The Surface Measure and Cone Measure on the sphere of ℓ_p^n

Assaf Naor *†

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Abstract

We prove a concentration inequality for the ℓ_q^n norm on the ℓ_p^n sphere for $p, q > 0$. This inequality, which generalizes results of Schechtman and Zinn, is used to study the distance between the cone measure and surface measure on the sphere of ℓ_p^n. In particular, we obtain a significant strengthening of the inequality derived in [NR], and calculate the precise dependence of the constants that appeared there on p.

1 Introduction

For every star shaped body $K \subset \mathbb{R}^n$, one can define two natural measures on the boundary of K. One is the regular surface measure, and the other is the "cone measure". The cone measure of a subset A of ∂K is the volume of $[0,1]A = \{ta : a \in A, 0 \le t \le 1\}$, i.e. the cone with base A and cusp 0. Both these measures have appeared in various contexts in the literature. For instance, the cone measure appears in the Gromov-Milman theorem for concentration of Lipschitz functions on uniformly convex bodies. This paper is devoted to the study of these measures for the particular case when K is the sphere of ℓ_p^n.

The cone measure arises naturally when one tries to generalize a theorem of Diaconis and Freedman [DF], which was originally stated for the Euclidean sphere, to the sphere of ℓ_p^n. This theorem states that the distribution of the first k coordinates of a random point on the Euclidean sphere, is close in total variation distance to the k dimensional Gaussian measure, as long as $k = o(n)$. Since the cone measure and the surface measure on the sphere of ℓ_p^n coincide only in the cases $p = 1, 2, \infty$, the Diaconis Freedman theorem could be generalized

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in two different ways: for the surface measure or for the cone measure. For the cone measure, such a generalization was obtained in [RaR], and for the surface measure it was obtained in [M]. The paper [NR] tackles this problem differently. It was proved there that the (normalized) surface measure on the sphere of ℓ_p^n is asymptotically close in total variation distance to the (normalized) cone measure. This allows one to transfer various results from the cone measure to the surface measure, and vice versa. The present paper develops this theme. We study the precise relation between the surface measure and cone measure on the ℓ_p^n sphere, and obtain results which, apart from their geometric interest, allow us to transfer several known theorems from the cone measure to the surface measure. In many situations, the cone measure turns out to be much easier to handle than the surface measure. This paper formulates a general principle which, essentially, shows that results for the cone measure automatically transfer to the surface measure.

If we denote by $\sigma_{p,n}$ and $\mu_{p,n}$ the normalized surface measure and cone measure on the sphere of ℓ_p^n, then in the paper [NR] it was proved that $||\mu_{p,n} - \sigma_{p,n}|| = O(1/\sqrt{n})$. Here $||\nu||$ is the total variation of a measure ν. In this paper we show that, in fact, $||\mu_{p,n} - \sigma_{p,n}|| \sim_p 1/\sqrt{n}$, where $\sim_p$ means equivalence up to universal constants which may depend on p. Since $\mu_{p,n}$ and $\sigma_{p,n}$ coincide when $p = 1, 2, \infty$, we also study the dependence on p in the above estimate, and show that there is a numerical constant C such that:

$$||\mu_{p,n} - \sigma_{p,n}|| \leq C\left(1 - \frac{1}{p}\right)\left|1 - \frac{2}{p}\right| \cdot \frac{\sqrt{np}}{n+p}.$$

We actually show that $\mu_{p,n}$ and $\sigma_{p,n}$ are close in a much stronger sense. There is a constant $C = C(p)$ such that for every Borel subset of the sphere of ℓ_p^n, A:

$$\left|\frac{\sigma_{p,n}(A)}{\mu_{p,n}(A)} - 1\right| \leq \frac{C}{\sqrt{n}} \cdot \left[\log\left(\frac{C}{\mu_{p,n}(A)}\right)\right]^{1-\min\{1/2,1/p\}}.$$

This inequality is tight when $p > 2$.

The above result is applied to concentration inequalities for $\sigma_{p,n}$. If (X,d) is a metric space, and ν is a Borel measure on X, then the concentration function of (X,d,ν) is defined as:

$$I_\nu^d(\epsilon) = \sup\left\{\nu(X \setminus A_\epsilon);\ \nu(A) \geq \frac{1}{2}\right\},$$

where $A_\epsilon = \{x;\ d(x, A) < \epsilon\}$.

The Gromov-Milman theorem [GM] states that:

$$I_{\mu_{p,n}}^{||\cdot||_p}(\epsilon) \leq Ce^{-cn\max\{2,p\}},$$

and a theorem of Schechtman and Zinn [SZ2], states that:

$$I_{\mu_{p,n}}^{||\cdot||_2}(\epsilon) \leq Ce^{-cn\min\{2,p\}},$$

(The case $p > 2$ in the above inequality was not stated in [SZ2], but it seems well known, and follows from the fact that the transportation function from the Gaussian measure to the p measure is Lipschitz.)

A simple corollary of our results is that for every metric d on the sphere of ℓ_p^n which induces the standard topology, and for every $\epsilon > 0$:

$$I_{\sigma_{p,n}}^{d}(\epsilon) \leq CI_{\mu_{p,n}}^{d}\left(\frac{\epsilon}{2}\right)\left[1 + \frac{1}{\sqrt{n}}\left|\log I_{\mu_{p,n}}^{d}\left(\frac{\epsilon}{2}\right)\right|^{1-\min\{1/2,1/p\}}\right].$$

This inequality essentially transfers both the Gromov-Milman and the Schechtman-Zinn inequalities from the cone measure to the surface measure.

In studying the relation between the cone measure and surface measure, one is naturally led to the study of the concentration of the ℓ_q^n norm on the ℓ_p^n sphere. Most of this paper is therefore devoted to deriving precise concentration inequalities in this situation. The problem of the concentration of the ℓ_q^n norm on the ℓ_p^n sphere, was studied for the case $q = 2, p = 1$ by Schechtman and Zinn [SZ2]. The tail behavior of the ℓ_q^n norm for the case $q > p$ was calculated by the same authors in [SZ1]. We prove here generalizations of the above results. In the case $q < p$ we prove that:

$$\mu_{p,n}\left(\left|||x||_q^q - \int_{S(\ell_p^n)} ||y||_q^q d\mu_{p,n}(y)\right| \geq t\int_{S(\ell_p^n)} ||y||_q^q d\mu_{p,n}(y)\right) \leq$$

$$\leq 12\exp\left(-Cn\min\left\{\left(\frac{t}{q}\right)^2, \frac{t}{q}\right\}\right).$$

This result reflects the fact that $||\cdot||_q^q$ tends to a constant when $q \to 0$. We also study what happens when $q \to p$. Our results seem to be the first time that deviation estimates were obtained for the ℓ_q^n norm on the ℓ_p^n sphere, which don't become trivial in the above limiting cases.

In the case $q > p$ we show that:

$$\mu_{p,n}\left(\left|\|x\|_q^p - \int_{S(\ell_p^n)} \|y\|_q^p d\mu_{p,n}(y)\right| \geq t\right) \leq Ce^{-c\psi(n,t)},$$

where:

$$\psi(n,t) = \begin{cases} nt, & \text{for } t > n^{-(1-p/q)} \\ n^{(2-p/q)p/q} t^{p/q}, & \text{for } n^{-(3q-p)(q-p)/[q(2q-p)]} < t \leq n^{-(1-p/q)} \\ n^{3-2p/q} t^2, & \text{for } 0 < t \leq n^{-(3q-p)(q-p)/[q(2q-p)]} \end{cases}$$

This estimate coincides in the case $p = 1$, $q = 2$ with the result in [SZ2], and is tight in the first two ranges of the definition of ψ.

It has been proved in [SZ1] and [RaR] that if G is a random vector of n i.i.d. random variables with density proportional to $\exp(-|t|^p)$, then $G/\|G\|_p$ generates the cone measure on the sphere of ℓ_p^n. It was also shown that $G/\|G\|_p$ is independent of $\|G\|_p$, although this property has been hardly used in the literature. In the present paper we shift the attention to the above independence, which turns out to be a very powerful and useful fact. It implies a fundamental negative correlation property (see Theorem 2), and essentially allows us to treat the random variable $\|G\|_p$ as if it were a constant (see for example Lemma 1 for one possible formulation of this fact. The proofs of Theorems 3 and 5 are also manifestations of this principle). Thus, many natural questions concerning the cone measure reduce to questions about i.i.d. random variables, which are better understood, and to which powerful tools such as the results of [L] may be applied. In addition to the new results presented here, this point of view leads to a new, more direct, proof of the result of [SZ1] (see Theorem 2) and has been applied to probabilistic problems in [NR], and to purely geometric questions in [BN].

2 Concentration on the ℓ_p^n sphere

Fix $p > 0$ and an integer n. Recall that the ℓ_p^n norm is defined by :

$$\|x\|_p = \left(\sum_{i=1}^n x_i^p\right)^{1/p}.$$

The ℓ_p^n sphere is defined by: $S(\ell_p^n) = \{x \in \mathbb{R}^n; \|x\|_p = 1\}$, and the ℓ_p^n ball is defined by $B(\ell_p^n) = \{x \in \mathbb{R}^n; \|x\|_p \leq 1\}$. We denote by $\sigma_{p,n}$ the normalized surface measure on $S(\ell_p^n)$, and by $\mu_{p,n}$ the normalized cone measure. In other words, for every measurable $A \subset S(\ell_p^n)$ we put :

$$\mu_{p,n}(A) = \frac{1}{\mathrm{vol}(B(\ell_p^n))} \cdot \mathrm{vol}([0,1]A).$$

Here vol refers to the Lebesgue measure on $\mathbb{R}^n$.

The measure $\mu_{p,n}$ has another useful probabilistic description. Let g be a random variable with density $1/(2\Gamma(1+1/p))e^{-|t|^p}$ ($t \in \mathbb{R}$). If $g_1, ..., g_n$ are i.i.d. copies of g, put :

$$S = \sum_{i=1}^{n} |g_i|^p.$$

We will also define a random vector in $\mathbb{R}^n$ by :

$$X = \left(\frac{g_1}{S^{1/p}}, ..., \frac{g_n}{S^{1/p}}\right).$$

The following result was proved in [SZ1], and later independently also in [RaR]:

Theorem 1 *The random vector X is independent of S. Moreover, For every measurable $A \subset S(\ell_p^n)$ we have :*

$$\mu_{p,n}(A) = P(X \in A).$$

Fix some $q > 0$ and write:

$$T = \sum_{i=1}^{n} |g_i|^q.$$

The following simple lemma will be crucial for our subsequent calculations :

Lemma 1 *For every $t, \alpha > 0$ and $\theta \in \mathbb{R}$ the following inequality holds :*

$$\mathbb{E}\exp\left(t\left|\frac{T^\alpha}{S^{\alpha q/p}} - \theta\right|\right) \le \mathbb{E}\exp\left(\frac{t}{\mathbb{E}S^{\alpha q/p}}\left|T^\alpha - \theta S^{\alpha q/p}\right|\right).$$

Proof: It follows from the previous theorem that S is independent of $T/S^{q/p}$. Hence, using Jensen's inequality we get:

$$\mathbb{E}\exp\left(\frac{t}{\mathbb{E}S^{\alpha q/p}}\left|T^\alpha - \theta S^{\alpha q/p}\right|\right) =$$

$$= \mathbb{E}\exp\left(\frac{tS^{\alpha q/p}}{\mathbb{E}S^{\alpha q/p}}\left|\left(\frac{T}{S^{q/p}}\right)^\alpha - \theta\right|\right) =$$

$$= \mathbb{E}_{T/S^{q/p}} \mathbb{E}_S \exp\left(\frac{tS^{\alpha q/p}}{\mathbb{E}S^{\alpha q/p}} \left| \left(\frac{T}{S^{q/p}} \right)^{\alpha} - \theta \right| \right) \geq$$

$$\geq \mathbb{E} \exp\left(t \left| \left(\frac{T}{S^{q/p}} \right)^{\alpha} - \theta \right| \right).$$

■

To illustrate the possible applications of Lemma 1 we will begin by giving a simple proof of the main result of [SZ1]. We will show later, in Corollary 3, that for every $q > p \geq 1$ there is a constant $C = C(p,q)$ such that for every $0 < t < 1/C$, $\mathbb{E}e^{tT^{p/q}} \leq n^{1-p/q}(1-Ct)^{-n^{p/q}}$.

Theorem 2 *For every $q \geq p$ there is a constant $C = C(p,q)$ such that for every $\lambda > C$:*

$$\mu_{p,n}\left(||x||_q > \frac{\lambda}{n^{1/p-1/q}} \right) \leq \exp\left(\frac{\lambda^p n^{p/q}}{C} \right).$$

Proof: Let C be as in the remark that preceded the statement of the theorem. Use Lemma 1 with $\alpha = p/q$, $\theta = 0$ and $t = n/(pC) - n/\lambda^p$ (which is positive as long as $\lambda > (pC)^{1/p}$) to get:

$$\mu_{p,n}\left(||x||_q > \frac{\lambda}{n^{1/p-1/q}} \right) = P\left(\frac{T^{p/q}}{S} > \frac{\lambda^p}{n^{1-p/q}} \right) \leq e^{-t\lambda^p/n^{1-p/q}} \mathbb{E}\exp\left(t\frac{T^{p/q}}{S} \right) \leq$$

$$\leq e^{-t\lambda^p/n^{1-p/q}} \mathbb{E}\exp\left(\frac{t}{\mathbb{E}S} T^{p/q} \right) = e^{-t\lambda^p/n^{1-p/q}} \mathbb{E}\exp\left(\frac{pt}{n} T^{p/q} \right) \leq$$

$$\leq \frac{n^{1-p/q} e^{-t\lambda^p/n^{1-p/q}}}{\left(1 - \frac{Cpt}{n}\right)^{n^{p/q}}} = n^{1-q/p} \left(\frac{e\lambda^p}{pC} \right)^{n^{p/q}} \exp\left(-\frac{n^{p/q}\lambda^p}{pC} \right).$$

And this clearly implies the required result. ■

Another useful property of the cone measure is the following negative correlation property. For the normalized volume measure on $B(\ell_p^n)$, such a statement was proved analytically in [BP] and geometrically in [ABP]. In our case, the concrete representation of the cone measure allows us to give a purely probabilistic proof.

