Scaled Enflo type is equivalent to Rademacher type

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Abstract

We introduce the notion of *scaled Enflo type* of a metric space, and show that for Banach spaces, scaled Enflo type p is equivalent to Rademacher type p.

1 Introduction

Recall that a Banach space X is said to have Rademacher type $p > 0$ (see [7]) if there exists a constant $T < \infty$ such that for every $x_1, \dots, x_n \in X$,

$$\mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|_X^p \le T^p \sum_{j=1}^n \|x_j\|_X^p, \tag{1}$$

where here, and in what follows, $\mathbb{E}_\varepsilon$ denotes the expectation with respect to uniformly chosen $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$. The infimum over all constants T for which (1) holds is denoted $T_p(X)$.

Motivated by the search for concrete versions of Ribe's theorem [12] for various fundamental local properties of Banach spaces (see the discussion in [2, 9, 8]), several researchers proposed non-linear notions of type, which make sense in the setting arbitrary metric spaces (see [5, 3, 1]). In particular, following Enflo [5] we say that a metric space $(\mathcal{M}, d_\mathcal{M})$ has *Enflo type p* if there exists a constant K such that for every $n \in \mathbb{N}$ and every $f : \{-1, 1\}^n \to \mathcal{M}$,

$$\mathbb{E}_\varepsilon\, d_\mathcal{M}(f(\varepsilon), f(-\varepsilon))^p \le T^p \sum_{j=1}^n \mathbb{E}_\varepsilon\, d_\mathcal{M}\Big(f(\varepsilon_1, \dots, \varepsilon_{j-1}, \varepsilon_j, \varepsilon_{j+1}, \dots, \varepsilon_n), f(\varepsilon_1, \dots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1}, \dots, \varepsilon_n)\Big)^p. \tag{2}$$

For Banach spaces (1) follows from (2) by considering the function $\varepsilon \mapsto \sum_{j=1}^n \varepsilon_j x_j$. The question whether in the category of Banach spaces Rademacher type p implies Enflo type p was posed by Enflo in [5], and in full generality remains open. In [11] Pisier showed that if a Banach space has Rademacher p then it has Enflo type p' for every $p' < p$ (see also the work of Bourgain, Milman and Wolfson [3] for a similar result which holds for a another notion of non-linear type). In [10] it was shown that for UMD Banach spaces (see [4]) Rademacher type p is equivalent to Enflo type p.

Motivated by our recent work on metric cotype [8], we introduce below the notion of *scaled Enflo type* of a metric space (which is, in a sense, "opposite" to the notion of metric cotype defined in [8]), and show that for Banach spaces, scaled Enflo type p is equivalent to Rademacher type p. This settles the long standing problem of finding a purely metric formulation of the notion of type (though Enflo's problem described above remains open). Modulo some of the results of [8], the proof of our main theorem is very simple.

Definition 1.1 (Scaled Enflo type)**.** Let $(\mathcal{M}, d_\mathcal{M})$ be a metric space and $p > 0$. We say that $\mathcal{M}$ has *scaled Enflo type p* with constant τ if for every integer n there exists an even integer m such that for every $f : \mathbb{Z}_m^n \to \mathcal{M}$,

$$\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} d_\mathcal{M}\left(f\left(x + \frac{m}{2}\varepsilon\right), f(x)\right)^p d\mu(x) \le \tau^p m^p \sum_{j=1}^n \int_{\mathbb{Z}_m^n} d_\mathcal{M}\Big(f(x + e_j), f(x)\Big)^p d\mu(x), \tag{3}$$

where μ is the uniform probability measure on $\mathbb{Z}_m^n$, and $\{e_j\}_{j=1}^n$ is the standard basis of $\mathbb{R}^n$. The infimum over all constants τ for which (3) holds is denoted $\tau_p(\mathcal{M})$.

Theorem 1.2. *Let X be a Banach space and $p \in [1,2]$. Then X has Rademacher type p if and only if X has scaled Enflo type p. More precisely,*

$$\frac{1}{2\pi} T_p(X) \leq \tau_p(X) \leq 5T_p(X).$$

2 Proof of Theorem 1.2

We start by showing that scaled Enflo type p implies Rademacher type p.

