*(y - z) > (1 - \eta)\|y - z\|$. If ϱ is small enough then $v(y)$ and $v(z)$ are close to $v(x^0) = \gamma$. From (B) of Lemma 9 we get

$$|f\big(u(z), v(y)\big) - f\big(u(z), v(z)\big)| < 40(\varepsilon/80)|v(y) - v(z)| < (\varepsilon/2)\|y - z\|,$$

and if ϱ is small enough then (D) of Lemma 9 gives

$$\begin{aligned} |f\big(u(y), v(y)\big) - f\big(u(z), v(y)\big)| &> \big((L + 2\varepsilon) - \varepsilon\big)|u(y) - u(z)| \\ &= (L + \varepsilon)|x_n^*(y) - x_n^*(z)| \\ &\geq (L + \varepsilon)(1 - \eta)\|y - z\|. \end{aligned}$$

Therefore

$$|F(y) - F(z)| = |f\big(u(y), v(y)\big) - f\big(u(z), v(z)\big)| \geq$$

$$\geq |f\big(u(y), v(y)\big) - f\big(u(z), v(y)\big)| - |f\big(u(z), v(y)\big) - f\big(u(z), v(z)\big)| \geq$$

$$\geq \big((L + \varepsilon)(1 - \eta) - \varepsilon/2\big)\|y - z\|.$$

For η small enough this gives $\mathrm{Lip}(F) > L$, and the lemma is proved. ∎

Remark 13 Let $X = \ell_1$, and let $e_1, e_2, \dots$ be the standard basis of ℓ_1. Let

$$C_0 = \left\{ \sum_j \lambda_j e_j \in \ell_1 : \quad 0 \le \lambda_j \le \frac{\lambda_1}{j} \right\}$$

$$C_n = \left\{ \sum_j \lambda_j e_j \in \ell_1 : \quad 0 \le \lambda_j, \quad \sum_j \lambda_j \le 2\lambda_n \le 2n \right\}$$

$$D = \left\{ \sum_{n=1}^{\infty} a_n : \quad a_n \in C_n, \sum_{n=1}^{\infty} a_n \in \ell_1 \right\}$$

$$K_0 = \{ x \in \ell_1 : \quad d(x, C_0) < 1 \}$$

$$K_1 = \{ x \in \ell_1 : \quad d(x, D) < 1 \}.$$

It is easy to see that K_0 and K_1 are open, unbounded convex sets, with $\overline{\text{span}}(C_0) = X$, $C_0 \subset K_0$ and $\overline{\text{span}}(K_1) = X$. However, K_1 does not contain any half-line, and conditions (i)–(iii) of Theorem 4 fail for K_0, but hold for K_1.

Proof: It is easy to check that for every $u = \sum \lambda_j e_j \in B(0, n+1) \cap C_0$ we have $0 \le \lambda_n \le 1$, and then $B(0, n+1) \cap K_0$ does not contain any u, v with $u - v = 4e_n$, which contradicts (ii) for K_0.

For every $x = d + z \in K_1$, $d \in D$, $\|z\| < 1$, $r > 0$, $n > \max(\|d\|, r)$ and $y = \sum_j y_j e_j \in S_X$ we have

$$2ne_{4n} + d + n \sum_{\{j : y_j \ge 0\}} y_j e_j \in C_{4n}$$

and

$$2ne_{4n} + d - n \sum_{\{j : y_j < 0\}} y_j e_j \in C_{4n},$$

therefore

$$u = 2ne_{4n} + x + n \sum_{\{j : y_j \ge 0\}} y_j e_j \in K_1 \cap B(x, 3n)$$

and

$$v = 2ne_{4n} + x - n \sum_{\{j : y_j < 0\}} y_j e_j \in K_1 \cap B(x, 3n),$$

and $u - v = ny$. Then $u' = x + (r/n)(u - x)$ and $v' = x + (r/n)(v - x)$ are both in $K_1 \cap B(x, 3r)$ and $u' - v' = ry$. That is, (ii) is satisfied for K_1 with $c = 1/3$.

It remains only to check that K_1 does not contain any half-line. Assume instead that K_1 contains a half-line of direction $y = \sum y_n e_n \in S_X$. It is clear that $y_n \geq 0$ for every n. Choose an $N \geq 4$ for which

$$\sum_{n=1}^{N-1} y_n > 2/3 \quad \text{and} \quad \sum_{n=N}^{\infty} y_n < 1/3.$$

For every $x \in K_1$ and $\lambda \geq 0$, $x + \lambda y \in K_1$. Indeed, fix some $v \in K_1$ such that $v + [0, \infty)y \subset K_1$. Since K_1 is open, there is an $\varepsilon > 0$ such that $x + \varepsilon(x - v) \in K_1$. By convexity:

$$x + \lambda y = \frac{1}{1+\varepsilon}\big(x + \varepsilon(x - v)\big) + \frac{\varepsilon}{1+\varepsilon}\left(v + \frac{1+\varepsilon}{\varepsilon} \cdot \lambda y\right) \in K_1.$$

In particular, $4N^2 y \in K_1$, and so $4N^2 \sum_n y_n e_n = \sum_n a_n + b$, where $a_n = \sum_j \lambda_{jn} e_j \in C_n$ for every n and $b = \sum_j b_j e_j \in B(0, 1)$. We also have $0 \leq \lambda_{jn}$ for every j and n, and $\sum_{j,n} \lambda_{jn} < \infty$. Since $C_n \subset B(0, 2n)$,

$$\sum_{n=1}^{N-1} \sum_{j=1}^{N-1} \lambda_{jn} \leq \sum_{n=1}^{N-1} 2n < N^2$$

follows. Hence

$$\begin{aligned} 4N^2 \left(\sum_{j=1}^{N-1} y_j - \sum_{j=N}^{\infty} y_j \right) &= \sum_{n=1}^{\infty} \left(\sum_{j=1}^{N-1} \lambda_{jn} - \sum_{j=N}^{\infty} \lambda_{jn} \right) + \sum_{j=1}^{N-1} b_j - \sum_{j=N}^{\infty} b_j \\ &\leq N^2 - \sum_{n=N}^{\infty} \left(\sum_{j=N}^{\infty} \lambda_{jn} - \sum_{j=1}^{N-1} \lambda_{jn} \right) + \sum_{j=1}^{N-1} b_j - \sum_{j=N}^{\infty} b_j, \end{aligned}$$

where for every $n \geq N$ we have

$$\sum_{j=N}^{\infty} \lambda_{jn} - \sum_{j=1}^{N-1} \lambda_{jn} = 2\sum_{j=N}^{\infty} \lambda_{jn} - \sum_{j=1}^{\infty} \lambda_{jn} \geq 2\sum_{j=N}^{\infty} \lambda_{jn} - 2\lambda_{nn} \geq 0$$

and

$$\sum_{j=1}^{N-1} b_j - \sum_{j=N}^{\infty} b_j \leq 1.$$

Therefore

$$4N^2\left(\sum_{j=1}^{N-1} y_j - \sum_{j=N}^{\infty} y_j\right) \leq N^2+1.$$

This is a contradiction, since $\sum_{j=1}^{N-1} y_j - \sum_{j=N}^{\infty} y_j > 1/3$. ∎

Remark 14 *Note that it does not make any difference if, instead of (i), we assume that for some (or every) $k \geq 2$, and for every convex functions $h_1, h_2, \ldots, h_k$ the functions must be Lipschitz whenever their sum is Lipschitz.*

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