Lemma 2 *Let $f_1, ..., f_n : \mathbb{R}_+ \to \mathbb{R}_+$ be nonnegative non-decreasing functions. Then:*

$$\int_{S(\ell_p^n)} \prod_{i=1}^n f_i(|x_i|) d\mu_{p,n}(x) \leq \prod_{i=1}^n \int_{S(\ell_p^n)} f_i(|x_i|) d\mu_{p,n}(x)$$

Proof: We have to show that:

$$\mathbb{E}\left[\prod_{i=1}^{n} f_i\left(\frac{|g_i|}{S^{1/p}}\right)\right] \leq \prod_{i=1}^{n} \mathbb{E} f_i\left(\frac{|g_i|}{S^{1/p}}\right).$$

We will prove this by induction on n. For $n = 1$ there is nothing to prove. The case $n = 2$ is based on the following standard trick: for every $x, y, a, b \geq 0$

$$\left[f_1\left(\frac{x}{(x^p + y^p)^{1/p}}\right) - f_1\left(\frac{a}{(a^p + b^p)^{1/p}}\right)\right] \cdot$$

$$\cdot \left[f_2\left(\frac{y}{(x^p + y^p)^{1/p}}\right) - f_2\left(\frac{b}{(a^p + b^p)^{1/p}}\right)\right] \leq 0.$$

Hence, if h_1, h_2 are independent copies of g_1, g_2 then:

$$0 \geq \mathbb{E}\left\{\left[f_1\left(\frac{|g_1|}{(|g_1|^p + |g_2|^p)^{1/p}}\right) - f_1\left(\frac{|h_1|}{(|h_1|^p + |h_2|^p)^{1/p}}\right)\right] \cdot\right.$$

$$\left.\cdot \left[f_2\left(\frac{|g_2|}{(|g_1|^p + |g_2|^p)^{1/p}}\right) - f_2\left(\frac{|h_2|}{(|h_1|^p + |h_2|^p)^{1/p}}\right)\right]\right\} =$$

$$= 2\mathbb{E}\left[f_1\left(\frac{|g_1|}{S^{1/p}}\right) f_2\left(\frac{|g_2|}{S^{1/p}}\right)\right] - 2\left[\mathbb{E} f_1\left(\frac{|g_1|}{S^{1/p}}\right)\right]\left[\mathbb{E} f_2\left(\frac{|g_2|}{S^{1/p}}\right)\right].$$

Assuming the required result for n, and denoting $S_k = \sum_{i=1}^{k} |g_i|^p$, we get:

$$\mathbb{E}\left[\prod_{i=1}^{n+1} f_i\left(\frac{|g_i|}{S_{n+1}^{1/p}}\right)\right] = \mathbb{E}\left[\mathbb{E}\left(\prod_{i=1}^{n+1} f_i\left(\frac{|g_i|}{S_{n+1}^{1/p}}\right) \middle| |g_{n+1}|\right)\right] =$$

$$= \int_0^\infty \frac{e^{-r^p}}{\Gamma(1+1/p)} \mathbb{E}\left\{\left[\prod_{i=1}^{n} f_i\left(\frac{|g_i|}{(S_n + r^p)^{1/p}}\right)\right] f_{n+1}\left(\frac{r}{(S_n + r^p)^{1/p}}\right)\right\} dr.$$

If we denote by $\psi(u)$ the density of S_n then by the independence of S_n and $(g_1, ..., g_n)/S_n^{1/p}$ we get:

$$\mathbb{E}\left\{\left[\prod_{i=1}^{n} f_i\left(\frac{|g_i|}{(S_n + r^p)^{1/p}}\right)\right] f_{n+1}\left(\frac{r}{(S_n + r^p)^{1/p}}\right)\right\} =$$

$$= \mathbb{E}\left[\mathbb{E}\left(f_{n+1}\left(\frac{r}{(S_n + r^p)^{1/p}}\right) \prod_{i=1}^{n} f_i\left(\frac{S_n^{1/p}}{(S_n + r^p)^{1/p}} \cdot \frac{|g_i|}{S_n^{1/p}}\right) \middle| S_n\right)\right] =$$

$$= \int_0^\infty \psi(u) f_{n+1}\left(\frac{r}{(u + r^p)^{1/p}}\right) \mathbb{E}\left[\prod_{i=1}^{n} f_i\left(\frac{u^{1/p}}{(u + r^p)^{1/p}} \cdot \frac{|g_i|}{S_n^{1/p}}\right)\right] du \leq$$

$$\leq \int_0^\infty \psi(u) f_{n+1}\left(\frac{r}{(u+r^p)^{1/p}}\right)\prod_{i=1}^n \mathbb{E}\left[f_i\left(\frac{u^{1/p}}{(u+r^p)^{1/p}}\cdot\frac{|g_i|}{S_n^{1/p}}\right)\right]du.$$

Define $h_u(r) = f_{n+1}\left(\frac{r}{(u+r^p)^{1/p}}\right)$ and:

$$k_u(r) = \prod_{i=1}^n \mathbb{E}\left[f_i\left(\frac{u^{1/p}}{(u+r^p)^{1/p}}\cdot\frac{|g_i|}{S_n^{1/p}}\right)\right].$$

Then h_u is increasing and k_u is decreasing, so that by the above argument:

$$\mathbb{E}\left[h_u(|g_{n+1}|)k_u(|g_{n+1}|)\right] \leq [\mathbb{E}h_u(|g_{n+1}|)]\cdot[\mathbb{E}k_u(|g_{n+1}|)].$$

Finally:

$$\mathbb{E}\left[\prod_{i=1}^{n+1} f_i\left(\frac{|g_i|}{S_{n+1}^{1/p}}\right)\right] \leq \int_0^\infty \frac{e^{-r^p}}{\Gamma(1+1/p)}\int_0^\infty \psi(u)h_u(r)k_u(r)dudr =$$

$$= \int_0^\infty \psi(u)\mathbb{E}\left[h_u(|g_{n+1}|)k_u(|g_{n+1}|)\right]du \leq$$

$$\leq \int_0^\infty \psi(u)[\mathbb{E}h_u(|g_{n+1}|)]\cdot[\mathbb{E}k_u(|g_{n+1}|)]du =$$

$$= \prod_{i=1}^n \mathbb{E}f_i\left(\frac{|g_i|}{S_{n+1}^{1/p}}\right).$$

■

In what follows we will derive concentration inequalities for the the ℓ_q^n norm on the ℓ_p^n sphere. By the above remarks, this amounts to the study of the concentration of the random variable $T/S^{q/p}$. It will be convenient to separate the calculations into two cases : $q \leq p$ and $q > p$.

2.1 Case 1 : $q \leq p$

For reasons that will become clear in the next section, we will take special care to study what happens when q tends to zero and q tends to p.

We will begin by deriving some elementary inequalities concerning the random variable g. To begin with, a simple change of variable shows that for $\lambda < 1$,

$$\mathbb{E}e^{\lambda|g|^p} = \frac{1}{(1-\lambda)^{1/p}}.$$

In what follows, we refer to [A] for the necessary background on the Gamma function.

Lemma 3 *For all* $0 < t \le 1/(2q)$,

$$\mathbb{E}e^{-t|g|^q} \le e^{-t(1-2q)}$$

Proof: Note that for every $\lambda > 0$:

$$P(|g| \le \lambda) = \frac{1}{\Gamma\left(1+\frac{1}{p}\right)} \int_0^\lambda e^{-x^p} dx \le \lambda.$$

Hence,

$$\mathbb{E}e^{-t|g|^q} = \int_0^1 P(e^{-t|g|^q} \ge x)dx = t\int_0^\infty e^{-tu} P(|g| \le u^{1/q})du \le$$

$$\le t\int_0^1 e^{-tu}u^{1/q}du + t\int_1^\infty e^{-tu}du = t\int_0^1 e^{-tu}u^{1/q}du + e^{-t}.$$

To estimate the above integral we will substitute $u = v^{2q}$ and get :

$$\int_0^1 e^{-tu}u^{1/q}du = 2q\int_0^1 e^{-tv^{2q}}v^{1+2q}dv \le 2q\int_0^1 e^{-tv^{2q}}vdv.$$

Since $t \le 1/(2q)$, for all $0 < v \le 1$:

$$e^{-tv^{2q}} = e^{-t}e^{(1-v^{2q})t} \le e^{-t}e^{(1-v^{2q})/(2q)} \le e^{-t}e^{-2q\log(v)/(2q)} = \frac{e^{-t}}{v}.$$

In the last estimate we used the inequality $1 - x \le -\log x$. Plugging this in the above integral we finally get :

$$\mathbb{E}e^{-tg^q} \le 2qte^{-t} + e^{-t} = e^{-t}(1+qt) \le e^{-t}e^{2qt}.$$

■

Remark: An inspection of the above proof shows that it remains true when $|g|$ is replaced by any non negative random variable with bounded density (and then the factor 2 is replaced by twice the bound of the density).

We will also need a similar estimate on the moment generating function:

Lemma 4 *Assume that* $p \ge 1$. *For every* $0 < t \le 1/(2q)$ *:*

$$\mathbb{E}e^{t|g|^q} \le e^{t(1+2q)}.$$

Proof: Fix $t \le 1/(2q)$. If $x > 1$ then:

$$P(|g| \ge x) = \frac{1}{\Gamma\left(1+\frac{1}{p}\right)} \int_x^\infty e^{-u^p}\, du \le \int_x^\infty \frac{pu^{p-1}}{px^{p-1}} e^{-u^p}\, du \le e^{-x^p}.$$

It follows that :

$$\mathbb{E}e^{t|g|^q} = 1 + t\int_0^\infty e^{tx} P(|g| \ge x^{1/q}) dx \le$$

$$\le 1 + t\int_0^1 e^{tx} dx + t\int_1^\infty e^{tx} e^{-x^{p/q}} dx = e^t + t\int_1^\infty e^{tx - x^{p/q}} dx.$$

Substituting $x = v^{q/p}$ in the last integral, and using the inequality $v^{q/p} \le 1 + qv/p$, which is true for $v \ge 1$ (since $q/p \le 1$), we get:

$$\int_1^\infty e^{tx - x^{p/q}} dx = \frac{q}{p}\int_1^\infty \frac{e^{tv^{q/p} - v}}{v^{1-q/p}} dv \le$$

$$\le q\int_1^\infty e^{t + tqv/p - v} dv \le qe^t \int_1^\infty e^{-v/2} dv < 2qe^t.$$

So that we finally get :

$$\mathbb{E}e^{t|g|^q} \le e^t + 2qte^t \le e^{t(1+2q)}.$$

■

A simple corollary of the previous two lemmas is:

Corollary 1 *If $p \ge 1$ and $|t| \le 1/(2q)$ then $\mathbb{E}e^{t|g|^q} \le e^{t+1}$.*

When $0 < p < 1$ we can prove the following weaker estimate :

Remark 1 *If $0 < p < 1$ and $0 < t \le p/2q$ then*

$$\mathbb{E}e^{t|g|^q} \le \frac{2^{1/p+1}}{\sqrt{p}} e^t.$$

Proof: Assume that $x^p > 1/p$ and put $\lambda = 1 - 1/(px^p) > 0$. Now:

$$P(|g| \ge x) \le e^{-\lambda x^p} \mathbb{E}e^{\lambda |g|^p} = \frac{e^{-\lambda x^p}}{(1-\lambda)^{1/p}} = (px^p)^{1/p} e^{-x^p + 1/p}.$$

This estimate is also true (trivially) for $0 < x \le 1/p$. An application of Stirling's formula gives:

$$P(|g| \ge x) \le \frac{\sqrt{2\pi}}{\sqrt{p}\,\Gamma\left(\frac{1}{p}+1\right)} x e^{-x^p}.$$

As in the proof of the previous lemma, we deduce that:

$$\mathbb{E}e^{t|g|^q} \leq e^t + t\frac{3}{\sqrt{p}\,\Gamma\left(\frac{1}{p}+1\right)}\int_1^\infty x^{1/q}e^{tx-x^{p/q}}dx$$

As before, by substituting $x = v^{q/p}$ we get:

$$\int_1^\infty x^{1/q}e^{tx-x^{p/q}}dx = \frac{q}{p}\int_1^\infty v^{(q+1)/p-1}e^{tv^{q/p}-v}dv \leq$$

$$\leq \frac{q}{p}e^t\int_1^\infty v^{(q+1)/p-1}e^{-v/2}dv \leq \frac{q2^{1/p}}{p}\Gamma\left(\frac{1}{p}+1\right)e^t.$$

And from this the required result follows. ■

Note that for every $\beta > -1$,

$$\mathbb{E}|g|^\beta = \frac{\Gamma\left(\frac{\beta+1}{p}\right)}{\Gamma\left(\frac{1}{p}\right)}$$

In particular $\mathbb{E}|g|^p = 1/p$.