Lemma 2.1. *Let X be a Banach space and $p \in [1,2]$. Then $T_p(X) \leq 2\pi\tau_p(X)$.*

Proof. Let X be a Banach space and assume that $\tau_p(X) < \infty$ for some $p \in [1,2]$. Fix $\tau > \tau_p(X)$, $v_1, \ldots, v_n \in X$, and let m be an even integer. Define $f : \mathbb{Z}_m^n \to X$ by $f(x_1, \ldots, x_n) = \sum_{j=1}^n e^{\frac{2\pi i x_j}{m}} v_j$. Then

$$\sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\| f\left(x + e_j\right) - f(x) \right\|_X^p d\mu(x) = \left| e^{\frac{2\pi i}{m}} - 1 \right|^p \cdot \sum_{j=1}^n \|v_j\|_X^p \leq \left(\frac{2\pi}{m}\right)^p \cdot \sum_{j=1}^n \|v_j\|_X^p, \tag{4}$$

and

$$\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\| f\left(x + \frac{m}{2}\varepsilon\right) - f(x) \right\|_X^p d\mu(x) = 2^p \int_{\mathbb{Z}_m^n} \left\| \sum_{j=1}^n e^{\frac{2\pi i x_j}{m}} v_j \right\|_X^p d\mu(x). \tag{5}$$

We recall the *contraction principle* (see [6]), which states that for every $a_1, \ldots, a_n \in \mathbb{R}$,

$$\mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j a_j v_j \right\|_X^p \leq \left(\max_{1 \leq j \leq n} |a_j| \right)^p \cdot \mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j v_j \right\|_X^p.$$

Thus,

$$\int_{\mathbb{Z}_m^n} \left\| \sum_{j=1}^n e^{\frac{2\pi i x_j}{m}} v_j \right\|_X^p d\mu(x) = \int_{\mathbb{Z}_m^n} \mathbb{E}_\varepsilon \left\| \sum_{j=1}^n e^{\frac{2\pi i}{m}\left(x_j + \frac{m(1-\varepsilon_j)}{4}\right)} v_j \right\|_X^p d\mu(x) = \int_{\mathbb{Z}_m^n} \mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j e^{\frac{2\pi i x_j}{m}} v_j \right\|_X^p d\mu(x) \geq \frac{1}{2^p} \mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j v_j \right\|_X^p. \tag{6}$$

Combining (4), (5) and (6) yields the required result. □

Let X be a Banach space with type p, m an integer divisible by 4, and k an odd integer. Fix $f : \mathbb{Z}_m^n \to X$ and $\varepsilon \in \{-1,1\}^n$. Define $\mathcal{A}^{(k)} f : \mathbb{Z}_m^n \to X$ by

$$\mathcal{A}^{(k)} f(x) = \frac{1}{k^n} \sum_{z \in (-k,k)^n \cap (2\mathbb{Z})^n} f(x+z).$$

Lemma 2.2. *For $p \geq 1$ and every $f : \mathbb{Z}_m^n \to X$*

$$\int_{\mathbb{Z}_m^n} \left\| \mathcal{A}^{(k)} f(x) - f(x) \right\|_X^p d\mu(x) \leq (k-1)^p n^{p-1} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|f(x+e_j) - f(x)\|_X^p d\mu(x).$$

Proof. For every $t \in \mathbb{R}$ let $s(t)$ be the sign of t (with convention that $s(0) = 0$). For every $z \in \mathbb{Z}_m^n$,

$$\|f(x+z) - f(x)\|_X^p \leq \|z\|_1^{p-1} \cdot \sum_{j=1}^n \sum_{\ell=1}^{|z_j|} \left\| f\left(x + \sum_{t=1}^{j-1} z_t e_t + \ell \cdot s(z_j) \cdot e_j\right) - f\left(x + \sum_{t=1}^{j-1} z_t e_t + (\ell - 1) \cdot s(z_j) \cdot e_j\right) \right\|_X^p.$$