The following elementary inequalities will be used several times in this paper:

Lemma 5 *The following inequalities hold:*

a) For every $0 < q \leq p$*:*

$$\frac{\mathbb{E}|g|^{2q}}{(\mathbb{E}|g|^q)^2} \leq 1 + \frac{q^2}{p} + \frac{2q^2(p-q)^2}{p^2}.$$

b) For every $2 \leq p \leq 5/2$ *put* $q = 2p-2$*. Then:*

$$\frac{\mathbb{E}|g|^{2q}}{(\mathbb{E}|g|^q)^2} \leq 1 + \frac{q^2}{p} + (p-2)^2.$$

c) For every $p, q > 0$*:*

$$\frac{\mathbb{E}|g|^{2q}}{(\mathbb{E}|g|^q)^2} \geq (1+q)\exp\left(-\frac{q(p-q)}{p(q+1)}\right) \geq 1 + \frac{q^2}{p}.$$

Proof: Recall the Weirstrass product formula for the gamma function :

$$\Gamma(x) = \frac{e^{-\gamma x}}{x}\prod_{k=1}^{\infty}\frac{e^{x/k}}{1+x/k}$$

Where γ is the Euler constant. Now :

$$\frac{\mathbb{E}|g|^{2q}}{(\mathbb{E}|g|^q)^2} = \frac{\Gamma\left(\frac{2q+1}{p}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{q+1}{p}\right)^2} = \prod_{k=0}^{\infty} \frac{(kp+q+1)^2}{(kp+1)(kp+2q+1)} =$$

$$= (q+1)\prod_{k=0}^{\infty} \frac{(kp+q+1)((k+1)p+q+1)}{((k+1)p+1)(kp+2q+1)} = \frac{1+q}{\prod_{k=0}^{\infty}\left(1+\frac{q(p-q)}{(kp+q+1)(kp+p+q+1)}\right)}.$$

To estimate this product, use the inequality $\log(1+a) \le a$, $a > -1$ to get that for $x > -1/2$:

$$\log(1+x) = -\log\left(1 - \frac{x}{1+x}\right) \ge \frac{x}{1+x} \ge x(1-x).$$

Hence for $k \ge 0$, if we put $x = \frac{q(p-q)}{(kp+q+1)(kp+p+q+1)}$ then our assumptions in $a)$ and $b)$ imply that $x > -1/2$, so that $1 + x \ge e^{x-x^2}$. Now,

$$\prod_{k=0}^{\infty}\left(1+\frac{q(p-q)}{(kp+q+1)(kp+p+q+1)}\right) \ge$$

$$\ge \exp\left[\sum_{k=0}^{\infty}\frac{q(p-q)}{(kp+q+1)(kp+p+q+1)}\right]\cdot$$

$$\cdot\exp\left[-\sum_{k=0}^{\infty}\frac{q^2(p-q)^2}{(kp+q+1)^2(kp+p+q+1)^2}\right] \ge$$

$$\ge \exp\left[\frac{q(p-q)}{p}\sum_{k=0}^{\infty}\left(\frac{1}{kp+q+1}-\frac{1}{kp+p+q+1}\right)\right]\cdot$$

$$\cdot\exp\left[-\frac{q^2(p-q)^2}{p^2}\left(\sum_{k=0}^{\infty}\left(\frac{1}{kp+q+1}-\frac{1}{kp+p+q+1}\right)\right)^2\right] =$$

$$= \exp\left[\frac{q(p-q)}{p(q+1)} - \frac{q^2(p-q)^2}{p^2(q+1)^2}\right] \ge 1 + \frac{q(p-q)}{p(q+1)} - \frac{q^2(p-q)^2}{p^2(q+1)^2}.$$

Hence, if we define $a = \frac{q(p-q)}{p(q+1)}$, then a little calculation gives:

$$\frac{\mathbb{E}|g|^{2q}}{(\mathbb{E}|g|^q)^2} \le (1+q)\left[1 + \frac{q(p-q)}{p(q+1)} - \frac{q^2(p-q)^2}{p^2(q+1)^2}\right]^{-1} =$$

$$= 1 + \frac{q^2}{p} + \frac{q^2(p-q)^2}{p^2(q+1)}\cdot\left[\frac{1}{1+a} + \frac{1}{(1+q)(1+a-a^2)}\right].$$

In the case $0 < q \le p$, $0 \le a \le 1$ and the result follows. In the case $2 \le p \le 5/2$ and $q = 2p - 2$, $-1/4 \le a \le 0$, from which it is easy to deduce the required inequality.

The proof of part c) runs along the same lines, and is simpler: one just estimates the product using the inequality $1 + x \le e^x$ (Part c was also proved in [Schm]). ■

Lemma 6 *For every* $0 < q < p$ *:*

$$Var(|g|^q) = \mathbb{E}|g|^{2q} - (\mathbb{E}|g|^q)^2 \le 3\max\left\{1, \frac{1}{p}\right\} \frac{q^2}{p^{2q/p}}.$$

Proof: By part a) of Lemma 5:

$$Var(|g|^q) = (\mathbb{E}(|g|^q))^2 \left[\frac{\Gamma\left(\frac{2q+1}{p}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{q+1}{p}\right)^2} - 1 \right] \le$$

$$\le (\mathbb{E}|g|^p)^{2q/p}) \left(1 + \frac{q^2}{p} + \frac{2q^2(p-q)^2}{p^2} - 1\right) \le$$

$$\le \frac{q^2}{p^{2q/p}} \left(\frac{1}{p} + 2\right),$$

and this is the required result. ■

Proposition 1 *Assume that* $p \ge 1$*. For* $0 < t \le 1/(4q)$ *the following inequality holds :*

$$\mathbb{E}e^{t\left|\,|g|^q - \mathbb{E}|g|^q\right|} \le e^{500q^2t^2}.$$

Proof: Let h be an independent copy of g. Now,

$$\mathbb{E}e^{t(|g|^q - |h|^q)} = \mathbb{E}e^{t(|g|^q - |h|^q)}1_{\left\{\left||g|^q - |h|^q\right| \le 1/t\right\}} + \mathbb{E}e^{t(|g|^q - |h|^q)}1_{\left\{\left||g|^q - |h|^q\right| \ge 1/t\right\}}.$$

To estimate the first summand note that for $|x| \le 1$, $e^x \le 1 + x + 2x^2$. Using this inequality and the fact that the random variable $(|g|^q - |h|^q)1_{\left\{\left||g|^q - |h|^q\right| \le 1/t\right\}}$ is symmetric we get :

$$\mathbb{E}e^{t(|g|^q - |h|^q)}1_{\left\{\left||g|^q - |h|^q\right| \le 1/t\right\}} \le$$

$$\le 1 + t\mathbb{E}(|g|^q - |h|^q)1_{\left\{\left||g|^q - |h|^q\right| \le 1/t\right\}} + 2t^2\mathbb{E}(|g|^q - |h|^q)^2 1_{\left\{\left||g|^q - |h|^q\right| \le 1/t\right\}} \le$$

$$\leq 1 + 4t^2 \mathrm{Var}(|g|^q) \leq 1 + 12q^2t^2.$$

The last inequality is an application of the previous lemma.

Now, using the independence of g and h, and previous inequalities, we get :

$$\mathbb{E}e^{t(|g|^q - |h|^q)} 1_{\left\{ \left| |g|^q - |h|^q \right| \geq 1/t \right\}} \leq$$

$$\leq \mathbb{E}e^{t(|g|^q - |h|^q)} 1_{\left\{ \left| |g|^q - 1 \right| \geq 1/(2t) \right\}} + \mathbb{E}e^{t(|g|^q - |h|^q)} 1_{\left\{ \left| |h|^q - 1 \right| \geq 1/(2t) \right\}} =$$

$$= \left(\mathbb{E}e^{t|g|^q} 1_{\left\{ \left| |g|^q - 1 \right| \geq 1/(2t) \right\}} \right) \mathbb{E}e^{-t|h|^q} +$$

$$+ \left(\mathbb{E}e^{-t|h|^q} 1_{\left\{ \left| |h|^q - 1 \right| \geq 1/(2t) \right\}} \right) \mathbb{E}e^{t|g|^q} \leq$$

$$\leq \left(\mathbb{E}e^{2t|g|^q} \right)^{1/2} \mathbb{E}e^{-t|h|^q} \sqrt{P\left(\left| |g|^q - 1 \right| \geq \frac{1}{2t} \right)} +$$

$$+ \left(\mathbb{E}e^{-2t|h|^q} \right)^{1/2} \mathbb{E}e^{t|g|^q} \sqrt{P\left(\left| |h|^q - 1 \right| \geq \frac{1}{2t} \right)} \leq$$

$$\leq \left[(e^{2t+1})^{1/2} e^{-t+1} + (e^{-2t+1})^{1/2} e^{t+1} \right] \sqrt{P\left(\left| |g|^q - 1 \right| \geq \frac{1}{2t} \right)} =$$

$$= 2e^{3/2} \cdot \sqrt{P\left(\left| |g|^q - 1 \right| \geq \frac{1}{2t} \right)} \leq$$

$$\leq 9 \left[P\left(|g|^q - 1 \geq \frac{1}{2t} \right) + P\left(1 - |g|^q \geq \frac{1}{2t} \right) \right]^{1/2} \leq$$

$$\leq 9 \left[P\left(\frac{q}{p}(|g|^p - 1) \geq \frac{1}{2t} \right) + P\left(|g|^q \leq 1 - \frac{1}{2t} \right) \right]^{1/2} \leq$$

$$\leq 9 \left[P\left(|g| \geq \left(1 + \frac{1}{2qt} \right)^{1/p} \right) + P\left(|g| \leq \left(1 - \frac{1}{2t} \right)^{1/q} \right) \right]^{1/2} \leq$$

$$\leq 9 \left[\exp\left(-\frac{1}{2qt} \right) + \left(1 - \frac{1}{2t} \right)^{1/q} \right]^{1/2} \leq 15 \exp\left(-\frac{1}{4qt} \right) \leq 220q^2t^2.$$

We used here the estimates that appeared in the beginning of the proofs of Lemma 3 and Lemma 4, and the inequality $e^{-1/x} \leq x^2$ which is true for all $0 < x \leq 1$ (to prove it note that the minimum of the function $f(x) = 1/x + 2\log x$, which is attained at $x = 1/2$, is positive).

Summing up, we have shown that for all $0 < t \leq 1/(4q)$, $\mathbb{E}\exp(t(|g|^q - |h|^q)) \leq 1 + 250q^2t^2$. If we now use the elementary inequality :

$$e^{|x|} \leq 1 + x + 2\left(\frac{e^x + e^{-x}}{2} - 1\right).$$

We get :

$$\mathbb{E}e^{t\left||g|^q - |h|^q\right|} \leq 1 + t\mathbb{E}(|g|^q - |h|^q) + 2\left(\frac{\mathbb{E}e^{t(|g|^q - |h|^q)} + \mathbb{E}e^{t(|h|^q - |g|^q)}}{2} - 1\right) \leq$$

$$\leq 1 + 500q^2t^2.$$

Finally, an application of Jensen's inequality gives:

$$\mathbb{E}e^{t\left||g|^q - \mathbb{E}|g|^q\right|} \leq \mathbb{E}e^{t\left||g|^q - |h|^q\right|} \leq 1 + 500q^2t^2 \leq e^{500q^2t^2}.$$

■

Remark: For $0 < p < 1$, similar reasoning shows that if $0 < t < p/(4q)$ then,

$$\mathbb{E}e^{t\left||g|^q - \mathbb{E}|g|^q\right|} \leq e^{C(p)q^2t^2}.$$

Where $C(p)$ depends only on p (and tends to infinity as p tends to zero).

The case $0 < p < 1$ will be very important for us later, and we will need another inequality, in addition to the above remark, to handle it:

Proposition 2 *For every $0 < q < p$ and $0 < t \leq p^{q/p}/(2q)$ we have :*

$$\mathbb{E}\exp\left(t\left||g|^q - \mathbb{E}|g|^q\right|\right) \leq 10\exp\left(\frac{2q^2t^2}{p^{1+2q/p}}\right).$$

Before passing to the proof, we will need some elementary inequalities, the short proofs of which we include for the sake of completeness.

Lemma 7 *For every $x, y > 0$,*

$$\frac{x^y\Gamma(x)}{\Gamma(x+y)} \leq 1 + \frac{y}{x} \leq e^{y/x}.$$

Proof: Using the formula:

$$\Gamma(t) = \lim_{m\to\infty} \frac{m^t m!}{t(t+1)\cdots(t+m)},$$

we get,

$$\frac{\Gamma(x)}{\Gamma(x+y)} = \lim_{m\to\infty} \frac{1}{m^y} \prod_{k=0}^{m} \frac{x+y+k}{x+k} = \lim_{m\to\infty} \frac{1}{m^y} \prod_{k=0}^{m} \left(1+\frac{y}{x+k}\right) \leq$$

$$\leq \left(1+\frac{y}{x}\right) \lim_{m\to\infty} \frac{1}{m^y} \exp\left(\sum_{k=1}^{m} \frac{y}{x+k}\right) \leq$$

$$\leq \left(1+\frac{y}{x}\right) \lim_{m\to\infty} \frac{1}{m^y} \exp\left(y\int_0^{x+m} \frac{du}{x+u}\right) =$$

$$= \left(1+\frac{y}{x}\right) \lim_{m\to\infty} \frac{1}{m^y} \exp\left(y\log\left(\frac{x+m}{x}\right)\right) =$$

$$= \frac{1+y/x}{x^y} \lim_{m\to\infty} \left(\frac{x+m}{m}\right)^y = \frac{1+y/x}{x^y}.$$

■

Corollary 2 *For every* $0 < q \leq p$*:*

$$\mathbb{E}|g|^q \leq \frac{1}{p^{q/p}} \leq (1+q)\mathbb{E}|g|^q.$$

Proof: Since $\mathbb{E}|g|^p = 1/p$, the left hand side follows from Hölder's inequality. To prove the second inequality just use the previous lemma with $x = 1/p$ and $y = q/p$:

$$\frac{1}{p^{q/p}\mathbb{E}|g|^q} = \frac{x^y\Gamma(x)}{\Gamma(x+y)} \leq 1+\frac{y}{x} = 1+q.$$

■

Lemma 8 *For every* $0 < \alpha \leq 1/2$*,*

$$\mathbb{E}\frac{1}{(p^{1/p}|g|)^\alpha} \leq 3e^{\frac{2\alpha^2}{p}}.$$

Proof: Put $x = (1-\alpha)/p$ and $y = \alpha/p$. Then:

$$\mathbb{E}\frac{1}{(p^{1/p}|g|)^\alpha} = \frac{\Gamma\left(\frac{1-\alpha}{p}\right)}{p^{\alpha/p}\Gamma\left(\frac{1}{p}\right)} = \frac{(x+y)^y\Gamma(x)}{\Gamma(x+y)} \leq$$

$$\leq \left(\frac{x+y}{x}\right)^y e^{y/x} = \left(1+\frac{\alpha}{1-\alpha}\right)^{\alpha/p} \exp\left(\frac{\alpha}{1-\alpha}\right) \leq \exp\left(1+\frac{2\alpha^2}{p}\right).$$

■

Proof of Proposition 2: To begin with,

$$\mathbb{E}\exp\left(t\left||g|^q - \frac{1}{p^{q/p}}\right|\right) \le \mathbb{E}\exp\left[t\left(|g|^q - \frac{1}{p^{q/p}}\right)\right] + \mathbb{E}\exp\left[t\left(\frac{1}{p^{q/p}} - |g|^q\right)\right] \le$$

$$\le 1 + \mathbb{E}\exp\left[t\left(|g|^q - \frac{1}{p^{q/p}}\right)\right] 1_{\{|g|^q \ge 1/p^{q/p}\}} + \mathbb{E}\exp\left[t\left(\frac{1}{p^{q/p}} - |g|^q\right)\right].$$

Now, if we put $\lambda = tq/p^{q/p} \le 1/2$ then,

$$\mathbb{E}\exp\left[t\left(|g|^q - \frac{1}{p^{q/p}}\right)\right] 1_{\{|g|^q \ge 1/p^{q/p}\}} \le \mathbb{E}\exp\left[\frac{tqp^{1-q/p}}{p}\left(|g|^p - \frac{1}{p}\right)\right] =$$

$$= e^{-\lambda/p}\mathbb{E}e^{\lambda|g|^p} = \frac{e^{-\lambda/p}}{(1-\lambda)^{1/p}} = \exp\left(\frac{1}{p}(-\lambda - \log(1-\lambda))\right) \le$$

$$\le e^{2\lambda^2/p} = \exp\left(\frac{2q^2t^2}{p^{1+2q/p}}\right)$$

In the last inequality we used the fact that for $0 < \lambda \le 1/2$, $\log(1-\lambda) \ge -\lambda - 2\lambda^2$ (to prove it just use the Taylor expansion of $\log(1+x)$).

Next, using Lemma 8 (remember that $\lambda \le 1/2$) and the inequality $\log(x) \le x - 1$ we get:

$$\mathbb{E}\exp\left[t\left(\frac{1}{p^{q/p}} - |g|^q\right)\right] = \mathbb{E}\exp\left[\frac{t}{p^{q/p}}\left(1 - (p^{1/p}|g|)^q\right)\right] \le$$

$$\le \mathbb{E}\exp\left(-\frac{tq}{p^{q/p}}\log(p^{1/p}|g|)\right) = \mathbb{E}\frac{1}{(p^{1/p}|g|)^\lambda} \le$$

$$\le 3e^{2\lambda^2/p} = 3\exp\left(\frac{2t^2q^2}{p^{1+2q/p}}\right).$$

Summing up we get:

$$\mathbb{E}\exp\left(t\left||g|^q - \frac{1}{p^{q/p}}\right|\right) \le 1 + 4\exp\left(\frac{2t^2q^2}{p^{1+2q/p}}\right) \le$$

$$\le 5\exp\left(\frac{2t^2q^2}{p^{1+2q/p}}\right).$$

Finally, using Corollary 2:

$$\mathbb{E}\exp\left(t\left||g|^q - \mathbb{E}|g|^q\right|\right) \le \exp\left(t\left|\frac{1}{p^{q/p}} - \mathbb{E}|g|^q\right|\right) \cdot \mathbb{E}\exp\left(t\left||g|^q - \frac{1}{p^{q/p}}\right|\right) \le$$

$$\le \exp\left(\frac{tq}{p^{q/p}}\right) \cdot 5\exp\left(\frac{2t^2q^2}{p^{1+2q/p}}\right) \le 5e^{1/2}\exp\left(\frac{2t^2q^2}{p^{1+2q/p}}\right).$$

■

We will now derive similar inequalities for the random variable S. It is easy to check that $|g|^p$ has $gamma(1/p, 1)$ distribution. In other words, the density of $|g|^p$ is :

$$\frac{1}{\Gamma\left(\frac{1}{p}\right)} x^{\frac{1}{p}-1} e^{-x} \text{ for } x > 0.$$

It is well known (see [D]) that $S = \sum_{k=1}^{n} |g_k|^p$ has $gamma(n/p, 1)$ distribution, i.e. the density of S is :

$$\frac{1}{\Gamma\left(\frac{n}{p}\right)} x^{\frac{n}{p}-1} e^{-x} \text{ for } x > 0.$$

By the above remarks, if we put $r = p/n$, and Z is a random variable with density $1/\Gamma(1 + 1/r)e^{-x^r}$ for $x > 0$, then S has the same distribution as $Z^{p/n}$. This is the reason why we took special care to study the case $p < 1$, although we are primarily interested in the case $p > 1$.

Lemma 9 *For every* $0 < t \leq \frac{p^{q/p}}{2q} n^{1-q/p}$,

$$\mathbb{E}e^{t|S^{q/p} - \mathbb{E}S^{q/p}|} \leq 10 \exp\left(\frac{2q^2t^2}{p^{1+2q/p}n^{1-2q/p}}\right).$$

Proof: In the above notation, $S^{q/p}$ has the same distribution as $Z^{q/n}$. Now all we need to do is to notice that the required result is just the statement of Proposition 2 where p is replaced by p/n and q is replaced by q/n. ■

We will need the following:

Lemma 10 *The following inequalities hold:*

a) $\frac{n}{p^{q/p}(1+q)} \leq \mathbb{E}T \leq \frac{n}{p^{q/p}}$.

b) $\frac{n^{p/q}}{p^{q/p}(1+q/n)} \leq \mathbb{E}S^{q/p} \leq \frac{n^{q/p}}{p^{q/p}}$.

c) $\left\|\frac{T}{S^{q/p}}\right\|_\infty \leq n^{1-q/p}$.

d) $\frac{n^{1-q/p}}{1+q} \leq \frac{\mathbb{E}T}{\mathbb{E}S^{q/p}} \leq n^{1-q/p}$.

Proof: Parts $a)$ and $b)$ are just a rephrasing of Corollary 2. Part $c)$ is also clear since:

$$\frac{T}{S^{q/p}} = \frac{||(g_1, ..., g_n)||_q^q}{||(g_1, ..., g_n)||_p} \leq n^{1-q/p}$$

The left hand inequality in $d)$ follows from $a)$ and $b)$. To prove the remaining inequality note that since $T/S^{q/p}$ and $S^{q/p}$ are independent:

$$\frac{\mathbb{E}T}{\mathbb{E}S^{q/p}} = \mathbb{E}\frac{T}{S^{q/p}} \leq \left\| \frac{T}{S^{q/p}} \right\|_\infty \leq n^{1-q/p}$$

■

We can now prove:

Theorem 3 *For every $0 < p < \infty$ there is constant $C = C(p) > 0$, such that for all $t > 0$:*

$$\mu_{p,n}\left(\left| ||x||_q^q - \int_{S(\ell_p^n)} ||y||_q^q d\mu_{p,n}(y) \right| \geq t \int_{S(\ell_p^n)} ||y||_q^q d\mu_{p,n}(y) \right) \leq$$

$$\leq 10 \exp\left(-Cn \min\left\{ \left(\frac{t}{q}\right)^2, \frac{t}{q} \right\} \right).$$

Proof: Fix some $\lambda > 0$. Using Lemma 1 and the remarks at the beginning of this section we get:

$$\mu_{p,n}\left(\left| ||x||_q^q - \int_{S(\ell_p^n)} ||y||_q^q d\mu_{p,n}(y) \right| \geq t \int_{S(\ell_p^n)} ||y||_q^q d\mu_{p,n}(y) \right) =$$

$$= P\left(\left| \frac{T}{S^{q/p}} - \frac{\mathbb{E}T}{\mathbb{E}S^{q/p}} \right| \geq t\frac{\mathbb{E}T}{\mathbb{E}S^{q/p}} \right) \leq$$

$$\leq \exp\left(-\lambda t \frac{\mathbb{E}T}{\mathbb{E}S^{q/p}} \right) \cdot \mathbb{E}\exp\left(\lambda \left| \frac{T}{S^{q/p}} - \frac{\mathbb{E}T}{\mathbb{E}S^{q/p}} \right| \right) \leq$$

$$\leq \exp\left(-\lambda t \frac{\mathbb{E}T}{\mathbb{E}S^{q/p}} \right) \cdot \mathbb{E}\exp\left(\frac{\lambda}{\mathbb{E}S^{q/p}} \left| T - \frac{\mathbb{E}T}{\mathbb{E}S^{q/p}} S^{q/p} \right| \right) \leq$$

$$\leq \exp\left(-\lambda t \frac{\mathbb{E}T}{\mathbb{E}S^{q/p}} \right) \cdot \mathbb{E}\exp\left(\frac{\lambda}{\mathbb{E}S^{q/p}} |T - \mathbb{E}T| + \frac{\lambda \mathbb{E}T}{(\mathbb{E}S^{q/p})^2} |S^{q/p} - \mathbb{E}S^{q/p}| \right) \leq$$

$$\leq \exp\left(-\lambda t \frac{\mathbb{E}T}{\mathbb{E}S^{q/p}} \right) \cdot$$

$$\cdot \left[\mathbb{E}\exp\left(\frac{2\lambda}{\mathbb{E}S^{q/p}} |T - \mathbb{E}T| \right) \cdot \mathbb{E}\exp\left(\frac{2\lambda \mathbb{E}T}{(\mathbb{E}S^{q/p})^2} |S^{q/p} - \mathbb{E}S^{q/p}| \right) \right]^{1/2}$$

Write $\lambda = ns\frac{\mathbb{E}S^{q/p}}{\mathbb{E}T}$, for some $s > 0$. Using Lemma 10 we see that:

$$\frac{2\lambda}{\mathbb{E}S^{q/p}} = \frac{2sn}{\mathbb{E}T} \leq 2sp^{q/p}(1+q) \leq 2sp^{q/p}(1+p).$$

And:

$$\frac{2\lambda\mathbb{E}T}{(\mathbb{E}S^{q/p})^2} = \frac{2sn}{\mathbb{E}S^{q/p}} \leq 2sp^{q/p}(1+p)n^{1-q/p}.$$

It follows that there is a constant $A(p)$ such that if $s \leq A(p)/q$ then :

$$\frac{2\lambda}{\mathbb{E}S^{q/p}} \leq \min\left\{\frac{1}{4q}, \frac{p}{4q}\right\} \quad \text{and} \quad \frac{2\lambda\mathbb{E}T}{(\mathbb{E}S^{q/p})^2} \leq \frac{p^{q/p}}{2q}n^{1-q/p}.$$

Combining Proposition 1, the remark following it and Proposition 9, we get that there are constants $B_1(p), B_2(p), B_3(p), B_4(p)$ such that if $s \leq A(p)/q$:

$$\mathbb{E}\exp\left(\frac{2\lambda}{\mathbb{E}S^{q/p}}|T - \mathbb{E}T|\right) = \mathbb{E}\exp\left(\frac{2\lambda}{\mathbb{E}S^{q/p}}\left|\sum_{k=0}^{n}|g_k|^q - n\mathbb{E}|g|^q\right|\right) \leq$$

$$\leq \mathbb{E}\left[\exp\left(\frac{2\lambda}{\mathbb{E}S^{q/p}}\left||g|^q - \mathbb{E}|g|^q\right|\right)\right]^n \leq \exp\left[nB_1(p)q^2\left(\frac{2\lambda}{\mathbb{E}S^{q/p}}\right)^2\right] \leq e^{B_2(p)nq^2s^2}.$$

And:

$$\mathbb{E}\exp\left(\frac{2\lambda\mathbb{E}T}{(\mathbb{E}S^{q/p})^2}|S^{q/p} - \mathbb{E}S^{q/p}|\right) \leq 10\exp\left[B_3(p)\frac{q^2}{n^{1-2q/p}}\left(\frac{2\lambda\mathbb{E}T}{(\mathbb{E}S^{q/p})^2}\right)^2\right] \leq$$

$$\leq 10\exp\left(B_4(p)q^2s^2\frac{n^{2(1-q/p)}}{n^{1-2q/p}}\right) = 10e^{B_4(p)nq^2s^2}.$$

Plugging this into what we derived before, we get the following statement: *there are constants $A(p)$ and $C(p)$ such that for all $s \leq A(p)/q$:*

$$\mu_{p,n}\left(\left|||x||_q^q - \int_{S(\ell_p^n)}||y||_q^q d\mu_{p,n}(y)\right| \geq t\int_{S(\ell_p^n)}||y||_q^q d\mu_{p,n}(y)\right) \leq 12e^{-nst+C(p)nq^2s^2}.$$

We can also clearly assume that $C(p) = 1/(2A(p))$. Indeed, replace $C(p)$ by $C(p) + 1/(2A(p))$ and $A(p)$ by $A(p)/(1+2C(p)A(p))$.

To finish the proof, assume first that $t \leq q$. Taking $s = t/(2C(p)q^2) \leq A(p)/q$ we get:

$$-st + C(p)q^2s^2 = -\frac{(t/q)^2}{4C(p)}.$$

If $t > q$ then take $s = A(p)/q$. Then:

$$-st + C(p)q^2s^2 = -\frac{A(p)t}{q} + C(p)A(p)^2 = -\frac{A(p)t}{q} + \frac{A(p)}{2} \leq$$

$$\leq -\frac{A(p)t}{q} + \frac{A(p)t}{2q} = -\frac{A(p)t}{2q}.$$

This proves the theorem.

■

Remark: An inspection of the preceding proofs shows that in the case $p > 1$, on can take $C(p) = c/p$ for some absolute constant c.

Remark: By Lemma 10:

$$0 \leq \frac{\mathbb{E}S^{q/p}}{\mathbb{E}T}\frac{T}{S^{q/p}} \leq (1+q).$$

So that in Theorem 3, the the probability estimated is nonzero only for $t \leq \max\{1, q\}$. Therefore, if we ignore the dependence on q, we proved a deviation inequality of the form:

$$\mu_{p,n}\left(\left| ||x||_q^q - \int_{S(\ell_p^n)} ||y||_q^q d\mu_{p,n}(y) \right| \geq t \int_{S(\ell_p^n)} ||y||_q^q d\mu_{p,n}(y)\right) \leq Ce^{-cnt^2}.$$

On the other hand if we assume that $p > 1$, then since the function $f(x) = ||x||_q / n^{1/q-1/p}$ is Lipschitz with respect to the ℓ_p^n metric (with constant independent of n), the Gromov-Milman theorem [GM] implies that:

$$\mu_{p,n}\left(\left| ||x||_q - \int_{S(\ell_p^n)} ||y||_q d\mu_{p,n}(y) \right| \geq tn^{1/q-1/p}\right) \leq C\exp\left(-cnt^{\max\{2,p\}}\right).$$

By standard arguments, this is equivalent to a concentration inequality for $||x||_q^q$, of the form given above, with the power 2 of t replaced by $\max\{2, p\}$, which is asymptoticly worse when $p > 2$. Reversing the argument, the concentration we got in the case $p > 2$ is better then the concentration that follows from the Gromov-Milman theorem. Moreover, the constants in the Gromov-Milman theorem tend to zero as p tends to 1, and this is not the case in our result (see [A-d-RV] and [Sche] for related results).

Theorem 4 is satisfactory when q tends to zero, since it reflects the fact that $||x||_q^q$ tends to a constant. We will now study what happens when q tends to

p. In this case we are only able to prove a less satisfactory inequality, which reflects the fact that $||x||_q^q$ tends to a constant when q tends to p, but which involves an apparently redundant $\log n$ term.