Observe that since k is odd, $|(-k,k)^n \cap (2\mathbb{Z})^n| = k^n$. Thus

$$
\begin{aligned}
\int_{\mathbb{Z}_m^n} \left\|\mathcal{A}^{(k)} f(x) - f(x)\right\|_X^p d\mu(x) &\le \frac{1}{k^n} \sum_{z\in(-k,k)^n\cap(2\mathbb{Z})^n} \int_{\mathbb{Z}_m^n} \|f(x+z)-f(x)\|_X^p d\mu(x) \\
&\le \frac{1}{k^n} \sum_{z\in(-k,k)^n\cap(2\mathbb{Z})^n} \int_{\mathbb{Z}_m^n} \|z\|_1^{p-1} \sum_{j=1}^n \sum_{\ell=1}^{|z_j|} \left\| f\Big(x+\sum_{t=1}^{j-1} z_t e_t + \ell s(z_j)e_j\Big) - f\Big(x+\sum_{t=1}^{j-1} z_t e_t + (\ell-1)s(z_j)e_j\Big)\right\|_X^p d\mu(x) \\
&\le \frac{1}{k^n} \sum_{z\in(-k,k)^n\cap(2\mathbb{Z})^n} \sum_{j=1}^n \|z\|_1^{p-1}|z_j| \int_{\mathbb{Z}_m^n} \|f(y+s(z_j)e_j)-f(y)\|_X^p d\mu(x) \\
&\le (k-1)^p n^{p-1} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|f(x+e_j)-f(x)\|_X^p d\mu(x).
\end{aligned}
$$

□

Proof of theorem 1.2. Fix an odd integer $k \in \mathbb{N}$, with $k < \frac{m}{2}$. As in [8], given $j \in \{1,\dots,n\}$ we define $S(j,k) \subseteq \mathbb{Z}_m^n$ by

$$
S(j,k) := \left\{ y \in [-k,k]^n \subseteq \mathbb{Z}_m^n :\ y_j \equiv 0 \mod 2 \text{ and } \forall\ \ell \neq j,\ y_\ell \equiv 1 \mod 2 \right\}.
$$

For $f : \mathbb{Z}_m^n \to X$ we define

$$
\mathcal{E}_j^{(k)} f(x) = \left(f * \frac{\mathbf{1}_{S(j,k)}}{\mu(S(j,k))} \right)(x) = \frac{1}{\mu(S(j,k))} \int_{S(j,k)} f(x+y) d\mu(y). \tag{7}
$$

In [8] (see equation (39) there) it is shown that for every $x \in \mathbb{Z}_m^n$ and $\varepsilon \in \{-1,1\}^n$,

$$
\left(\frac{k}{k+1}\right)^{n-1} \left(\mathcal{A}^{(k)} f(x+\varepsilon) - \mathcal{A}^{(k)} f(x-\varepsilon)\right) = \sum_{j=1}^n \varepsilon_j \left[\mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x-e_j)\right] + U(x,\varepsilon) + V(x,\varepsilon),
$$

where, by inequalities (41) and (42) in [8], for every $\varepsilon \in \{-1,1\}^n$,

$$
\max\left\{ \int_{\mathbb{Z}_m^n} \|U(x,\varepsilon)\|_X^p d\mu(x), \int_{\mathbb{Z}_m^n} \|V(x,\varepsilon)\|_X^p d\mu(x) \right\} \le \frac{8^p n^{2p-1}}{k^p} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\| f(x+e_j) - f(x) \right\|_X^p.
$$

Thus, for every $T > T_p(X)$,

$$
\begin{aligned}
&\left(\frac{k}{k+1}\right)^{p(n-1)} \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\|\mathcal{A}^{(k)} f(x+\varepsilon) - \mathcal{A}^{(k)} f(x-\varepsilon)\right\|_X^p d\mu(x) \\
&\le 3^{p-1}\, \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\| \sum_{j=1}^n \varepsilon_j \left[\mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x-e_j)\right] \right\|_X^p d\mu(x) + \frac{24^p n^{2p-1}}{k^p} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|f(x+e_j)-f(x)\|_X^p d\mu(x) \\
&\le 3^{p-1} T^p \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\|\mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x-e_j)\right\|_X^p d\mu(x) + \frac{24^p n^{2p-1}}{k^p} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|f(x+e_j)-f(x)\|_X^p d\mu(x) \\
&\le 3^{p-1} T^p \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\| f(x+e_j) - f(x-e_j)\right\|_X^p d\mu(x) + \frac{24^p n^{2p-1}}{k^p} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|f(x+e_j)-f(x)\|_X^p d\mu(x) \qquad (8) \\
&\le \left(\frac{6^p}{3} T^p + \frac{24^p n^{2p-1}}{k^p}\right) \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|f(x+e_j)-f(x)\|_X^p d\mu(x), \qquad (9)
\end{aligned}
$$

where in (8) we used the fact that $\mathcal{E}_j^{(k)}$ is an averaging operator, and hence has norm 1.