Theorem 4 *For every $p \geq 1$, $0 < q \leq p$ and $u > 0$:*

$$\mu_{p,n}\left(\left|||x||_q^q - \int_{S(\ell_p^n)} ||y||_q^q d\mu_{p,n}(y)\right| \geq t \int_{S(\ell_p^n)} ||y||_q^q d\mu_{p,n}(y)\right) \leq$$

$$\leq \exp\left[-\frac{n}{600p}\min\left\{\left(\frac{u(n-1)}{p(n-n^{q/p})}\right)^2, \frac{u(n-1)}{p(n-n^{q/p})}\right\}\right] \leq$$

$$\leq \exp\left[-\frac{n}{600p}\min\left\{\left(\frac{u}{(p-q)\log n}\right)^2, \frac{u}{(p-q)\log n}\right\}\right].$$

Proof: For every $x \in S(\ell_p^n)$,

$$0 \leq 1 - \frac{1}{n^{1-q/p}}\sum_{i=1}^n |x_i|^q = \frac{1}{n}\sum_{i=1}^n \left[(n^{1/p}|x_i|)^p - (n^{1/p}|x_i|)^q\right] \leq$$

$$\leq \frac{1}{n}\sum_{i=1}^n \left[(n^{1/p}|x_i|)^p - (n^{1/p}|x_i|)^q\right] 1_{\{n^{1/p}|x_i| \geq 1\}} = \frac{1}{n}\sum_{i=1}^n f(n^{1/p}|x_i|),$$

where for $x > 0$, $f(x) = (x^p - x^q)1_{\{x \geq 1\}}$. Simple differentiation shows that f is nondecreasing on $[0,\infty)$. By Jensen's inequality and Lemma 2, for every $\theta > 0$:

$$\int_{S(\ell_p^n)} \exp\left(\frac{\theta}{n^{1-q/p}}\left|||x||_q^q - \int_{S(\ell_p^n)} ||z||_q^q d\mu_{p,n}(z)\right|\right) d\mu_{p,n}(x) \leq$$

$$\leq \int_{S(\ell_p^n)} \exp\left(\frac{\theta}{n^{1-q/p}}\left|||x||_q^q - ||y||_q^q\right|\right) d\mu_{p,n}(x) d\mu_{p,n}(y) \leq$$

$$\leq \int_{S(\ell_p^n)} \exp\left(2\theta\left|\frac{||x||_q^q}{n^{1-q/p}} - 1\right|\right) d\mu_{p,n}(x) \leq$$

$$\leq \int_{S(\ell_p^n)} \prod_{i=1}^n \exp\left(\frac{2\theta f(n^{1/p}|x_i|)}{n}\right) d\mu_{p,n}(x) \leq$$

$$\leq \left[\int_{S(\ell_p^n)} \exp\left(\frac{2\theta f(n^{1/p}|x_1|)}{n}\right) d\mu_{p,n}(x)\right]^n =$$

$$= \left[\mathbb{E}\exp\left(\frac{2\theta f(n^{1/p}Z^{1/p})}{n}\right)\right]^n,$$

where Z denotes the random variable $|g_1|^p/S^p$.

Note that the function $a(x) = \frac{x - x^{q/p}}{x-1}$ is increasing on $(1, \infty)$. Indeed, since $(x-1)^2 a'(x) = x^{q/p-1}\left(x\left(1-\frac{q}{p}\right)+\frac{q}{p}\right) - 1$, it is enough to show that for $x \geq 1$, $b(x) = x\left(1-\frac{q}{p}\right)+\frac{q}{p} - x^{1-q/p} \geq 0$. Now, $b(1) = 0$ and $b'(x) = \left(1-\frac{q}{p}\right)\left(1-\frac{1}{x^{q/p}}\right) \geq 0$.

Since $Z \leq 1$,

$$f(n^{1/p}Z^{1/p}) = \big(nZ - (nZ)^{q/p}\big)1_{\{nZ\geq 1\}} \leq \frac{n-n^{q/p}}{n-1}\cdot |nZ-1|.$$

Arguing the same as in the proof of Lemma 1 we see that for every $\lambda > 0$:

$$\mathbb{E}e^{\lambda|nZ-1|} = \mathbb{E}\exp\left(n\lambda\cdot\left|\frac{|g_1|^p}{S^p}-\frac{1}{n}\right|\right) \leq$$

$$\leq \mathbb{E}\exp\left(\frac{n\lambda}{\mathbb{E}S^p}\cdot\left||g|^p - \frac{1}{n}\cdot\mathbb{E}S^p\right|\right) = \mathbb{E}e^{p\lambda\left||g|^p-\mathbb{E}|g|^p\right|}.$$

Now, as in the proof of Proposition 1, if h is an independent copy of g then for every $0 < t < 1$:

$$\mathbb{E}e^{t\left||g|^p-\mathbb{E}|g|^p\right|} \leq \mathbb{E}e^{t\left||g|^p-|h|^p\right|} \leq$$

$$\leq \mathbb{E}\left[1 + t(|g|^p - |h|^p) + 2\left(\frac{e^{t(|g|^p-|h|^p)}+e^{-t(|g|^p-|h|^p)}}{2} - 1\right)\right] =$$

$$= 2\big(\mathbb{E}e^{t|g|^p}\big)\cdot\big(\mathbb{E}e^{-t|g|^p}\big) - 1 = \frac{2}{(1-t^2)^{1/p}} - 1.$$

It is elementary to check that for $0 < t \leq 1/2$, $2(1-t^2)^{-1/p} - 1 \leq e^{8t^2/p}$. Hence, if we set $a = \frac{2p}{n}\cdot\frac{n-n^{q/p}}{n-1}$ then we have shown that for $0 < \theta \leq \frac{1}{2a}$:

$$\mathbb{E}\exp\left(\frac{\theta}{n^{1-q/p}}\cdot\left|\frac{T}{S^{q/p}} - \mathbb{E}\left(\frac{T}{S^{q/p}}\right)\right|\right) \leq \exp\left(\frac{8na^2\theta^2}{p}\right).$$

By standard arguments, this gives that for $u > 0$,

$$P\left(\left|\frac{T}{S^{q/p}} - \mathbb{E}\left(\frac{T}{S^{q/p}}\right)\right| \geq n^{1-q/p}u\right) \leq \exp\left(-\min\left\{\frac{pu^2}{32na^2}, \frac{u}{4a}\right\}\right).$$

This yields the required result since by Lemma 10, $2p\mathbb{E}\big(T/(S^{q/p})\big) \geq n^{1-q/p}$.

■

2.2 Case 2: $q > p$

We shall begin by stating a result from [L]. Let X be a random variable with finite moments. If $X_1, ..., X_n, Y_1, ..., Y_n$ are i.i.d. copies of X and $\alpha \geq 2$ then:

$$\left(\mathbb{E}\left|\sum_{k=1}^{n}(X_k - Y_k)\right|^{\alpha}\right)^{1/\alpha} \sim$$

$$\sim \sup\left\{\frac{\alpha}{s}\left(\frac{n}{\alpha}\right)^{1/s}(\mathbb{E}|X|^s)^{1/s};\ \max\left\{2, \frac{\alpha}{n}\right\} \leq s \leq \alpha\right\}.$$

Where the symbol $\sim$ means equivalence up to universal constants independent of n, α and the distribution of X.

If in addition we assume that X is non-negative then for $\alpha \geq 1$:

$$\left[\mathbb{E}\left(\sum_{k=1}^{n} X_k\right)^{\alpha}\right]^{1/\alpha} \sim$$

$$\sim \sup\left\{\frac{\alpha}{s}\left(\frac{n}{\alpha}\right)^{1/s}(\mathbb{E}|X|^s)^{1/s};\ \max\left\{1, \frac{\alpha}{n}\right\} \leq s \leq \alpha\right\}.$$

By Stirling's formula, for any $q \geq p \geq 1$ and $s \geq 1$,

$$(\mathbb{E}|g|^{qs})^{1/s} = \left[\frac{\Gamma\left(\frac{1+qs}{p}\right)}{\Gamma\left(\frac{1}{p}\right)}\right]^{1/s} \sim \frac{(1+qs)^{q/p}}{(ep)^{q/p}(p^{1/2}q^{1/2-1/p})^{1/s}},$$

where we have used also the fact that $\Gamma(1/p) \sim p$.

Proposition 3 *For $\alpha \geq 1$ and $q \geq p \geq 1$ the following equivalence holds:*

$$(\mathbb{E}T^{\alpha})^{1/\alpha} \sim \left(\frac{1+q}{ep}\right)^{q/p} \cdot \frac{nq^{1/p}}{\sqrt{pq}} \max\left\{1, \left(\frac{\sqrt{pq}}{nq^{1/p}}\right)^{1-1/\alpha} \cdot \alpha^{q/p}\right\},$$

Proof: Assume first that $q \geq 6p$. By the preceding remarks:

$$(\mathbb{E}T^{\alpha})^{1/\alpha} \sim$$

$$\sim \frac{\alpha}{(ep)^{q/p}} \sup\left\{\frac{1}{s}\left(\frac{n}{\alpha p^{1/2}q^{1/2-1/p}}\right)^{1/s}(1+qs)^{q/p};\ \max\left\{1, \frac{\alpha}{n}\right\} \leq s \leq \alpha\right\}.$$

Define:

$$f(s) = \frac{1}{s}\left(\frac{n}{\alpha p^{1/2}q^{1/2-1/p}}\right)^{1/s}(1+qs)^{q/p},$$

and $g(s) = \log f(s)$. It is strait-forward to check that:

$$\frac{sg''(s)}{2} + g'(s) = \frac{sq(q-p)(sq+2)-p}{2ps(1+qs)^2}.$$

In particular, for $q \geq 6p$ and $s \geq 1$, $sg''(s)/2+g'(s) > 0$ so that a local extremum of g (and hence also of f) must be a minimum. It follows that:

$$(\mathbb{E}T^\alpha)^{1/\alpha} \sim \frac{\alpha}{(ep)^{q/p}} \cdot \max\left\{ f\left(\max\left\{1, \frac{\alpha}{n}\right\}\right), f(\alpha)\right\}.$$

We claim that when $\alpha \geq n$, f is increasing. Indeed, it is enough to show that g is increasing. Now,

$$g'(s) = -\frac{1}{s} - \frac{1}{s^2}\log\left(\frac{n}{\alpha p^{1/2} q^{1/2-1/p}}\right) + \frac{q^2}{p(1+qs)} \geq$$

$$\geq -\frac{1}{s} + \frac{1}{s^2}\left[\frac{1}{2}\log p + \left(\frac{1}{2} - \frac{1}{p}\right)\log q\right] + \frac{q^2}{p(1+qs)} \geq$$

$$\geq \left(\frac{q}{2p} - 1\right)\frac{1}{s} - \frac{\log q}{ps^2} \geq \frac{q - 2\log q}{2p} - 1 \geq$$

$$\geq \frac{6p - 2\log(6p)}{2p} - 1 \geq \frac{6p - 2\log 6 - 2(p-1)}{2p} - 1 =$$

$$= 1 - \frac{\log 6 - 1}{p} \geq 2 - \log 6 > 0,$$

where we have used the fact that the function $x - 2\log x$ is increasing for $x \geq 2$, $q \geq 6p \geq 6$ and the inequality $\log x \leq x - 1$.

Summarizing, we have shown that for $q \geq 6p$:

$$(\mathbb{E}T^\alpha)^{1/\alpha} \sim \frac{\alpha}{(ep)^{q/p}} \cdot \max\{f(1), f(\alpha)\} \sim$$

$$\sim \left(\frac{1+q}{ep}\right)^{q/p} \cdot \frac{nq^{1/p}}{\sqrt{pq}} \max\left\{1, \left(\frac{\sqrt{pq}}{nq^{1/p}}\right)^{1-1/\alpha}\left(\frac{1+q\alpha}{1+q}\right)^{q/p}\right\},$$

which is equivalent to the required formula since

$$\alpha^{q/p} \geq \left(\frac{1+q\alpha}{1+q}\right)^{q/p} \geq \left(\frac{\alpha}{1+1/q}\right)^{q/p} \geq \frac{\alpha^{q/p}}{e}.$$

The case $p \leq q \leq 6p$ is much simpler. In this case it is clear that for every $s \geq 1$:

$$(\mathbb{E}|g|^{q/s})^{1/s} \sim \left(\frac{qs}{p}\right)^{q/p} \cdot \frac{1}{p^{1/s}}.$$

Hence,

$$(\mathbb{E}T^\alpha)^{1/\alpha} \sim \frac{\alpha q^{q/p}}{p^{q/p}} \sup\left\{ s^{q/p-1}\left(\frac{n}{\alpha p}\right)^{1/s} ;\ \max\left\{1, \frac{\alpha}{n}\right\} \le s \le \alpha \right\}.$$

If we define $f(s) = s^{q/p-1}\left(\frac{n}{\alpha p}\right)^{1/s}$, then it is easy to verify that f is increasing when $\alpha \ge n$, and the only local extremum of f, which is attained at $s = (q/p-1)^{-1}\log\left(n/(\alpha p)\right)$, is a minimum. The rest of the proof is as above. ■

Let T' be an independent copy of T. Since for $\alpha \ge 1$, $\left(\mathbb{E}|T - \mathbb{E}T|^\alpha\right)^{1/\alpha} \sim \left(\mathbb{E}|T - T'|^\alpha\right)^{1/\alpha}$, similar reasoning gives:

Proposition 4 *For $\alpha \ge 2$ and $q \ge p \ge 1$:*

$$\left(\mathbb{E}|T - \mathbb{E}T|^\alpha\right)^{1/\alpha} \sim \left(\frac{1+2q}{ep}\right)^{q/p} \max\left\{ \left(\frac{\alpha n q^{1/p}}{\sqrt{pq}}\right)^{1/2}, \left(\frac{n q^{1/p}}{\sqrt{pq}}\right)^{1/\alpha} \left(\frac{\alpha}{2}\right)^{q/p} \right\}.$$

We record here two simple corollaries, the first of which was used in the proof of Theorem 2:

Corollary 3 *There is a constant $C = C(p,q)$ such that for all $\lambda \le 1/C$:*

$$\mathbb{E}e^{\lambda T^{p/q}} \le \frac{n^{1-p/q}}{(1 - C\lambda)^{n^{p/q}}}.$$

Proof: Set $m = [n^{p/q}]$. Let ζ be an exponential random variable and ξ be the sum of m i.i.d. copies of ζ. By Proposition 3, for every $k \ge 1$,

$$\mathbb{E}T^{kp/q} \le C^k \max\left\{n^{kp/q}, nk^k\right\},$$

$$\mathbb{E}\xi^k \ge c^k \max\left\{m^k, mk^k\right\} \ge (c')^k \max\left\{n^{kp/q}, n^{p/q}k^k\right\},$$

where, c, C are absolute constants depending on p and q.

It follows that $\mathbb{E}T^{kp/q} \le (C')^k n^{1-p/q}\mathbb{E}\xi^k$. Hence,

$$\mathbb{E}e^{\lambda T^{p/q}} \le n^{1-p/q}\mathbb{E}e^{C'\lambda\xi} \le n^{1-p/q}\left(\mathbb{E}e^{C'\lambda\zeta}\right)^{n^{p/q}},$$

and the result follows. ■

Corollary 4 *There is a constant $C = C(p,q)$ such that for all $\alpha \ge 2$:*

$$\mathbb{E}\left|T - \mathbb{E}T\right|^\alpha \le \left(C \max\left\{n^{1/\alpha}\alpha^{q/p}, n^{1/2}\alpha^{1/2}\right\}\right)^\alpha.$$

To use this inequality we will require a simple numerical result, the proof of which we include for the sake of completeness.

Lemma 11 *For every* $0 < \theta \leq 1$ *and* $a, b > 0$ *:*

$$|a-b|^\theta \leq \max\{1, 2\theta\} \frac{|a-b|}{a^{1-\theta} + b^{1-\theta}}.$$

Proof: We can assume that $a > b$. If $\theta \leq 1/2$ then:

$$(a^{1-\theta} + b^{1-\theta})(a^\theta - b^\theta) = a - b + a^\theta b^\theta (b^{1-2\theta} - a^{1-2\theta}) \leq a - b.$$

If $\theta \geq 1/2$, by the above identity we have to show that:

$$a^\theta b^{1-\theta} - b^\theta a^{1-\theta} \leq (2\theta - 1)(a - b).$$

Putting $t = a/b \geq 1$, we have to show that:

$$f(t) = (2\theta - 1)(t - 1) - t^\theta + t^{1-\theta} \geq 0.$$

Now, $f'(1) = 0$ and $f''(t) = \theta(1-\theta)t^{-\theta-1}(t^{2\theta-1}-1) \geq 0$, so that $f(t) \geq f(1) = 0$. ∎

We can now generalize a result of Schechtman and Zinn [SZ2]:

Theorem 5 *For* $0 < p < q$ *let*

$$\psi(n,t) = \begin{cases} nt, & \text{for } t > n^{-(1-p/q)} \\ n^{(2-p/q)p/q} t^{p/q}, & \text{for } n^{-(3q-p)(q-p)/[q(2q-p)]} < t \leq n^{-(1-p/q)} \\ n^{3-2p/q} t^2, & \text{for } 0 < t \leq n^{-(3q-p)(q-p)/[q(2q-p)]} \end{cases}$$

Then, for some absolute constants $0 < c, C < \infty$*, which depend only on* p *and* q*,*

$$\mu_{p,n}\left(\left| ||x||_q^p - \int_{S(\ell_p^n)} ||y||_q^p d\mu_{p,n}(y) \right| \geq t \right) \leq C e^{-c\psi(n,t)}.$$

Proof: In what follows C, c are constants which depends only on p and q. Fix some $\alpha \geq 2$. Then, by the independence of S and $T/S^{q/p}$:

$$\left(\mathbb{E} \left| \frac{T^{p/q}}{S} - \frac{(\mathbb{E}T)^{p/q}}{\mathbb{E}S} \right|^\alpha \right)^{1/\alpha} = \frac{1}{(\mathbb{E}S^\alpha)^{1/\alpha}} \left(\mathbb{E} \left| T^{p/q} - \frac{(\mathbb{E}T)^{p/q}}{\mathbb{E}S} S \right|^\alpha \right)^{1/\alpha} \leq$$

$$\leq \frac{1}{\mathbb{E}S} \left[\left(\mathbb{E} \left| T^{p/q} - (\mathbb{E}T)^{p/q} \right|^\alpha \right)^{1/\alpha} + \left(\mathbb{E} \left| (\mathbb{E}T)^{p/q} - \frac{(\mathbb{E}T)^{p/q}}{\mathbb{E}S} S \right|^\alpha \right)^{1/\alpha} \right] \leq$$

$$\leq \frac{C}{n}\left[\frac{2}{(\mathbb{E}T)^{1-p/q}}\left(\mathbb{E}|T-\mathbb{E}T|^{\alpha}\right)^{1/\alpha}+\frac{(\mathbb{E}T)^{p/q}}{\mathbb{E}S}\left(|\mathbb{E}S-S|^{\alpha}\right)^{1/\alpha}\right] \leq$$

$$\leq \frac{C}{n^{2-p/q}}\left(\max\{n^{1/2}\alpha^{1/2}, n^{1/\alpha}\alpha^{q/p}\}+\max\{n^{1/2}\alpha^{1/2}, n^{1/\alpha}\alpha\}\right) \leq$$

$$\leq \frac{C}{n^{2-p/q}}\max\{n^{1/2}\alpha^{1/2}, n^{1/\alpha}\alpha^{q/p}\}.$$

For the sake of simplicity put $a = (\mathbb{E}T)^{p/q}/(\mathbb{E}S) = Cn^{-(1-p/q)}$. If we denote by X the random variable $T^{p/q}/S$, and Y is an independent copy of X then:

$$\mathbb{E}|X-\mathbb{E}X|^{\alpha} \leq \mathbb{E}|X-Y|^{\alpha} \leq 2^{\alpha}\mathbb{E}|X-a|^{\alpha}.$$

So that we proved the inequality:

$$\mathbb{E}\left|\frac{T^{p/q}}{S}-\mathbb{E}\left(\frac{T^{p/q}}{S}\right)\right|^{\alpha} \leq \left[\frac{C}{n^{2-p/q}}\max\{n^{1/2}\alpha^{1/2}, n^{1/\alpha}\alpha^{q/p}\}\right]^{\alpha}.$$

If $t \geq C/n^{1-p/q} \geq \mathbb{E}T^{p/q}/(\mathbb{E}S)$ then by Theorem 2:

$$P\left(\left|\frac{T^{p/q}}{S}-\frac{\mathbb{E}T^{p/q}}{\mathbb{E}S}\right| \geq t\right) \leq \mu(||x||_q \geq t^{1/p}) \leq e^{-cnt}.$$

Assume therefore that $t < C/n^{1-p/q}$. Fix some $\alpha > 2$ to be chosen momentarily. Now,

$$P\left(\left|\frac{T^{p/q}}{S}-\mathbb{E}\left(\frac{T^{p/q}}{S}\right)\right| \geq t\right) \leq \frac{1}{t^{\alpha}}\left|\mathbb{E}\frac{T^{p/q}}{S}-\mathbb{E}\left(\frac{T^{p/q}}{S}\right)\right|^{\alpha} \leq$$

$$\leq \left[\frac{K\max\{n^{1/2}\alpha^{1/2}, n^{1/\alpha}\alpha^{q/p}\}}{n^{2-p/q}t}\right]^{\alpha}.$$

Where K is a universal constant.

If $t > n^{-(3q-p)(q-p)/[q(2q-p)]}$ then take $\alpha = (eK)^{-2}n^{(2-p/q)p/q}t^{p/q}$, so that $\alpha > 2$ for n large enough, and:

$$n^{1/\alpha}\alpha^{q/p} \geq \alpha^{1/2}\left(\frac{n^{(2-p/q)p/q}t^{p/q}}{e^2K^2}\right)^{q/p-1/2} \geq$$

$$\geq \frac{\alpha^{1/2}}{(eK)^{2q/p-1}}n^{(2-p/q)(q/p-1/2)p/q}n^{-(3q-p)(q-p)(q/p-1/2)p/[q^2(2q-p)]} =$$

$$= \frac{\alpha^{1/2}n^{1/2}}{(eK)^{2q/p-1}}.$$

So that:

$$P\left(\left|\frac{T^{p/q}}{S}-\mathbb{E}\left(\frac{T^{p/q}}{S}\right)\right|\geq t\right)\leq\left[\frac{K(eK)^{2q/p-1}n^{1/\alpha}\alpha^{q/p}}{n^{2-p/q}t}\right]^{\alpha}=$$

$$=n\left[\frac{K(eK)^{2q/p-1}}{(eK)^{2q/p}}\right]^{(eK)^{-2}n^{(2-p/q)p/q}t^{p/q}}\leq e^{-cn^{(2-p/q)p/q}t^{p/q}}.$$

This proves the required inequality in the range: $n^{-(3q-p)(q-p)/[q(2q-p)]}<t\leq C/n^{1-p/q}$.

Finally, if $t\leq n^{-(3q-p)(q-p)/[q(2q-p)]}$ then take:

$$\alpha=\max\left\{\frac{t^2n^{3-2p/q}}{e^{4q/p}K^2},2\right\}.$$

First, since the only extremum of the function $f(\beta)=n^{1/\beta}\beta^{q/p-1/2}$ is a minimum, we have:

$$f(\alpha)\leq\max\left\{f(2),f\left(\frac{n^{-2(3q-p)(q-p)/[q(2q-p)]}n^{3-2p/q}}{e^{4q/p}K^2}\right)\right\}=$$

$$=\max\left\{2^{q/p-1/2}n^{1/2},\frac{n^{1/2}}{(e^{4q/p}K^2)^{q/p-1/2}}\exp\left(\frac{e^{4q/p}K^2\log n}{n^{p/(2q-p)}}\right)\right\}\leq$$

$$\leq n^{1/2}e^{q/p}\exp\left(\frac{e^{4q/p}K^2\log n}{n^{p/(2q)}}\right).$$

In other words, we have proved that:

$$n^{1/\alpha}\alpha^{q/p}\leq n^{1/2}\alpha^{1/2}e^{q/p}\exp\left(\frac{e^{4q/p}K^2\log n}{n^{p/(2q)}}\right).$$

Hence, assuming that $\alpha>2$ (otherwise the required estimate is trivial):

$$P\left(\left|\frac{T^{p/q}}{S}-\mathbb{E}\left(\frac{T^{p/q}}{S}\right)\right|\geq t\right)\leq\left[\frac{Ke^{q/p}\exp\left(\frac{e^{4q/p}K^2\log n}{n^{p/(2q)}}\right)n^{1/2}\alpha^{1/2}}{n^{2-p/q}t}\right]^{\alpha}=$$

$$=\exp\left[-\left(\frac{q}{pe^{4q/p}K^2}-\frac{\log n}{n^{p/2q}}\right)\psi(n,t)\right].$$

Which gives the required result for n large enough (the result is trivial for bounded n). ■

Remark: Note that for $p=1$, $q=2$, the result coincides with Proposition 5.1 in [SZ2]. Just as remarked there, it is best possible in the first two ranges in the definition of ψ. The proof of this is exactly the same as in [SZ2], and it is based on the fact that using the equivalence in Proposition 4, in the first two ranges all our estimates can be reversed since the estimate for $T^{p/q}$ is asymptotically larger then the estimate for S.

3 Surface measure vs. cone measure on $S(\ell_p^n)$

In this section we will apply the results of the previous sections to study the distance between the surface measure and the cone measure on the sphere of ℓ_p^n. The total variation distance between these two measures was estimated in [NR], where it was shown that it is bounded by $C_p/\sqrt{n}$. Before proceeding to strengthen this result, we will begin by studying the dependence of C_p on p .

We remind some basic facts on the total variation distance. For P, Q probability measures on a measurable space $(\Omega, \mathcal{F})$, the total variation distance between them is defined as $||P - Q|| = 2\sup\{|P(A) - Q(A)| : A \subset \mathcal{F}\}$. If P, Q are absolutely continuous with respect to some reference measure λ, with respective densities f and g, then the total variation distance is known to be equal to $\int_\Omega |f - g| d\lambda$.

Fix some $p \geq 1$. It has been proved in [NR] that:

$$\frac{d\sigma_{p,n}}{d\mu_{p,n}}(x) = C_{p,n} \cdot \left(\sum_{i=1}^{n} |x_i|^{2p-2} \right)^{1/2}.$$

Put $q = 2p - 2$. It has been proved in [NR] that the following estimate holds:

$$||\mu_{p,n} - \sigma_{p,n}|| \leq 2 \cdot \frac{(\mathbb{E}|g|^q)^{1/2}}{\mathbb{E}|g|^{q/2}} \sqrt{1 - \frac{1}{n} + \frac{\mathbb{E}|g|^{2q}}{n(\mathbb{E}|g|^q)^2} - \frac{\mathbb{E}S^{2q/p}}{(\mathbb{E}S^{q/p})^2}}.$$

Note that:

$$\frac{(\mathbb{E}|g|^q)^{1/2}}{\mathbb{E}|g|^{q/2}} = \frac{\sqrt{\Gamma\left(\frac{1+q}{p}\right)\Gamma\left(\frac{1}{p}\right)}}{\Gamma\left(\frac{1+q/2}{p}\right)} = \sqrt{\Gamma\left(2 - \frac{1}{p}\right) p\Gamma\left(1 + \frac{1}{p}\right)} \leq \sqrt{p}.$$

Assume first that $1 < p < 2$. In this case $0 < q < p$ so that by Lemma 5:

$$||\mu_{p,n} - \sigma_{p,n}|| \leq 2\sqrt{p} \cdot \sqrt{1 - \frac{1}{n} + \frac{1}{n} \cdot \left(1 + \frac{q^2}{p} + \frac{2q^2(p-q)^2}{p^2}\right) - \left(1 + \frac{(q/n)^2}{p/n}\right)} =$$

$$= 2\sqrt{p} \cdot \sqrt{\frac{8(p-1)^2(2-p)^2}{np^2}} \le \frac{4\sqrt{2} \cdot (p-1)(2-p)}{\sqrt{n}}.$$

In the case $2 < p < 5/2$, we can apply part $b)$ of Lemma 5 and get similarly:

$$||\mu_{p,n} - \sigma_{p,n}|| \le \frac{C(p-2)}{\sqrt{n}},$$

where C is some universal constant.

It remains to deal with the case $p \to \infty$. The proof of the result in [NR] actually shows that if we denote $W = T/(S^{q/p})$ then:

$$||\mu_{p,n} - \sigma_{p,n}|| = \mathbb{E}\left|\frac{\sqrt{W}}{\mathbb{E}\sqrt{W}} - 1\right| \le$$

$$\le 2\frac{\sqrt{\mathbb{E}W}}{\mathbb{E}\sqrt{W}} \cdot \sqrt{\frac{\mathbb{E}W^2}{(\mathbb{E}W)^2} - 1} \le$$

$$\le 2\frac{\sqrt{\mathbb{E}W^2}}{\mathbb{E}W} \cdot \sqrt{\frac{\mathbb{E}W^2}{(\mathbb{E}W)^2} - 1},$$

where in the last line we used Hölder's inequality.