On the other hand

$$\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\| f\left(x + \frac{m}{2}\varepsilon\right) - f(x)\right\|_X^p d\mu(x) \le 3^{p-1}\,\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\| \mathcal{A}^{(k)} f\left(x+\frac{m}{2}\varepsilon\right) - \mathcal{A}^{(k)} f(x)\right\|_X^p d\mu(x) +$$

$$3^{p-1}\,\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\| f\left(x + \frac{m}{2}\varepsilon\right) - \mathcal{A}^{(k)} f\left(x+\frac{m}{2}\varepsilon\right)\right\|_X^p d\mu(x) + 3^{p-1}\,\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\|\mathcal{A}^{(k)} f(x) - f(x)\right\|_X^p d\mu(x)$$

$$\le \quad 3^{p-1}\left[\left(\frac{m}{4}\right)^{p-1} \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \sum_{t=1}^{m/4} \left\|\mathcal{A}^{(k)} f\,(x+2t\varepsilon) - \mathcal{A}^{(k)} f(x+(2t-2)\varepsilon)\right\|_X^p d\mu(x) + 2\,\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\|\mathcal{A}^{(k)} f(x) - f(x)\right\|_X^p d\mu(x)\right]$$

$$\le \quad 3^{p-1}\left[\left(\frac{m}{4}\right)^{p} \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\|\mathcal{A}^{(k)} f\,(x+\varepsilon) - \mathcal{A}^{(k)} f(x-\varepsilon)\right\|_X^p d\mu(x) + 2k^p n^{p-1} \sum_{j=1}^{n} \int_{\mathbb{Z}_m^n} \|f(x+e_j) - f(x)\|_X^p d\mu(x)\right] \tag{10}$$

$$\le \quad \left[3^{p-1}\left(\frac{m}{4}\right)^p \left(1+\frac{1}{k}\right)^{p(n-1)} \left(\frac{6^p}{3}T^p + \frac{24^p n^{2p-1}}{k^p}\right) + \frac{2(3kn)^p}{3n}\right] \sum_{j=1}^{n} \int_{\mathbb{Z}_m^n} \|f(x+e_j) - f(x)\|_X^p d\mu(x) \tag{11}$$

$$\le \quad 5^p m^p T^p \sum_{j=1}^{n} \int_{\mathbb{Z}_m^n} \|f(x+e_j) - f(x)\|_X^p d\mu(x), \tag{12}$$

where in (10) we used Lemma 2.2, in (11) we used (8), and (12) is true if $4n^{2-1/p} \le k \le \frac{3m}{2n^{1-1/p}}$, which is a valid choice of k if $m \ge 3n^{3-2/p}$. □

Remark 2.3. If a metric space has Enflo type p then it also has scaled Enflo type p. This follows from a straightforward modification of Lemma 2.4 in [8]. We do not know if scaled Enflo type p implies Enflo type p. In the category of Banach spaces, a positive answer to this question would show that Enflo type p is equivalent to Rademacher type p, resolving positively Enflo's problem [5]. We do know that for Banach spaces, scaled Enflo type p implies Enflo type p' for all $p' < p$, and that scaled Enflo type and Enflo type coincide for UMD Banach spaces.

Remark 2.4. The idea of scaling by $\frac{m}{2}$ in the definition of scaled Enflo type originates from the definition of *metric cotype* introduced in [8], which involves a similar scaling procedure. In the case of non-linear type it is possible that this scaling is not necessary, i.e. that Enflo type is equivalent to Rademacher type. However, as shown in [8], in the context of metric cotype the scaling *is necessary*- we refer to [8] for more details.

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