By the remark following Lemma 13 in Section 4, as long as $p \ge 10$,

$$\frac{\mathbb{E}W^2}{(\mathbb{E}W)^2} \le 1 + \frac{Cpn}{(p+n)^2} \le 1 + \frac{C}{4},$$

for some numerical constant C. Hence

$$||\mu_{p,n} - \sigma_{p,n}|| \le \frac{C'\sqrt{np}}{n+p}.$$

Summarizing, we have proved

Theorem 6 *There is an absolute constant $C > 0$ such that for all $p \ge 1$ and for every n:*

$$||\mu_{p,n} - \sigma_{p,n}|| \le C\left(1 - \frac{1}{p}\right)\left|1 - \frac{2}{p}\right| \cdot \frac{\sqrt{np}}{n+p}.$$

If P and Q are two probability measures on Ω, then the fact that $||P - Q|| < \epsilon$ means that for every measurable $A \subset \Omega$,

$$\left|\frac{Q(A)}{P(A)} - 1\right| < \frac{\epsilon}{2P(A)}.$$

For the cone measure and surface measure on $S(\ell_p^n)$ we have in fact a much stronger inequality:

Theorem 7 *Assume that $1 < p < 2$. There is an absolute constant $C > 0$ such that for every measurable $A \subset S(\ell_p^n)$:*

$$\left|\frac{\sigma_{p,n}(A)}{\mu_{p,n}(A)} - 1\right| \le \frac{C}{\sqrt{n}} \cdot \sqrt{\log\left(\frac{100}{\mu_{p,n}(A)}\right)}.$$

Proof: Let $q = 2p-2$ and define $W = T^{1/2}/(S^{q/(2p)})$. If V is an independent copy of W then for every $\lambda > 0$:

$$\mathbb{E}e^{\lambda|W-\mathbb{E}W|} \le \mathbb{E}e^{\lambda|W-V|} \le \mathbb{E}e^{\lambda|W-\sqrt{\mathbb{E}W^2}|+\lambda|V-\sqrt{\mathbb{E}V^2}|} \le$$

$$\le \mathbb{E}e^{2\lambda|W-\sqrt{\mathbb{E}W^2}|} = \mathbb{E}\exp\left(\frac{2\lambda|W^2-\mathbb{E}W^2|}{W+\sqrt{\mathbb{E}W^2}}\right) \le$$

$$\le \mathbb{E}\exp\left(\frac{2\lambda|W^2-\mathbb{E}W^2|}{\sqrt{\mathbb{E}W^2}}\right) =$$

$$= 1 + 2\lambda\sqrt{\mathbb{E}W^2}\int_0^\infty P\left(|W^2-\mathbb{E}W^2| \ge x\mathbb{E}W^2\right) \cdot e^{2\lambda x\sqrt{\mathbb{E}W^2}}\,dx.$$

Note that $W^2 = T/(S^{q/p})$, so that by the remark following Theorem 3 there is a universal constant $C > 0$ such that $P(|W^2 - \mathbb{E}W^2| > x\mathbb{E}W^2) \le 10e^{-Cnx^2}$. A simple calculation now gives:

$$\mathbb{E}e^{\lambda|W-\mathbb{E}W|} \le 100\exp\left(\frac{2\lambda^2\mathbb{E}W^2}{nC}\right).$$

It is easy to see that $\mathbb{E}W^2 \le 4(\mathbb{E}W)^2$, hence if we define for $x \in S(\ell_p^n)$

$$f(x) = \frac{d\sigma_{p,n}}{d\mu_{p,n}}(x) = \left(\int_{S(\ell_p^n)} ||y||_q^{q/2}\,d\mu_{p,n}(y)\right)^{-1} \cdot ||x||_q^{q/2},$$

then we have shown that there is an absolute constant c such that for all $\lambda > 0$:

$$\int_{S(\ell_p^n)} e^{|f-1|}\,d\mu_{p,n} \le 100e^{c\lambda^2/n}.$$

Finally, for $A \subset S(\ell_p^n)$ and every $\lambda > 0$:

$$|\sigma_{p,n}(A) - \mu_{p,n}(A)| \le \int_A |f-1|\,d\mu_{p,n} = \frac{\mu_{p,n}(A)}{\lambda}\int_A \log(e^{\lambda|f-1|})\frac{d\mu_{p,n}}{\mu_{p,n}(A)} \le$$

$$\le \frac{\mu_{p,n}(A)}{\lambda}\log\left(\frac{1}{\mu_{p,n}(A)}\int_{S(\ell_p^n)} e^{\lambda|f-1|}\,d\mu_{p,n}\right) \le$$

$$\leq \frac{\mu_{p,n}(A)}{\lambda} \log\left(\frac{100 e^{c\lambda^2/n}}{\mu_{p,n}(A)}\right).$$

Choosing $\lambda = \sqrt{\frac{n}{c} \cdot \log\left(\frac{100}{\mu_{p,n}(A)}\right)}$ gives the required result.

■

For $p > 2$ we have:

Theorem 8 *Assume that $2 < p < \infty$. For every measurable $A \subset S(\ell_p^n)$:*

$$\left|\frac{\sigma_{p,n}(A)}{\mu_{p,n}(A)} - 1\right| \leq \frac{C}{\sqrt{n}} \cdot \left[\log\left(\frac{C}{\mu_{p,n}(A)}\right)\right]^{1-\frac{1}{p}}.$$

Here $C = C(p)$ is an absolute constant (which may depend on p).

Proof: As before, set $q = 2p-2 > p$ and $W = T^{1/2}(S^{q/(2p)})$. In what follows C denotes a constant depending on p, which may change in each particular occurrence. By the previous methods, it is easy to see that

$$\mathbb{E}W \leq \frac{C}{n^{\frac{1}{2}(\frac{q}{p}-1)}} = \frac{C}{n^{1/2-1/p}} \quad \text{and} \quad \mathbb{E}W^{2p/q} \geq \frac{C}{n^{1-p/q}}.$$

Since $q/(2p) < 1$,

$$\left|W - (\mathbb{E}W^{2p/q})^{q/(2p)}\right| \leq \min\left\{\frac{2|W^{2p/q} - \mathbb{E}W^{2p/q}|}{(\mathbb{E}W^{2p/q})^{1-q/(2p)}}, |W^{2p/q} - \mathbb{E}W^{2p/q}|^{q/(2p)}\right\} \leq$$

$$\leq \min\left\{Cn^{(1-p/q)(1-q/2p)}|W^{2p/q} - \mathbb{E}W^{2p/q}|, |W^{2p/q} - \mathbb{E}W^{2p/q}|^{q/(2p)}\right\}.$$

Hence, since $q = 2p-2$ we get that for $t < 1$:

$$P\left(|W - (\mathbb{E}W^{2p/q})^{2/(2p)}| \geq \frac{t}{n^{1/2-1/p}}\right) \leq$$

$$\leq P\left(|W^{2p/q} - \mathbb{E}W^{2p/q}| \geq Ctn^{-(p-2)/(2p-2)}\right),$$

and for $t \geq 1$:

$$P\left(|W - (\mathbb{E}W^{2p/q})^{2/(2p)}| \geq \frac{t}{n^{1/2-1/p}}\right) \leq$$

$$\leq P\left(|W^{2p/q} - \mathbb{E}W^{2p/q}| \geq Ct^{p/(p-1)}n^{-(p-2)/(2p-2)}\right).$$

Note that $W^{2p/q} = T^{p/q}/S$, so that by Theorem 5 we get:

$$P\left(|W - (\mathbb{E}W^{2p/q})^{2/(2p)}| \geq \frac{t}{n^{1/2-1/p}}\right) \leq Ce^{-c\varphi(n,t)},$$

where:

$$\varphi(n,t) = \begin{cases} n^{p/(2p-2)} t^{p/(p-1)}, & \text{for } t > 1 \\ n^{p/(2p-2)} t^{p/(2p-2)}, & \text{for } n^{-(p-2)/(3p-4)} < t \le 1 \\ nt^2, & \text{for } 0 < t \le n^{-(p-2)/(3p-4)} \end{cases}$$

By standard arguments (since in the proof of Theorem 5 we have only used moment estimates), this implies:

$$P\left(\left|\frac{W}{\mathbb{E}W} - 1\right| \ge t\right) \le P\left(|W - \mathbb{E}W| \ge \frac{Ct}{n^{1/2-1/p}}\right) \le Ce^{-c\varphi(n,t)}.$$

Hence, if f is as in the proof of Theorem 7 then for every $\lambda > 0$:

$$\int_{S(\ell_p^n)} e^{\lambda|f-1|} d\mu_{p,n} = 1 + \lambda \int_0^\infty P\left(\left|\frac{W}{\mathbb{E}W} - 1\right| \ge t\right) \cdot e^{\lambda t} dt \le$$

$$\le 1 + C\lambda \int_0^\infty \exp\left(\lambda t - cn^{p/(2p-2)} t^{p/(p-1)}\right) dt =$$

$$= 1 + \frac{C\lambda}{\sqrt{n}} \int_0^\infty \exp\left(\frac{\lambda u}{\sqrt{n}} - cu^{p/(p-1)}\right) du \le$$

$$\le 1 + \frac{C\lambda}{\sqrt{n}} \int_0^\infty \exp\left(\frac{1}{p} \cdot \frac{\lambda^p}{c^{p-1} n^{p/2}} + \frac{p-1}{p} cu^{p/p-1} - cu^{p/(p-1)}\right) du \le$$

$$\le C \exp\left[c\left(\frac{\lambda}{\sqrt{n}}\right)^p\right],$$

where we have used the fact that for $\alpha, \beta > 0$, $\alpha\beta \le \frac{\alpha^p}{p} + \frac{(p-1)\beta^{p/(p-1)}}{p}$. The rest of the proof is now just as in the proof of Theorem 7.

■

Remark: Theorem 8 is tight in the following sense. Using the preceding notation, fix some $\epsilon > 0$, and A be the set $\{W \ge (1+\epsilon)\mathbb{E}W\}$. Clearly:

$$\sigma_{p,n}(A) = \int_A W d\mu_{p,n} \ge (1+\epsilon)\mu_{p,n}(A).$$

Since in the range $t \ge n^{-(1-p/q)}$, the statement of Theorem 5 is tight, there are constants c_1, C_1, c_2, C_2 (which depend on p) such that:

$$c_1 \exp\left(-C_1 n^{p/(2p-2)} \epsilon^{p/(p-1)}\right) \le \mu_{p,n}(A) \le C_2 \exp\left(-c_2 n^{p/(2p-2)} \epsilon^{p/(p-1)}\right).$$

Now,

$$\frac{\sigma_{p,n}(A)}{\mu_{p,n}(A)} - 1 \ge \epsilon \ge \frac{1}{C_1^{1-1/p}\sqrt{n}} \left[\log\left(\frac{c_2}{\mu_{p,n}(A)}\right)\right]^{1-\frac{1}{p}}.$$

We now show how a simple application of Theorems 7 and 8 gives the inequality for concentration functions mentioned in the introduction. Recall that the concentration function of a probability metric space (X, d, ν) is defined by:

$$I_\nu^d(\epsilon) = \sup\left\{\nu(X \setminus A_\epsilon);\ \nu(A) \geq \frac{1}{2}\right\},$$

where $A_\epsilon = \{x;\ d(x, A) < \epsilon\}$.

It is easy to see that for any Borel $A \subset X$, $\nu(A_\epsilon) \leq \frac{2I_\nu^d(\epsilon/2)}{\nu(A)}$. Indeed, when $\nu(A) > 0$, define $\delta = (I_\nu^d)^{-1}(\nu(A)/2)$. Then $\nu(A_\delta) \geq 1/2$, since otherwise, using the fact that $(X \setminus A_\delta)_\delta \cap A = \emptyset$ we get:

$$1 - \nu(A) \geq \nu\big((X \setminus A_\delta)_\delta\big) \geq 1 - I_\nu^d(\delta) = 1 - \frac{\nu(A)}{2},$$

which is a contradiction. Hence, for $\epsilon \geq 2\delta$, since $(A_\delta)_{\epsilon/2} \subset A_\epsilon$, $\nu(X \setminus A_\epsilon) \leq I_\nu^d(\epsilon/2)$. For $\epsilon \leq 2\delta$ our estimate is trivial, since $\nu(X \setminus A_\epsilon) \leq 1 \leq \frac{I_\nu^d(\epsilon/2)}{I_\nu^d(\delta)} = \frac{2I_\nu^d(\epsilon/2)}{\nu(A)}$.

Returning to our setting, let d be a metric on $S(\ell_p^n)$ which induces the standard topology. Fix a Borel $A \subset S(\ell_p^n)$ with $\sigma_{p,n}(A) \geq 1/2$. Theorems 7 and 8 imply that:

$$\frac{1}{2} \leq \sigma_{p,n}(A) \leq C\mu_{p,n}(A) \log\left(\frac{C}{\mu_{p,n}(A)}\right),$$

so that $\mu_{p,n}(A) \geq c$, for some absolute constant $c = c(p)$. By our previous remarks, $\mu_{p,n}(X \setminus A_\epsilon) \leq CI_{\mu_{p,n}}^d(\epsilon/2)$, and another application of Theorems 7 and 8 gives

$$\sigma_{p,n}(X \setminus A_\epsilon) \leq CI_{\mu_{p,n}}^d\left(\frac{\epsilon}{2}\right)\left[1 + \frac{1}{\sqrt{n}}\left|\log I_{\mu_{p,n}}^d\left(\frac{\epsilon}{2}\right)\right|^{1-\min\{1/2,1/p\}}\right].$$

This proves the following:

Corollary 5 *For every $p \geq 1$ there is a constant $C = C(p)$ such that if d is a metric on $S(\ell_p^n)$ which induces the standard topology, then for every $\epsilon > 0$:*

$$I_{\sigma_{p,n}}^d(\epsilon) \leq CI_{\mu_{p,n}}^d\left(\frac{\epsilon}{2}\right)\left[1 + \frac{1}{\sqrt{n}}\left|\log I_{\mu_{p,n}}^d\left(\frac{\epsilon}{2}\right)\right|^{1-\min\{1/2,1/p\}}\right].$$

4 Lower bounds and further remarks

In this section we will prove several estimates which show that some of our preceding results are tight. We will begin with:

Lemma 12 *Assume that $p \geq 1$ and $0 < q \leq p$. Then for $n \geq 16(1+q)$,*

$$\int_{S(\ell_p^n)} \left(||x||_q^q - \int_{S(\ell_p^n)} ||y||_q^q d\mu_{p,n}(y) \right)^2 d\mu_{p,n}(x) \geq$$

$$\geq \frac{q^2(p-q)^2}{10np^2(1+q)} \left(\int_{S(\ell_p^n)} ||y||_q^q d\mu_{p,n}(y) \right)^2 .$$

Proof: As we have done many times before,

$$\int_{S(\ell_p^n)} \left(||x||_q^q - \int_{S(\ell_p^n)} ||y||_q^q d\mu_{p,n}(y) \right)^2 d\mu_{p,n}(x) = \mathbb{E}\left[\frac{T}{S^{q/p}} - \mathbb{E}\left(\frac{T}{S^{q/p}} \right) \right]^2 =$$

$$= \frac{\mathbb{E}T^2}{\mathbb{E}S^{2q/p}} - \frac{(\mathbb{E}T)^2}{(\mathbb{E}S^{q/p})^2} = \frac{(\mathbb{E}T)^2}{\mathbb{E}S^{2q/p}} \left[\frac{n\mathbb{E}|g|^{2q} + n(n-1)(\mathbb{E}|g|^q)^2}{n^2(\mathbb{E}|g|^q)^2} - \frac{\mathbb{E}S^{2q/p}}{(\mathbb{E}S^{q/p})^2} \right].$$

If we put $x = \frac{q(p-q)}{p(q+1)}$ then $x < 1$, so that $e^{-x} \geq \left(1 - \frac{x}{2}\right)^2 = 1 - x + \frac{x^2}{4}$. Hence, by Lemma 5,

$$1 - \frac{1}{n} + \frac{\mathbb{E}|g|^{2q}}{n(\mathbb{E}|g|^q)^2} - \frac{\mathbb{E}S^{2q/p}}{(\mathbb{E}S^{q/p})^2} \geq$$

$$\geq \left[1 - \frac{1}{n} + \frac{1+q}{n} \exp\left(-\frac{q(p-q)}{p(q+1)} \right) \right] - \left[1 + \frac{(q/n)^2}{p/n} + \frac{2(q/n)^2(p/n - q/n)^2}{(p/n)^2} \right] \geq$$

$$\geq 1 - \frac{1}{n} + \frac{1+q}{n} \left(1 - \frac{q(p-q)}{p(q+1)} + \frac{q^2(p-q)^2}{4p^2(q+1)^2} \right) - 1 - \frac{q^2}{np} - \frac{2q^2(p-q)^2}{n^2p^2} =$$

$$= \frac{q^2(p-q)^2}{np^2} \left(\frac{1}{4(1+q)} - \frac{2}{n} \right) \geq \frac{q^2(p-q)^2}{8np^2(1+q)}.$$

Finally, using Lemma 10 we get:

$$\int_{S(\ell_p^n)} \left(||x||_q^q - \int_{S(\ell_p^n)} ||y||_q^q d\mu_{p,n}(y) \right)^2 d\mu_{p,n}(x) \geq$$

$$\geq \frac{q^2(p-q)^2}{8np^2(1+q)} \cdot \frac{(\mathbb{E}S^{q/p})^2}{\mathbb{E}S^{2q/p}} \cdot \left(\frac{\mathbb{E}T}{\mathbb{E}S^{q/p}} \right)^2 \geq$$

$$\geq \frac{q^2(p-q)^2}{8np^2(1+q)(1+q/n)^2} \cdot \left(\frac{\mathbb{E}T}{\mathbb{E}S^{q/p}} \right)^2 \geq$$

$$\geq \frac{q^2(p-q)^2}{10np^2(1+q)} \left(\int_{S(\ell_p^n)} ||y||_q^q d\mu_{p,n}(y) \right)^2 .$$

■

Proposition 5 *For every* $1 < p < 2$ *there is a constant* $C = C(p) > 0$ *such that for every* n, $||\mu_{p,n} - \sigma_{p,n}|| \geq \frac{C}{\sqrt{n}}$.

Proof: Define $W = T/S^{q/p}$ where $q = 2p - 2$. We have to show that

$$\mathbb{E}\left|\frac{\sqrt{W}}{\mathbb{E}\sqrt{W}} - 1\right| \geq \frac{C}{\sqrt{n}}.$$

If V is an independent copy of W then:

$$\mathbb{E}|W - \mathbb{E}W| \leq \mathbb{E}|W - V| = \mathbb{E}\left[\left(\sqrt{W} + \sqrt{V}\right)\left|\sqrt{W} - \sqrt{V}\right|\right] \leq$$

$$\leq \sqrt{\mathbb{E}\left(\sqrt{W} + \sqrt{V}\right)^2}\sqrt{\mathbb{E}\left|\sqrt{W} - \sqrt{V}\right|^2} =$$

$$= 2\sqrt{\mathbb{E}W + \left(\mathbb{E}\sqrt{W}\right)^2}\sqrt{\mathbb{E}W - \left(\mathbb{E}\sqrt{W}\right)^2} \leq$$

$$\leq 2^{3/2}\sqrt{\mathbb{E}W}\sqrt{\mathbb{E}\left(\sqrt{W} - \mathbb{E}\sqrt{W}\right)^2}.$$

By Theorem 3, there are absolute constants $C, c > 0$ such that:

$$\mathbb{E}\left|\frac{W}{\mathbb{E}W} - 1\right|^4 = 4\int_0^\infty t^3 P\left(\left|\frac{W}{\mathbb{E}W} - 1\right| \geq t\right) \leq$$

$$\leq 4\int_0^\infty t^3 \cdot Ce^{-cnt^2}dt \leq \frac{C'}{n^2}.$$

By Lemma 12 there is a constant $C = C(p)$ such that

$$\mathbb{E}\left(\frac{W}{\mathbb{E}W} - 1\right)^2 \geq \frac{C}{n},$$

and using Hölder's inequality we deduce that:

$$\mathbb{E}\,|W - \mathbb{E}W| \geq \frac{\left[\mathbb{E}(W - \mathbb{E}W)^2\right]^{3/2}}{\left[\mathbb{E}(W - \mathbb{E}W)^4\right]^{1/2}} \geq \frac{C}{\sqrt{n}} \cdot \mathbb{E}W,$$

and it follows that:

$$\mathbb{E}\left(\sqrt{W} - \mathbb{E}\sqrt{W}\right)^2 \geq \frac{C}{n} \cdot \mathbb{E}W.$$

As we have done many times before:

$$\mathbb{E}\left(\sqrt{W} - \mathbb{E}\sqrt{W}\right)^4 \leq 8\mathbb{E}\left(\sqrt{W} - \sqrt{\mathbb{E}W}\right)^4 \leq$$

$$\leq \frac{8}{(\mathbb{E}W)^2}\mathbb{E}(W-\mathbb{E}W)^4 \leq \frac{C}{n^2}\cdot(\mathbb{E}W)^2.$$

Finally, another application of Hölder's inequality gives:

$$\mathbb{E}\left|\sqrt{W}-\mathbb{E}\sqrt{W}\right| \geq \frac{\left[\mathbb{E}\left(\sqrt{W}-\mathbb{E}\sqrt{W}\right)^2\right]^{3/2}}{\left[\mathbb{E}\left(\sqrt{W}-\mathbb{E}\sqrt{W}\right)^4\right]^{1/2}} \geq$$

$$\geq \frac{C}{\sqrt{n}}\cdot\sqrt{\mathbb{E}W} \geq \frac{C}{\sqrt{n}}\cdot\mathbb{E}\sqrt{W}.$$

■

For the case $p>2$ we will need the following:

Lemma 13 *There are absolute constants $c, C>0$ such that for every $p>2$ and $n>C$:*

$$\int_{S(\ell_p^n)}\left(||x||_{2p-2}^{2p-2}-\int_{S(\ell_p^n)}||y||_{2p-2}^{2p-2}d\mu_{p,n}(y)\right)^2 d\mu_{p,n}(x) \geq$$

$$\geq \frac{c(p-2)^2 n}{p(n+p)^2}\left(\int_{S(\ell_p^n)}||y||_{2p-2}^{2p-2}d\mu_{p,n}(y)\right)^2.$$

Proof: Arguing the same as in Lemma 12 we get:

$$\mathbb{E}\left|\frac{T/(S^{q/p})}{\mathbb{E}(T/(S^{q/p})^2)}-1\right|^2 =$$

$$= \left[1-\frac{1}{n}+\frac{\Gamma\left(4-\frac{3}{p}\right)\Gamma\left(\frac{1}{p}\right)}{n\Gamma\left(2-\frac{1}{p}\right)^2}\right]\cdot\frac{\Gamma\left(\frac{n-2}{p}+2\right)^2}{\Gamma\left(\frac{n-4}{p}+4\right)\Gamma\left(\frac{n}{p}\right)}-1.$$

Just as in the proof of Lemma 12, an application of Lemma 5 gives the result for bounded p, say $p\leq 10$.

Assume that $p>10$. Since for $x>0$ and $0<y<1$, $\Gamma(x+y)\leq x^y\Gamma(x)$, the following inequalities hold:

$$\frac{\Gamma\left(4-\frac{3}{p}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(2-\frac{1}{p}\right)^2} \geq \frac{\Gamma\left(4+\frac{1}{p}\right)\Gamma\left(\frac{1}{p}\right)}{\left(4-\frac{3}{p}\right)^{4/p}\Gamma\left(2+\frac{1}{p}\right)^2} =$$

$$= \frac{\left(3+\frac{1}{p}\right)\left(2+\frac{1}{p}\right)\left(1+\frac{1}{p}\right)\frac{1}{p}}{\left(4-\frac{3}{p}\right)^{4/p}\left(1+\frac{1}{p}\right)^2\frac{1}{p^2}} \geq \left(1-\frac{8}{p}\right)\cdot\frac{(3p+1)(2p+1)}{p+1},$$

and using also Lemma 7

$$\frac{\Gamma\left(\frac{n-2}{p}+2\right)^2}{\Gamma\left(\frac{n-4}{p}+4\right)\Gamma\left(\frac{n}{p}\right)} \geq \frac{\Gamma\left(\frac{n}{p}+2\right)^2\left(\frac{n-4}{p}+4\right)^{4/p}}{\left(\frac{n-2}{p}+2\right)^{4/p}\left(1+\frac{4/p}{(n-4)/p+4}\right)\Gamma\left(\frac{n}{p}+4\right)\Gamma\left(\frac{n}{p}\right)} \geq$$

$$\geq \left(1-\frac{2}{n+p}\right)\frac{n(p+n)}{(n+3p)(n+2p)}.$$

Plugging these estimates into what we derived before, it is easy to verify the required inequality. ■

Remark: The above proof actually shows that for $p \geq 10$

$$\mathbb{E}\left|\frac{T/(S^{q/p})}{\mathbb{E}(T/(S^{q/p})^2)}-1\right|^2 \sim \frac{np}{(n+p)^2},$$

where the equivalence is up to numerical constants.

Using Lemma 13, and arguing just as in the proof of Proposition 5 we get:

Theorem 9 $||\mu_{p,n}-\sigma_{p,n}|| \sim_p 1/\sqrt{n}$, *where* $\sim_p$ *means equivalence up to constants which may depend on* p.

We will end by stating some conjectures and open problems. It is an unfortunate fact that in the concentration inequalities that appear in the literature, the estimates become worse as $q \to p$. This absurdity seems to be a fundamental weakness in the known techniques. It requires considerable effort to deal with the limiting cases; we have managed to deal with the case $q \to 1$ in Theorem 3 and we partially deal with the case $q \to p^-$ in Theorem 4. We conjecture that the following improvement of Theorem 4 holds:

Conjecture 1: *For every* $0 < q < p$ *and* $t > 0$*:*

$$\mu_{p,n}\left(\left|||x||_q^q - \int_{S(\ell_p^n)}||y||_q^q d\mu_{p,n}(y)\right| \geq t\int_{S(\ell_p^n)}||y||_q^q d\mu_{p,n}(y)\right) \leq$$

$$\leq C\exp\left(-cn\min\left\{\left(\frac{t}{q(p-q)}\right)^2, \frac{t}{q(p-q)}\right\}\right).$$

As in the proof of Proposition 12, this would imply that for $1 < p < 2$, $||\mu_{p,n} - \sigma_{p,n}||$ is equivalent, up to universal constants, to $(p-1)(2-p)/\sqrt{n}$. Our proof gives the required behavior when $q \to 1$, and by Theorem 4 we also know, up to a $\log n$ factor, that this is the behavior for $q \to p^-$.

For general p we conjecture the following:

Conjecture 2:

$$||\mu_{p,n} - \sigma_{p,n}|| \sim \left(1 - \frac{1}{p}\right) \left|1 - \frac{2}{p}\right| \cdot \frac{\sqrt{np}}{n+p}.$$

Our proof shows that this equivalence is true if we replace the total variation distance by $\left(\int (f-1)^2 d\mu_{p,n}\right)^{1/2}$, where $f = \frac{d\mu_{p,n}}{d\sigma_{p,n}}$. To prove this conjecture, using the method of Proposition 12, would require obtaining exact constants in Theorem 5, which seems to be a difficult problem. Such exact deviation inequalities are also required to prove the following:

Conjecture 3: *There is an absolute constant $C > 0$ such that for every Borel $A \subset S(\ell_p^n)$:*

$$\left|\frac{\sigma_{p,n}(A)}{\mu_{p,n}(A)} - 1\right| \leq C \left(1 - \frac{1}{p}\right) \left|1 - \frac{2}{p}\right| \cdot \frac{\sqrt{np}}{n+p} \left[\log\left(\frac{C}{\mu_{p,n}(A)}\right)\right]^{1-\min\{1/2,1/p\}}.$$

Theorem 3 gives the above behavior for $p \to 1$, and up to a $\log n$ factor, this inequality holds also for $p \to 2^-$, by Theorem 4.

The question whether the results of Theorem 3, and the last range of the definition of ψ in Theorem 5, are tight, is also interesting. Since the moment estimates we derived in these cases coincide asymptoticly for T and $S^{q/p}$, it is unclear how to reverse our argument as in the remark following Theorem 5. Even precise results such as in Propositions 3 and 4 seem insufficient. A proof that Theorem 5 is tight would imply also that Theorem 7 is tight, as in the Remark following Theorem 8.

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A. Naor,
Department of Mathematics,
Hebrew University, Givaat-Ram,
Jerusalem, Israel.
naor@math.huji.ac.il